Discrete and Continuous Laplace transform

Proseminar on Numerical Convolution

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October 26, 2009

Laplace transform - Definition

Definition

Let $u : [0, \infty] \to \mathbb{R}$ and piecewise continuous. Then the **Laplace transform** of u is given by:

$$\mathcal{L}(u) = \int_0^\infty e^{-st}\, u(t)\, dt \tag{1}$$

Comments:

- $\mathcal{L}(u)$ is a function of the **complex variable** $s = x + iy$

$$\begin{aligned} \mathcal{L}(u) &= \int_0^\infty e^{-xt}(cos(yt) - i\, sin(yt))u(t)dt \\ &= F(x, y) + i\, G(x, y) \end{aligned} \tag{2}$$

$\to$ When does the integral (1) **exist**?

Existence conditions

To ensure that $\mathcal{L}(u) = \int_0^\infty e^{-st}\, u(t)\, dt$ [$\rightarrow$ (1)] exists, we impose the following **conditions**:

Let $u(t)$ be a **piecewise continuous** *function on* $[0, \infty)$

1. *Let $c_1, c_2 \in \mathbb{R}$ s.t. for $t \rightarrow \infty$*

$$|u(t)| < c_1 e^{c_2 t} \tag{3}$$

2. *For any finite T*

$$\int_0^T |u(t)|\, dt < \infty \tag{4}$$

$\rightarrow$ (1) **converges absolutely and uniformly** for $Re(s) > c_2$, since

$$\int_0^\infty \left|e^{-st} u(t)\right| dt \leq c_1 \int_0^\infty \left|e^{(c_2 - Re(s))t}\right| dt < \infty \tag{5}$$

Comparison: Fourier transform

Definition

The **Fourier transform** of a function is given by:

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \mathrm{e}^{-\mathrm{i}\,\omega t} \mathrm{d}t \qquad (6)$$

where f belongs to the so called **Schwartz space**

$$\mathcal{S}\left(\mathbb{R}^n\right) = \{ f \in C^{\infty}(\mathbb{R}^n) \mid \|f\|_{\alpha,\beta} < \infty \, \forall \, \alpha, \beta \} \qquad (7)$$

where

$$\|f\|_{\alpha,\beta} = \sup_{x \in \mathbf{R}^n} |x^{\alpha} D^{\beta} f(x)|$$

$\rightarrow$ Laplace transform is much more powerful than Fourier transform

Properties of Laplace transform - 1

- Linearity

$$\mathcal{L}(u+v) = \mathcal{L}(u) + \mathcal{L}(v) \tag{8}$$
$$\mathcal{L}(\lambda u) = \lambda \mathcal{L}(u)$$

- Uniqueness [Lerch's Theorem]

  Distinct continuous functions on $[0, \infty)$ have distinct LTs
  ($\rightarrow$ Be careful transforming functions with discontinuities)

- Translation in s- and t-space

$$\mathcal{L}(e^{-bt}u(t)) = F(s+b) \tag{9}$$
$$\int_1^\infty e^{-st}u(t-1)dt = e^{-s}L(u) \tag{10}$$

Properties of Laplace transform - 2

- Laplace transform of integral

$$\mathcal{L}\left(\int_0^T f(t)dt\right) = F(s)/s. \tag{11}$$

- Multiplication by t

$$\mathcal{L}\left(t\, f(t)\right) = -\frac{dF(s)}{ds} \tag{12}$$

- Division by t

$$\mathcal{L}\left(\frac{f(t)}{t}\right) = \int_0^s F(p)dp \tag{13}$$

Some transformed functions

For basic functions, the Laplace transform has been calculated using equation (1) and properties (5) to (13).

f(t)	F(s)	f(t)	F(s)
t^n $(n \in N)$	$\frac{n!}{s^{n+1}}$	$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
e^{-at}	$\frac{1}{s+a}$	$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
te^{-at}	$\frac{1}{(s+a)^2}$	$t^n e^{-at}$	$\frac{n!}{(s+a)^{n+1}}$
$\delta(t)$ (Dirac delta)	1	$\delta^{(n)}(t)$	s^n

- Laplace transform of n^{th} derivative

$$\mathcal{L}\left(\frac{d^n u}{dt^n}\right) = s^n \mathcal{L}(u) - s^{n-1} u(0) - \ldots - u^{(n-1)}(0) \qquad (14)$$

Proof.

<u>**n=1**</u>: Integrating by parts yields:

$$\mathcal{L}\left(\frac{du}{dt}\right) = \int_0^\infty e^{-st}\left(\frac{du}{dt}\right) dt \overset{PI}{=} e^{-st}u(t)\Big|_0^\infty - \int_0^\infty (-s)e^{-st}u(t)dt$$
$$= \underbrace{\lim_{t\to\infty} e^{-st}u(t)}_{\to 0,\ see\ (5)} - u(0) + s\,\mathcal{L}(u) = s\,\mathcal{L}(u) - u(0)$$

<u>**n-1**$\to$ **n**</u>: Similarly, we get:

$$\mathcal{L}\left(\frac{d^n u}{dt^n}\right) = -u^{(n-1)}(0) + s\,\mathcal{L}\left(\frac{d^{n-1}u}{dt^{n-1}}\right) = s^n\mathcal{L}(u) - s^{n-1}u(0) - \ldots - u^{(n-1)}(0) \quad \square$$

Solving ODEs

The Laplace transform turns **ODEs** into simple **algebraic expressions**.

Example (1)

ODE: $\boxed{\mathrm{du/dt = au(t) + v(t), \quad u(0) = c_1}}$

Applying the Laplace transform on both sides gives:

$$\mathcal{L}(du/dt) = \mathcal{L}(au) + \mathcal{L}(v) \qquad \text{[Linearity, (5)]}$$

$$s\,\mathcal{L}(u) - u(0) = a\,\mathcal{L}(u) + \mathcal{L}(v) \qquad \text{[Transf. of derivatives, (14)]}$$

$$(s-a)\,\mathcal{L}(u) = c_1 + \mathcal{L}(v)$$

$$\Rightarrow \mathcal{L}(u) = \frac{c_1}{(s-a)} + \frac{\mathcal{L}(v)}{(s-a)} \tag{15}$$

→ To find solution u(t), existence of **inverse LT** is necessary.

Inverse by Partial fraction expansion

- **Partial fraction expansion**
  ($\rightarrow$ Inverse Transformations for most rational functions easy thanks to known results)

f(t)	F(s)	f(t)	F(s)
t^n $(n \in N)$	$\frac{n!}{s^{n+1}}$	$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
e^{-at}	$\frac{1}{s+a}$	$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
te^{-at}	$\frac{1}{(s+a)^2}$	$t^n e^{-at}$	$\frac{n!}{(s+a)^{n+1}}$

Bromwich Integral

Another method for inversion of the Laplace transform is provided by the **Bromwich Integral** formula (Fourier–Mellin integral; Mellin's inverse formula).

Let F(s) be a function which satisfies the following conditions:

$$(a) \quad F(s) \text{ is analytic for } Re(s) > \sigma_0 \tag{16}$$

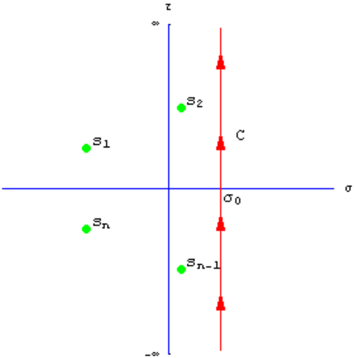
$$(b) \quad F(s) = \frac{c_i}{s} + O\left(\frac{1}{|s|^2}\right) \text{ as } |s| \to \infty \text{ along } s = b + it,\ b > \sigma_0 \tag{17}$$

Let σ_0 be greater than the real part of all Singularities of F(s). Then the inverse Laplace transformation is given by the line integral

$$\mathcal{L}^{-1}\{F(s)\} = f(t) = \frac{1}{2\pi i} \lim_{T\to\infty} \int_{\sigma_0 - iT}^{\sigma_0 + iT} e^{st}\, F(s)\, ds \tag{18}$$

Calculation of Inverse LT

- **Complex Analysis - Calculus of Residues**



Recall: Residue theorem

Theorem

Let $D \subset \mathbb{C}$ be a domain and $s_1, ..., s_n \in D$ be finite many (pairwise disjoint) points. Further let $f : D \setminus \{s_1, ..., s_n\} \to \mathbb{C}$ be an analytic function and $\Gamma : [a, b] \to D \setminus \{s_1, ..., s_n\}$ be a closed contour. Then

$$\int_\Gamma f(\zeta)\, d\zeta = 2\pi i \sum_{j=1}^{n} Res_{s_j}(f)\, \chi_{s_j}(\Gamma) \tag{19}$$

where $Res_{s_j}(f)$ denotes the Residue of f at point s_j.
$\chi_{s_j}(\Gamma)$ is called winding number.

Some Facts about Residues

Basically, there are **three types** of Singularities in Complex Analysis:

1. removable singularities

$$\text{e.g.}\quad f(z) = \frac{sin(z)}{z}$$

2. poles

$$\text{e.g.}\quad f(z) = \frac{z}{(z-a)^5}$$

3. essential singularities

$$\text{e.g.}\quad f(z) = e^{\frac{1}{z}} = \sum_{k=0}^{\infty} \frac{1}{z^k\, k!}$$

Calculating Residues

Let $D = \{\, z \mid 0 < |z - c| < R \}$ a punctured disc in the complex plane. f is a holomorphic function defined at least on D.

The residue $\text{Res}_c(f)$ of f at singularity c is the **coefficient** a_{-1} in the Laurent Series expansion of f $\left(\text{i.e. } f(z) = \sum_{n=-\infty}^{\infty} a_n (z-c)^n \right)$

Example

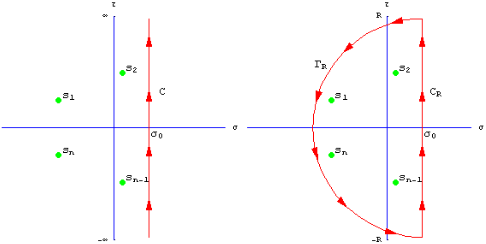
1. removable singularities

$$Res_c(f) = a_{-1} = 0$$

2. poles of n^{th} order

$$Res_c(f) = \frac{1}{(n-1)!} \lim_{z \to c} \frac{d^{n-1}}{dz^{n-1}} \left((z-c)^n f(z) \right)$$

Remember: Bromwich Integral $\frac{1}{2\pi i} \lim_{T \to \infty} \int_{\sigma_0 - iT}^{\sigma_0 + iT} e^{st} F(s)\, ds$



Residue theorem can be used to calculate the integral along $\Gamma_R \cup C_R$:

$$\frac{1}{2\pi i} \int_{\Gamma_R \cup C_R} e^{st} F(s)\, ds = \sum_{s_1, \ldots, s_n} Res_{s_i}(e^{st} F(s))$$

Depending on f we (hopefully!) can choose Γ_R s.t. $\int_{\Gamma_R} e^{st} F(s)\, ds \to 0$ as $R \to \infty$

Example (2)

ODE: $\boxed{\mathrm{du/dt = au(t) + v(t), \quad u(0) = c_1}}$

As we derived in (15), we get the following algebraic expression:

$$\begin{aligned} &\Rightarrow \mathcal{L}(u) = \frac{c_1}{(s-a)} + \frac{\mathcal{L}(v)}{(s-a)} \\ &\Rightarrow u(t) = \mathcal{L}^{-1}\left\{\frac{c_1}{(s-a)}\right\} + \mathcal{L}^{-1}\left\{\frac{\mathcal{L}(v)}{(s-a)}\right\} \end{aligned} \tag{20}$$

Using prior results, (e.g. compare basic Laplace transform table), we know:

$$\mathcal{L}^{-1}\left\{\frac{c_1}{(s-a)}\right\} = \underline{\underline{c_1\, e^{at}}} \tag{21}$$

$\rightarrow$ We still cannot handle the 2^{nd} part. What is $\mathcal{L}^{-1}(f \cdot g)$?

Convolution Theorem

Theorem (Convolution Theorem)

Let f and g be piecewise continuous on $[0, \infty]$ *and of exponential order* α *(cf. (3)), then*

$$\mathcal{L}\left[(f * g)(t)\right] = \mathcal{L}(f(t)) \cdot \mathcal{L}(g(t)), \qquad (Re(s) > \alpha) \tag{22}$$

where $f * g$ *denotes the* **convolution of f and g** *which is given by*

$$(f * g)(t) = \int_0^t f(\tau)\, g(t - \tau)\, d\tau \tag{23}$$

Proof.

$$\mathcal{L}(f)\,\mathcal{L}(g) = \left(\int_0^\infty e^{-s\tau} f(\tau)\,d\tau\right)\left(\int_0^\infty e^{-s\,u} g(u)\,du\right)$$
$$= \int_0^\infty \left(\int_0^\infty e^{-s(\tau+u)} f(\tau)\,g(u)\,du\right)\,d\tau$$

Substituting $t = \tau + u$ we get: (τ is fixed in the inner integral and $g(t) = 0$ for $t < 0$ implies $g(t-\tau) = 0$ for $t < \tau$)

$$\mathcal{L}(f)\,\mathcal{L}(g) = \int_0^\infty \left(\int_0^\infty e^{-st} f(\tau)\,g(t-\tau)\,dt\right)\,d\tau$$

Since the Laplace integrals of f and g converge abolutely we are allowed to reverse the order of integration, so that

$$\mathcal{L}(f)\,\mathcal{L}(g) = \int_0^\infty \left(\int_0^\infty e^{-st} f(\tau)\,g(t-\tau)\,d\tau\right)\,dt$$
$$= \int_0^\infty e^{-st}\left(\int_0^t f(\tau)\,g(t-\tau)\,d\tau\right)\,dt = \mathcal{L}\left[(f*g)(t)\right] \qquad \square$$

Example (3)

ODE: $\boxed{\mathrm{du/dt = au(t) + v(t), \quad u(0) = c_1}}$

As we derived in (20) and (21):

$$\Rightarrow u(t) = c_1\, e^{at} + \mathcal{L}^{-1}\left\{\frac{\mathcal{L}(v)}{(s-a)}\right\}$$

Using the Convolution Theorem (22) we get:

$$\mathcal{L}^{-1}\left\{\frac{\mathcal{L}(v)}{(s-a)}\right\} = \mathcal{L}^{-1}\left\{\mathcal{L}(v)\right\} * \mathcal{L}^{-1}\left\{\frac{1}{(s-a)}\right\} = \underline{\underline{\int_0^t v(\tau)\, e^{a(t-\tau)}\, d\tau}}$$

$\rightarrow$ **Solution of ODE:**

$$\boxed{\mathrm{u(t) = c_1}\, e^{at} + \int_0^t v(\tau)\, e^{a(t-\tau)}\, d\tau}$$

Stability of LT

Let v(t) be a function such that

$$\int_0^\infty |v(t)|\; e^{-kt} dt < \epsilon \tag{24}$$

Then, for $Re(s) \geq k$

$$|\mathcal{L}(u+v) - \mathcal{L}(u)| = \left| \int_0^\infty v(t)\, e^{-st}\, dt \right| < \int_0^\infty |v(t)|\; e^{-Re(s)\, t}\, dt < \epsilon$$

So, a small change in u(t) produces an equally small change in L(u).

| $\rightarrow$ L(u) is **stable** under perturbations of type (24) |

Instability of Inverse LT - 1

The inverse Laplace transform is **not stable** under reasonable perturbations.

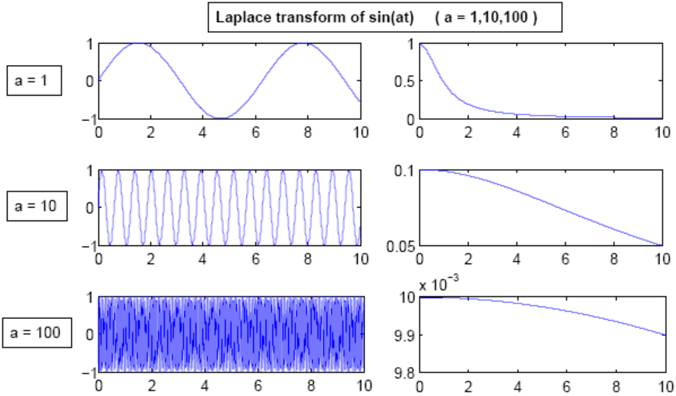
Example

Take as an example the transformation:

$$\mathcal{L}(sin(at)) = \frac{a}{(s^2 + a^2)}$$

As a increases...

- ... $sin(at)$ oscillates more and more rapidly, but remains of constant amplitude.
- ... The LT is uniformly bounded by $1/a$ for $s \geq 0$, thus approaches 0 uniformly.



Consequence:

Impossibility of usable universal algorithms for Inverse LT!

Discrete Laplace transform (z-Transform)

In many discrete systems, the signals flowing are considered at **discrete values of t**, e.g. at $nT, \; n = 0, 1, 2, ...$, where T is called the **sampling period**.

So we are looking at a **sequence of values** f_n.

Here: $f_n = f(nT)$

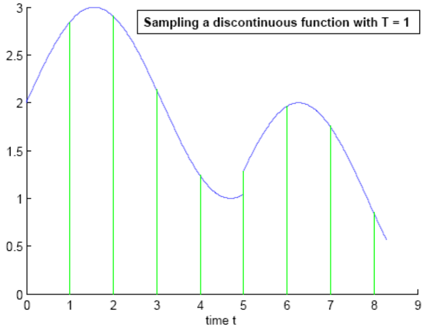
Definition

Let $T > 0$ be fixed, f(t) be defined for $t \geq 0$. The z-Transformation of f(t) is the function

$$\mathcal{Z}[f] = \mathcal{F}(z) = \sum_{n=0}^{\infty} f(nT)\, z^{-n} \tag{25}$$

of the complex variable z, $\;for\, |z| > R = \frac{1}{\rho}$ where ρ denotes the radius of convergence of the series.

Existence of z-Transform



If f(t) has a jump discontinuity at some nT, we interpret $f(nT)$ as the limit of $f(t)$ as $t \to nT^+$. To ensure existence of the z-Transform, assume existence of this limit for $n = 0, 1, 2, ...$ for all $f(t)$ considered.

Properties of z-Transform - 1

- Linearity

$$\mathcal{Z}(af + bg) = a\mathcal{Z}(f) + b\mathcal{Z}(g) \tag{26}$$

- Shifting theorem

$$\mathcal{Z}(f(t + mT)) = z^m \left[\mathcal{F}(z) - \sum_{k=0}^{m-1} f(kT)\, z^{-k} \right] \tag{27}$$

- Corollary of Shifting theorem

$$\mathcal{Z}(f(t - nT)\, u(t - nT)) = z^{-n}\, \mathcal{F}(z) \tag{28}$$

where u(t) denotes the unit step function.

Properties of z-Transform - 2

- Complex scale change

$$\mathcal{Z}(e^{-at}f(t)) = \mathcal{F}(e^{aT}z) \tag{29}$$

- Complex differentiation or multiplication by t

$$\mathcal{Z}(t\,f) = -T\,z\,\frac{d}{dz}\mathcal{F}(z) \tag{30}$$

Convolution

Definition

The **convolution of two sequences** $\{f_n\}$ **and** $\{g_n\}$ is given by the sequence $\{h_n\}$, where its n^{th} element is given by:

$$h_n = \sum_{k=0}^{n} f_k \, g_{n-k} \tag{31}$$

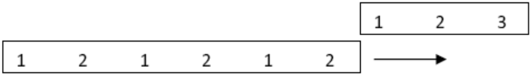
Algorithmic calculation of the discrete convolution

Let f_n and g_n be the following two sequences:

g_n =	2	1	2	1	2	1

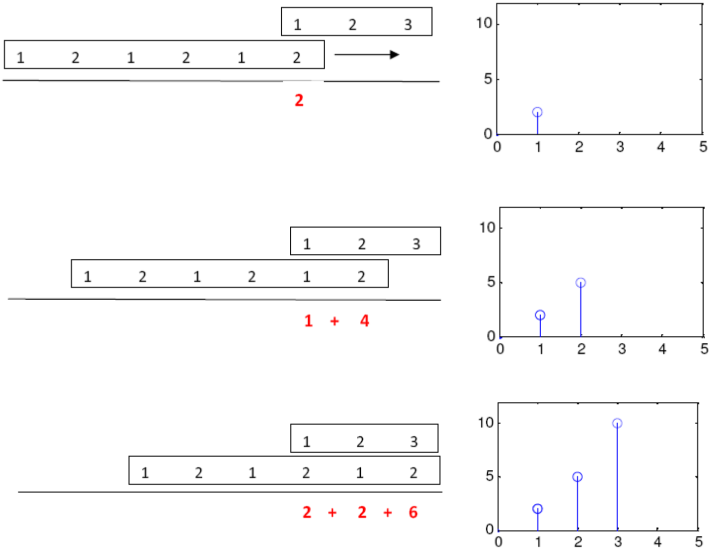
f_n =	1	2	3

First step: Change order of one sequence



Second step: Multiply elements below each other and add them together.

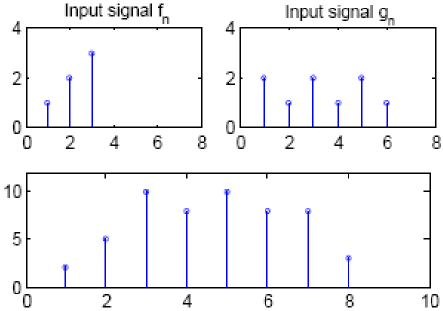
Third step: Move sequence by one position and start again at second step.



1 2 3
1 2 1 2 1 2
2
1 2 3
1 2 1 2 1 2
1 + 4
1 2 3
1 2 1 2 1 2
2 + 2 + 6
10
5
0
0 1 2 3 4 5

Example: Discrete Convolution

The convolution (MATLAB: `conv(f,g)`) of the two sequences $f_n = (1, 2, 3)$ and $g_n = (2, 1, 2, 1, 2, 1)$ is given by:



Convolution theorem for z-Transform

Theorem

If there exist the transform $\mathcal{Z}(f_1) = \mathcal{F}_1(z)$ for $|z| > 1/R_1$ and $\mathcal{Z}(f_2) = \mathcal{F}_2(z)$ for $|z| > 1/R_2$, then the transform $\mathcal{Z}(f_1 * f_2)$ also exists and we have for $|z| > max(1/R_1, 1/R_2)$,

$$\mathcal{Z}(f_1 * f_2) = \mathcal{Z}\left[\sum_{k=0}^{n} f_1(kT)\, f_2((n-k)T)\right] = \mathcal{F}_1(z)\,\mathcal{F}_2(z) \qquad (32)$$

Proof.

First remark, that (28) implies, that:

$$z^{-k}\mathcal{F}_2(z) = \mathcal{Z}\left[f_2(t-kT)\right], \quad \text{if } f_2((n-k)T) = 0 \text{ for } n < k$$

Hence,

$$\begin{aligned}
\mathcal{F}_1(z)\,\mathcal{F}_2(z) &= \sum_{k=0}^{\infty} f_1(kT)\,z^{-k}\,\mathcal{F}_2(z) = \sum_{k=0}^{\infty} f_1(kT)\,\mathcal{Z}\left[f_2(t-kT)\right] \\
&= \sum_{k=0}^{\infty} f_1(kT) \sum_{n=0}^{\infty} f_2\left[(n-k)T\right] z^{-n} \\
&= \sum_{n=0}^{\infty} \left\{ \sum_{k=0}^{\infty} f_1(kT)\, f_2\left[(n-k)T\right] \right\} z^{-n}
\end{aligned}$$

but $f_2((n-k)T) = 0$ for $n < k$. Therefore we get:

$$\mathcal{F}_1(z)\,\mathcal{F}_2(z) = \mathcal{Z}(f_1 * f_2) = \mathcal{Z}\left[\sum_{k=0}^{n} f_1(kT)\, f_2((n-k)T)\right]$$

Inverse z-Transform

We are interested in retrieving the values f(nT) from a given transform $\mathcal{F}(z)$, so symbolically we write:

$$f(nT) = \mathcal{Z}^{-1}\left[\mathcal{F}(z)\right]$$

There are three typical methods:

- Partial fraction expansion
- Power series method
- Solving complex integrals

Power series method

Let $\mathcal{F}(z)$ be given as a function analytic for $|z| > R$ and at $z = \infty$, then the value of **f(nT)** can be obtained as the **coefficient of $\mathbf{z}^{-n}$** in the power series expansion of $\mathcal{F}(z)$ as a function of z^{-1}.

Assume that $\mathcal{F}(z)$ is given as a rational function in z^{-1}:

$$\mathcal{F}(z) = \frac{p_0 + p_1\, z^{-1} + \ldots + p_n\, z^{-n}}{q_0 + q_1\, z^{-1} + \ldots + q_n\, z^{-n}} = f(0T) + f(1T)\, z^{-1} + \ldots \qquad (33)$$

where by **comparison of coefficients:**

$$\begin{aligned} p_0 &= f(0T)\, q_0 \\ p_1 &= f(1T)\, q_0 + f(0T)\, q_1 \\ &\ldots \\ p_n &= f(nT)\, q_0 + f\left[(n-1)T\right]\, q_1 + f\left[(n-2)T\right]\, q_2 + \ldots + f(0T)\, q_n \end{aligned}$$

Complex integral formula

The coefficient f(nT) can also be **expressed as a complex integral**.

We need the following result:

$$\int_{|z|=r} z^n \, dz = \begin{cases} 2\pi i, & n = -1 \\ 0, & n \neq -1 \end{cases}$$

By multiplying $\mathcal{F}(z)$ by z^{n-1} and integrating, we get:

$$\oint_\Gamma \mathcal{F}(z) \, z^{n-1} \, dz = f(nT) \cdot 2\pi i \tag{34}$$

So, using again the Residue theorem (19) we get

$$f(nT) = \frac{1}{2\pi i} \oint_\Gamma \mathcal{F}(z) \, z^{n-1} \, dz = \sum (\text{Residues of } \mathcal{F}(z) \, z^{n-1}) \tag{35}$$

Of course choose Γ s.t. all residues lie **inside the contour**

Problem: Branch points

If we use the complex integral formula we have to be careful because of **branch points** in the integrand.

Example

1. **complex logarithm**

$$\mathrm{Log}(z) = \ln|z| + i\mathrm{Arg}\ z \tag{36}$$

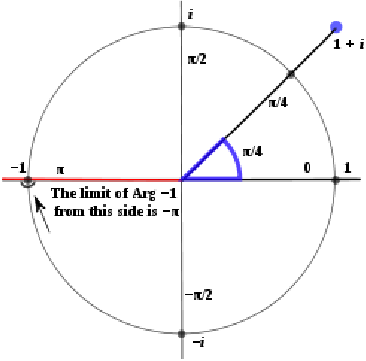
2. **roots**
   Let F(z) be given as:

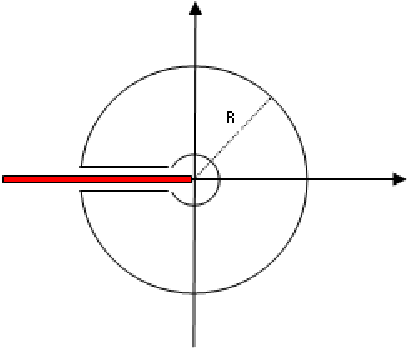
$$\mathcal{F}(z) = z^x, \quad x \in \mathbb{R} \backslash \mathbb{N}$$

   We can rewrite this as

$$\mathcal{F}(z) = e^{Log(z)\,x}$$

As we meet such branch cuts, we have to be careful choosing our contour Γ:





Comparison between Laplace and z-Transform

Goal:

Develop a transformation to switch between z-Transform and Laplace Transform.

Recall:

- **Laplace transform** (1)

$$\mathcal{L}(u) = \int_0^\infty e^{-st}\, u(t)\, dt$$

- **z-transform** (25)

$$\mathcal{Z}[f] = \mathcal{F}(z) = \sum_{n=0}^{\infty} f(nT)\, z^{-n}$$

Define the impulse function:

$$f^*(t) = \sum_{n=0}^{\infty} f(nT)\delta(t - nT) \tag{37}$$

Using that $\mathcal{L}(\delta(t)) = 1$ and (10) we get

$$\mathcal{L}(\delta(t - kT)) = e^{-kTs} \tag{38}$$

We obtain:

$$\begin{aligned} F^*(s) &= \mathcal{L}\left[f^*(t)\right] = \mathcal{L}\left(\sum_{n=0}^{\infty} f(nT)\delta(t - nT)\right) \\ &= \sum_{n=0}^{\infty} f(nT)\mathcal{L}(\delta(t - nT)) = \sum_{n=0}^{\infty} f(nT)\, e^{-nTs} \end{aligned}$$

which actually is the z-Transform with $z = e^{Ts}$

Relationship between z-Transform and Laplace Transform

Using the prior results we can deduct the following relationship:

$$\boxed{\mathcal{Z}(f) = \mathcal{L}(\mathrm{f}^*(t)), \quad \text{evaluated at: } s = T^{-1} ln(z)} \tag{39}$$