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CONVOLUTION QUADRATURE REVISITED ⋆

Dedicated to Michel Crouzeix and Syvert Nørsett
on the occasion of their sixtieth birthdays

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Abstract.

This article reviews convolution quadrature and its uses, extends the known approximation results for the case of sectorial Laplace transforms to finite-part convolutions with non-integrable kernel, and gives new, unified proofs of the optimal error bounds for both locally integrable and non-integrable convolution kernels.

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1 Introduction.

1.1 Convolution quadrature.

A convolution quadrature approximates the continuous convolution

$$\int_0^t f(\tau)\, g(t-\tau)\, \mathrm{d}\tau, \quad t > 0, \tag{1.1}$$

by a discrete convolution with a step size $h > 0$,

$$\sum_{0 \le jh \le t} \omega_j\, g(t - jh), \quad t = h, 2h, 3h, \ldots, \tag{1.2}$$

where the convolution quadrature weights ω_j are determined from their generating power series as

$$\sum_{j=0}^{\infty} \omega_j \zeta^j = F\left(\frac{\delta(\zeta)}{h}\right). \tag{1.3}$$

Here $F(s)$ is the Laplace transform of $f(t)$, and $\delta(\zeta)$ is a given rational function, chosen as the quotient of the generating polynomials of a linear multistep method.

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Such convolution quadratures were proposed, implemented and analyzed twenty years ago in my report [23], later published in [24, 25]. A forerunner to these methods appears in the (non-numerical) analytical work of Post [39] from 1930, which can be interpreted as dealing with the limit for $h \to 0$ of (1.2)–(1.3) for the backward Euler method, i.e., for $\delta(\zeta) = 1 - \zeta$. In an applied, empirical context of control engineering there is Tustin's digital filter [52], proposed in 1947, which corresponds to the choice of the trapezoidal rule, $\delta(\zeta) = 2(1-\zeta)/(1+\zeta)$. In numerical analysis, the interpretation of stable linear multistep methods as convolution quadrature for pure integration (i.e., $f(t) = 1$) with weights given by (1.3) for $F(s) = s^{-1}$ was put forward in [37].

A complete error analysis of (1.2)–(1.3) for the case $F(s) = s^{-\mu}$ with real μ, which is known as fractional integration and differentiation, was given in [19]. That paper prepared the ground for the extension to general Laplace transforms $F(s)$, analyzed in [24] for $F(s)$ which are analytic and polynomially bounded outside a sector with an acute angle to the negative real axis, and in [27, 28] for $F(s)$ which are polynomially bounded only in a half-plane. An alternative approach to the error analysis for the sectorial case was presented in [7]. The general result is that a pth order multistep method with appropriate stability properties, such as $A(\alpha)$-stability, gives a convolution quadrature approximation of the full order p provided that sufficiently many derivatives of g vanish at 0. The above-mentioned papers also describe modifications of (1.2) by correction terms involving a few grid values of g near 0, which improve the accuracy to the full order for integrands g that are not small near 0. Convolution quadrature based on Runge–Kutta instead of linear multistep methods is proposed and studied in [29]. There, the basic formula is still (1.3) but $\delta(\zeta)$ becomes a low-dimensional matrix-valued function instead of a scalar function.

Attractive features of convolution quadratures are that they work well in the following situations:

- singular kernels $f(t)$ [19, 25],
- kernels with multiple time scales [25],
- highly oscillatory kernels [27],
- only the Laplace transform $F(s)$, but not the convolution kernel $f(t)$ is known analytically [25, 28, 45, 46].

The last item is obvious, but of importance in many applications where the model equations are derived via a frequency-domain fundamental solution or transfer function, whereas the time-domain fundamental solution or impulse response is not available.

From the beginning, a main motivation for considering these convolution quadrature methods was that they enjoy excellent stability properties when used for the discretization of integral equations or integro-differential equations of convolution type, in a way often strikingly opposed to standard quadrature formulas using values of $f(t)$ or product integration formulas using moments of $f(t)$ over short intervals. The stability aspect of convolution quadrature methods was emphasized in [20, 22] and [21] for Abel–Volterra integral equations of the second and first kind, respectively, for a large class of nonlinear convolution equations in [8], for Wiener–Hopf integral equations in [9, 25].

Further developments of convolution quadrature came up in their application to integro-partial differential equations [40, 18, 34, 35, 53–55, 3, 4] and further Volterra integral and integro-differential equations [13, 27, 17, 6, 38], to non-reflecting boundary conditions [41, 42] and to boundary integral equations for time-dependent problems [28, 33]. The latter class of applications made these methods enter the mechanical engineering literature in the late 1990s [46, 12], where they have since been successfully applied to problems of wave propagation in viscoelastic and poroelastic continua [43–45, 47, 48, 50, 15, 16] and of dynamic crack analysis [56–60, 51].

On the other hand, convolution quadrature also turned out to be an extremely useful theoretical tool in analyzing standard numerical time discretization methods (linear multistep methods, Runge–Kutta methods) for stiff ordinary differential equations [26, 14] as well as for parabolic partial differential equations [26, 29–32] and hyperbolic initial-boundary value problems [28]. This is because convolution quadrature provides an appropriate framework for the discrete variation-of-constants formula needed in studying these problems.

1.2 Discretized operational calculus.

To a large extent, the good stability properties of the convolution quadrature discretizations rely on a simple operational relation. To formulate this relation, it is convenient to introduce the operational calculus *notation*, cf. [11, 36, 49], which for $F(s)$ denoting again the Laplace transform of the convolution kernel $f(t)$ and with ∂ symbolizing differentiation sets

$$F(\partial)g(t) := \int_0^t f(t)\, g(t-\tau)\, \mathrm{d}\tau, \quad t > 0. \tag{1.4}$$

This notation is motivated by the facts that then $\partial^{-1}g(t) = \int_0^t g(\tau)\,\mathrm{d}\tau$ and, by the associativity of convolution,

$$F_2(\partial)F_1(\partial)g(t) = (F_2F_1)(\partial)g(t). \tag{1.5}$$

From the very construction of the convolution quadrature (1.2)–(1.3), we have for

$$F(\partial_h)g(t) := \sum_{0\le jh\le t} \omega_j\, g(t-jh), \quad t > 0, \tag{1.6}$$

that $\partial_h^{-1}g(t)$ is the result of the underlying multistep method applied to the differential equation $y' = g(t)$ (with the special choice of starting values $y_j = 0$ for $j < 0$), and again by (1.3) and the associativity of convolution,

$$F_2(\partial_h)F_1(\partial_h)g(t) = (F_2F_1)(\partial_h)g(t). \tag{1.7}$$

This is a basic relation which is at the heart of stability of convolution quadrature discretizations of integral equations [8, 21, 28, 33], together with the fact that the range of the generating function of the weights (1.3) for $|\zeta| < 1$ is in the range of the Laplace transform $F(s)$. The relation (1.7) does not hold for more traditional discretizations based on values or short-time moments of $f(t)$. The convolution quadrature method is therefore sometimes called *operational quadrature method*.

The operational notation is useful in that it emphasizes the role of $F(s)$ rather than the convolution kernel $f(t)$ and suggests representations and interpretations that would not appear obvious otherwise. For example, the error bounds in [24] are based on comparing

$$F(\partial_h)g(t) = \frac{1}{2\pi i}\int_\Gamma F(\lambda)(\partial_h - \lambda)^{-1} g(t)\,\mathrm{d}\lambda$$

for a suitable complex contour Γ with the analogous formula for the continuous case (i.e., with ∂ in place of ∂_h), and the error analysis in [27] is based on

$$F(\partial_h)g(t) = \int_0^\infty f(\tau)\mathrm{e}^{-\tau\partial_h} g(t)\,\mathrm{d}\tau.$$

1.3 Scope of the present paper.

This paper closes a gap that was left in the theory of [24] in the case of finite-part convolution integrals (or in other terms, of derivatives of convolution integrals). The motivation for this extension of the theory comes from recent work on nonsmooth-data error bounds for discretizations of integro-partial differential equations [5], where such results are needed. The results are stated in Section 2 and proved in Sections 3 and 4.

2 Statement of results.

The framework is that of a *sectorial* Laplace transform $F(s)$:

(2.1) $F(s)$ is analytic in a sector $|\arg(s-c)| < \pi - \varphi$ with $\varphi < \frac{1}{2}\pi$ and real c, and $|F(s)| \le M|s|^{-\mu}$ for some real μ and M.

The inverse Laplace transform is then given by

$$f(t) = \frac{1}{2\pi i}\int_\Gamma F(\lambda)\,\mathrm{e}^{\lambda t}\,\mathrm{d}\lambda, \quad t > 0, \tag{2.2}$$

with Γ a contour in the sector of analyticity, parallel to its boundary and oriented with increasing imaginary part. The function $f(t)$ is analytic in $t > 0$ and satisfies

$$|f(t)| \le C\,t^{\mu-1}\,\mathrm{e}^{ct}, \quad t > 0, \tag{2.3}$$

and is therefore locally integrable if $\mu > 0$. For $\mu \le 0$ the convolution (1.1) is interpreted as a Hadamard finite-part integral, or equivalently for $\widetilde{F}(s) = F(s)s^{-k}$ with the integer k chosen such that $\widetilde{\mu} = \mu + k > 0$, as the derivative

$$F(\partial)g(t) = \partial^k \widetilde{F}(\partial)g(t) = \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^k \int_0^t \widetilde{f}(\tau)\,g(t-\tau)\,\mathrm{d}\tau \tag{2.4}$$

for k-times continuously differentiable functions $g(t)$, where $\widetilde{f}(t)$ is the inverse Laplace transform of $\widetilde{F}(s)$.

The convolution quadrature approximation is done by (1.2) with (1.3). For the linear multistep method we make the assumptions of strong $A(\alpha)$-stability and order p as in [24, (1.10)]:

$$
\begin{aligned}
&\delta(\zeta) \text{ is analytic and without zeros in a neighbourhood of the}\\
&\text{closed unit disk } |\zeta| \le 1, \text{ with the exception of a zero at } \zeta = 1,\\
&|\arg \delta(\zeta)| \le \pi - \alpha \quad \text{for } |\zeta| < 1, \text{ for some } \alpha > \varphi,\\
&\frac{1}{h}\delta(\mathrm{e}^{-h}) = 1 + \mathrm{O}(h^p) \quad \text{with the order } p \ge 1.
\end{aligned}
\tag{2.5}
$$

Well-known examples are the backward differentiation formulas of order $p \le 6$, given by $\delta(\zeta) = \sum_{k=1}^{p} \frac{1}{k}(1-\zeta)^k$, with $\alpha = 90°, 90°, 86°, 73°, 51°, 17°$ for $p = 1, \dots, 6$, respectively.

The first result of this paper is concerned with the quadrature weights divided by the step size h,

$$f_n = \omega_n / h,$$

which are shown to be pth order approximations to the inverse Laplace transform $f(t)$ for $t = nh$ bounded away from 0.

THEOREM 2.1. *Under the conditions* (2.1) *and* (2.5) *we have*

$$|f_n - f(nh)| \le C\, t^{\mu-1-p} h^p \quad (t = nh),$$

where the constant C does not depend on $h \in (0, \overline{h}]$ and $t \in (0, \overline{t}]$ with fixed $\overline{t} < \infty$. If $c \le 0$ in (2.1), *then the error bound holds uniformly for $t > 0$.*

For $\mu > 0$ in (2.1) this is Theorem 4.1 of [24]. The contribution of the present paper is the extension to $\mu \le 0$. As an immediate but useful consequence of Theorem 2.1. and (2.3) we note the bound

$$|f_n| \le C\, t^{\mu-1} \quad (t = nh), \tag{2.6}$$

and for $f_0 = \frac{1}{h} F(\delta(0)/h)$ condition (2.1) implies directly $|f_0| \le C\, h^{\mu-1}$.

The second result bounds the error of the convolution quadrature (1.2) for $g(\tau) = \tau^{\beta-1}$ with real $\beta > 0$. (In the case of $\beta < 1$ the sum in (1.6) is understood to be over $0 \le jh \le t - h$ to avoid the evaluation of $\tau^{\beta-1}$ for $0 \le \tau < h$.)

THEOREM 2.2. *Under the conditions* (2.1) *and* (2.5) *we have, for real $\beta > 0$,*

$$|F(\partial_h)\tau^{\beta-1}(t) - F(\partial)\tau^{\beta-1}(t)| \le \begin{cases} C\, t^{\mu-1} h^{\beta} & \text{for } 0 < \beta \le p,\\ C\, t^{\mu-1+\beta-p} h^p & \text{for } \beta \ge p, \end{cases}$$

where the constant C does not depend on $h \in (0, \overline{h}]$ and $t \in (0, \overline{t}]$ with fixed $\overline{t} < \infty$. If $c \le 0$ in (2.1), *then the error bound holds uniformly for $t > 0$.*

For $\mu > 0$ in (2.1) and β an integer this is Theorem 3.1 of [24]. For $\mu \leq 0$ and β an integer it is Theorem 5.1 of [24], for which, however, only an outline of the proof is given in [24] (the full proof is in the unpublished report [23]). For $\mu > 0$ and non-integral β this is Theorem 5.2 of [24] (whose proof is based on Theorem 5.1). The novelty of the present result lies thus in the remaining case of $\mu \leq 0$ and non-integral β, and in a new, self-contained proof based on Theorem 2.1, which thus assumes a key role in the theory.

In view of the relations (with $*$ denoting continuous convolution)

$$\begin{aligned} F(\partial)(g_1 * g_2)(t) &= (F(\partial)g_1) * g_2(t), \\ F(\partial_h)(g_1 * g_2)(t) &= (F(\partial_h)g_1) * g_2(t), \end{aligned}$$

which express the associativity of convolution, Theorem 2.2 yields immediately error bounds for functions $g(\tau)$ that can be written as a linear combination of powers of τ and a remainder term that can be expressed as a convolution of a power function with an integrable function (in particular, this holds for a Taylor expansion with remainder term in integral form). Cf. the formulation of Theorems 3.1 and 5.1 in [24].

3 Proof of Theorem 2.1.

The proof extends the proof of Theorem 4.1 in [24]. We write $\widetilde{F}(s) = s^{-k}F(s)$ with the integer k chosen such that $\widetilde{\mu} = \mu + k > 0$, so that $\widetilde{F}(s)$ satisfies (2.1) with $\widetilde{\mu}$ instead of μ. Then $f_n = \omega_n/h$ is the nth coefficient of

$$\begin{aligned} \sum_{n=0}^{\infty} f_n \zeta^n &= \frac{1}{h}\left(\frac{\delta(\zeta)}{h}\right)^k \widetilde{F}\left(\frac{\delta(\zeta)}{h}\right) = \left(\frac{\delta(\zeta)}{h}\right)^k \frac{1}{2\pi i}\int_\Gamma \widetilde{F}(\lambda)\frac{1}{\delta(\zeta)-h\lambda}\,\mathrm{d}\lambda \\ &= \frac{1}{2\pi i}\int_\Gamma F(\lambda)\left(\frac{\delta(\zeta)}{h\lambda}\right)^k \frac{1}{\delta(\zeta)-h\lambda}\,\mathrm{d}\lambda. \end{aligned}$$

We introduce $e_n(z)$ as the nth coefficient of the series

$$\sum_{n=0}^{\infty} e_n(z)\,\zeta^n = \left(\frac{\delta(\zeta)}{z}\right)^k \frac{1}{\delta(\zeta)-z}$$

so that

$$f_n = \frac{1}{2\pi i}\int_\Gamma F(\lambda)\,e_n(h\lambda)\,\mathrm{d}\lambda, \tag{3.1}$$

which is to be compared with (2.2) for $t = nh$. We thus need to study $e_n(z) - \mathrm{e}^{nz}$ for $z = h\lambda$ with $\lambda \in \Gamma$.

By (2.5) and the inverse function theorem, the equation $\delta(\zeta) = z$ has a unique solution $\zeta = 1/R(z)$ with $R(0) = 1$ for z in some disk around 0, and

$$R(z) = \mathrm{e}^z + \mathrm{O}(z^{p+1}). \tag{3.2}$$

For any such approximation to the exponential, there exists $r > 0$ such that

$$\begin{aligned} |R(z)| &\leq |\mathrm{e}^{z/2}| \quad \text{for } |z| \leq r,\ |\arg(-z)| \leq \alpha' < \tfrac{1}{2}\pi, \\ \mathrm{e}^{-2|z|} \leq |R(z)| &\leq \mathrm{e}^{2|z|} \quad \text{for } |z| \leq r, \end{aligned} \tag{3.3}$$

and hence we have for $R(z)^n - e^{nz} = (R(z) - e^z)(R(z)^{n-1} + R(z)^{n-2}e^z + \cdots + e^{(n-1)z})$, for $n \ge 1$,

$$\begin{aligned} &|R(z)^n - e^{nz}| \le C|z^p e^{nz/2}| \quad \text{for } |z| \le r,\ |\arg(-z)| \le \alpha' < \tfrac{1}{2}\pi, \\ &|R(z)^n - e^{nz}| \le C|z^p| e^{2|nz|} \quad \text{for } |z| \le r. \end{aligned} \tag{3.4}$$

We write

$$\left(\frac{\delta(\zeta)}{z}\right)^k \frac{1}{\delta(\zeta) - z} = \gamma(z,\zeta)\frac{1}{1 - R(z)\zeta}, \quad \gamma(z,\zeta) = \sum_{n=0}^{\infty} \gamma_n(z)\zeta^n,$$

where now $\gamma(z,\zeta)$ is analytic in a disk $|\zeta| \le 1/\rho$ of radius $1/\rho > 1$ and bounded by $M/|z|^k$ with some constant M for $|z| \le r$. By Cauchy's estimate, the coefficients $\gamma_n(z)$ thus satisfy

$$|\gamma_n(z)| \le \frac{M}{|z|^k}\rho^n \quad \text{for } |z| \le r.$$

We have

$$e_n(z) = \sum_{j=0}^{n} \gamma_j(z) R(z)^{n-j} = R(z)^n \sum_{j=0}^{\infty} \gamma_j(z) R(z)^{-j} - \sum_{j=n+1}^{\infty} \gamma_j(z) R(z)^{n-j}.$$

The last sum is $O(\rho^n/|z|^k)$ for $|z| \le r$ provided that r is chosen so small that $\rho e^{2r} < 1$. For the other sum we have, by $\delta(1/R(z)) = z$ and de L'Hospital's rule, and by (3.2),

$$\begin{aligned} \sum_{j=0}^{\infty} \gamma_j(z) R(z)^{-j} &= \gamma(z, 1/R(z)) = \left(\frac{\delta(\zeta)}{z}\right)^k \frac{1 - R(z)\zeta}{\delta(\zeta) - z}\bigg|_{\zeta = 1/R(z)} = \frac{-R(z)}{\delta'(1/R(z))} \\ &= \frac{R'(z)}{R(z)} = \frac{e^z + O(z^p)}{e^z + O(z^{p+1})} = 1 + O(z^p). \end{aligned}$$

This is the key relation of the proof. Together with (3.3) and (3.4) it gives us

$$\begin{aligned} &|e_n(z) - e^{nz}| \le C(|z^p e^{nz/2}| + \rho^n/|z|^k) \quad \text{for } |z| \le r,\ |\arg(-z)| \le \alpha' < \tfrac{1}{2}\pi, \\ &|e_n(z) - e^{nz}| \le C(|z^p| e^{2|nz|} + \rho^n/|z|^k) \quad \text{for } |z| \le r. \end{aligned} \tag{3.5}$$

On the other hand, uniformly for $|z| \ge r$, $|\arg(-z)| \le \alpha' < \alpha$ with the angle α of $A(\alpha)$-stability given in (2.5), we have that there is a $\rho < 1$ such that

$$\left|\left(\frac{\delta(\zeta)}{z}\right)^k \frac{1}{\delta(\zeta) - z}\right| \le \frac{M}{|z|^{k+1}} \quad \text{for } |\zeta| \le \frac{1}{\rho}.$$

By Cauchy's estimates, this yields

$$|e_n(z)| \le \frac{M}{|z|^{k+1}}\rho^n \quad \text{for } |z| \ge r,\ |\arg(-z)| \le \alpha' < \alpha. \tag{3.6}$$

The result now follows upon forming the difference of (2.2) and (3.1) and using the bounds (3.5), (3.6) and (2.1).

4 Proof of Theorem 2.2.

The proof uses Theorem 2.1 and the following lemma, in which $\delta_n^{(\kappa)}$ is the nth coefficient of the power series

$$\sum_{n=0}^{\infty} \delta_n^{(\kappa)} \zeta^n = \delta(\zeta)^\kappa.$$

LEMMA 4.1. *Under condition* (2.5) *there is the asymptotic expansion*

$$\frac{n^{\beta-1}}{\Gamma(\beta)} = \delta_n^{(-\beta)} + c_p \delta_n^{(p-\beta)} + c_{p+1}\delta_n^{(p+1-\beta)} + \cdots + c_{N-1}\delta_n^{(N-1-\beta)} + \mathrm{O}(n^{\beta-1-N})$$

for $\beta \neq 0, -1, -2, \ldots$, *where the coefficients* c_k *and the constant symbolized by* O *do not depend on* $n \geq 1$.

PROOF. From Stirling's formula we get the asymptotic expansion (cf. [10, p. 47])

$$(4.1)\quad (-1)^n \binom{-\alpha}{n} = \frac{\Gamma(n+\alpha)}{\Gamma(n+1)\Gamma(\alpha)} = \frac{n^{\alpha-1}}{\Gamma(\alpha)} + a_1 \frac{n^{\alpha-2}}{\Gamma(\alpha-1)} + \cdots + a_{N-1}\frac{n^{\alpha-N}}{\Gamma(\alpha-N+1)} + \mathrm{O}(n^{\alpha-1-N})$$

for $\alpha \neq 0, -1, -2, \ldots$ (the expansion terminates after a finite sum for integer α).

By assumption (2.5) we have $\delta(\zeta) = (1-\zeta)\phi(\zeta)$, where $\phi(\zeta)$ is analytic and without zeros in a disk of radius strictly greater than 1, and $\phi(1) = 1$. From a Taylor expansion of $\phi(\zeta)$ at $\zeta = 1$ we thus obtain for any real α

$$\delta(\zeta)^{-\alpha} = (1-\zeta)^{-\alpha}\big(1 + d_1(1-\zeta) + \cdots + d_{N-1}(1-\zeta)^{-(N-1)}\big) + r(\zeta),$$

where the remainder $r(\zeta)$ is sufficiently often differentiable on the unit circle so that its coefficients r_n are bounded by $r_n = \mathrm{O}(n^{\alpha-1-N})$. Hence we have

$$\delta_n^{(-\alpha)} = (-1)^n \binom{-\alpha}{n} + d_1(-1)^n\binom{1-\alpha}{n} + \cdots + d_{N-1}(-1)^n \binom{N-1-\alpha}{n} + \mathrm{O}(n^{\alpha-1-N}).$$

With (4.1) this gives us the expansion

$$(4.2)\quad \delta_n^{(-\alpha)} = \frac{n^{\alpha-1}}{\Gamma(\alpha)} + b_1\frac{n^{\alpha-2}}{\Gamma(\alpha-1)} + \cdots + b_{N-1}\frac{n^{\alpha-N}}{\Gamma(\alpha-N+1)} + \mathrm{O}(n^{\alpha-1-N}).$$

Moreover, Theorem 2.1 for $F(s) = s^{-\alpha}$ shows that $b_1 = \cdots = b_{p-1} = 0$. With $\alpha = \beta, \beta - p, \beta - p - 1, \ldots$ this yields, conversely, the stated expansion of $n^{\beta-1}/\Gamma(\beta)$. □

We now turn to the proof of Theorem 2.2. With $f_n^{(\kappa)}$ denoting the nth coefficient of

$$\sum_{n=0}^{\infty} f_n^{(\kappa)} \zeta^n = \frac{1}{h} F\left(\frac{\delta(\zeta)}{h}\right) \cdot \left(\frac{\delta(\zeta)}{h}\right)^{\kappa}$$

we obtain from Lemma 4.1. that we have at $t = nh$, for sufficiently large N,

$$\begin{aligned} (4.3) \quad F(\partial_h) \frac{\tau^{\beta-1}}{\Gamma(\beta)}(t) &= h^{\beta} \sum_{j=0}^{n} f_{n-j} \frac{j^{\beta-1}}{\Gamma(\beta)} \\ &= f_n^{(-\beta)} + h^p\, c_p\, f_n^{(p-\beta)} + \cdots + h^{N-1}\, c_{N-1}\, f_n^{(N-1-\beta)} + \\ &\quad + \mathrm{O}(t^{\mu-1} h^{\beta}). \end{aligned}$$

Here the $\mathrm{O}(t^{\mu-1}h^{\beta})$ remainder term arises from the bound (2.6) of f_n and the fact that the discrete convolution of an $\mathrm{O}(n^{\mu-1})$ sequence and an $\mathrm{O}(n^{\beta-1-N})$ sequence is again $\mathrm{O}(n^{\mu-1})$, if $\beta - N < \min(0, \mu)$.

On the other hand, $f^{(-\beta)}(t) := F(\partial)\frac{\tau^{\beta-1}}{\Gamma(\beta)}(t)$ is the inverse Laplace transform of $F(s)s^{-\beta}$. Hence the error bound of Theorem 2.1,

$$|f_n^{(-\beta)} - f^{(-\beta)}(nh)| \le C t^{\beta+\mu-1-p} h^p \quad (t = nh),$$

and the bounds (2.6),

$$|f_n^{(-\alpha)}| \le C t^{\alpha+\mu-1} \quad (t = nh),$$

for $\alpha = \beta - p, \beta - p - 1, \ldots$ inserted into (4.3) give the stated result for $t = nh$.

Finally, for values $t = (n + \theta)h$ between grid points $(0 < \theta < 1)$ the result is obtained by noting

$$F(\partial)\tau^{\beta-1}(nh + \theta h) = F(\partial)(\tau + \theta h)^{\beta-1}(nh) + \mathrm{O}((nh)^{\mu-1}h^{\beta})$$

and using the above arguments together with the binomial expansion for $(n + \theta)^{\beta-1}$ in both the discrete and the continuous convolution.

Acknowledgement.

My interest in the numerical solution of integral equations was first aroused in October 1980 when Ernst Hairer handed to me, then a young undergraduate student, the preprint of his paper with Hermann Brunner and Syvert Nørsett on the Runge–Kutta theory of Volterra integral equations [1]. The forming influence of this paper on my subsequent work is gratefully acknowledged.

In 1984, after analyzing the error of convolution quadrature for sectorial Laplace transforms, I became aware of the relationship (and with some relief,

of the differences) to the error analysis of linear multistep methods for parabolic differential equations given in [2]. Since then, Michel Crouzeix's varied mathematical works have been for me an enduring inspiration and challenge.

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