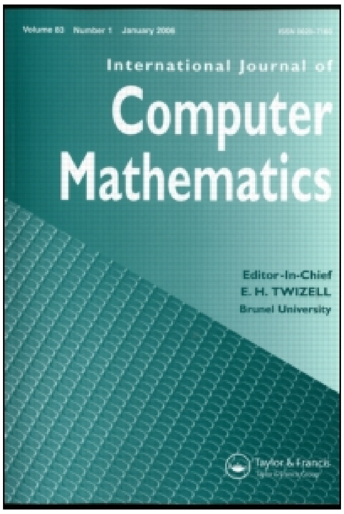

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AN EXPANSION METHOD FOR IRREGULAR OSCILLATORY INTEGRALS

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Many current problems in applied mathematics require the numerical integration of irregular oscillatory integrals. Few methods have been specifically found for these problems. An alternative method based on expansion is presented here and this method is compared with other approaches. Tests are carried out on a representative set of examples, and the algorithms are applied to problems with large oscillatory factors.

Keywords: Irregular oscillatory quadrature; numerical integration

1. INTRODUCTION

There has been considerable recent interest in the numerical integration of irregular oscillatory integrals of the form

$$I = \int_a^b f(x) \begin{matrix}\sin \\ \cos\end{matrix} \omega q(x)\, dx \tag{1.1}$$

as integrals of this type arise commonly in the analysis of water waves in a wide variety of situations including waves on a sloping beach (Ehrenmark [1]). Such integrands also arise in integrals which exhibit two of the major difficulties, namely both being singular and regularly oscillatory. This is because one of the best devices for the singular problem is to use one of the effective transformation methods, such as Takahasi and Mori [2] or Aihie and Evans [3]. Unfortunately the regular oscillations are then replaced with irregular terms, for example $\sin \omega x$ becomes $\sin(\omega \tanh(t^n))$ for $n = 3$ or $n = 5$ with the tanh transformation. Clearly for small ω, conventional methods may be used such as Clenshaw-Curtis [4] or Patterson [5], but for large ω

specialized methods akin to those for the regular problem, Evans [6], are required.

Of the methods for irregular oscillatory integrals, the most successful has been that of Levin [7], which handles varying oscillations such as

$$I = \int_a^b f(x)e^{iq(x)}\,dx \tag{1.2}$$

by finding a solution $p(x)$ to the ordinary differential equation

$$f(x) = iq'(x)p(x) + p'(x) \tag{1.3}$$

so that the required integral (1.2) can be integrated analytically to give

$$I = p(b)e^{iq(b)} - p(a)e^{iq(a)}. \tag{1.4}$$

However there are no boundary conditions for (1.3), and hence Levin singles out the slowly oscillatory solution by the device of trying the expansion

$$p(x) = \sum_{k=1}^{n} \alpha_k u_k(x) \tag{1.5}$$

to yield a set of linear algebraic equations for the α's. The usual difficulty of stability of these linear equations arises for too large a value of n, and tests seem to indicate that a practical limit of around $n = 16$ should be adhered to. It is also clear that the method requires the analytic form of $q(x)$ so that $q'(x)$ in (1.3) can be utilized. This is a common requirement in most of the methods for the irregular problem.

Other methods for irregular oscillatory integrals include that of Ehrenmark [8] in which *both* the non-oscillatory and the oscillatory parts of the integrand are sampled to give a low order rule which is made exact for 1, sin kx and cos kx. The constant k is chosen so that kx is the average of $\omega q(x)$ across the current sub-division of integration, so extending the method to the irregular oscillatory case. It is also feasible to approximate both $f(x)$ and $q(x)$ by low degree polynomials to obtain the irregular oscillatory equivalents of the trapezoidal rule and Simpson's rule, Evans [9], for which q' is not required.

A further method is the overlooked but obvious device of making the transformation $y = q(x)$ to yield the conventional oscillatory integral

$$\int_{q(a)}^{q(b)} \frac{f(x)}{q'(x)} \begin{matrix}\sin\\\cos\end{matrix} \omega y \, dy \tag{1.6}$$

with $x = q^{-1}(y)$. Again this method requires $q'(x)$ and also $q^{-1}(x)$, and a form of Newton's method is the basis for finding the inverse function, a linear interpolator being used to obtain the initial point. The resulting standard integral is then evaluated using a method such as that of Patterson [10] or Alaylioglu, Evans and Hyslop [11]. This method has been presented in Evans [12], and has some similarities with the method being proposed here, which will be considered later.

2. EXPANSION METHOD

The problem with irregular oscillatory integrals is that almost any attempt at approximating $f(x)$ or $q(x)$ by polynomials or Chebyshev functions results in analytic integrations which have no easy closed form so precluding the generation of a quadrature rule. One possibility that will overcome this difficulty is to expand $f(x)$ as

$$\frac{f(x)}{q'(x)} = \sum_{i=0}^{N} a_i g_i(q(x)) \tag{2.1}$$

with the choice of $g_i(q)$ left free for the time being. The required integral then reduces to

$$I = \int_a^b f(x) \begin{matrix}\sin\\\cos\end{matrix} \omega q(x) dx$$

$$= \sum_{i=0}^{N} a_i \int_{q(a)}^{q(b)} g_i(q) \begin{matrix}\sin\\\cos\end{matrix} \omega q \, dq \tag{2.2}$$

which can then be either evaluated analytically for $g_i(q) = q^i$, or by using a regular oscillatory method such as Patterson [10] or Alaylioglu, Evans and Hyslop [11] for a general $g_i(q)$. The fit can be carried out by collocating $f(x)$

with the series in (2.1) for a set of x_i's possibly equally spaced across the interval $[a, b]$, yielding a set of linear algebraic equations for the coefficients a_i.

Hence the expansion in (2.1) can be compared with the inverse method in which the expansion

$$\frac{f(q^{-1}(y))}{q'(q^{-1}(y))} = \sum_{j=0}^{N} c_j T_j(y) \tag{2.3}$$

is used. The current proposal avoids the use of q^{-1} which would need a technique such as that in Evans [12], although the choice of $g = q^{-1}$ ensures the expansion in (2.1) is in powers of x. The inverse method is reproduced using $g_i = T_i(q^{-1})$, but not only does this invoke the q^{-1} complication but now the resulting integrals will require the Patterson recurrence in [13] or an equivalent routine.

The choice of the function $g_i(q)$ is made to avoid $q'(x)$ being zero in (2.1), which would invalidate the expansion. To illustrate this situation consider the case when $q(x) = \cos x$ for small x, which arises in example f 3 to be considered later. Then $\cos x$ behaves like $1 - O(x^2)$ and hence expanding in effectively powers of unity will not yield a collocating expansion. In this example, a sensible choice of $g_i(q)$ would be

$$g_i(q) = (\sqrt{(1-q^2)})^{i-1} \tag{2.4}$$

which then gives an expansion in powers of sin x, or equivalently in powers of x at least for small x, with the expected improvement in the obtained fit. More examples of this sort will be given in the results section, and the error analysis highlights how a zero in $q'(x)$ will give unbounded errors. Commonly the choice $g_i(q) = q^i$ will be used, and this allows the integral in equation (2.2) to be found using the analytic formulae

$$\int_{-1}^{1} x^r \cos\omega x \, dx = \sum_{l=0}^{r} l! \binom{r}{l} \frac{x^{r-1}}{\omega^{l+1}} \sin\left(\omega x + \frac{1}{2} l\pi\right)\Bigg|_{-1}^{1} \tag{2.5}$$

$$\int_{-1}^{1} x^r \sin\omega x \, dx = -\sum_{l=0}^{r} l! \binom{r}{l} \frac{x^{r-l}}{\omega^{l+1}} \cos\left(\omega x + \frac{1}{2} l\pi\right)\Bigg|_{-1}^{1} \tag{2.6}$$

As in most methods for the irregular problem, a combination of the underlying method with subdivision is employed, allowing N to be kept to low order to avoid any stability difficulties.

3. ERROR ANALYSIS

The underlying process here is basically a collocation in which the expansion

$$\frac{f(x)}{q'(x)} = \sum_{i=0}^{N} a_i g_i(q(x)) + E_1(x) \tag{3.1}$$

is made exact for the set of selected points $x = x_i$. In all the examples used the form of $g_i(q(x))$ is

$$g_i(q(x)) = h^i(x) \tag{3.2}$$

to give an expansion in terms of powers of $h(x)$. This is a transformation of Lagrange interpolation based on letting $y = h(x)$ to give

$$F(y) = \frac{f(h^{-1}(y))}{q'(h^{-1}(y))} = \sum_{i=0}^{N} a_i y^i - E_2(y) \tag{3.3}$$

The summation over i can be rewritten in terms of transformed Lagrange coefficients, and the classical error formula yields $E_2(y)$ in the form

$$E_2(y) = \frac{F^{(N+1)}(\xi)}{(N+1)!} \prod(y) \tag{3.4}$$

with ξ being an unknown point in the range of interpolation, and

$$\prod(y) = (y - y_0) \cdots (y - y_N) \tag{3.5}$$

with $y_i = h(x_i)$. In order to find the quadrature error, $E_1(x)$ is required and will have the form

$$E_1(x) = \frac{\frac{d^{(N+1)}F}{dy^{(N+1)}}(\xi)}{(N+1)!} \prod(h(x)) \tag{3.6}$$

Expressing $h^{-1}(y) = H(y)$ which will exist for the relatively simple functions $y = h(x)$ being employed, successive differentiation yields

$$\frac{d^{(N+1)}F}{dy^{(N+1)}} = \sum_{j=1}^{N+1} b_j \frac{d^j F}{dx^j} \tag{3.7}$$

where b_i is a function of $H(y)$ and its derivatives. Typically

$$\frac{d^3F}{dy^3}=\frac{d^3F}{dx^3}H'^3+3\frac{d^2F}{dx^2}H'H''+\frac{dF}{dx}H''' \tag{3.8}$$

Hence if the function F becomes unbounded in any of its derivatives then there is a possibility of large errors arising in the fitting process. This encompasses the cases when zeros of $q'(x)$ occur in the range of interpolation, and the situation mentioned in equation (2.3) where $q(x)$ becomes almost constant. Clearly in this latter case, the integral is no longer highly oscillatory and resort to conventional quadrature methods may be made with no difficulty.

Hence the error in the computed integral, E_I, is given by

$$E_I=\int_a^b E(x)q'(x)e^{i\omega q(x)}dx \tag{3.9}$$

where

$$E(x)=\frac{\prod(h(x))}{(N+1)!}\sum_{j=1}^{N+1} b_i\frac{d^jF}{dx^j}\bigg|_{\eta(x)} \tag{3.10}$$

for some $\eta(x)$ in the interval of integration. But the value of I itself behaves like $1/\omega$. Hence integrating (3.10) by parts gives

$$E_I=\frac{E(x)}{i\omega}e^{i\omega q(x)}-\int_a^b\frac{E'(x)}{i\omega}e^{i\omega q(x)}dx \tag{3.11}$$

Then

$$|E_I|\leqslant\frac{1}{\omega}\left[\,|E(x)|+\int_a^b|E'(x)|\,\right]dx \tag{3.12}$$

which implies that the relative error is independent of ω. This effect is demonstrated in practice in the results section.

As ever, the error formulae are of limited computational use as the higher derivatives of F cannot be easily found in such an environment. But nevertheless these formulae provide a basis for the method, and emphasise the problem areas.

4. TESTS AND RESULTS

A set of test examples has been compiled from the examples employed by previous authors to demonstrate their methods. In addition some further examples have been included here to extend the effectiveness of the test examples in showing up weaknesses in the proposed methods. The method of comparison will need to consider not only the number of function evaluations of $f(x)$, but also the number of evaluations of $q(x)$ and $q'(x)$. These test cases are shown in Table I, where $\beta = 0.72$.

The results are shown in Table II, where a comparison is made between the inverse method, Levin's method and the proposed method. For Levin's method and the inverse method a triple of results gives the values of the counts for the functions f, q and q' respectively. For the inverse method, the inverse function $q^{-1}(y)$ was found numerically even though analytic inverses were available. The object was to not bias these results by effectively using analytic processes, as in general such processes would not be available. Where $q'(x)$ has a zero, so introducing a singularity, the offending region is integrated by the application of Clenshaw-Curtis quadrature on the whole

TABLE I

f	*limits*	*integrand*	ω	*value*
f1	(0, 1)	$e^x \cos \omega x$	10	− 0.1788996028768
f2	(100, 200)	$(1 + \ln x)\cos(\omega x \ln x)$	1	− 1.774298974906
f3	$(0, \beta)$	$\cos x \cos(\omega \cos x)/\beta$	40	0.05019445610620
f4	(0, 1)	$\sin x \cos(\omega x(1 - x))$	500	0.000459859397840
f5	(0, 1)	$\cos x \cos(\omega \sqrt{(1 - x^2})$	10	− 0.39615562798520
f6	(0, 1)	$e^x \sin(\omega \cosh x)$	10	−0.2556593290493
f7	(0, 1)	$\sin^2 x \cos(\omega \tanh x)$	10	0.13411649903305

TABLE II

f	*Levin*	*Inverse*	*Expansion*
f1	(20, 2, 20)	(17, 19, 51)	16
f2	(5, 2, 5)	(9, 31, 67)	2
f3	(80, 8, 80) (7)	(33, 130, 281) + 16	4*16 + 16
f4	(80, 8, 80)	(65, 262, 577)	8*16(10)
f5	fail	16 + (33, 135, 291) + 16	16 + (16*12) + 8
f6	(286, 38, 286)	(65, 267, 587) + 16	(4*12) + 16
f7	(94, 14, 94)	(33, 129, 279)	4*12

integrand. This is effective as the integrand is not oscillatory in this region. Counts for applications of Clenshaw-Curtis appear in the table separated by +.

The final column gives the counts for the expansion method. A pair such as 4*16 indicates 4 subdivisions and the 16 point rule. The number of f, q and q' function evaluations are identical and hence a single count is quoted. The object was to achieve 12 figures and compare the work involved for each method. Where this accuracy was not achieved for a reasonable effort, the number of figures actually found appears in round brackets after the count.

A range of combinations of expansion order N, and numbers of subdivisions were employed. Like many of these methods, too large a choice of N gives numerical instability, and $N < 16$ was found to be a safe limit in practice. It also has to be remembered that the function being expanded is effectively $f(x)/q'(x)$ from (2.1) and hence any example which exhibits zeros of $q'(x)$ in the interval of integration has to be treated as singular at this point. As this singular behaviour is artificially introduced as part of the method, it is straightforward to complete the part of the integrand containing that point by using a general quadrature method on the whole integrand. Hence for $f3$, $q'(x) = -\sin x$ and hence the integral is split into the range $[0, \varepsilon]$ which is integrated by Clenshaw-Curtis and $[\varepsilon, \beta]$ using the expansion method. The constant ε is chosen to give about half a cycle of the oscillatory part which enables Clenshaw-Curtis to be effective even for large ω. The number of points used in the Clenshaw-Curtis rule again appears after the + sign where appropriate.

The method is clearly effective and competitive, especially for large ω. The efficiency of the method can be enhanced by the surreptitious choice of function $g_i(q)$. Example $f2$ was invented to give an analytic result for comparison. The suggested method becomes exact in this case as $f(x)$ is $q'(x)$. On the other hand there are potential problems with $f3$ beyond the zero of $q'(x)$ at $x = 0$. The expansion involves trying to express cot x in powers of cos x. This is the case referred to earlier in which cos x behaves like $1 - O(x^2)$ and the g_i of equation (2.3) was used. This g was also used for $f5$, and by the same reasoning for $f6$ the form

$$g_i(q) = (\sqrt{(q^2 - 1)})^{i-1} \tag{4.1}$$

was used to convert an expansion in cosh x to one in sinh x, in each case with excellent results.

As a final test the method was used on three selected examples, $f5, f6$ and $f7$, to confirm its continuing accuracy for large ω. Again Clenshaw-Curtis quadrature is used on the whole integrand in the singular regions for a half-cycle range. The results are in agreement with the error analysis and show independence of the method to changes of ω ranging from 10.0 to 10000.0.

In addition, some experiments were carried out with different choices for the collocation points x_i, including the use of cosine weighted abscissae and open rules. The results showed little significant difference to those presented.

Two further examples

$$\int_0^1 \sin\left(\frac{x}{1-x}\right)dx = 0.3433779615564 \tag{4.2}$$

and

$$\int_0^{2\pi} \ln(x)\sin 30x\,dx = -0.1938772750999 \tag{4.3}$$

were considered in Levin [7]. These examples involve two simultaneous quadrature difficulties. In (4.2) the frequency of oscillations becomes infinite at $x = 1$. This type of problem arises when an infinite range oscillatory integral is transformed onto a finite range, in this case the original integral being

$$\int_0^\infty \frac{\sin x}{(1+x)^2}dx \tag{4.4}$$

where the combination of infinite range and oscillatory integrand constitutes the double problem. The transformation $x = y/(1-y)$ has been used to convert the infinite range to finite range, and the conventional approach to (4.4) is the method of Alaylioglu, Evans and Hyslop [13] in which cycles of the oscillatory integrand are evaluated and the resulting sequence is accelerated by a Shanks' process. Nine cycles yields 8 figure accuracy and requires 72 function evaluations.

In (4.3) the combination of difficulties is that of an oscillatory integrand combined with a singularity. The use of transformation methods to alleviate the singularity as in Evans, Forbes and Hyslop [14], Squire [15] and

Takahasi and Mori [2] will result in a regular oscillatory problem becoming irregular.

Levin generates relatively low accuracy values to these two examples. In the case of (4.2) the resulting ordinary differential equation can be multiplied through by $(1-x)$ so avoiding the singularity at $x=1$. A 10-point rule then yields 5 correct figures and a 15-point rule yields 7 figures. In the case of (4.3), Levin uses an expansion set which not only includes powers of x but also $x^i \ln x$ for non-negative i. Even with the ln function so built-in, it takes 7, 10-point rules to yield 5 figure accuracy.

By using an open version of the current method on (4.2), the four point rule yields 7 figures which is a considerable gain over the Levin results. In fact for the fitting process only a second order formula is needed as $f(x)=1$ and so the expansion using $g(q)=q/(1+q)$ is only required to fit $1/q'(x)=(1-x)^2$.

The method was also used to compute a sequence using the initial range $[0, y_1]$, with $x_1=[25\pi]/\pi$ and $x=y/(1-y)$, and then successive cycles in order to apply Aitken acceleration. An accuracy of 12 figures was obtained with just 9 sequence elements, each of which again only required a 2-point rule, to total just 18 function evaluations.

On (4.3), the transformation $y=x^N$ with $N=5$ was employed to remove the singularity as in Evans, Forbes and Hyslop [14]. A direct application of the above method then gave 6 figures with 64 points, 8 figures with 128 and 10 figures with 256.

The complication here was that $g_i(q)=q^{1/N}$ was used to get the expansion in powers of x. The underlying integrals (2.2) involved fractional powers of q with associated singular higher derivatives which gave slow convergence to the Patterson integrator though this does not affect the function count. Hence integration by parts was employed to alleviate this problem.

The most effective general approach to this type integral seems to be to split the range into $[0,\varepsilon]$ and $[\varepsilon, 2\pi]$ with the first interval ranging over say one half-cycle. Then the above transformation can be used on the range $[0,\varepsilon]$ which now has just one half-cycle allowing easy integration by a general purpose method such as Clenshaw-Curtis, which yields 14 figure accuracy with 32 points. The remaining range is then treated as an irregular oscillatory quadrature which gives 10 correct figures in 128 points.

To give a fair comparison with the Levin expansion which involved a basis set $x^i \ln x$, the alterative choice of $g_i(q)=q^i \ln q$ was employed here which of course gives the full accuracy at $n=2$. Though such a surreptious choice of g cannot be used in a general manner, for particular problems in

TABLE III

n	I
8*8	−8.597842027116(−5)
16*16	−8.597841103167(−5)
32*32	−8.597841100637(−5)
64*64	−8.597841100636(−5)

which efficiency is paramount then highly rapid evaluation may be achieved.

As a final test, a two dimensional example used by Levin [7] was attempted, namely

$$I = \int_0^1 \int_0^1 \cos(x+y)\cos[\omega(x+y+c(x^2+y^2)]\,dx\,dy \qquad (4.2)$$

The expansion method was applied as a product rule, which presents no new problems. The method is applied to the y integral whose integrand definition contains a further call to the method for the x integral with a specific y value fixed in the outer block. The results are shown in Table III for the specific values of $c = 1$ and $\omega = 100$.

These results compare with Levin's method in which a 25*25 rule gives $-8.595(-5)$, showing an impressive gain in accuracy, with the full 13 figures obtained for the 32*32 rule.

In conclusion the presented method seems to be competitive and simple to implement. The choice of g_i appears non-critical except when $q'(x)$ is close to zero, and in this case the integral is no longer highly oscillatory, and can be tackled using conventional means.

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