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Polynomial Invariants

We consider two classes of problems with polynomial invariants of degree higher than two:

1. linear problems

2. isospectral flows

The Determinant as an invariant

$\dot{Y} = A(Y)Y, \quad Y(0) = Y_0 \qquad (1.1)$
where Y and $A(Y)$ are $n \times n$ matrices.

Definition: **trace** $A = \sum_{i=1}^{n} a_{ii}$

Definition: A non constant function $I(Y)$ is called a first invariant if $\quad I'(Y)A(Y)Y = 0$ for all Y

$\Rightarrow$ every solution $Y(t)$ satisfies $I(Y(t)) = I(Y_0) = const.$

Lemma 1.1 If trace $A(Y) = 0$ for all $Y \Rightarrow g(Y) := \mathbf{det}Y$ is an invariant.

$Proof : g(Y) = \mathbf{det}Y$. We show that $g'(Y)(AY) = \mathbf{trace}A \cdot \mathbf{det}Y$. Hence the determinant is an invariant if $\mathbf{trace}\, A(Y) = 0$ for all Y.

$\mathbf{det}Y$ represents the volume of the parallelepiped:
Conservation of the invariant $\Rightarrow$ volume preservation.

$\mathbf{det}Y$: a polynomial invariant of degree n

Question: Can Runge-Kutta methods conserve this invariant for $n \geq 3$?

Let's see...

Theorem 1.3 For $n \geq 3$, no Runge-Kutta method can conserve all polynomial invariants of degree n.

A negative result!

Isospectral Flows

$\dot{L} = [B(L), L], \quad L(0) = L_0 \quad (1.2)$

L_0 : a symmetric matrix, $B(L)$: skew-symmetric,
$[B, L] = BL - LB$: commutator of B and L.

Lemma 1.4 L_0 symmetric, $B(L)$ skew-symmetric for all L
$\Rightarrow L(t)$ symmetric, $\lambda(t) =$ **const.**

$Proof - scetch$: We define the function $U(t)$ by
$\dot{U} = B(L(t))U, \quad U(0) = I. \quad (1.3)$

In the characteristic polynomial
$\mathbf{det}\,(L - \lambda I) = \sum_{i=0}^{n} a_i \lambda^i$
the coefficients a_i also are independent of t.

Coefficients as: $a_0 = \mathbf{det}\, L$,
$a_{n-1} = \pm\mathbf{trace}\, L$ are polynomial invariants.

Because of Theorem 1.3
$\Rightarrow$ no hope that Runge-Kutta methods can conserve invariants automatically for $n \geq 3$.

Isospectral Methods

$\dot{L} = [B(L), L], \quad L(0) = L_0:$
solve $\quad \dot{U} = B(UL_0U^{-1})U, \quad U(0) = I \quad$ up to time t

$\Rightarrow \; \tilde{L}(t) = \tilde{U}(t)L_0\tilde{U}(t)^{-1},$

$B(L)$ skew-symmetric for all L, then U^TU is a quadratic invariant of $\dot{U} = B(L)U$, $U(0) = I$ and the methods of Sect. $IV.2$ (i.e. Gauss) will produce an orthogonal $\tilde{U}$.

$\Rightarrow \; spec(\tilde{L}(t)) = spec(L_0)$ and $\tilde{L}(t)^T = \tilde{L}(t)$ for all t.

$\Longrightarrow$ isospectral methods conserve polynomial invariants.

Example: Toda Lattice $(n = 3)$ $\dot{L} = [B, L]$

$$B(t) = \begin{pmatrix} 0 & b_1(t) & -b_3(t) \\ -b_1(t) & 0 & b_2(t) \\ b_3(t) & -b_2(t) & 0 \end{pmatrix}$$

$$L(t) = \begin{pmatrix} a_1(t) & b_1(t) & b_3(t) \\ b_1(t) & a_2(t) & b_2(t) \\ b_3(t) & b_2(t) & a_3(t) \end{pmatrix}$$

$$a_1(0) = \tfrac{3}{4}\ ,\ a_2(0) = -\tfrac{1}{2}\ ,\ a_3(0) = -\tfrac{1}{4}$$

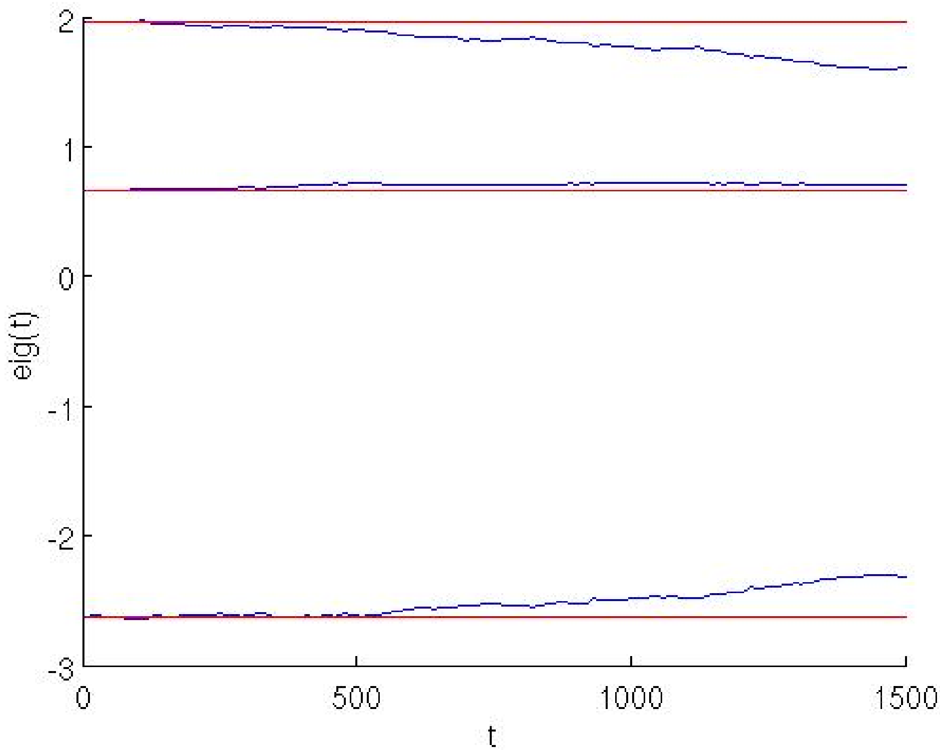
$$b_1(0) = \tfrac{1}{2}\, e^{-\frac{1}{2}}\ ,\ b_2(0) = \tfrac{1}{2}\, e^{\frac{3}{2}}\ ,\ b_3(0) = \tfrac{1}{2}\, e^{-1}$$

<u>study $\lambda_1(t)\,, \lambda_2(t)\,, \lambda_3(t)$</u>

2 **Methods:**

A) RK: ODE 45 (Matlab)
on $\dot{L} = [B, L]\,,\quad L(0) = L_0$

B) impl. midpoint on $\dot{U} = B(UL_0U^{-1})U,\; U(0) = I$
then $L(t) = U(t)L_0U^{-1}(t)$



2
1
0
-1
-2
-3
eig(t)
0
500
1000
1500
t