ELSEVIER

Applied Numerical Mathematics 14 (1994) 383–395

APPLIED NUMERICAL MATHEMATICS

Two robust methods for irregular oscillatory integrals over a finite range

G.A. Evans

Department of Mathematical Sciences, Loughborough University of Technology, Loughborough, Leicestershire LE11 3TU, UK

Abstract

Two quadrature methods are developed for irregular oscillatory integrands. They are the equivalents of the well-known trapezoidal and Simpson's rules for regular quadrature, and involve linear and quadratic fitting of the defining functions respectively. The quadratic form is the equivalent of Filon's method for regular oscillatory quadrature. The resulting errors are analysed and a comparison is made with other methods which do not involve the derivative of the oscillatory function.

Key words: Irregular oscillatory quadrature; Numerical integration; Simpson's rule; Trapezoidal rule; Filon's method

1. Introduction

A number of practical problems such as the analysis of water waves on sloping beaches (see [6]) have encouraged the study of irregular oscillatory integrands and there are now some alternative approaches to this problem. The conventional methods for the integration of regular oscillatory integrands deal with integrals of the form

$$I = \int_a^b f(x) \begin{matrix}\sin\\ \cos\end{matrix} \omega x \, \mathrm{d}x. \tag{1}$$

Methods for the numerical solution of this integral are thoroughly summarised in [9]. The earliest methods involve approximating $f(x)$ as a polynomial in x and then completing the resulting integrals analytically to generate quadrature rules as in [2]. However the most effective group of methods uses a Chebyshev polynomial fit to the non-oscillatory part of the integral, $f(x)$, on the transformed interval $[-1, 1]$. The problem then reduces to finding an algorithm for

$$\int_{-1}^{1} T_n(x) \begin{matrix}\sin\\ \cos\end{matrix} \omega x \, \mathrm{d}x \tag{2}$$

SSDI 0168-9274(94)00003-Y

and the various ways of achieving this differentiate the three methods of Patterson [16], Alaylioglu, Evans and Hyslop [3] and Littlewood and Zakian [14]. Clearly with integrals of the form

$$\int_a^b f(x) \frac{\sin}{\cos} \omega q(x)\,\mathrm{d}x \tag{3}$$

even simple forms for the irregular contribution $q(x)$ will render the auxiliary integral equivalent to equation (2) quite intractable.

As a result, some specialized methods have been proposed, and the object here is to introduce two robust methods, and compare their performance with existing approaches. To this end a set of test examples will be used, taken partly from the test sets of other authors, and partly constructed to illustrate specific points.

2. Survey of existing methods

The simplest method for irregular oscillations is that of Ehrenmark [7] in which the somewhat unconventional formula with the form

$$\int_a^b g(x)\,\mathrm{d}x = \sum_{i=1}^{N} W_i g(x_i) + E \tag{4}$$

is used with $g(x)$ representing *both* the non-oscillatory and the oscillatory parts of the integrand being defined in the first instance by

$$g(x) = f(x) \frac{\sin}{\cos} kx. \tag{5}$$

This method is unusual for oscillatory problems because the complete integrand is sampled for the evaluation of the integral. Indeed Ehrenmark forces the error E to be zero when $g(x) = 1$, $\sin kx$ or $\cos kx$ to yield three equations for the three unknown weights W_i in equation (4), taking only the simple case of $N = 3$. The constant k is chosen so that kx is the average of $\omega q(x)$ across the current subinterval of integration, subdivision being used to give sufficient abscissae for convergence. The outcome is demonstrated to be fairly insensitive to the choice of k, and the use of the maximum frequency in the subinterval can be employed as another strategy.

This method lends itself to extension to higher point numbers N, and a choice of which $g(x)$ should be used to make the formula exact then arises. A number of tests have been made in extending the basis set of exact functions beyond the three original functions, 1, $\sin kx$ and $\cos kx$, to yield formulae of the form of (4).

A number of such sets $\{C_j(x)\}$ were considered, but the most productive sets were the three extensions:

(i) $\sin kx$, $\cos kx$, 1, x, $x^2, \ldots,$
(ii) 1, $\sin kx$, $\cos kx$, $\sin 2kx$, $\cos 2kx$, $\sin 3kx, \ldots,$ and
(iii) 1, $\sin kx/M$, $\cos kx/M$, $\sin 2kx/M$, $\cos 2kx/M$, $\sin 3kx/M, \ldots,$

where $M = [\frac{1}{2}N]$. Furthermore two choices of abscissac were also used, namely equally-spaced to yield Newton–Cotes-like formulae, and cosine-weighted points to yield formulae in the spirit of Clenshaw and Curtis [5]. The best of these choices based on some preliminary tests was employed in the results section. Choice (ii) proved the best of the extensions of the original algorithm with cosine-weighted abscissae in the form

$$x_j = \tfrac{1}{2}(b-a)\cos(i\pi/N) + \tfrac{1}{2}(b-a) \tag{6}$$

for $i = 0, \ldots, N$. In all cases the implementation was very simple. As the abscissae are fixed, equation (4) reduces to a set of linear algebraic equations

$$\int_a^b C_j(x)\,\mathrm{d}x = c_j = \sum_{i=1}^{N} A_i C_j(x_i), \quad j = 1, \ldots, n \tag{7}$$

for the weights A_i which can then be applied trivially to the test examples. There is a large economy to be gained by ensuring that the set of exact functions includes both sin nkx and cos nkx, so that the *same* weights A_i remain valid for all subintervals after subdivision, requiring only one set of linear algebraic equations to be solved. This is valid because the linear change of variable required to convert any of the subintervals to $[0, h]$, say, requires the integration of terms like sin $nk(\alpha x + \beta)$ for suitable α and β. Hence expanding the sin term it is clear that the formula is still exact under the transformation as long as both sin $nk\alpha$ and cos $nk\alpha$ are exact. Because of this feature formulae with odd n are considered exclusively.

Two stability problems arise in the setting up and use of these formulae. The linear equations for the weights A_i may exhibit ill-conditioning and yield inaccurate values as occurs in the derivation of Gaussian formulae, and the resulting integration formula may itself be unstable. The former problem is easily monitored in one of two ways. The condition factor $\kappa(A)$ of the matrix of coefficients A may be computed using an eigenvalue routine, or the computations can be repeated at double precision to compare the numbers of digits in error. The second problem is more straightforward to resolve, in that the integration formula will be stable if all the weights A_i are positive. Even if some of the weights vary in sign, the formula is not necessarily unstable, as long as the alternations are not large, and therefore the cause of cancellation in the summation of equation (4).

A very different approach is that of Levin [13], which again handles varying oscillations of the form

$$I = \int_a^b f(x)\mathrm{e}^{\mathrm{i}q(x)}\,\mathrm{d}x. \tag{8}$$

The basis of the method is that if $f(x)$ were of the form

$$f(x) = \mathrm{i}q'(x)p(x) + p'(x), \tag{9}$$

then the integral in (8) could be evaluated directly to give

$$I = p(b)\mathrm{e}^{\mathrm{i}q(b)} - p(a)\mathrm{e}^{\mathrm{i}q(a)}. \tag{10}$$

Eq. (9) is considered as a differential equation for the function $p(x)$ which is solved for $p(x)$ and used in (10). The general solution is

$$p(x) = \mathrm{e}^{-\mathrm{i}q(x)}\left[\int_a^x f(t)\mathrm{e}^{\mathrm{i}q(t)}\,\mathrm{d}t + c\right], \tag{11}$$

which means that in general the solution for $p(x)$ is as oscillatory as the original integrand, but as f and q' are slowly oscillatory there exists a slowly oscillatory solution p_0. Hence writing

$$p(x) = p_0(x) + c\mathrm{e}^{-\mathrm{i}q(x)} \tag{12}$$

and using this form in (9) gives

$$I = p_0(b)\mathrm{e}^{\mathrm{i}q(b)} - p_0(a)\mathrm{e}^{\mathrm{i}q(a)}, \tag{13}$$

and so $p_0(a)$ and $p_0(b)$ are all that are needed. To single out $p_0(x)$ from other highly oscillatory solutions, a representation is tried in which n-point collocation to the basis set $\{u_k(x)\}$ for $k = 1, \ldots, n$ is set up to give

$$p_0(x) = \sum_{k=1}^{n} \alpha_k u_k(x). \tag{14}$$

This form is substituted into (9) and a set of linear algebraic equations is generated by taking a set of x-values $\{x_j\}$ for $j = 1, \ldots, n$. This set of equations is solved for the α's. Hence I can be easily evaluated from (13). Once again stability difficulties with large linear systems seem to arise for too great a value of n and the suggestion is to once again subdivide. It is also noted that the method requires the evaluation of the derivative q' at the points $\{x_j\}$. Hence the method requires more stringent conditions for its implementation than the more robust methods being considered here, and indeed care is needed to avoid any of the set $\{x_j\}$ corresponding to a singular point of q'.

There are other possible approaches to the irregular problem which require the availability of $q'(x)$, the most obvious being the use of the transformation $y = q(x)$ to yield an integral of the form

$$\int_{q(a)}^{q(b)} \frac{f(q^{-1}(y)) \genfrac{}{}{0pt}{}{\sin}{\cos} \omega y}{q'(q^{-1}(y))} \,\mathrm{d}y. \tag{15}$$

This method is straightforward if $q(x)$ is simple enough to allow $q^{-1}(x)$ to be found analytically, otherwise some robust numerical device is required for this inverse. The method also injects a possible new difficulty if $q'(x)$ has a zero in the interval of integration, the resulting problem being now both singular and oscillatory. Both these difficulties are addressed in [10].

3. Two new methods

The first of these methods is very easy to derive. The method is equivalent to that of Clendenin [4] for regular oscillatory quadrature, or the trapezoidal rule for general-purpose quadrature. Both $f(x)$ and $q(x)$ are approximated by linear forms and the resulting approximation can be integrated analytically to yield a quadrature rule. Hence

$$I = \int_a^b f(x) \genfrac{}{}{0pt}{}{\sin}{\cos} \omega q(x) \,\mathrm{d}x, \tag{16}$$

and subdivision gives

$$I=\sum_{i=1}^{N}\int_{a+(i-1)h}^{a+ih} f(x)\,{\sin \atop \cos}\,\omega q(x)\,\mathrm{d}x=\sum_{i=1}^{N} I_i. \tag{17}$$

Then it is easy to establish

$$I_i=\int_{a+(i-1)h}^{a+ih}(mx+c)\,{\sin \atop \cos}\,\omega(\alpha x+\beta)\,\mathrm{d}x \tag{18}$$

with the constants m, c, α and β taking the values

$$m=(f(a+ih)-f(a+(i-1)h))/h, \tag{19}$$

$$c=f(a+ih)-m(a+ih), \tag{20}$$

$$\alpha=(q(a+ih)-q(a+(i-1)h))/h, \tag{21}$$

$$\beta=q(a+ih)-\alpha(a+ih). \tag{22}$$

Expanding the sine and cosine in (18) gives the following form for the quadrature rule

$$\begin{aligned} I_i&=\int_{a+(i-1)h}^{a+ih}(mx+c)\begin{Bmatrix}\sin\,\omega\alpha x\,\cos\,\omega\beta+\cos\,\omega\alpha x\,\sin\,\omega\beta\\ \cos\,\omega\alpha x\,\cos\,\omega\beta+\sin\,\omega\alpha x\,\sin\,\omega\beta\end{Bmatrix}\mathrm{d}x\\ &=\begin{Bmatrix}\cos\,\omega\beta(mI_{1\mathrm{s}}+cI_{0\mathrm{s}})+\sin\,\omega\beta(mI_{1\mathrm{c}}+cI_{0\mathrm{c}})\\ \cos\,\omega\beta(mI_{1\mathrm{c}}+cI_{0\mathrm{c}})-\sin\,\omega\beta(mI_{1\mathrm{s}}+cI_{0\mathrm{s}})\end{Bmatrix}\Bigg|_{a+(i-1)h}^{a+ih}\end{aligned} \tag{23}$$

and the quantities $I_{i\mathrm{s}}$ and $I_{i\mathrm{c}}$ are standard integrals

$$I_{i\mathrm{s}}=\int x^i\,\sin\,kx\,\mathrm{d}x, \tag{24}$$

$$I_{i\mathrm{c}}=\int x^i\,\cos\,kx\,\mathrm{d}x. \tag{25}$$

The second method is the equivalent of Filon's rule [11] for the regular oscillatory problem and of Simpson's rule for general quadrature. Now a quadratic is approximated to both $f(x)$ and $q(x)$. The resulting integrals are less straightforward to compute analytically and use Fresnel integrals. In particular one specific representation is required to achieve a fast algorithm for these integrals.

Instead of (18), I_i is now given by

$$I_i=\int_{a+(i-1)h}^{a+ih}(ax^2+bx+c)\,{\sin \atop \cos}\,\omega(\alpha x^2+\beta x+\gamma)\,\mathrm{d}x, \tag{26}$$

which reduces to

$$I_i=\int_{a+(i-1)h+A}^{a+ih+A}\left[a(y-A)^2+b(y-A)+c\right]{\sin \atop \cos}\,\omega\alpha(y^2+D)\,\mathrm{d}y \tag{27}$$

under the substitution $y = x + \beta/2\alpha = x + A$ where $A = \beta/2\alpha$ and $D = \gamma/\alpha - \beta^2/4\alpha^2$. Hence I_i becomes

$$I_i = \int_{a+(i-1)h+A}^{a+ih+A} (Py^2 + Qy + R) \begin{Bmatrix} \sin \omega\alpha y^2 \cos \omega\alpha D + \cos \omega\alpha y^2 \sin \omega\alpha D \\ \cos \omega\alpha y^2 \cos \omega\alpha D - \sin \omega\alpha y^2 \sin \omega\alpha D \end{Bmatrix} \mathrm{d}y \tag{28}$$

with

$$P = a, \quad Q = -2aA + b, \quad R = aA^2 - bA + c. \tag{29}$$

If $\alpha = 0$, then the formulae above are not valid and the quadrature rule then follows the lines for the trapezoidal case of (23) but now with I_{2s} and I_{2c} terms giving the formula

$$\begin{aligned} I_i &= \int_{a+(i-1)h}^{a+ih} (ax^2 + bx + c) \begin{Bmatrix} \sin \omega\beta x \cos \omega\gamma + \cos \omega\beta x \sin \omega\gamma \\ \cos \omega\beta x \cos \omega\gamma + \sin \omega\beta x \sin \omega\gamma \end{Bmatrix} \mathrm{d}x \\ &= \begin{Bmatrix} \cos \omega\gamma(aI_{2s} + bI_{1s} + cI_{0s}) + \sin \omega\gamma(aI_{2c} + bI_{1c} + cI_{0c}) \\ \cos \omega\gamma(aI_{2c} + bI_{1c} + cI_{0c}) - \sin \omega\gamma(aI_{2s} + bI_{1s} + cI_{0s}) \end{Bmatrix} \Bigg|_{a+(i-1)h}^{a+ih} \end{aligned} \tag{30}$$

The general rule of equation (28) requires the evaluation of integrals of the form

$$\int_c^d y^{2N} \begin{matrix}\sin \\ \cos\end{matrix} \omega x^2 \, \mathrm{d}x \tag{31}$$

and successive integration by parts gives the series

$$\begin{aligned} &-\frac{1}{2\omega} x^{2N-1} \cos \omega x^2 + \frac{2N-1}{(2\omega)^2} x^{2N-3} \sin \omega x^2 + \frac{(2N-1)(2N-3)}{(2\omega)^3} x^{2N-5} \cos \omega x^2 \\ &\quad - \frac{(2N-1)(2N-3)(2N-5)}{(2\omega)^4} x^{2N-7} \sin \omega x^2 - \cdots \end{aligned} \tag{32}$$

in the sine case and

$$\begin{aligned} &-\frac{1}{2\omega} x^{2N-1} \sin \omega x^2 + \frac{2N-1}{(2\omega)^2} x^{2N-3} \cos \omega x^2 - \frac{(2N-1)(2N-3)}{(2\omega)^3} x^{2N-5} \sin \omega x^2 \\ &\quad - \frac{(2N-1)(2N-3)(2N-5)}{(2\omega)^4} x^{2N-7} \cos \omega x^2 - \cdots \end{aligned} \tag{33}$$

in the cosine case, each taken between the limits c and d. The final integral in these series will be either

$$\int_c^d \sin \omega x^2 \, \mathrm{d}x = \sqrt{\frac{\pi}{2\omega}} \left[S\left(d\sqrt{\frac{2\omega}{\pi}} \right) - S\left(c\sqrt{\frac{2\omega}{\pi}} \right) \right] \tag{34}$$

for the sine case with N even and for the cosine case with N odd, or

$$\int_c^d \cos \omega x^2 \, \mathrm{d}x = \sqrt{\frac{\pi}{2\omega}} \left[C\left(d\sqrt{\frac{2\omega}{\pi}} \right) - C\left(c\sqrt{\frac{2\omega}{\pi}} \right) \right] \tag{35}$$

for the sine case with N odd and the cosine case with N even. The functions S and C are the usual Fresnel integrals as defined by Abramowitch and Stegun [13], and the following formulae are used for their evaluation in this algorithm.

For $x > 5.5$ the definitions

$$C(x) = \tfrac{1}{2} + f(x) \sin\left(\tfrac{1}{2}\pi x^2\right) - g(x) \cos\left(\tfrac{1}{2}\pi x^2\right), \tag{36}$$

$$S(x) = \tfrac{1}{2} + f(x) \cos\left(\tfrac{1}{2}\pi x^2\right) - g(x) \sin\left(\tfrac{1}{2}\pi x^2\right) \tag{37}$$

are used with the asymptotic forms

$$\pi x f(x) \sim 1 + \sum_{m=1}^{\infty} (-1)^m \frac{1 \cdot 3 \cdots (4m-1)}{\left(\pi x^2\right)^{2m}}, \tag{38}$$

$$\pi x g(x) \sim 1 + \sum_{m=0}^{\infty} (-1)^m \frac{1 \cdot 3 \cdots (4m+1)}{\left(\pi x^2\right)^{2m+1}}. \tag{39}$$

For $x < 2.5$ series representations give rapid and stable convergence and the formulae

$$\begin{aligned} C(x) = {} & \cos\left(\tfrac{1}{2}\pi x^2\right) \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{1 \cdot 3 \cdots (4n+1)} x^{4n+1} \\ & + \sin\left(\tfrac{1}{2}\pi x^2\right) \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{1 \cdot 3 \cdots (4n+3)} x^{4n+3}, \end{aligned} \tag{40}$$

$$\begin{aligned} S(x) = {} & -\cos\left(\tfrac{1}{2}\pi x^2\right) \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{1 \cdot 3 \cdots (4n+3)} x^{4n+3} \\ & + \sin\left(\tfrac{1}{2} x^2\right) \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{1 \cdot 3 \cdots (4n+1)} x^{4n+1} \end{aligned} \tag{41}$$

are employed. The more awkward case occurs between these extremes and for this region the expansions

$$C_2(x) = J_{1/2}(x) + J_{5/2}(x) + J_{9/2}(x) + \cdots, \tag{42}$$

$$S_2(x) = J_{3/2}(x) + J_{7/2}(x) + J_{11/2}(x) + \cdots \tag{43}$$

are employed with

$$C(x) = C_2\left(\tfrac{1}{2}\pi x^2\right), \qquad S(x) = S_2\left(\tfrac{1}{2}\pi x^2\right). \tag{44}$$

The spherical Bessel functions are evaluated by the usual reverse recurrence [8,15], which automatically fixes the number of terms required in (42) and (43). As the terms are then generated from some asymptotically determined $J_{N/2}$ for N sufficiently large, the sums required in (42) and (43) can be accumulated as the reverse recurrence generates the spherical Bessel functions in turn. The terms are alternately accumulated into the sums for C_2 and S_2.

4. Error analysis and stability considerations

The Ehrenmark algorithm is exact for a given set of functions, and therefore has error terms which follow from the analysis of Ghizzetti and Ossicini [12]. The new rules involve two approximate functions and this general approach breaks down. The errors involved in approximating both $f(x)$ and $q(x)$ can be expressed for both the trapezoidal and the Simpson equivalent rules by considering the general case that

$$f(x)=p(x)+\varepsilon(x), \tag{45}$$

where $p(x)$ is either a linear or quadratic approximation. Similarly let

$$q(x)=\hat{p}(x)+\delta(x), \tag{46}$$

where $f(x)$ and $q(x)$ are assumed differentiable to orders two and three respectively. Then the error in the integral, $E(x)$, satisfies

$$E(x)=\int_a^b\left[f(x)\,{\textstyle\frac{\sin}{\cos}}\,\omega q(x)-p(x)\,{\textstyle\frac{\sin}{\cos}}\,\omega\hat{p}(x)\right]\,\mathrm{d}x, \tag{47}$$

that is

$$E(x)=\int_a^b\left\{f(x)\,{\textstyle\frac{\sin}{\cos}}\,\omega q(x)-[f(x)-\varepsilon(x)]\,{\textstyle\frac{\sin}{\cos}}\,[\omega(q(x)-\delta(x))]\right\}\,\mathrm{d}x. \tag{48}$$

Hence collecting terms for the sine case gives

$$\begin{aligned}E(x)=\int_a^b[&f(x)\sin\omega q(x)(1-\cos\omega\delta(x))+f(x)\cos\omega q(x)\sin\omega\delta(x)\\&+\varepsilon(x)\sin\omega q(x)\cos\omega\delta(x)-\varepsilon(x)\cos\omega q(x)\sin\omega\delta(x)]\,\mathrm{d}x\end{aligned} \tag{49}$$

and in the cosine case

$$\begin{aligned}E(x)=\int_a^b[&f(x)\cos\omega q(x)(1-\cos\omega\delta(x))-f(x)\sin\omega q(x)\sin\omega\delta(x)\\&+\varepsilon(x)\cos\omega q(x)\cos\omega\delta(x)+\varepsilon(x)\sin\omega q(x)\sin\omega\delta(x)]\,\mathrm{d}x.\end{aligned} \tag{50}$$

In order to express the terms of (49) and (50) in terms of small quantities it is necessary for $\omega\delta(x)$ to satisfy the condition

$$\omega\delta(x)<\tfrac{1}{2},\ \text{say,}\quad \text{for } x\in[a,b], \tag{51}$$

so allowing expansions of cos $\omega\delta(x)$ and sin $\omega\delta(x)$ to be made. This type of condition arises also in Ehrenmark's work and arises whenever the oscillatory function $q(x)$ is approximated. Effectively, if ω is large then to get sufficient accuracy to evaluate either sin or cos requires ever-increasing accuracy in $q(x)$. This effect can be seen in the results, and will be shown to be less restricting in the quadratic case than the trapezoidal case or indeed Ehrenmark's algorithm.

Hence in the sine case

$$E(x)=\int_a^b\left[\omega^2\delta^2(x)f(x)\frac{\sin\,\omega q(x)}{2.0}+\omega\delta(x)f(x)\cos\,\omega q(x)\right.$$
$$\left.+\varepsilon(x)\sin\,\omega q(x)-\omega\delta(x)\varepsilon(x)\cos\,\omega q(x)\right]\mathrm{d}x, \tag{52}$$

and in the cosine case

$$E(x)=\int_a^b\left[\omega^2\delta^2(x)f(x)\cos\,\omega q(x)-\omega\delta(x)f(x)\sin\,\omega q(x)\right.$$
$$\left.+\varepsilon(x)\cos\,\omega q(x)+\omega\delta(x)\varepsilon(x)\sin\,\omega q(x)\right]\mathrm{d}x. \tag{53}$$

However in the trapezoidal case both $\varepsilon(x)$ and $\delta(x)$ have the forms

$$\varepsilon(x)=\frac{f''(\xi_f)}{2!}(x-a-ih)(x-a-ih+h), \tag{54}$$

$$\delta(x)=\frac{q''(\xi_q)}{2!}(x-a-ih)(x-a-ih+h), \tag{55}$$

for the usual Lagrange error on the interval $(a+(i-1)h,\ a+ih)$, with ξ_f and ξ_q both lying in this interval and both dependent on x.

In the quadratic case the equivalent formulae are

$$\varepsilon(x)=\frac{f'''(\eta_f)}{3!}(x-a-ih)\left(x-a-ih+\tfrac{1}{2}h\right)(x-a-ih+h), \tag{56}$$

$$\delta(x)=\frac{q'''(\eta_q)}{3!}(x-a-ih)\left(x-a-ih+\tfrac{1}{2}h\right)(x-a-ih+h), \tag{57}$$

and because of the usual problem of not knowing η_f or η_q and possibly not having higher derivatives available, the importance of these formulae lies in the order of the error. Hence in the linear case $|\varepsilon(x)|\leqslant\frac{1}{4}h^2\cdot f''(\xi_f)/2!$ and a similar expression for δ. In the quadratic case $|\varepsilon(x)|\leqslant(h^3/12\sqrt{3})\cdot f'''(\eta_f)/3!$ and again a similar term for δ. Here upper bounds have been taken for the factors $(x-a-ih)(x-a-ih+h)$ and $(x-a-ih)(x-a-ih+\frac{1}{2}h)(x-a-ih+h)$. Hence the dominant terms in the linear case are

$$\begin{aligned}|E(x)| &\sim \int_a^b[\omega\delta(x)f(x)+\varepsilon(x)]\,\mathrm{d}x\\ &=\sum_{i=1}^{N}\int_{a+(i-1)h}^{a+ih}[\omega\delta(x)f(x)+\varepsilon(x)]\,\mathrm{d}x\\ &=\sum_{i=1}^{N}\int_{a+(i-1)h}^{a+ih}\omega\cdot\tfrac{1}{8}h^2q''\left(\xi_q^{(i)}\right)f(x)+\tfrac{1}{8}h^2f''\left(\xi_f^{(i)}\right)\mathrm{d}x\\ &\leqslant\omega\cdot\tfrac{1}{8}h^2q_M+\tfrac{1}{8}h^2f_M=h^2\cdot\tfrac{1}{8}K_1.\end{aligned} \tag{58}$$

The quantities $\xi_q^{(i)}$ and $\xi_f^{(i)}$ are now dependent on i, and q_M and f_M are upper bounds satisfying

$$q_M \leqslant \sum_{i=1}^{N} \int_{a+(i-1)h}^{a+ih} |q''(\xi_q^{(i)})f(x)| \, dx, \tag{59}$$

$$f_M \leqslant \sum_{i=1}^{N} \int_{a+(i-1)h}^{a+ih} |f''(\xi_f^{(i)})| \, dx. \tag{60}$$

By a similar argument

$$|E(x)| \sim \frac{h^3}{6} \frac{K_2}{12\sqrt{3}} \tag{61}$$

in the quadratic case.

Like the Ehrenmark algorithm there is some dependence on ω as given by (51), but for the Simpson equivalent case as h depends on the cube root of ω this is unlikely to be a serious limitation. This effect can be seen in the results which follow.

Stability problems can arise for a very large number of subdivisions when h is so small that m and α in (19) and (21) involve the difference of very close values of f and q respectively. Unlike the non-oscillatory trapezoidal rule, m and α are required explicitly. If this extreme situation arises Taylor expansions of f and q could be used, but then derivatives are required.

In the quadratic formula, care is needed if $q(x)$ approaches a straight line, when α in (26) will be very small with the effect that D and A will be very large. Hence errors will arise in computing $\sin \omega\alpha D$ and the equivalent cosine term in (28) as well as in evaluating the integral at two large but very close limits. The suggestion is that if α becomes smaller than some threshold such as 10^{-3} that a linear fit is used for $q(x)$, namely equation (30).

5. Tests and results

A set of test examples has been compiled from those employed by previous authors to demonstrate their methods. In addition some further examples have been included here to extend the effectiveness of the test examples in showing up weaknesses in the proposed methods. The method of comparison is based on the number of function evaluations of $f(x)$ and $q(x)$ required for a given accuracy. The Ehrenmark method combines the evaluation of $f(x)$ with $q(x)$ into $g(x)$ and so one evaluation of g is roughly equivalent to one of f and one of q. The test set is shown in Table 1 together with the accurate values which have been computed in an expensive manner using high-order Gauss formulae with subdivision, where analytic values are not available.

Two approaches were used to gain insight into the relative performances of the methods. The first was to see what accuracy results for a given number of function evaluations, in this case 256. In the Ehrenmark case, a count of the number of evaluations of the complete integrand $g(x)$ is made. In the trapezoidal and Simpson equivalents, the number of evaluations of pairs of $f(x)$ and $q(x)$ is used. The second approach was to see how many function

Table 1
Test set

f	Limits	Integrand	ω	Value
$f1$	(0, 1)	$e^x \cos \omega x$	10	−0.1788996028768
$f2$	(100, 200)	$(1+\ln x)\cos(x \ln x)$	1.0	−1.774298974906
$f3$	(0, β)	$\cos x \cos(\omega \cos x)/\beta$	40	0.05019445610620
$f4$	(0, 1)	$\sin x \cos(\omega x(1+x))$	500	4.59859397840(−4)
$f5$	(0, 1)	$\cos x \cos(\omega\sqrt{1-x^2})$	10.0	−3.9615562798520(−1)
$f6$	(0, 1)	$\cos x \cos(\omega \sin x)$	10.0	8.468680691183(−2)
$f7$	(0, 1)	$e^x \sin(\omega \cosh x)$	10.0	−2.556593290493(−1)
$f8$	(0, 1)	$\sin^2 x \cos(\omega \tanh x)$	10.0	0.13411649903305

Table 2
Number of correct figures for 256 function evaluations

f	3-point Ehrenmark	5-point Ehrenmark	Trapezoidal	Simpson
$f1$	8	11	6	12
$f2$	0	0	6	8
$f3$	3	4	4	10
$f4$	0	0	2	9
$f5$	6		3	5
$f6$	8	10	5	8
$f7$	6		5	11
$f8$	8	12	6	9

Table 3
Number of function evaluations for 8 correct figures

f	3-point Ehrenmark	5-point Ehrenmark	Trapezoidal	Simpson
$f1$	300	120	512	32
$f2$	9600		4096	1024
$f3$	12144	2560	16384	128
$f4$	38400	8000	32768	256
$f5$	1200	960	32768	1024
$f6$	480	200	8192	256
$f7$	1440		2048	64
$f8$	180	150	1024	32

evaluations were needed to gain a given accuracy, in this case to get 8 correct figures. The first of these tests is shown in Table 2 and the second in Table 3.

The gaps in the tables under the 5-point Ehrenmark rule indicate instabilities arising in defining linear equations for these particular test functions. It is clear that even with enhancements the method of Ehrenmark is in general only as effective as the trapezoidal or linear rule, and very much less effective than the quadratic or Simpson method. This is despite having a

Table 4
Number of correct figures for 250 function evaluations (T: linear or trapezoidal case; S: quadratic or Simpson case)

f	$\omega = 10.0$		$\omega = 100.0$		$\omega = 250.0$		$\omega = 500.0$		$\omega = 1000.0$	
	T	S	T	S	T	S	T	S	T	S
$f5$	3	6	2	3	3	3	3	3	3	4
$f6$	5	8	4	7	3	5	3	5	3	5
$f7$	5	11	4	9	4	8	4	8	4	8
$f8$	6	9	5	8	4	7	4	6	4	6

very small test value for ω which would favour the Ehrenmark rule. It is the sampling of the whole integrand which reduces the power of the method and causes an increase in required point number with increasing frequency of oscillation. It is of note that on $f2$, $f3$ and $f4$ which have high oscillatory factors (due to the limits in $f2$), such a sampling approach is highly expensive. As expected the piecewise fitting is particularly suited to examples $f1$ and $f4$, which therefore perform well. The error analysis demonstrates that whenever $q(x)$ is approximated there will be an ω-dependence for h, but it is clear from the tables, particularly for $f2$ and $f4$, how the dependence of h on the cube root of ω (Eq. (51)) in the quadratic case relaxes any restrictions so that the required h are not too small as to be impractical except for very large ω indeed. In the Ehrenmark case the number of required function evaluations rises linearly with ω, and in the linear or trapezoidal case as $\sqrt{\omega}$. These effects are illustrated in Table 4 where increasing ω-values are employed on a selection of test functions.

For ω as large as 1000.0 the Ehrenmark algorithm becomes quite untenable, but the new rules, even though slightly degraded, lose their accuracy very slowly. It is of note that $f5$ seems to be a particularly difficult integral. The error analysis shows that the estimate will be unbounded as the singularity in the higher derivatives of $q(x)$ will cause the problems.

If $q'(x)$ is also available, then an improvement in accuracy can be obtained by using the method of Levin [13]. This method also has reduced accuracy for an example such as $f5$, and of course suffers from the first derivative being singular, as this is now used directly. The Levin method involves solving linear equations for the fit to the function $p_0(x)$ of (14), and these equations show signs of instability for the larger n-values. Hence like the rules proposed here relatively low-order methods need to be employed with subdivision being used to achieve high accuracy. It is well known that subdivision is far less effective a method to increase accuracy than the use of a high-order rule directly, [9].

Levin [13] also attempts a two-dimensional integral, namely

$$I = \int_0^1 \int_0^1 \cos(x+y) \cos\left[\omega\left(x+y+c(x^2+y^2)\right)\right] \, dx \, dy, \tag{62}$$

and uses a 25×25 rule to obtain $-8.595(10^{-5})$. By employing the irregular Simpson rule recursively to this integral the value $-8.59783(10^{-5})$ was obtained with a 32×32 rule, which is correct to one place in the sixth digit, and does not employ any derivatives of $q(x)$. Even the 16×16 point rule gives five correct figures.

It appears that the new rules provide competitive methods for irregular oscillatory integrals, especially if $q'(x)$ is not available. For low accuracy but with a simple algorithm, the linear rule is effective, and for higher accuracy the quadratic rule may be gainfully employed.

References

[1] M. Abramowitz and I.A. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1964).

[2] A. Alaylioglu, G.A. Evans and J. Hyslop, Automatic generation of quadrature formulae for oscillatory integrals, *Comput. J.* 18 (2) (1975) 173–176.

[3] A. Alaylioglu, G.A. Evans and J. Hyslop, The use of Chebeshev series for the evaluation of oscillatory integrals, *Comput. J.* 19 (3) (1976) 258–267.

[4] W.W. Clendenin, A method for the numerical calculation of Fourier integrals, *Numer. Math.* 8 (1966) 422–436.

[5] C.W. Clenshaw and A.R. Curtis, A method for the numerical integration on an automatic computer, *Numer. Math.* 12 (1960) 197–205.

[6] U.L. Ehrenmark, Far field asymptotics of the two-dimensional linearised sloping beach problem, *SIAM J. Appl. Math.* 47 (5) (1987) 965–981.

[7] U.T. Ehrenmark, A three-point formula for numerical quadrature of oscillatory integrals with variable frequency, *J. Comput. Appl. Math.* 21 (1988) 87–99.

[8] G.A. Evans, A note on an algorithm for generating Bessel functions of integer and half order, *Math. Res.* 53 (1975).

[9] G.A. Evans, *Practical Numerical Integration* (Wiley, New York, 1993).

[10] G.A. Evans, An alternative method for irregular oscillatory integrals over a finite range, *Internat. J. Comput. Math.* (to appear).

[11] L.N.G. Filon, On a quadrature formula for trigonometric integrals, *Proc. Roy. Soc. Edinburgh* 49 (1928) 38–47.

[12] A. Ghizzetti and A. Ossicini, *Quadrature Formulae*, International Series of Numerical Analysis (Birkhäuser, Basel, 1970).

[13] D. Levin, Procedures for computing one and two dimensional integrals of functions with rapid irregular oscillations, *Math. Comp.* 38 (158) (1982) 531–538.

[14] R.K. Littlewood and V. Zakian, Numerical evaluation of Fourier integrals, *J. Inst. Math. Appl.* 18 (1976) 331–339.

[15] J.C.P. Miller, On the choice of standard solutions to Weber's equation, *Proc. Camb. Phil. Soc.* 48 (1952) 428–435.

[16] T.N.L. Patterson, On high precision methods for the evaluation of Fourier integrals with finite and infinite limits, *Numer. Math.* 27 (1976) 41–52.