



A sparse \mathcal{H} -matrix arithmetic: general complexity estimates

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Abstract

In a preceding paper (Hackbusch, *Computing* 62 (1999) 89–108), a class of matrices (\mathcal{H} -matrices) has been introduced which are data-sparse and allow an approximate matrix arithmetic of almost linear complexity. Several types of \mathcal{H} -matrices have been analysed in Hackbusch (*Computing* 62 (1999) 89–108) and Hackbusch and Khoromskij (Preprint MPI, No. 22, Leipzig, 1999; *Computing* 64 (2000) 21–47) which are able to approximate integral (nonlocal) operators in FEM and BEM applications in the case of quasi-uniform unstructured meshes. In the present paper, the general construction of \mathcal{H} -matrices on rectangular and triangular meshes is proposed and analysed. First, the reliability of \mathcal{H} -matrices in BEM is discussed. Then, we prove the optimal complexity of storage and matrix–vector multiplication in the case of rather arbitrary admissibility parameters $\eta < 1$ and for finite elements up to the order 1 defined on quasi-uniform rectangular/triangular meshes in \mathbb{R}^d , $d = 1, 2, 3$. The almost linear complexity of the matrix addition, multiplication and inversion of \mathcal{H} -matrices is also verified. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

A class of hierarchical (\mathcal{H}) matrices has been recently introduced in [5]. They are shown to provide an efficient tool for a data-sparse approximation to large and fully populated stiffness matrices arising in BEM and FEM applications. In fact, the storage and matrix–vector multiplication complexity of the rank- k \mathcal{H} -matrices associated with quasi-uniform grids are estimated by $O(kn \log n)$, where n is the problem size, see [5,6]. Moreover, these matrices also allow the arithmetic of optimal complexity. In particular, the “formatted” matrix–matrix addition, product as well as the inversion for a class of \mathcal{H} -matrices were proven to have almost linear complexity $O(n \log^q n)$ with moderate $q \geq 0$. In this way the approach may be applied for the data-sparse approximation and fast

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solution of the linear integral/pseudodifferential equations which arise in the FE/BE methods for elliptic problems.

First, we discuss the principal ingredients of the \mathcal{H} -matrix techniques. We then show the existence of optimal order approximations by \mathcal{H} -matrices for a class of integral operators in FEM/BEM applications. We prove the almost linear complexity of various \mathcal{H} -matrix operations. In particular, we study the complexity of hierarchical matrices in the following cases:

- (i) arbitrary constant $\eta < 1$ in the admissibility criterion;
- (ii) quasi-uniform quadrangular/triangular meshes in \mathbb{R}^d , $d = 1, 2, 3$;
- (iii) piecewise constant/linear/bilinear elements.

Our results for the storage and matrix–vector multiplication expenses are given with asymptotically sharp constants which depend explicitly upon the spatial dimension d , the parameter η and the problem size. We prove the linear-logarithmic complexity of the formatted *addition, multiplication and inverse* of \mathcal{H} -matrices.

We also stress that our constructions apply to unstructured quasi-uniform meshes as well, using the techniques from [6]. The extension to the case of graded meshes was discussed in [7]. A class of \mathcal{H}^2 -matrices having the linear complexity $O(n)$ was developed in [9]. The systematic approach to build optimal order degenerate approximations (wire-basket expansions of the order $O(\log^{d-1} n)$) for a class of kernels in FEM and BEM applications has been considered in [8].

2. Introduction to \mathcal{H} -matrices

2.1. A motivation for data-sparse approximations in BEM

In this section, we discuss simple examples illustrating the principal ideas of \mathcal{H} -matrix approximations in BEM. The nonlocal operators to be approximated arise in both FEM and BEM applications. FE/FD approximations of elliptic PDEs result in sparse stiffness matrices. In such applications, we are interested in the data-sparse approximation of the inverse to discrete elliptic operators or to the Schur-complement matrices with respect to a certain subset of degrees of freedom. In both cases, we actually deal with a discretisation of an integral (pseudodifferential) operator with implicitly given Schwartz kernel. Below, we consider three examples of integral operators

$$(A_\gamma u)(x) = \int_{\Sigma} s_\gamma(x, y) u(y) dy, \quad x \in \Sigma := [0, 1] \quad (1)$$

with $\gamma = 1, 2, 3$, where

$$s_1(x, y) := \log(1 + (x - y)^2), \quad s_2(x, y) := \log(x + y), \quad s_3(x, y) := \log|x - y|. \quad (2)$$

The FE Galerkin discretisation of (1) with piecewise constant basis functions defined for the uniform grid (partitioning)

$$X_i = [(i - 1)h, ih], \quad h := n^{-1}, \quad 1 \leq i \leq n$$

leads to the full stiffness matrix

$$M = (m_{ij})_{i,j \in I}, \quad m_{ij} := \int_{X_i \times X_j} s_\gamma(x, y) dx dy,$$

where $I = \{1, \dots, n\}$ is the corresponding index set of the Galerkin ansatz functions $\{\varphi_i\}_{i \in I}$. Assume a hierarchical p -level structure of the grid by imposing $n = 2^p$. The \mathcal{H} -matrix approximation to M will provide a matrix $M_{\mathcal{H}}$ such that the error $M - M_{\mathcal{H}}$ is of the same order $\varepsilon = h^\delta$, $\delta > 0$, as for the Galerkin error related to M . However, both the storage and the matrix–vector multiplication costs for $M_{\mathcal{H}}$ will amount to $O(n \log^q n)$ instead of $O(n^2)$, with a moderate $q \geq 0$ discussed below.

For the first example in (2) the desired approximation $M_{\mathcal{H}}$ can be obtained exploiting the global smoothness of the kernel in the product domain $\Sigma \times \Sigma$. Due to classical approximation theory there exists a simple approximation of $s_1(x, y)$ by a short sum $\tilde{s}_1 := \sum_{\beta=1}^k a_\beta(x) c_\beta(y)$ of separable functions (e.g., by the Taylor expansion or by the ortho–projection onto polynomials, see also Section 3) such that

$$\left| s_1(x, y) - \sum_{\beta=1}^k a_\beta(x) c_\beta(y) \right| \lesssim \varepsilon \tag{3}$$

with $k = O(\log \varepsilon^{-1})$. The corresponding stiffness matrix

$$M_{\mathcal{H}} := (\tilde{m}_{ij})_{i,j \in I}, \quad \tilde{m}_{ij} := \int_{X_i \times X_j} \tilde{s}_1(x, y) \, dx \, dy,$$

provides the required approximation of M on the one hand, and also it has the data-sparse structure (indeed, it is a matrix of rank k) of the complexity $O(kn)$, on the other hand. Therefore, the global smoothness of s_1 allows a data-sparse approximation of M by an $n \times n$ low-rank matrix.

The singular kernels in the second and third examples allow instead of a global only blockwise degenerate approximations. In this way, the above construction is applied locally in a hierarchical manner and it is based on an *admissible partitioning* of the product index set $I \times I$. Such an admissible partitioning is described below using hierarchical cluster trees of I and $I \times I$.

2.2. The cluster trees of I and $I \times I$

Starting with the full index set $I_1^0 := I$ of level 0, we then split it into two equal subsets I_1^1 and I_2^1 and then apply this procedure to each part successively such that at level p , we reach the one-element sets $I_1^p = \{1\}, \dots, I_n^p = \{n\}$. In general, at level ℓ , we have the set of tree vertices (clusters)

$$I_j^\ell := \{(j - 1)2^{p-\ell} + 1, \dots, j2^{p-\ell}\} \quad \text{for } 0 \leq \ell \leq p, \quad 1 \leq j \leq 2^\ell.$$

In the following, the vertices are called the *clusters*. Each cluster $\tau = I_j^\ell$ has exactly two sons, $I_{j'}^{\ell+1}$ and $I_{j''}^{\ell+1}$ with $j' = 2j + 1$ and $j'' = 2j + 2$, obtained by halving the parent vertex. The set of all clusters I_j^ℓ together with the tree structure is called the *cluster tree* $T(I)$. In this example, $T(I)$ is a binary tree of depth p . I is the root of $T(I)$ and the sets I_i^p , $i = 1, \dots, n$, are the leaves of $T(I)$ (one-element vertices). Introducing the isomorphism between the index set I and the interval decomposition $\{X_i\}_{i \in I}$ by $i \leftrightarrow J_i$, one can define diameters and the distance between two clusters τ and σ just measuring the Euclidean diameter $\text{diam}(X(\tau))$ and the distance $\text{dist}(X(\tau), X(\sigma))$, where $X(\tau) := \bigcup \{X_\alpha : \alpha \in \tau\}$.

Having in hands the cluster tree $T_1 := T(I)$, we then construct the corresponding hierarchical tree $T_2 := T(I \times I)$ on the product index-set $I \times I$ and with the same number p of levels. In our particular case, we have the following set of vertices:

$$I_{ij}^\ell := I_i^\ell \times I_j^\ell \quad \text{for } 0 \leq \ell \leq p, \quad 1 \leq i, j \leq 2^\ell.$$

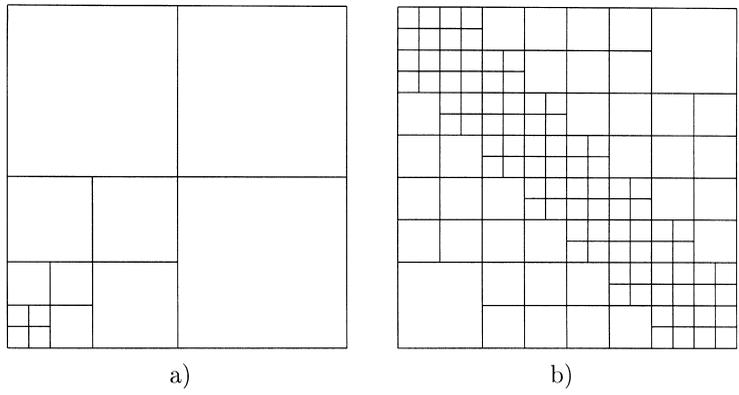


Fig. 1. Block-structure for \mathcal{M}_N (a) and \mathcal{M}_G (b) formats.

The set of sons $S_2(t)$ of $t = I_{ij}^\ell \in T_2$ is given by $S_2(t) := \{\tau \times \sigma : \tau \in S_1(I_i^\ell), \sigma \in S_1(I_j^\ell)\}$, where $S_1(f)$ is the set of sons belonging the parent cluster $f \in T_1$. This construction inherits the hierarchical structure of $T(I)$ and provides the recursive data access of optimal complexity. The tree T_2 contains a variety of large and small blocks. The block decomposition described later on will use only blocks contained in T_2 . Note that the general construction of hierarchical trees $T_1 = T(I)$ and $T_2 = T(I \times I)$ for an arbitrary index set I is introduced in [5,6]. Here we concentrate only on the particular examples which, however, illustrate the main features of the general framework.

The hierarchical format of an \mathcal{H} -matrix is based on a particular partitioning P_2 of $I \times I$ satisfying certain admissibility conditions. The latter will guarantee the optimal approximation.

2.3. Admissible block partitionings P_2 and \mathcal{H} -matrices

A partitioning $P_2 \subset T_2$ is a set of disjoint blocks $b \in T_2$ such that the union of all blocks from P_2 yields $I \times I$. The partitioning P_2 is usually built by a recursive construction involving implicitly an admissibility condition. The latter incorporates characteristics of the singularity locations of the kernel function $s(x, y)$, $x, y \in \Sigma$, and provides the balance between the size of matrix blocks and their distance from the singularity points.

For a globally smooth kernel as the *first example* s_1 in (2), we need no admissibility restriction; therefore the biggest block $I \times I$ is already admissible resulting in the simplest partitioning $P_2 = \{I \times I\}$. As we have seen above, this block will be filled by a *rank-k* matrix.

In the *second example* (kernel s_2), we use the following admissibility condition: a block $\tau \times \sigma$ with $\tau, \sigma \in T_1$ belongs to P_2 if

$$\min\{\text{diam}(\tau), \text{diam}(\sigma)\} \leq 2\eta \max(\text{dist}(\tau, 0), \text{dist}(\sigma, 0)), \tag{4}$$

where $\eta \leq 1$ is a given threshold parameter responsible for the approximation. Let, e.g., $\eta = \frac{1}{2}$. The block $I \times I$ is not admissible and must be decomposed into its four sons (see Fig. 1a). Three of them already satisfy (4), and only one must be refined further on. Finally, we obtain the block partitioning

$$P_2 = \{I_{ij}^\ell \in T_2: 0 < \ell < p, \max\{i, j\} = 2\} \cup \{I_{11}^p\}.$$

The block-matrix corresponding to $b \in P_2$ is denoted by $M^b := (m_{\alpha\beta})_{(\alpha,\beta) \in b}$. The level number ℓ of a block b is written as $\text{level}(b)$.

In the case of $s_3(x, y)$, the admissibility condition is more restrictive because we have the singularity of the kernel in each diagonal point $x = y$ of the product domain $\Sigma \times \Sigma$. Now $\tau \times \sigma$ belongs to P_2 if

$$\min\{\text{diam}(\tau), \text{diam}(\sigma)\} \leq 2\eta \text{dist}(\tau, \sigma), \quad \eta < 1. \tag{5}$$

For the choice $\eta = \frac{1}{2}$, we obtain a block partitioning $P_2 := \bigcup_{\ell=2}^p P_2^\ell$, where $P_2^\ell = \{I_{14}^2\} \cup \{I_{41}^2\}$ and

$$P_2^\ell = \{I_{ij}^\ell \in T_2: |i - j| \geq 1 \text{ and } I_{ij}^\ell \cap P_2^{\ell'} = \emptyset, \ell' < \ell\} \quad \text{for } \ell = 3, \dots, p.$$

So far, we have given an explicit definition of the partitioning P_2 . In the following, we describe a recursive definition¹ which leads to the same partitioning.

Now, we consider families of three different matrix formats: \mathcal{R} , \mathcal{N} , and \mathcal{D} which correspond to P_2 -partitionings in the above-mentioned examples. Here “ \mathcal{D} ” is the abbreviation for the case with diagonal singularities. \mathcal{R} -matrices are matrices of rank $\leq k$. The value of k is thought to be much less than the problem (or block) size, in particular, the choice $k = O(\log n)$ is sufficient for the optimal order approximation. The \mathcal{R} -matrices can be represented in the form

$$\sum_{i=1}^k [a_i, c_i] \quad \text{where } [a_i, c_i] := a_i * c_i^H,$$

with column vectors a_i and row vectors c_i^H . We abbreviate by $n_\ell = 2^{p-\ell}$ the problem size on the level ℓ . The set of real \mathcal{R} -matrices of the size n_ℓ is denoted by $\mathcal{M}_{\mathcal{R}} \subset \mathbb{R}^{n_\ell \times n_\ell}$. This class gives the trivial example of \mathcal{H} -matrices of the rank k .

The class $\mathcal{M}_{\mathcal{N}} \subset \mathbb{R}^{n_\ell \times n_\ell}$, $\ell = p, \dots, 1$, of \mathcal{N} -matrices serves for the approximation of the operators with the kernel $s_2(x, y)$ having only one singularity point $x = y = 0$ in $\Sigma \times \Sigma$. For $\ell = p$, \mathcal{N} -matrices are simple 1×1 matrices. Then we define the \mathcal{N} -format recursively for the levels $\ell = p - 1, \dots, 1$. An $n_\ell \times n_\ell$ matrix M has the \mathcal{N} -format if

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \quad \text{with } \frac{n_\ell}{2} \times \frac{n_\ell}{2}\text{-blocks } M_{ij}, \quad i, j = 1, 2, \tag{6}$$

where $M_{11}, M_{12}, M_{22} \in \mathcal{M}_{\mathcal{R}}$ and $M_{21} \in \mathcal{M}_{\mathcal{N}}$. Similarly, we define the transposed format: M is an \mathcal{N}^* -matrix if M^T has the \mathcal{N} -format. This format may be applied in the case of one singular point of $s(x, y)$ at $x = y = 1$. The sets of \mathcal{N} - and \mathcal{N}^* -matrices are denoted by $\mathcal{M}_{\mathcal{N}}$ and $\mathcal{M}_{\mathcal{N}^*}$, respectively.

Finally, the class $\mathcal{M}_{\mathcal{D}}$ of \mathcal{H} -matrices of the \mathcal{D} -format is defined by the following recursion. Let $M \in \mathbb{R}^{n_\ell \times n_\ell}$ with $\ell = p, \dots, 1$. For $\ell = p$, $\mathcal{M}_{\mathcal{D}}$ contains all 1×1 matrices. For $\ell = p - 1, \dots, 1$, an $n_\ell \times n_\ell$ -matrix M belongs to $\mathcal{M}_{\mathcal{D}}$ if

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \quad \text{with } M_{11}, M_{22} \in \mathcal{M}_{\mathcal{D}}, \quad M_{12} \in \mathcal{M}_{\mathcal{N}}, \quad M_{21} \in \mathcal{M}_{\mathcal{N}^*}, \tag{7}$$

where all block-matrices M_{ij} are of the size $n_\ell/2 \times n_\ell/2$. In the case of $p = 4$, the resulting block structure of an 16×16 matrix is given in Fig. 1. The partitionings defined above correspond to the

¹ The explicit and recursive definitions are possible for the model problems discussed. In general, there is an algorithm for computing the minimal admissible partitioning (see [3]).

choice $\eta = \frac{1}{2}$ in the related admissibility conditions (4) and (5). This provides the approximation order $O(\eta^m)$ with the appropriate choice $m = O(\log n)$, see Section 3. Note that if the partitioning P_2 is given a priori, then, we obtain the following explicit definition of \mathcal{H} -matrices.

Definition 1. Let a block partitioning P_2 of $I \times I$ and $k < n = 2^p$ be given. The set of real \mathcal{H} -matrices induced by P_2 and k is

$$\mathcal{M}_{\mathcal{H},k}(I \times I, P_2) := \{M \in \mathbb{R}^{I \times I} : \forall b \in P_2, \text{ there holds } \text{rank}(M^b) \leq k\}. \tag{8}$$

Note that \mathcal{H} -matrices with block-dependent rank (e.g., $k(b) := a_1 \text{level}(b) + a_2$) can also be considered, cf. [9]. In [9], a special hierarchical construction of bases $\{a_i\}, \{c_i\}$ for the block-matrices M^b leads to an $O(n)$ complexity of both the memory and the matrix–vector multiplication.

3. Reliability of \mathcal{H} -matrix approximations in BEM

The \mathcal{H} -matrices provide sparse discretisations of integral operators. In this section, we show that the hierarchical matrices are also dense enough, i.e., they lead to the same asymptotically optimal approximations as the exact FE/BE Galerkin schemes. We consider the typical BEM applications, where integral operators of the form

$$(Au)(x) = \int_{\Sigma} s(x, y)u(y) \, dy, \quad x \in \Sigma$$

occur with s being the fundamental solution (singularity function) associated with the p.d.e. under consideration or with s replaced by a suitable directional derivatives Ds of s . Here Σ is either a bounded $(d - 1)$ -dimensional manifold (surface) or a bounded domain in \mathbb{R}^d , $d = 2, 3$. The \mathcal{H} -matrix techniques exploit the block-wise approximation of s by a degenerate kernel based on the smoothness properties of the singularity function s (cf. [4, Definition 3.3.3]). This holds for s as well as for $\partial s(x, y)/\partial n(x)$ or $\partial s(x, y)/\partial n(y)$ (double-layer kernel and its adjoint; cf. [4, (8.1.31a,b)]) even if the normal vector n is nonsmooth (because of the nonsmoothness of the surface Σ). More precisely, we assume that the singularity function s satisfies²

$$|\partial_x^\alpha \partial_y^\beta s(x, y)| \leq c(|\alpha|, |\beta|)|x - y|^{-|\alpha| - |\beta|} g(x, y) \quad \text{for all } |\alpha|, |\beta| \leq m \tag{9}$$

and for all $x, y \in \mathbb{R}^d$, $x \neq y$, where α, β are multi-indices with $|\alpha| = \alpha_1 + \dots + \alpha_d$. We consider two particular choices of the (singular) function $g \geq 0$ defined also on $\Gamma \times \Gamma$. The first case $g(x, y) = |s(x, y)|$ is discussed in [6]. The second choice to be discussed is $g(x, y) = |x - y|^{1-d-2r}$. Here $2r \in \mathbb{R}$ is the order of the integral operator $A : H^r(\Gamma) \rightarrow H^{-r}(\Gamma)$. Similar smoothness prerequisites are usually required in the wavelet or multi-resolution techniques (cf. [1,13]). We shall give a simple example how the above assumption on the kernel implies the local expansions of the form

$$s_{\tau, \sigma} = \sum_{j=1}^k a_j(x)c_j(y), \quad (x, y) \in \tau \times \sigma \tag{10}$$

²In the case $g(x, y) = |s(x, y)|$, estimate (9) is a bit simplified. It covers most of the situations, e.g., the case of the singularity function $(1/4\pi)|x - y|^{-1}$ for $d = 3$. As soon as logarithmic terms appear (as for $d = 2$; $s(x, y) = \log(x - y)/2\pi$), one has to modify (9). A simple modification is also required for the single-layer potential on polyhedrons.

for each cluster $\tau \times \sigma \in P_2$, where k is the order of expansion. Then, we prove the consistency error estimate. We refer to [2] on the familiar multipole expansions of the form (10) applied in the case of the Laplace equation.

By Definition 1, \mathcal{H} -matrices are composed locally (blockwise) of rank- k matrices. These low rank matrices can be constructed by means of separable representations (10). In turn, the latter can be obtained, for example, by polynomial approximation with the Taylor expansion³ of $s(x, y)$. Alternatively, the local L^2 -projection onto the set of polynomials as well as the multipole-type expansions (the latter are only available for special kernels like $(1/4\pi)|x - y|^{-1}$ for $d = 3$) may be also applied.

Let x, y vary in the respective sets $X(\tau)$ and $X(\sigma)$ corresponding to the admissible clusters $\tau, \sigma \in T_1$ (cf. Section 2.2) and assume, without loss of generality, that $\text{diam}(X(\sigma)) \leq \text{diam}(X(\tau))$. The optimal centre of expansion is the Chebyshev centre⁴ y_* of $X(\sigma)$, since then $\|y - y_*\| \leq \frac{1}{2} \text{diam}(X(\sigma))$ for all $y \in X(\sigma)$. The Taylor expansion reads $s(x, y) = \tilde{s}(x, y) + R$ with the polynomial

$$\tilde{s}(x, y) = \sum_{|v|=0}^{m-1} \frac{1}{v!} (y_* - y)^v \frac{\partial^v s(x, y_*)}{\partial y^v} \tag{11}$$

and the remainder R , which can be estimated by

$$|R| = |s(x, y) - \tilde{s}(x, y)| \leq \frac{1}{m!} \|y_* - y\|^m \max_{\zeta \in X(\sigma), |\gamma|=m} \left| \frac{\partial^\gamma s(x, \zeta)}{\partial \zeta^\gamma} \right|. \tag{12}$$

Below, we recall the familiar approximation results based on the Taylor expansions (see, e.g., [6] for the proof).

Lemma 2. *Assume that (9) is valid and that the admissibility condition (5) holds with η satisfying $c(0, 1)\eta < 1$. Then for $m \geq 1$, the remainder (12) satisfies the estimate*

$$|s(x, y) - \tilde{s}(x, y)| \leq \frac{c(0, m)}{m!} \eta^m \max_{y \in X(\sigma)} |g(x, y)|, \quad x \in X(\tau), \quad y \in X(\sigma). \tag{13}$$

Let $A_{\mathcal{H}}$ be the integral operator with s replaced by $\tilde{s}(x, y)$ for $(x, y) \in X(\tau) \times X(\sigma)$ provided that $\tau \times \sigma \in P_2$ is an admissible block and no leaf. Construct the Galerkin system matrix from $A_{\mathcal{H}}$ instead of A . The perturbation of the matrix induced by $A_{\mathcal{H}} - A$ yields a perturbed discrete solution of the initial variational equation

$$\langle (\lambda I + A)u, v \rangle = \langle f, v \rangle \quad \forall v \in W := H^r(\Sigma), \quad r \leq 1,$$

where $\lambda \in \mathbb{R}$ is a given parameter. The effect of this perturbation in the panel clustering methods is studied in several papers (cf. [10–12]). Here we give the consistency error estimate for the \mathcal{H} -matrix approximation. Define the integral operator \hat{A} with the kernel

$$\hat{s}(x, y) := \begin{cases} \max_{y \in \sigma} |g(x, y)| & \text{for } (x, y) \in X(\tau) \times X(\sigma), \quad \tau \times \sigma \in P_2, \\ 0 & \text{otherwise.} \end{cases} \tag{14}$$

³ This does not require that the practical implementation has to use the Taylor expansion. If the singular-value decomposition technique from [5] is applied, the estimates are at least as good as the particular ones for the Taylor expansion.

⁴ Given a set X , the Chebyshev sphere is the minimal one containing X . Its centre is called the *Chebyshev centre*.

For the given ansatz space $W_h \subset W$ of piecewise constant/linear FEs, consider the perturbed Galerkin equation for $u_{\mathcal{H}} \in W_h$,

$$\langle (\lambda I + A_{\mathcal{H}})u_{\mathcal{H}}, v \rangle = \langle f, v \rangle \quad \forall v \in W_h.$$

In the following we use a bound on the discrete operator norm $\|\hat{A}\|_{W_h \rightarrow W'_h}$ appearing in

$$|\langle \hat{A}u, v \rangle| \leq \|\hat{A}\|_{W_h \rightarrow W'_h} \|u\|_W \|v\|_W, \quad \forall u, v \in W_h. \tag{15}$$

Lemma 3. *Assume that (9) is valid. Suppose that the operator $\lambda I + A \in \mathcal{L}(W, W')$ is W -elliptic. Then there holds*

$$\|u - u_{\mathcal{H}}\|_W \leq c \left\{ \inf_{v_h \in V_h} \|u - v_h\|_W + \frac{c(0, m)}{m!} \eta^m \|\hat{A}\|_{W_h \rightarrow W'_h} \|u\|_W \right\}.$$

The norm of \hat{A} is estimated by

$$\|\hat{A}\|_{W_h \rightarrow W'_h} \lesssim \begin{cases} \|A\| & \text{if } g = s(x, y) \wedge s(x, y) \geq 0, \\ \delta(d, r) h^{\min\{0, r\}} & \text{if } g = |x - y|^{1-d-2r}, \end{cases} \tag{16}$$

where (with $\varepsilon = 1 - d - 2r$)

$$\delta(d, r) := \left(\sum_{l=0}^p 2^{2(l-p)\varepsilon} \right)^{1/2} = \begin{cases} O(1), & \varepsilon > 0, \\ O(p), & \varepsilon = 0, \\ O(h^\varepsilon), & \varepsilon < 0. \end{cases}$$

Proof. The continuity and strong ellipticity of A imply

$$\|u - u_{\mathcal{H}}\|_W \lesssim \inf_{v \in W_h} \|u - v\|_W + \sup_{u, v \in W_h} \frac{|\langle (A - A_{\mathcal{H}})u, v \rangle|}{\|u\|_W \|v\|_W} \|u_{\mathcal{H}}\|_W$$

(cf. first Strang Lemma). On the other hand, under assumption (9), Lemma 2 yields

$$|\langle (A - A_{\mathcal{H}})u, v \rangle| \lesssim \frac{c(0, m)}{m!} \eta^m \|\hat{A}\|_{W_h \rightarrow W'_h} \|u\|_W \|v\|_W, \quad u, v \in W_h.$$

Indeed,

$$\begin{aligned} |\langle (A - A_{\mathcal{H}})u, v \rangle| &\leq \frac{c(0, m)}{m!} \eta^m \sum_{\tau \times \sigma \in P_2} \int_{X(\tau) \times X(\sigma)} |\hat{s}(x, y) u(y) v(x)| \, dx \, dy \\ &\lesssim \frac{c(0, m)}{m!} \eta^m \|\hat{A}\|_{W_h \rightarrow W'_h} \|u\|_W \|v\|_W. \end{aligned} \tag{17}$$

Now, assuming that $(c(0, m)/m!) \eta^m \|\hat{A}\|_{W_h \rightarrow W'_h}$ is sufficiently small, estimate (16) and $\eta < 1$ imply the strong ellipticity of the discrete Galerkin operator yielding the stability $\|u_{\mathcal{H}}\|_W \leq c \|u\|_W$. Note that in the case $g = s(x, y)$, the first assertion in (16) follows from the bound $\| |u| \|_W \leq \|u\|_W$ for all $u \in W_h$. In the case $g = |x - y|^{1-d-2r}$ and $r \geq 0$, bound (16) follows from the direct estimate based on the essential properties of the admissible partitioning P_2 : $\text{diam}(\tau) = O(2^{-\ell})$, $\tau \in P_2^\ell$ and $\#P_2^\ell = O(2^{d\ell})$. In the case $r < 0$, we first obtain an estimate with the constant $\delta(d, r)$ in the L^2 -norm. Then, applying the inverse inequality $\|v\|_{L^2(\Gamma)} \lesssim h^r \|v\|_{H^r(\Gamma)}$, $v \in W_h$ completes our proof. \square

The block Rk -approximation in the Galerkin method may be computed as the block entry $\mathcal{A}_{\mathcal{H}}^{\tau \times \sigma}$ of the stiffness matrix $\mathcal{A}_{\mathcal{H}} := \{\langle A_{\mathcal{H}} \varphi_i, \varphi_j \rangle\}_{i,j=1}^N$ associated with each cluster $\tau \times \sigma$ on the level ℓ may be presented as a rank- k matrix

$$\mathcal{A}_{\mathcal{H}}^{\tau \times \sigma} = \sum_{|v|=0}^{m-1} a_v * b_v^T \quad \text{where } k := \binom{d_{\Sigma} + m - 1}{m - 1} = O((m - 1)^{d_{\Sigma}})$$

is the number of terms and

$$a_v = \left\{ \int_{X(\tau)} (y - y_*)^v \varphi_i(y) \, dy \right\}_{i=1}^{N_{\tau}}, \quad b_v = \left\{ \int_{X(\sigma)} \frac{\partial^v s(x, y_*)}{\partial y^v} \varphi_j(x) \, dx \right\}_{j=1}^{N_{\sigma}}.$$

Here $N_{\tau} = \#\tau = O(2^{d_{\Sigma}(p-\ell)})$ (resp. $N_{\sigma} = \#\sigma = O(2^{d_{\Sigma}(p-\ell)})$) is the cardinality of τ (resp. σ). Note that in BEM applications, we have $d_{\Sigma} = d - 1$, while for volume integral calculations there holds $d_{\Sigma} = d$.

4. \mathcal{H} -matrices on tensor-product meshes

4.1. Partitioning of tensor-product index set]Proof

In $\Omega = (0, 1)^d$ with $d = 1, 2, 3$, we consider the regular grid

$$I = \{\mathbf{i} = (i_1, \dots, i_d) : 1 \leq i_k \leq N, k = 1, \dots, d\}, \quad N = 2^p. \tag{18}$$

We define the norms $|\mathbf{i}|_{\infty} = \max_{1 \leq n \leq d} |i_n|$ and $|\mathbf{i}|_1 = \sum_{n=1}^d |i_n|$. Each index $\mathbf{i} \in I$ is identified with the (collocation) point $\xi_{i_1 \dots i_d} = ((i_1 - \frac{1}{2})h, \dots, (i_d - \frac{1}{2})h) \in \mathbb{R}^d$, where $h := 1/N$ and the value $\xi_{\mathbf{i}} = \xi_{i_1 \dots i_d}$ is the midpoint of the support $X_{\mathbf{i}}$ of the basis function $\varphi_{\mathbf{i}}$ in the FE or BE method considered (cf. (19) below).

The cluster tree $T_1 = T(I)$ of I uses a division of the underlying cubes into 2^d subcubes. The blocks

$$t_{\mathbf{j}}^{\ell} = \{\mathbf{i} : 2^{p-\ell} j_1 + 1 \leq i_1 \leq 2^{p-\ell} (j_1 + 1), \dots, 2^{p-\ell} j_d + 1 \leq i_d \leq 2^{p-\ell} (j_d + 1)\}$$

for $\mathbf{j} \in \{0, \dots, 2^{\ell} - 1\}^d$ belong to level ℓ . $S_1(t_{\mathbf{j}}^{\ell-1}) := \{t_{\mathbf{j}'}^{\ell} : 0 \leq 2j'_k - i_k \leq 1 (1 \leq k \leq d)\}$ defines the set of sons of the cluster $t_{\mathbf{j}}^{\ell-1}$. Hence, the tree T_1 consisting of all blocks at all levels $\ell \in \{0, \dots, p\}$ is a binary, quad- or octree for $d = 1, 2, 3$, respectively. The number of clusters on level ℓ equals $O(2^{d\ell})$.

Each index $\mathbf{i} \in I$ is associated with the d -dimensional cube⁵

$$X_{\mathbf{i}} := \{(x_1, \dots, x_d) : (i_1 - 1)h \leq x_1 \leq i_1 h, \dots, (i_d - 1)h \leq x_d \leq i_d h\}, \tag{19}$$

which may be considered as the support of the piecewise constant function for the index \mathbf{i} . Using the Euclidean norm, we obtain the diameter $\text{diam}(t) = \sqrt{d} 2^{p-\ell} h = \sqrt{d}/2^{\ell}$ for blocks of level ℓ . Let τ, σ be two blocks of level ℓ characterised by \mathbf{j} and \mathbf{j}' , i.e., $\tau = t_{\mathbf{j}}^{\ell}$, $\sigma = t_{\mathbf{j}'}$. Then

$$\text{dist}(\tau, \sigma) = 2^{-\ell} \sqrt{\delta(j_1 - j'_1)^2 + \dots + \delta(j_d - j'_d)^2} \tag{20}$$

⁵ The grid can also be associated with a regular *triangulation* and, e.g., the supports $X_{\mathbf{i}}$ of piecewise linear functions, see Section 5. The asymptotic complexities turn out to be the same as for the present choice.

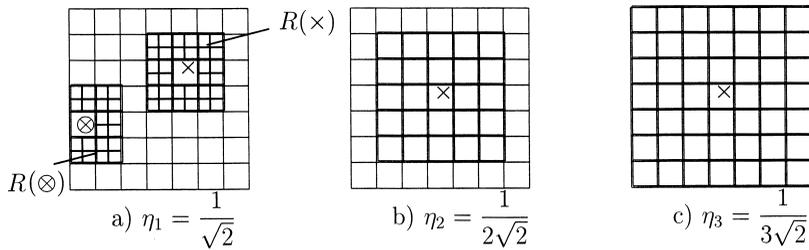


Fig. 2. Unacceptable region for the given clusters “x”, “⊗” depending on the threshold constant η .

with $\delta(\xi) := \max\{0, |\xi| - 1\}$. Let the block-cluster tree $T_2 = T(I \times I)$ be defined in accordance with the cluster tree $T_1 = T(I)$ (see [5] for more details). An important property is stated in

Remark 4. Let $\tau \times \sigma \in T(I \times I)$. Then $\tau, \sigma \in T(I)$ belong to the same level $\ell \in \{0, \dots, p\}$.

In view of this remark, for $\ell \in \{0, \dots, p\}$, we denote by T_2^ℓ the set of clusters $\tau \times \sigma \in T_2$ such that blocks τ, σ belong to level ℓ . In particular, $T_2^0 = \{I \times I\}$ is the root of T_2 and $T_2^p = \{(x, y) : x, y \in I\}$ is the set of leaves. The set of clusters $t \in T(I)$ from level ℓ is called T_1^ℓ . In the following we consider the choice

$$\eta = \eta_\mu = \frac{\sqrt{d}}{2\mu}, \quad \mu \in \mathbb{N} \tag{21}$$

of η . Note that increasing μ yields arbitrarily small values of η .

Using $\min\{\text{diam}(t_1), \text{diam}(t_2)\} = \sqrt{d}/2^\ell$ and $\text{dist}(t_1, t_2)$ from (20), we observe that $t \in T(I \times I)$ is admissible for the choice (21) if the squares $X_1 = X(\tau)$, $X_2 = X(\sigma)$, $\tau, \sigma \in T(I)$ have a relative position as indicated in Figs. 2a–c corresponding to $\mu = 1, 2$ and 3 , respectively, with $d = 2$. The square $X(\tau)$ corresponding to τ is the crossed square, while $X(\sigma)$ must be outside the bold area. In the case of $d = 2$ and $\eta = 1/\sqrt{2}$, i.e., for $\mu = 1$, the admissible T_2 -partitioning P_2 was described in details in [6]. Note that the general Definition 5 of $\mathcal{M}_p^\square(p, \eta)$ -formats given below generalises the particular examples for $d = 1, 2, 3$ from [5,6].

4.2. Basic definitions

In this section, we introduce the general formats for matrices operating in the vector space \mathbb{K}^I for the cell-centred tensor product grid $I = I_h^d$ in $\Omega = (0, 1)^d$ with the mesh-size $h = 2^{-p}$, $\#I = 2^{dp}$ and $d = 1, 2, 3$. The natural notation of indices from $I = I_h^d$ is by multiindices $\mathbf{i} = (i_1, \dots, i_d)$ with $1 \leq i_n \leq N = 2^p$.

As in the particular cases in [5,6], we can describe the partitioning by a number of formats $\mathcal{M}_q^j = \mathcal{M}_q^j(p, \eta)$, where $q \in \{0, \dots, p\}$, η is parametrised by (21) and the multiindex $\mathbf{j} = (j_1, \dots, j_d)$ with $|\mathbf{j}|_\infty \leq \mu$ indicates a translation in the following sense. Let $b = \tau \times \tau' \in T_2^\ell$ be a block of level $\ell = p - q$. If $\tau = \tau'$, we have a diagonal block corresponding to the vanishing shift, i.e., $\mathbf{j} = \mathbf{0} = (0, \dots, 0)$. For these blocks we shall introduce the top format $\mathcal{M}_q^0 = \mathcal{M}_q^0(p, \eta)$. In general, let $\tau = (i_{01}, \dots, i_{0d}) + \{(i_1, \dots, i_d) : 1 \leq i_n \leq 2^\ell\}$ and $\tau' = (i_{01} + j_1 2^\ell, \dots, i_{0d} + j_d 2^\ell) + \{(i_1, \dots, i_d) : 1 \leq i_n \leq 2^\ell\}$ be two clusters (cubes of length 2^ℓ). Then their relation is given by the translation in direction

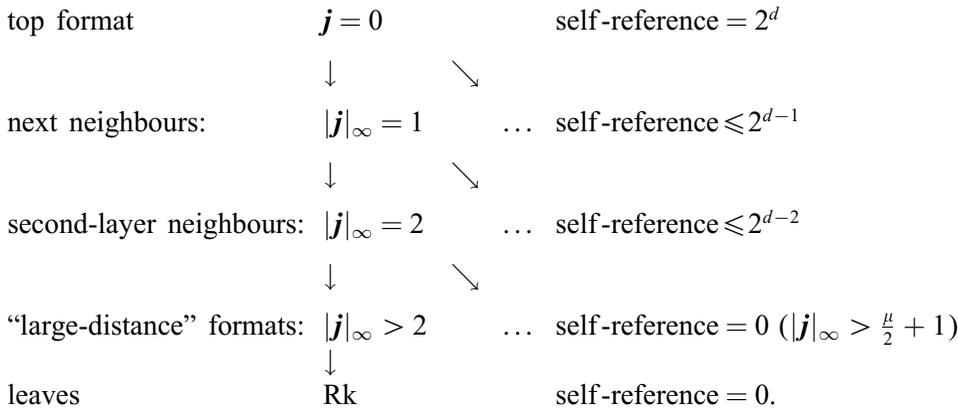
$\vec{j} = (j_1, \dots, j_d)^T$. We write $\tau' = \mathcal{T}_\ell^j \tau$, where \mathcal{T}_ℓ^j is the translation operator with respect to the vector $h_\ell \cdot \vec{j}$, $|\mathbf{j}|_\infty \leq 2^\ell - 1$ (due to (21), we actually have the bound $|\mathbf{j}|_\infty \leq \mu$ for non-admissible clusters), where $h_\ell = 2^{-\ell}$.

Let $\tau \in T_1$ be a cluster from level ℓ . The corresponding set of sons, $S_1(\tau) = \{\sigma_i\}_{i \in I_d}$ is associated with the set of multiindices I_d , where

$$I_d = \{\mathbf{k} \in \mathbb{N}^d : |\mathbf{k}|_\infty = 1 \wedge |\mathbf{k}|_1 = d\}, \quad \dim I_d = 2^d,$$

as depicted in Fig. 3a for $d = 3$, and Fig. 3b for $d = 2$. Equivalently, $S_1(\tau) = \{a, b, c, d, e, f, g, h\}$. This multiindex block numbering indicates the location of sons with respect to the centre of gravity of the parent cluster: $\text{cent}(\sigma_i) = \text{cent}(\tau) + \frac{1}{2} h_{\ell+1} \mathbf{i}$ for $\sigma_i \in S_1(\tau)$. For example, there holds $\sigma_{i_a} = a$ and $\sigma_{i_b} = b$ with the vector notations $\mathbf{i}_a = (-1, 1, 1)$, $\mathbf{i}_b = (1, 1, 1)$. The block-matrix with columns from a and rows from b is denoted by $A_{i_a i_b} \in \mathbb{K}^{a \times b}$. The examples of two-dimensional vectors are drawn in Fig. 3b, where, e.g., $\mathbf{i}_1 = (-1, 1)$, $\mathbf{i}_2 = (1, 1)$.

For block-clusters $\sigma \times \sigma' \in T_2^\ell$ from level $\ell = p - q$, where $\sigma' = \mathcal{T}_\ell^j \sigma$, $|\mathbf{j}|_\infty \leq \mu$, we define recursively for $q = 0, \dots, p$ the formats $\mathcal{M}_q^j = \mathcal{M}_q^j(p, \eta)$ of \mathcal{H} -matrices from $\mathbb{K}^{\sigma \times \sigma'}$ starting from $q = 0$ and ending with $q = p$. In this way, a family of auxiliary formats \mathcal{M}_q^j , with $|\mathbf{j}|_\infty \neq 0$ is involved, e.g., “next neighbours” ($|\mathbf{j}|_\infty = 1$), “2-layer neighbours” ($|\mathbf{j}|_\infty = 2$) and so on. In Definition 5 below these formats contain the same construction at the next level (“self-reference”) and other formats as depicted in the graph generalising the corresponding picture from [6]:



We underline that the matrix format \mathcal{M}_q^j does not depend on the particular choice of the cluster σ but it is only determined by the translation operator \mathcal{T}_{p-q}^j . Roughly speaking, each *format* under consideration actually specifies (in general, recursively) the location and size of Rk -blocks in the matrix array from $\mathbb{K}^{I \times I}$ corresponding to the given admissible partitioning P_2 of $I \times I$. The partitioning P_2 itself is generated implicitly by Definition 5 below. Here the basic parameters $p \in \mathbb{N}$ and $\mu \in \mathbb{N}$ are both fixed, so, we may skip them in the notation \mathcal{M}_q^j without ambiguity.

We recall that \mathcal{M}_q^R is a set of Rk -matrices of the size $2^{dq} \times 2^{dq}$, $q = 0, 1, \dots, p$. Now, we define our format in the following range of parameters: $q = 0, 1, \dots, p$ and $|\mathbf{j}|_\infty \leq 2^\ell - 1$, where $\ell = p - q$.

Definition 5. (a) For $q = 0, \dots, p$ and for all $|\mathbf{j}|_\infty \geq \mu + 1$, define the format \mathcal{M}_q^j by $\mathcal{M}_q^j = \mathcal{M}_q^R$.
 (b) For $q = 0$, define \mathcal{M}_0^j as the set of 1×1 matrices for all $|\mathbf{j}|_\infty \leq \mu$.

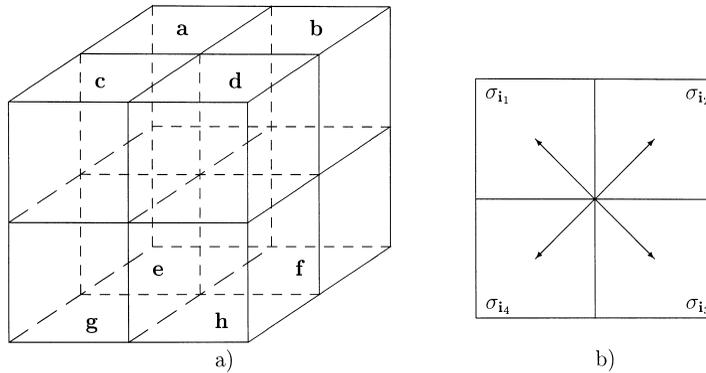


Fig. 3. (a) Multiindex labelling of sons of the 3D cluster, where $a=(-1, 1, 1)$, $b=(1, 1, 1)$, $c=(-1, -1, 1)$, $e=(-1, 1, -1)$, $d=(1, -1, 1)$, $f=(1, 1, -1)$, $g=(-1, -1, -1)$, $h=(1, -1, -1)$. (b) The ordering by local translations for 2D cell.

(c) Consider the case $q=1, \dots, p$ and $1 \leq |\mathbf{j}|_\infty \leq \mu$. To describe the recursion step, assume that for each $q \leq q_0$ with some $q_0 \geq 0$, the format $\mathcal{M}_q^{\mathbf{j}}$ is already defined for all $1 \leq |\mathbf{j}|_\infty \leq \mu$. In the following we define the format $\mathcal{M}_q^{\mathbf{j}}$ for $q = q_0 + 1$. Consider indices \mathbf{j} with $1 \leq |\mathbf{j}|_\infty \leq \mu$ and blocks $\sigma \times \sigma' \in T_2$ of level $l = p - q$ such that $\sigma' = \mathcal{F}_l^{\mathbf{j}} \sigma$.⁶

For the matrices from $\mathbb{K}^{\sigma \times \sigma'}$, we say that $A_{\sigma, \sigma'} = \{A_{i\bar{i}'}\}_{i \in I_d, i' \in I'_d}$ belongs to $\mathcal{M}_q^{\mathbf{j}}$, if $A_{i\bar{i}'} \in \mathcal{M}_{q-1}^{i'-i+2\mathbf{j}}$, where, due to the admissibility condition there holds $\mathcal{M}_{q-1}^{i'-i+2\mathbf{j}} \in \mathcal{M}_{q-1}^R$ for all indices from the range $|i' - i + 2\mathbf{j}|_\infty \geq \mu + 1$.

(d) Finally, for $\mathbf{j}=\mathbf{0}$, define the top formats $\mathcal{M}_q^{\mathbf{0}}$ for $q=1, \dots, p$. Let $\sigma \in T_1$ be from level $\ell = p - q$. Then we say that $A_{\sigma, \sigma} = \{A_{i\bar{i}}\}_{i \in I_d, i' \in I_d}$ belongs to $\mathcal{M}_q^{\mathbf{0}}$ if there holds $A_{i\bar{i}} \in \mathcal{M}_{q-1}^{\mathbf{0}}$ and $A_{i\bar{i}'} \in \mathcal{M}_{q-1}^{i'-i}$ for $i' \neq i$, where the auxiliary formats are already defined in item (c).

Note that the format $\mathcal{M}_p^{\mathbf{0}}$ introduced by Definition 5 reproduces (with different abbreviations) the particular constructions from [5,6] given for $d = 1, 2, 3$ and for $\mu = 1$.

4.3. Complexity estimates

In the following, we discuss the storage requirements \mathcal{N}_{st} and the cost \mathcal{N}_{MV} of the matrix–vector multiplication for the general $\mathcal{M}_p^{\mathbf{0}}(p, \eta)$ formats. The corresponding results for the particular cases $\mathcal{M}_p^{\mathbf{0}}(p, \sqrt{d}/2)$ were presented in [5,6].

Note that the maximal level number p is $\leq O(|\log h|)$. In the following, we call a pair of one addition and one multiplication a *coupled operation*.

Theorem 6. Let $d \in \{1, 2, 3\}$, $A \in \mathcal{M}_{p,k}^{\mathbf{0}}(p, \eta)$ and $\eta = \eta_\mu := \sqrt{d}/2\mu$, $\mu \in \mathbb{N}$. Then the matrix–vector multiplication complexity is bounded by

$$\mathcal{N}_{MV} \leq (2^d - 1)(\sqrt{d}\eta^{-1} + 1)^d pkn \tag{22}$$

⁶ As above, we use the local numbering of sons $S(\sigma) = \{\sigma_i\}_{i \in I_d}$ and $S(\sigma') = \{\sigma'_{i'}\}_{i' \in I'_d}$, where $I'_d = \mathcal{F}_l^{\mathbf{j}} I_d$.

coupled operations. There holds

$$\mathcal{N}_{st} \leq (2^d - 1)(\sqrt{d}\eta^{-1} + 1)^d pkn \tag{23}$$

for the storage requirements. Both estimates are asymptotically sharp.

Proof. Recall that the matrix–vector multiplication with matrices from \mathcal{M}_p^R costs $2kn$ multiplications and kn additions. For each $\tau \in T_1^\ell$, we introduce the set of nonadmissible clusters $R(\tau)$ by

$$R(\tau) := \{\tau' \in P_1^\ell, \tau' \neq \tau : \text{diam}(\tau') > 2\eta_\mu \text{dist}(\tau', \tau)\}.$$

For any son $\sigma \in S(\tau)$, the number $Q_\sigma := \#\{b \in P_2 : b = \tau \times \sigma\}$ of Rk -blocks in the block-matrix row of A and associated with a cluster position σ is majorised by the corresponding one for the case of purely “interior” cluster τ .⁷ Particularly, Q_σ equals the number of sons $\sigma_{\tau'} \in S(\tau')$ from the set of clusters $\tau' \in T_1^\ell$, which are neighboured to τ and satisfy the admissibility condition with σ ,

$$Q_\sigma := \#\{\sigma_{\tau'} : \sigma_{\tau'} \in S(\tau'), \tau' \in R(\tau), \sigma \times \sigma_{\tau'} \text{ satisfies (5)}\}.$$

In the case of purely “interior” clusters, the direct calculation shows that this number is equal to $Q_\tau = (2^d - 1)(2\mu + 1)^d$. Now the multiplication complexity of all Rk -blocks from the given level ℓ amounts to $2^\ell Q_\tau$ multiplications of Rk -blocks with a vector of the dimension $n2^{-\ell}$. Moreover, we have the summation of intermediate results located in the block columns which costs $(k - 1)Q_\tau$ additions of full n -dimensional vectors. This exactly results in the constant 2 for counting the *coupled* operations. To prove the sharpness of this bound, we note that the number of “nearly boundary” clusters⁸ on each level ℓ is estimated by $O(2^{(d-1)\ell})$. Thus, the complexity count for the corresponding matrix blocks is dominated by the value $O(\sum_{\ell=0}^p 2^{-\ell} kn) = O(kn)$ which shows that (22) is asymptotically sharp. The bound (23) is proven along the same line. \square

Remark 7. It is clear by the construction that using linear/bilinear elements disturbs the parameter η only slightly. In fact, the perturbed parameter is estimated by $\eta_{\text{new}} = \eta + ch < 1$ for small enough h . Then all the previous constructions remain verbatim with the corresponding modifications.

In view of above remark, we need also the construction based on the truncated tree. For a level number $p_0 \in \{0, \dots, p\}$, we call the T_2 -partitioning P_2^* a p_0 -truncation of P_2 if it is obtained from the smaller tree $T_2^* \subset T_2$ by deleting all vertices belonging to levels $\ell > p - p_0$ and inserting the sons of size 1×1 (leaves) for all nonadmissible blocks of the initial tree T_2 at level $\ell = p - p_0$, i.e., $\tau \in T_2^{p-p_0}$ has the sons $S(\tau) = \{\{i\} : i \in \tau\}$. By assumption, all nonadmissible blocks of level $\ell = p - p_0$ are full submatrices. Clearly, a treatment of these blocks costs $2^{dp_0}(2\mu + 1)^d n$ operations. This yields the following estimates (24) and (25) for the p_0 -truncated partitioning: the matrix–vector product costs

$$\mathcal{N}_{MV} \leq (2^d - 1)(\sqrt{d}\eta^{-1} + 1)^d (p - p_0)kn + 2^{dp_0-1}(\sqrt{d}\eta^{-1} + 1)^d n \tag{24}$$

⁷ The purely “interior” cluster $\sigma \in T_1$ from level ℓ is defined to satisfy $\text{dist}(\sigma, \partial\Omega) \geq \mu 2^{-\ell}$, see an example with the cluster “ \times ” in Fig. 2.

⁸ The “nearly boundary” cluster $\sigma \in T_1$ from level ℓ is defined to satisfy $\text{dist}(\sigma, \partial\Omega) \leq (\mu - 1)2^{-\ell}$, see an example in Fig. 2.

coupled operations; for the storage needs there holds

$$\mathcal{N}_{st} \leq (2^d - 1)(\sqrt{d}\eta^{-1} + 1)^d (p - p_0)kn + 2^{dp_0}(\sqrt{d}\eta^{-1} + 1)^d n. \tag{25}$$

Remark 8. Bounds (24) and (25) allow the optimal choice $p_0 = O(\log k)$ of the parameter, which provides a balance between both summands on the right-hand sides. On the other hand, along the line of Section 3.6 in [6] and taking into account Theorem 6, we conclude that $\eta = O(1) < 1$ and $k = \log^\beta n$ with some $\beta = \beta(d)$ would be the optimal choice retaining the approximation order $O(h^\alpha)$, $\alpha > 0$, of the exact Galerkin scheme.

5. \mathcal{H} -matrices on triangular meshes

5.1. Translation operators on the index set I_Δ

The computational domain Ω is assumed to be composed of a finite number M of macrotriangles $\Omega_1, \dots, \Omega_M$. For the ease of presentation, we restrict our considerations to $\Omega = \Omega_1$, i.e., $M = 1$. We consider the index set $I = I_\Delta$ associated with the supports of piecewise constant elements. The index structure for the hierarchical triangulation is defined in accordance with Fig. 4. Fig. 4c illustrates the non admissible clusters with respect to $\tau_1 \in T(I)$ taken as a crossed triangle. Here all admissible clusters τ_2 must be outside the bold area restricted by Γ_μ and composed of μ cluster layers, where η is parametrised by $\eta = \eta_\mu = 2/3\mu$ with $\mu = 1, 2, 3, \dots$

The cluster tree $T_1 = T(I)$ is defined by a subdivision of each triangle into four equal parts. The admissible partitionings from the block cluster tree $T(I \times I)$ are determined by (4) with the constant $\eta = \eta_\mu$, see also Figs. 4b and c.

We identify the sons of a cluster $\sigma \in T_1^\ell$ in accordance with their relative locations, which will be described by the proper translation/reflection operators. In this way, we introduce the oriented clusters from $T_1 = \mathcal{A} \cup \mathcal{T}$: the subset \mathcal{A} contains clusters with the “standard” orientation, see Fig. 4a, while \mathcal{T} contains the set of reflected clusters with respect to the centre of gravity (e.g., σ_4 in Fig. 4a). Accordingly, we write $\mathcal{A}^\ell = \mathcal{A} \cap T_1^\ell$ and $\mathcal{T}^\ell = \mathcal{T} \cap T_1^\ell$. We also distinct the orientationally dependent and orientationally invariant transforms. The latter include simple translations π to be specified later on. The orientationally dependent (converting) maps include the identity operators $E_\mathcal{A}$ and $E_\mathcal{T}$ in the classes \mathcal{A} and \mathcal{T} , respectively, as well as reflection operators $\mathcal{S}_m : \mathcal{A} \rightarrow \mathcal{T}$, $\mathcal{S}_m^\mathcal{T} : \mathcal{T} \rightarrow \mathcal{A}$, $m = 1, 2, 3$

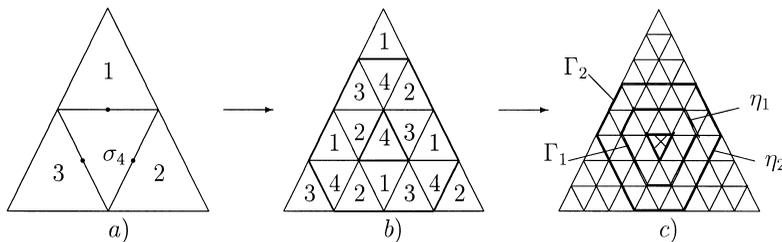


Fig. 4. The hierarchical triangulation: local ordering, nonadmissible clusters; $\eta_1 = \frac{2}{3}$, $\eta_2 = \frac{1}{3}$.

defined below. We shall also distinguish the mapping classes \mathcal{T}_A and \mathcal{T}_γ containing the maps from $A^\ell \rightarrow T_1^\ell$ and $\gamma^\ell \rightarrow T_1^\ell$, respectively.

Assume that the target cluster σ belongs to A . The son $\sigma_4 \in S(\sigma)$ (see Fig. 4a) belongs to $\gamma^{\ell+1}$ and it corresponds to the trivial translation operator E_γ , while $\sigma_i, i = 1, 2, 3$ belong to $A^{\ell+1}$. Let $\xi_i, i = 1, \dots, 4$, be the centres of gravity for the corresponding clusters σ_i providing $\xi_4 = \text{cent}(\sigma)$. Introduce the vectors $\mathbf{j}_{nm} = \xi_m - \xi_n$ and reflection transforms \mathcal{S}_m and \mathcal{S}_m^T with $n, m = 1, 2, 3$. \mathcal{S}_m maps the cluster σ_m into its symmetric image σ_4 with respect to the centre of common edge. Similarly, the transposed (inverse) mapping $\mathcal{S}_m^T: \sigma_4 \rightarrow \sigma_m$ may be introduced. The general translation $\pi = \pi^{j_\alpha}$ is defined as a shift by the vector $h_\alpha \mathbf{j}_\alpha$. Here, $\alpha \in \mathbb{N}_0^3$ such that $\mathbf{j}_\alpha := \alpha_1 \mathbf{j}_{13} + \alpha_2 \mathbf{j}_{21} + \alpha_3 \mathbf{j}_{32}$, where $\mathbf{j}_{13} + \mathbf{j}_{21} + \mathbf{j}_{32} = \mathbf{0}$. The general transforms $\mathcal{T}_1 \in \mathcal{T}_A, \mathcal{T}_2 \in \mathcal{T}_\gamma$ now take the form

$$\mathcal{T}_1 := \pi^{j_\alpha} (\mathcal{S}_m)^\beta, \quad \mathcal{T}_2 := \pi^{j_\alpha} (\mathcal{S}_m^T)^\beta, \quad \beta \in \{0, 1\}, \quad m \in \{1, 2, 3\}. \tag{26}$$

We call $\mathcal{T} \in \mathcal{S}_\mu^\ell$ if $|\mathcal{T}| \leq \mu$, where the ‘‘norm’’ is defined by $|\mathcal{T}| = \max\{|\beta|, |\alpha|_\infty\}$. This value measures the translation distance (shift) between σ and $\sigma' = \mathcal{T}\sigma$. Note that the transposed transform is defined by (say, for $\mathcal{T} \in \mathcal{T}_A$)

$$\mathcal{T}^T := (\mathcal{S}_m^T)^\beta \pi^{-j_\alpha}$$

yielding $\mathcal{T} \mathcal{T}^T = E_\gamma, \mathcal{T}^T \mathcal{T} = E_A$.

With the given $\mu \geq 1$, the nonadmissible area for the underlying cluster σ is then defined by

$$R(\sigma) := \{\mathcal{T}\sigma : 1 \leq |\mathcal{T}| \leq \mu\}.$$

For example, let $\sigma = \sigma_4 \in A^\ell$ be the smallest triangle located in the centre of the reference triangle drawn in Fig. 4b and choose $\eta_1 = \frac{2}{3}$. Then, nonadmissible clusters within the bold area $R(\sigma_4)$ are associated with the set of transforms $\{\pi^{\pm j_1}, \pi^{\pm j_2}, \pi^{\pm j_3}, \mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, \pi^{j_1} \mathcal{S}_1, \pi^{j_2} \mathcal{S}_2, \pi^{j_3} \mathcal{S}_3\} \in \mathcal{S}_1^\ell \setminus E_A$ corresponding to $\mu = 1$. Let $\sigma, \sigma' \in T_1^\ell$ with $\sigma' = \mathcal{T}\sigma$, where $\mathcal{T} \in \mathcal{S}_\mu^\ell$. For the matrix block $\sigma \times \sigma' \in \mathbb{K}^{\sigma \times \sigma'}$, we construct the family of formats $\mathcal{M}_{p-\ell}^{\mathcal{T}}(p, \mu) = \mathcal{M}_{p-\ell}^{\mathcal{T}}$, where the case $|\mathcal{T}| = 0$, i.e., $\mathcal{T} \in \{E_A, E_\gamma\} \in \mathcal{S}_0^\ell$, corresponds to the top format $\mathcal{M}_{p-\ell}^\Delta(p, \mu) = \mathcal{M}_{p-\ell}^\Delta$ if $\mathcal{T} = E_A$ and $\mathcal{M}_{p-\ell}^\nabla(p, \mu) = \mathcal{M}_{p-\ell}^\nabla$ if $\mathcal{T} = E_\gamma$.

To have a constructive definition, we need the recursive representation of $\mathcal{M}_{p-\ell}^{\mathcal{T}}$ in terms of matrices with smaller subindex $p - \ell - 1$. To that end, with each $\sigma' = \mathcal{T}_\ell \sigma$, we associate a 4×4 matrix of transforms on level $\ell + 1$ generated from \mathcal{T}_ℓ by a lifting mapping

$$\mathcal{L}^\ell : \mathcal{T}_\ell \rightarrow \{\mathcal{T}_{\ell+1}^{jj'}\}_{j,j'=1}^4, \quad \mathcal{T}_{\ell+1}^{jj'} \in \mathcal{S}_v^{\ell+1}, \quad 0 \leq v \leq 2|\mathcal{T}_\ell| + 1,$$

where $\mathcal{T}_{\ell+1}^{jj'} : \sigma_j \rightarrow \sigma_{j'}, \sigma_j \in S(\sigma), \sigma_{j'} \in S(\sigma')$. All the transforms $\mathcal{T}_{\ell+1}^{jj'}$ belong to class (26), where the specific parameters α, β and m are uniquely determined by the corresponding characteristics of \mathcal{T}_ℓ and by the choice of j and j' . In particular, according to Fig. 4a, the matrix-valued operator $\mathcal{L}^1(E_A) := \{\mathcal{T}^{jj'}\}$ has the form

$$\mathcal{L}^\ell(E_A) := \begin{array}{c|ccc} E_A & \pi^{j_2} & \pi^{-j_1} & \mathcal{S}_1 \\ \hline \pi^{-j_2} & E_A & \pi^{j_3} & \mathcal{S}_2 \\ \hline \pi^{j_1} & \pi^{-j_3} & E_A & \mathcal{S}_3 \\ \hline \mathcal{S}_1^T & \mathcal{S}_2^T & \mathcal{S}_3^T & E_\gamma \end{array}, \quad \mathcal{L}^\ell(E_\gamma) := \begin{array}{c|ccc} E_\gamma & \pi^{-j_2} & \pi^{j_1} & \mathcal{S}_1^T \\ \hline \pi^{j_2} & E_\gamma & \pi^{-j_3} & \mathcal{S}_2^T \\ \hline \pi^{-j_1} & \pi^{j_3} & E_\gamma & \mathcal{S}_3^T \\ \hline \mathcal{S}_1 & \mathcal{S}_2 & \mathcal{S}_3 & E_A \end{array}, \tag{27}$$

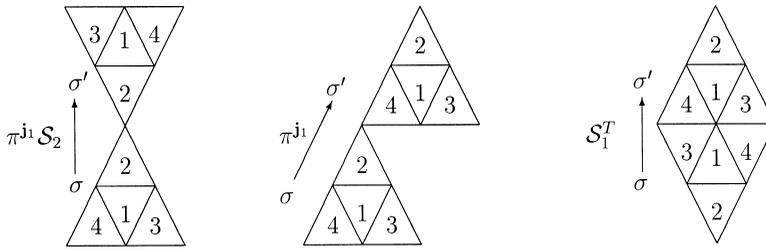


Fig. 5. Coupling of clusters corresponding to typical translations for $\mu = 1$.

where $\mathcal{T}^{jj'} \in \mathcal{S}_1^{\ell+1}$, $j, j' = 1, \dots, 4$. Having defined the lifting mapping \mathcal{L}^ℓ , we are looking for the recursive representation of the matrix structure (format) of the block $b = \sigma \times \mathcal{T} \sigma$ for $l = 0, \dots, p - 1$ and $\sigma \in T_1^\ell$,

$$A_{\sigma, \sigma'} = \{A_{jj'}\}_{j, j'=1, \dots, 4} \in \mathcal{M}_{p-\ell}^{\mathcal{T}_\ell} \quad \text{if} \quad A_{jj'} \in \mathcal{M}_{p-\ell-1}^{\mathcal{T}_{\ell+1}^{jj'}}.$$

While all the far-distance formats with $|\mathcal{T}_{\ell+1}^{jj'}| > \mu$ are supposed to have a $2^{p-\ell-1} \times 2^{p-\ell-1}$ Rk -matrix structure, the blocks corresponding to nonadmissible area $|\mathcal{T}_{\ell+1}^{jj'}| \leq \mu$ are to be defined in the next recurrence steps.

For example, let us consider the recursive block structure of a particular format $\mathcal{M}_p^\Delta(p, \frac{2}{3})$, where the initial index set belongs to the class \mathcal{A} . We choose $\eta_1 = 2/3$ and build the matrix-valued lifting transforms $\mathcal{L}(\mathcal{T}_\ell)$, $\mathcal{T}_\ell \in \mathcal{S}_1^\ell$ for typical neighbouring translations with $|\mathcal{T}_\ell| = 1$. Here and in the following, R denotes the class of translations with $|\mathcal{T}| \geq \mu + 1$ resulting in the Rk -matrix blocks on the corresponding level. The following schemes illustrate typical lifting transforms in the case of two clusters with one common vertex, see Fig. 5 (left and middle):

$$\mathcal{L}^\ell(\pi^{j_1} \mathcal{S}_2) := \begin{array}{c|c|c} \pi^{j_1} \mathcal{S}_2 & R & R & R \\ \hline R & R & R & R \\ \hline R & R & R & R \\ \hline R & R & R & R \end{array}, \quad \mathcal{L}^\ell(\pi^{j_1}) := \begin{array}{c|c|c|c} R & R & \pi^{j_1} & R \\ \hline R & R & R & R \\ \hline R & R & R & R \\ \hline R & R & R & R \end{array}. \tag{28}$$

The translation of sons for two adjacent clusters with one common edge has the following block (recursive) structure, see Fig. 5(right):

$$\mathcal{L}^\ell(\mathcal{S}_1^T) := \begin{array}{c|c|c} R & R & R \\ \hline R & \pi^{-j_3} \mathcal{S}_1^T & \mathcal{S}_1^T & \pi^{j_3} \\ \hline R & \mathcal{S}_1^T & \pi^{j_3} \mathcal{S}_1^T & \pi^{j_2} \\ \hline R & \pi^{j_1} & \pi^{-j_2} & \pi^{j_1} \mathcal{S}_2 \end{array}. \tag{29}$$

Using the nondiagonal lifting transforms defined by (28) and (29), we can describe the recursion for the identity transforms, see (27), which then generates the top formats, see the diagram in Fig. 6.

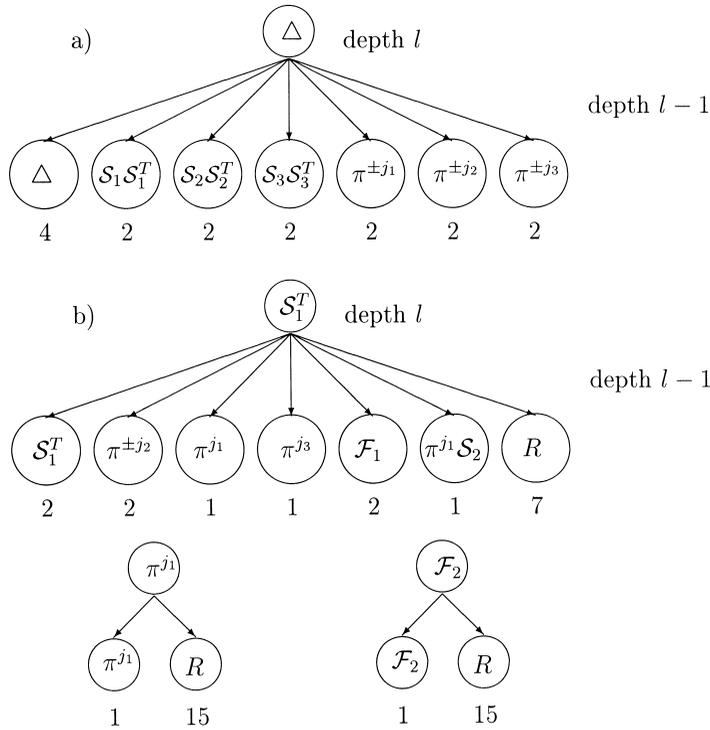


Fig. 6. The subtrees of the diagonal and typical auxiliary formats on I_Δ , where $\mathcal{F}_1 = \pi^{\pm j_3} \mathcal{S}_1^T$, $\mathcal{F}_2 = \pi^{j_1} \mathcal{S}_2$.

5.2. General definition and complexity of \mathcal{M}_p^Δ -formats

Corresponding to the case $\Omega \in \mathcal{A}$, we introduce the general \mathcal{M}_p^Δ -format, where the level number $p \in \mathbb{N}$ is a fixed parameter. If the target domain $\Omega \in \mathcal{Y}$, the format \mathcal{M}_p^∇ may be defined along the same line. Recall \mathcal{M}_q^R as a set of Rk -matrices of the size $4^q \times 4^q$, $q = 0, 1, \dots, p$.

- Definition 9.** (a) For $q = 0, \dots, p$ define $\mathcal{M}_q^\mathcal{T} = \mathcal{M}_q^R$ for all $|\mathcal{T}| \geq \mu + 1$.
 (b) For $q = 0$, define $\mathcal{M}_0^\Delta(p, \eta)$ and $\mathcal{M}_0^\nabla(p, \eta)$ as the sets of 1×1 matrices for all $|\mathcal{T}| \leq \mu$.
 (c) Consider the case $q = 1, \dots, p$ and $1 \leq |\mathcal{T}| \leq \mu$. Assume that for each $q \leq q_0$ the format $\mathcal{M}_q^\mathcal{T}$ is already defined for all $\mathcal{T} : |\mathcal{T}| \leq \mu$, and define the formats $\mathcal{M}_q^\mathcal{T}$ for $q = q_0 + 1$. Consider translation $\mathcal{T} \in \mathcal{S}_\mu^\ell$ with $|\mathcal{T}| = \mu, \mu - 1, \dots, 1$ and blocks $\sigma, \sigma' \in T_1^\ell$ of level $\ell = p - q$ such that $\sigma' = \mathcal{T}\sigma$. For the matrices from $\mathbb{R}^{\sigma \times \sigma'}$, we say that $A = \{A_{jj'}\}_{\sigma_j \in S(\sigma), \sigma'_j \in S(\mathcal{T}\sigma)}$ belongs to $\mathcal{M}_q^\mathcal{T}$ if

$$A_{jj'} \in \mathcal{M}_{q-1}^{\mathcal{T}_{l+1}^{jj'}} \quad \text{for } |\mathcal{T}_{l+1}^{jj'}| \leq \mu$$

and (due to item (a))

$$A_{jj'} \in \mathcal{M}_{q-1}^R \quad \text{for } |\mathcal{T}_{l+1}^{jj'}| \geq \mu + 1,$$

where $\mathcal{T}_{l+1}^{jj'} := (\mathcal{L}^\ell(\mathcal{T}))_{jj'}$.

(d) Finally, define the top format \mathcal{M}_q^Δ for $q = 1, \dots, p$. Let $\sigma \in \mathcal{A}^\ell$ be from level $\ell = p - q$ and set $\mathcal{T} = E_\mathcal{A} \in \mathcal{S}_0^\ell$. Then we say that $A = \{A_{jj'}\}_{\sigma_j, \sigma_{j'} \in S(\sigma)}$ belongs to \mathcal{M}_q^Δ if there holds $A_{jj} \in \mathcal{M}_{q-1}^\Delta$ and $A_{jj'} \in \mathcal{M}_{q-1}^{\mathcal{L}^\ell(E_\mathcal{A})_{jj'}}$ for $j \neq j'$. The same construction is applied for $\sigma \in \mathcal{Y}^\ell$, $\mathcal{T} = E_\mathcal{Y}$.

The following statement gives sharp complexity bounds for the above-defined family of formats. Here we use the generalised construction based on the p_0 -truncated partitioning as in Section 4.

Theorem 10. *Let $A \in \mathcal{M}_p^\Delta(p, \eta)$ with $\eta_\mu = 2/3\mu$, $\mu = 1, 2, \dots$. With given $p_0 \in \{0, \dots, p-1\}$, suppose that \mathcal{M}_p^Δ corresponds to the p_0 -truncated partitioning P_2^* . Then the complexity of the matrix–vector multiplication is bounded by*

$$\mathcal{N}_{MV}^\Delta \leq 6(6\mu^2 + 6\mu + 1)(p - p_0)kn + 2^{2p_0-1}(6\mu^2 + 6\mu + 1)kn$$

coupled operations. Moreover,

$$\mathcal{N}_{st}^\Delta \leq 3(6\mu^2 + 6\mu + 1)(p - p_0)kn + 4^{p_0}(6\mu^2 + 6\mu + 1)kn.$$

The constants in both relations are asymptotically sharp.

Proof. The proof is similar to those from Theorem 6. In fact, let $\sigma \in \mathcal{S}(\tau)$ be an arbitrary son for each “purely interior” cluster $\tau \in T_1^\ell$. Then, the number of sons $\sigma' \in \mathcal{S}(\tau')$ from the set of neighbouring to τ clusters $\tau' \in T_1^\ell$, i.e., $\tau' \in R(\tau)$, and satisfying with σ the admissibility condition (4) on level $\ell + 1$, is equal to $Q_\sigma = (2^2 - 1)((1 + 3\mu)^2 - 3\mu^2)$. Then the assertions follow. \square

Remark 11. Combining Definition 9 with the corresponding results from Section 4, we obtain formats of the optimal complexity for the right triangular prism elements in 3D. Further extensions of construction from above to the 3D case are based on breaking the tetrahedron into 8 or 27 parts.

Remark 12. Due to larger nonadmissible area in the construction of the $\mathcal{M}_p^\Delta(p, \eta)$ -format, see Theorem 10, the corresponding constants in \mathcal{N}_{st}^Δ and \mathcal{N}_{MV}^Δ appear to be bigger than in the case of \mathcal{M}_p^\square -formats.

When using grid (18) for finite difference or finite element discretisations of the second-order PDEs, we obtain a five-, seven-, or nine-point formula as discretisation matrix for $d = 2$ (similar for $d = 3$). The next lemma implies that such a matrix can be represented *exactly* as an \mathcal{H} -matrix, see [6] for the proof in the case of \mathcal{M}_p^\square -format.

Lemma 13. *The FE stiffness matrix A_h is in the set⁹ $\mathcal{M}_{\mathcal{H},k}(I \times I, P_2)$ for any $k \geq 1$.*

As a consequence, the approximate inverse of $A_h \in \mathcal{M}_p^\square$ as well as of $A_h \in \mathcal{M}_p^\Delta$ can be computed with the complexity $O(p^2k^2n)$, where $n = \#I$, see Section 6.3.

⁹ If I is as in (18) with fixed p , $\mathcal{M}_{\mathcal{H},k}(I \times I, P_2)$ equals $\mathcal{M}_p^\square(p, \frac{1}{\sqrt{2}})$. However, this lemma holds for rather general \mathcal{H} -partitionings.

6. Matrix addition, multiplication and inversion

6.1. Matrix addition

In this Section, we study the complexity of matrix addition, multiplication and inverse-to-matrix operations for the principal case $\mu=1$ and with $d=2$. As in [5], one can introduce the approximate addition $+\square$, multiplication $*\square$, and inversion to the matrices from $\mathcal{M}_p^\square = \mathcal{M}_p^0(p, \frac{1}{\sqrt{2}})$. The complexity analysis of formatted addition $+\square$ is rather simple (it operates with the same types of formats in a blockwise sense) and yields $\mathcal{N}_{\square+\square}(p) = O(pn)$, where $n = 2^{d p}$. Indeed, let us denote by symbols \circ and \times each set of formats \mathcal{M}_p^j where $|\mathbf{j}|_1 = 1$ and $|\mathbf{j}|_1 = 2 \wedge |\mathbf{j}|_\infty = 1$, respectively. Then the recursion

$$\mathcal{N}_{\square+\square}(p) = 4\mathcal{N}_{\square+\square}(p-1) + 8\mathcal{N}_{\circ+\circ}(p-1) + 4\mathcal{N}_{\times+\times}(p-1) \tag{30}$$

follows from (28), see Fig. 6a. In turn, the recursive Definition 5 easily implies

$$\mathcal{N}_{\circ+\circ}(p) = 2\mathcal{N}_{\circ+\circ}(p-1) + 2\mathcal{N}_{\times+\times}(p-1) + 12\mathcal{N}_{R1+R1}(p-1),$$

$$\mathcal{N}_{\times+\times}(p) = \mathcal{N}_{\times+\times}(p-1) + 15\mathcal{N}_{R1+R1}(p-1).$$

The latter two relations lead to the bounds

$$\mathcal{N}_{\times+\times}(p) = O(n), \quad \mathcal{N}_{\circ+\circ}(p) = O(n). \tag{31}$$

Substitution of (31) into (30) implies the desired complexity estimate for $\mathcal{N}_{\square+\square}(p)$, taking into account $\mathcal{N}_{R+R}(p) = 21n + O(1)$, see [5].

6.2. Complexity of matrix multiplication

The proof of $\mathcal{N}_{\square*\square}(p) = O(p^2 k^2 n)$ is more lengthy, since various combinations of factors occur. First, we introduce the formatted matrix–matrix multiplication procedure. The recursive definition of formatted multiplication of two matrices A and B from \mathcal{M}_p^\square is similar to Definition 5 above. For the precise description, we use the following notations and remark. We call $\mathbf{j}_1 \prec \mathbf{j}_2$ if either $|\mathbf{j}_1|_\infty < |\mathbf{j}_2|_\infty$ or $|\mathbf{j}_1|_\infty = |\mathbf{j}_2|_\infty \wedge |\mathbf{j}_1|_1 < |\mathbf{j}_2|_1$ and define $\mathbf{j}_1 \approx \mathbf{j}_2$, otherwise.

Remark 14. Any Rk -matrix of the size $2^{dq} \times 2^{dq}$ may be exactly converted to each of the formats \mathcal{M}_q^j , $|\mathbf{j}|_\infty \leq 1$, so we have the imbedding $\mathcal{M}_q^R \hookrightarrow \mathcal{M}_q^j$. We also assume that either $\mathcal{M}_q^{j_1} \hookrightarrow \mathcal{M}_q^{j_2}$ if $\mathbf{j}_2 \prec \mathbf{j}_1 \vee \mathbf{j}_1 \approx \mathbf{j}_2$ or (if the above imbedding is not the case) $\mathcal{M}_q^{j_1}$ may be approximately converted to the format $\mathcal{M}_q^{j_2}$ with almost linear cost, where $\mathcal{M}_q^j \hookrightarrow \mathcal{M}_q^R$ for $|\mathbf{j}|_\infty > 1$. This assumption is based on the properties of the particular format \mathcal{M}_p^\square under consideration.

Definition 15 (Recursion step). Assume that for some $q < p$ and for each $A \in \mathcal{M}_q^{j_1}$, $B \in \mathcal{M}_q^{j_2}$ the $\mathcal{M}_q^{j_3}$ -formatted product $C = A *_{\mathcal{H}} B \in \mathcal{M}_q^{j_3}$ is already defined for $|\mathbf{j}_m| \geq 0$, $m = 1, 2, 3$.

Then, for each matrix $A \in \mathcal{M}_{q+1}^{j_1}$ and $B \in \mathcal{M}_{q+1}^{j_2}$ with the recursive block structure $A = \{A_{ij}\}_{i,j \in I_d}$, $B = \{B_{ij}\}_{i,j \in I_d}$, we define $C = A *_{\mathcal{H}} B := \{C_{km}\}_{k,m \in I_d}$ with $C_{km} \in \mathcal{M}_{q+1}^{k-m+2j_3}$ by

$$C_{km} = \sum_{i \in I_d} A_{ki} *_{\mathcal{H}} B_{im}, \quad A_{ki} \in \mathcal{M}_q^{k-i+2j_1}, \quad B_{im} \in \mathcal{M}_q^{i-m+2j_2}.$$

Here the formatted addition $+\mathscr{H}$ is understood as the operation within the format $\mathscr{M}_q^{k-m+2j_3}$ in view of Remark 14. In particular, if $\mathbf{j}_1 = \mathbf{j}_2 = \mathbf{j}_3 = \mathbf{0}$, we obtain the multiplication procedure for the top format.

In view of Definition 15 and taking into account the particular structure of \mathscr{M}_p^\square -format, the complexity estimate $\mathcal{N}_{\square*\square}(p)$ on the level p is reduced recursively to the following operation counts: $\mathcal{N}_{\square*\square}(p-1)$, $\mathcal{N}_{\square*\circ}(p-1)$, $\mathcal{N}_{\square*\times}(p-1)$, $\mathcal{N}_{\circ*\times}(p-1)$, $\mathcal{N}_{\circ*\circ}(p-1)$ and $\mathcal{N}_{\times*\times}(p-1)$. The latter may be further reduced to the already known estimates for $\mathcal{N}_{R*R}(p-2)$ and $\mathcal{N}_{R+R}(p-2)$, see the proof of Lemma 16.

Lemma 16. *The following complexity bounds hold:*

$$\mathcal{N}_{\square+\square}(p) = O(pkn), \quad \mathcal{N}_{\square*\square}(p) = O(p^2k^2n) + O(k^3n). \tag{32}$$

Proof. The first assertion is proved in Section 6.1. The bound for $\square*\square$ is based on the recurrence

$$\begin{aligned} \mathcal{N}_{\square*\square}(p) &= 4\mathcal{N}_{\square*\square}(p-1) + 16\mathcal{N}_{\square*\circ}(p-1) + 8\mathcal{N}_{\square*+}(p-1) \\ &\quad + 16\mathcal{N}_{\circ*\circ}(p-1) + 16\mathcal{N}_{\circ*+}(p-1) + 4\mathcal{N}_{++}(p-1) + \sum_{\alpha,\beta \in \mathbb{N}} \mathcal{N}_{\alpha+\beta}(p-1), \end{aligned} \tag{33}$$

where $\mathbb{N} := \{\square, \circ, \times, R\}$. To proceed with, we then estimate the remaining terms on the right-hand side above. In this way we use the relations

$$\begin{aligned} \mathcal{N}_{\square*\circ}(p) &= 2\mathcal{N}_{\square*\circ}(p-1) + 2\mathcal{N}_{\square*+}(p-1) \\ &\quad + 12\mathcal{N}_{\square*R}(p-1) + 4\mathcal{N}_{\circ*\circ}(p-1) + 24\mathcal{N}_{\circ*R}(p-1) \\ &\quad + 6\mathcal{N}_{\circ*+}(p-1) + 2\mathcal{N}_{++}(p-1) + 12\mathcal{N}_{+*R}(p-1) + \sum_{\alpha,\beta \in \mathbb{N}} \mathcal{N}_{\alpha+\beta}(p-1), \end{aligned}$$

$$\begin{aligned} \mathcal{N}_{\square*+}(p) &= \mathcal{N}_{\square*+}(p-1) + 2\mathcal{N}_{\circ*+}(p-1) + \mathcal{N}_{++}(p-1) \\ &\quad + 15(\mathcal{N}_{\square*R}(p-1) + 2\mathcal{N}_{\circ*R}(p-1) + \mathcal{N}_{+*R}(p-1)) + \sum_{\alpha,\beta \in \mathbb{N}} \mathcal{N}_{\alpha+\beta}(p-1), \end{aligned}$$

$$\mathcal{N}_{\circ*\circ}(p) = 4\mathcal{N}_{\circ*R}(p-1) + 12\mathcal{N}_{+*R}(p-1) + 40\mathcal{N}_{R*R}(p-1) + \sum_{\alpha,\beta \in \mathbb{N}} \mathcal{N}_{\alpha+\beta}(p-1),$$

$$\mathcal{N}_{\circ*+}(p) = 4\mathcal{N}_{\circ*R}(p-1) + 8\mathcal{N}_{+*R}(p-1) + 52\mathcal{N}_{R*R}(p-1),$$

$$\mathcal{N}_{++}(p) = 7\mathcal{N}_{+*R}(p-1) + 57\mathcal{N}_{R*R}(p-1) + \sum_{\alpha,\beta \in \mathbb{N}} \mathcal{N}_{\alpha+\beta}(p-1),$$

$$\mathcal{N}_{+*R}(p) = 4\mathcal{N}_{+*R}(p-1) + 60\mathcal{N}_{R*R}(p-1) + \sum_{\alpha,\beta \in \mathbb{N}} \mathcal{N}_{\alpha+\beta}(p-1).$$

Note that $\mathcal{N}_{\alpha+\beta}(p) = O(n)$ for $\alpha, \beta \in \{\circ, \times, R\}$, while $\mathcal{N}_{\square+\alpha}(p) = O(pn)$ for $\alpha \in \{\circ, \times, R\}$. Substituting these results into above recurrences and taking into account $\mathcal{N}_{R*R}(p) = 3n - 1$, see [5], we obtain

$$\mathcal{N}_{\alpha*\beta}(p) = O(n), \quad \mathcal{N}_{\square*\alpha}(p) = O(pn), \quad \alpha, \beta \in \{\circ, \times, R\}.$$

Finally, Eq. (33) results in the recursion $\mathcal{N}_{\square*\square}(p) = 4\mathcal{N}_{\square*\square}(p - 1) + O(pn)$, which yields the desired assertion. In fact, the term $O(k^3n)$ results from the cost of eigenvalue problem solvers (or the singular-value decomposition) within the implementation of Rk -matrix arithmetic, see [5]. \square

6.3. Matrix inversion

The recursive inversion is based on blockwise transformations and the Schur-complement calculations involving the addition and multiplication addressed above, see [5] for more details. While in [5] the \mathcal{H} -matrix was treated as a 2×2 block matrix, now the refinement format has a 4×4 block pattern. This does not change the complexity order $\mathcal{N}_{\text{Inversion}}(p) = O(p^2n)$ obtained there with $k = O(1)$.

As an alternative, here we discuss in more details the nonrecursive construction of the inverse of an \mathcal{H} -matrix based on the iterative correction and formatted matrix–matrix multiplication. We propose to apply the nonlinear iterations for computation of A^{-1} . The proper initial guess X_0 may be obtained by the recursive Schur-complement algorithm from [5]. Assume that A is invertible. Let us solve the nonlinear operator equation in the corresponding normed space $\mathbf{Y} := \mathbb{R}^{n \times n}$ of square matrices

$$F(X) := X^{-1} - A = 0, \quad X \in \mathbf{Y}$$

by the Newton’s method, which results in the iterations

$$X_{i+1} = X_i(2I - A \cdot X_i), \quad X_0 \text{ given}, \quad i = 1, 2, \dots \tag{34}$$

For this scheme, which is well known from the literature, we give a simple direct convergence analysis.

Lemma 17. *Let $A \in \mathbf{Y}$ be invertible and assume that the initial guess in (34) satisfies*

$$\|A\| \|X_0 - A^{-1}\| = q < 1. \tag{35}$$

Then iteration (34) converges quadratically,

$$\|X_{i+1} - A^{-1}\| \leq c q^2, \quad i = 1, 2, \dots \tag{36}$$

Suppose that A and X_0 are both the s.p.d. matrices and X_0 satisfies $0 < X_0 < A^{-1}$. Then the iteration (34) yields $X_i = X_i^T > 0$ for all $i = 1, 2, \dots$.

Proof. Denote $X_i = A^{-1} - \delta_i$. By definition

$$X_{i+1} = 2(A^{-1} - \delta_i) - (A^{-1} - \delta_i)A(A^{-1} - \delta_i) = A^{-1} - \delta_i A \delta_i,$$

which implies

$$\delta_{i+1} = \delta_i A \delta_i, \quad i = 1, 2, \dots \tag{37}$$

Therefore, the first assertion follows:

$$\|\delta_{i+1}\| \leq \|A\|^{(1+2+2^2+2^3+\dots+2^{i-1})} \|\delta_0\|^2 \leq c q^2.$$

In the case of s.p.d. matrices, we have $A^{-1} > \delta_0 > 0$ by assumption. Furthermore, assume by induction, that $A^{-1} > \delta_i > 0$. Then X_{i+1} is symmetric and (37) yields $\delta_{i+1} = A^{-1} - X_{i+1} > 0$. Moreover, the inequality

$$A^{1/2} \delta_{i+1} A^{1/2} = (A^{1/2} \delta_i A^{1/2})^2 < I$$

implies $\delta_{i+1} < A^{-1}$ yielding $X_{i+1} > 0$. This proves the induction step. \square

Due to the quadratic convergence of the scheme proposed, we need only $\log \log \varepsilon^{-1}$ iterative steps which results in the $O(k^2 p^2 n \log \log n)$ complexity of the iterative correction algorithm.

A specific truncation error analysis of the \ast_{\square} -multiplication and of the inversion will not be considered in this paper. However, the background to create efficient calculus of \mathcal{H} -matrices is based on the observation that for many practically important problem classes the product or the sum of pseudodifferential operators A and B as well as the inverse operator A^{-1} have the integral representations which ensure the existence of the proper \mathcal{H} -matrix approximations to $A + B$, $B \ast A$ and A^{-1} themselves. Having in hands the linear complexity multiplication/inversion algorithms, one may use then two basic strategies for fast solution of the operator equation $Au = f$:

- (a) Direct method based on the \mathcal{H} -matrix approximation to the operator A^{-1} by the recursive Schur-complement scheme. Here the approximation of A^{-1} must be sufficiently good.
- (b) Computation of a rather rough inverse $B \approx A^{-1}$ and correction by few steps of $u^{i+1} = u^i - B \ast_{\mathcal{H}} (Au^i - f)$.

Both approaches provide almost linear complexity algorithms for solving a wide class of integral/pseudodifferential equations.

To complete the discussion, we note that all the \mathcal{H} -matrix formats considered may be extended to the case of quasi-uniform unstructured meshes. A possible construction is based on the *fictitious* uniform tensor-product or triangular grids discussed in the previous section, see [6]. We do not claim that such a construction is optimal, but it leads to a straightforward proof of the almost linear complexity bounds. The \mathcal{H} -matrices on graded meshes have been analysed in [7]. Numerical experiments mainly confirm the approximation and complexity results for the \mathcal{H} -matrix techniques applied to the boundary integral operators in 3D as well as for the data-sparse approximation to inverse of the discrete Laplacian. These results will be reported in a forthcoming paper.

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