Numerical Methods Based On Local Coordinates

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Outline

Definitions of Manifolds

Definition (Local constraints)

Let U be a neighbourhood of $a \in \mathbb{R}^n$, $g : U \mapsto \mathbb{R}^m$ a differentiable map with $g(a) = 0$ and assume that $g'(a)$ has full rank m. Then a manifold $\mathcal{M}$ is locally given by

$$\mathcal{M} := \{y \in U;\ g(y) = 0\}\ .$$

Definition (Local parameters)

Let $V \subset \mathbb{R}^{n-m}$ be a neighbourhood of 0, $\Psi : V \mapsto \mathbb{R}^n$ a differentiable map with $\Psi(0) = a$ and assume that $\Psi'(0)$ has full rank $n - m$. Then a manifold $\mathcal{M}$ is locally given by

$$\mathcal{M} := \{y = \Psi(z);\ z \in V\}\ .$$

Definition of Tangent Space

A tangent to a curve (or the tangent plane to a surface) is an affine space passing through the contact point $a \in \mathcal{M}$.

Definition (Tangent space)

For a manifold $\mathcal{M}$ we define the *tangent space* at $a \in \mathcal{M}$ by

$$T_a\mathcal{M} := \left\{ v \in \mathbb{R}^n ; \begin{array}{l} \exists \text{ differentiable path } \gamma : (-\varepsilon, \varepsilon) \mapsto \mathbb{R}^n \\ \text{with } \gamma(t) \in \mathcal{M} \ \forall \ t \,,\ \gamma(0) = a \,,\ \dot{\gamma}(0) = v \end{array} \right\} .$$

Characterization of Tangent Space

Lemma

If the manifold $\mathcal{M}$ is locally given by a constraint $g : U \mapsto \mathbb{R}^m$, wich is differentiable with $g(a) = 0$ and $\operatorname{rank} g'(a) = m$, then we have

$$T_a\mathcal{M} = \operatorname{Ker} g'(a)\,.$$

If the manifold $\mathcal{M}$ is locally given by a parametrization $\Psi : V \mapsto \mathbb{R}^n$, which is differentiable with $\Psi(0) = a$ and $\operatorname{rank} \Psi'(0) = n - m$, then we have

$$T_a\mathcal{M} = \operatorname{Im} \Psi'(0)\,.$$

Definition of Differential Equations on Manifolds

Suppose we have an $(n-m)$-dimensional submanifold of $\mathbb{R}^n$,

$$\mathcal{M} := \{y \in \mathbb{R}^n;\ g(y) = 0\}\ , \quad g : \mathbb{R}^n \mapsto \mathbb{R}^m\,,$$

and a differential equation $\dot{y} = f(y)$ with the property that

$$y_0 \in \mathcal{M} \quad \text{implies} \quad y(t) \in \mathcal{M} \text{ for all } t\,.$$

In this situation we call $g(y)$ a *weak invariant*, and we say that $\dot{y} = f(y)$ is a differential equation on the manifold.

Characterization of Differential Equations on Manifolds

Theorem

Let $\mathcal{M}$ be manifold of $\mathbb{R}^n$. The problem $\dot{y} = f(y)$ is a differential equation on the manifold $\mathcal{M}$, if and only if

$$f(y) \in T_y\mathcal{M} \text{ for all } y \in \mathcal{M}\,.$$

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The proof implies that locally $\dot{y} = f(y)$ is equivalent to the differential equation

$$\dot{z} = \Psi'(z)^+ f(\Psi(z))$$

in the parameter domain, where

$$\Psi'(z)^+ := \left(\Psi'(z)^\top \Psi'(z)\right)^{-1} \Psi'(z)^\top\,.$$

Algorithm

Assume that $y_n \in \mathcal{M}$ and that Ψ is a local parametrization of the manifold $\mathcal{M}$ satsifying $\Psi(z_n) = y_n$. One step $y_n \mapsto y_{n+1}$ is defined as follows:

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1. Compute $z_{n+1} = \Phi_h(z_n)$, the result of the method Φ_h applied to

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It is important to remark that the parametrization $y = \Psi(z)$ can be changed at every step.

Generalized Coordinate Partitioning

Wehage & Haug 1982

Assume that the manifold $\mathcal{M}$ is given a constraint $g : \mathbb{R}^n \mapsto \mathbb{R}^m$. If the jacobian $g'(y)$ has full rank m at $y = a$, then we can find a partition $y = (y_1, y_2)$, such that

$$\frac{\partial g}{\partial y_2}(a)$$

is invertible. In this case we can choose $z = y_1$ as local coordinates and the function $y = \Psi(z)$ is defined by $y_1 = z$ and $y_2 = \Psi_2(z)$, where $\Psi_2(z)$ is implicitly given by

$$g(z, \Psi_2(z)) = 0\,.$$

The partition can be obtained from a QR decomposition applied to the matrix $g'(a)$.

Tangent Space Parametrization

Potra & Rheinboldt 1991

Assume again that the manifold $\mathcal{M}$ is given a constraint $g : \mathbb{R}^n \mapsto \mathbb{R}^m$, and collect the vectors of an orthonormal basis of $T_a\mathcal{M}$ in the matrix Q. We consider the parametrization

$$\Psi(z) := a + Qz + g'(a)^\top u(z)\,,$$

where the function $u : \mathbb{R}^{n-m} \mapsto \mathbb{R}^n$ is defined by $g(\Psi(z)) = 0$. Since $Q^\top Q = I$ and $g'(a)^\top Q = 0$ the differential equation in the parameter domain reduces to

$$\dot{z} = Q^\top f(\Psi(z))\,.$$