Numerical integration of highly oscillatory functions based on analytic continuation

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Outline

1. Introduction

a one-dimensional oscillatory integral

The model problem

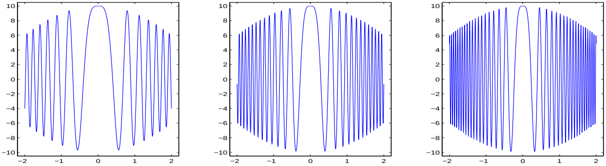
$$I := \int_a^b f(x) e^{i\omega g(x)} \, \mathrm{d}x$$

- with...
  - ω: *frequency parameter*; f: *amplitude*; g: *oscillator*
  - f, g: smooth real functions
- classical quadrature *deteriorates* rapidly as ω increases
  - take *fixed* number of points *per oscillation*
  - amount of operations scales *linearly* with ω
- new oscillatory quadrature methods
  - **asymptotic, Filon, Levin, numerical steepest descent**

1. Introduction

a one-dimensional oscillatory integral

What determines the value of the integral? Example: $(\mathbf{10} - \mathbf{x}^2)\mathbf{e}^{\mathbf{i}\,\omega\,\mathbf{x}^2}$



- regions where the oscillations do not cancel:
  $\Rightarrow$ **boundary points**
- regions where the integrand is (locally) not oscillatory:
  $\Rightarrow$ **stationary points**: solutions to $\mathbf{g}'(\mathbf{x}) = \mathbf{0}$

1. Introduction

a one-dimensional oscillatory integral

What is the size of the integral (asymptotically) ?

$$|I| = \mathcal{O}(\omega^{-1/(r+1)})$$

with r: the largest order of any stationary point ξ in $[a, b]$
$g^{(j)}(\xi) = 0$, $j = 1, \ldots, r$; $g^{(r+1)}(\xi) \neq 0$

Our goal: to find a decomposition of the integral

$$\int_a^b f(x)e^{i\omega g(x)}dx = F(a) + F(b) + \sum_i F(\xi_i)$$

with $\xi_i \in [a, b]$ the *stationary points*

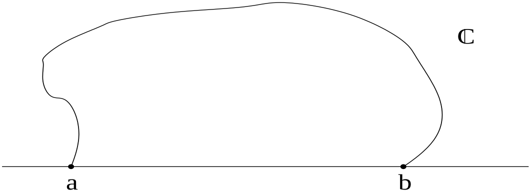
2. The steepest descent approach

an integration path in the complex plane

Basic idea: select a new integration path

- assume f and g are analytic
- *Cauchy's theorem*:

  the value of I does not depend on the complex path taken



2. The steepest descent approach

an integration path in the complex plane

The path of "steepest descent"
(Cauchy(1827), Riemann (1863), Debye (1909))

$$\mathbf{e}^{i\omega g(x)} = e^{i\omega(\Re g(x) + i\Im g(x))} = e^{-\omega \Im g(x)} e^{i\omega \Re g(x)}$$

1. The function $e^{i\omega g(x)}$ does not oscillate if $\Re g(x)$ is fixed
2. The function $e^{i\omega g(x)}$ decays exponentially fast if $\Im g(x) > 0$

$\Rightarrow$ new path at point a: $h_a(p)$ such that

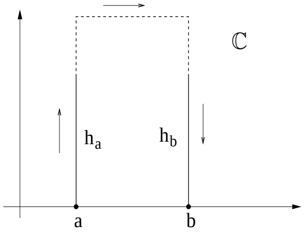
$$\mathbf{g(h_a(p)) = g(a) + p\,i}, \; p \geq 0$$

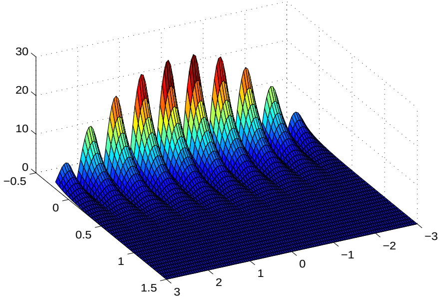
2. The steepest descent approach

an integration path in the complex plane

Example 1: Fourier oscillator $g(x) = x \quad (f(x) = 10 - x^2)$

$\mathbf{g(h_a(p)) = g(a) + pi} \Rightarrow h_a(p) = a + pi$





2. The steepest descent approach

an integration path in the complex plane

Decomposition: $I = \int_a^b f(x) e^{i\omega g(x)}\, dx = F(a) - F(b)$

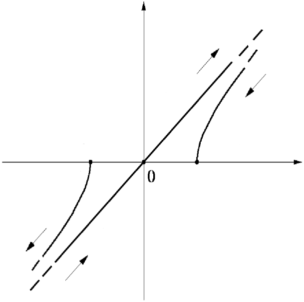
$$
\begin{aligned}
F(a) &= \int_0^\infty f(h_a(p)) e^{i\omega g(h_a(p))} h_a'(p) dp \\
&= \int_0^\infty f(a + pi) e^{i\omega(a+pi)} i\, dp \\
&= e^{i\omega a} \int_0^\infty f(a + pi) e^{-\omega p} i\, dp
\end{aligned}
$$

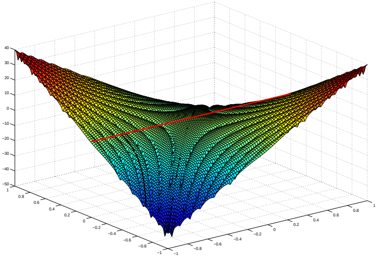
2. The steepest descent approach

an integration path in the complex plane

Example 2: Quadratic oscillator $g(x) = x^2$ **(**$f(x) = 10 - x^2$**)**

$$\mathbf{g(h_a(p)) = g(a) + pi} \Rightarrow h_a(p) = \pm\sqrt{a^2 + pi}$$





2. The steepest descent approach

an integration path in the complex plane

Decomposition: $I = F_1(a) - F_1(\xi) + F_2(\xi) - F_2(b)$

$$\int_{-1}^{1} f(x) e^{i\omega x^2} \mathrm{d}x =$$

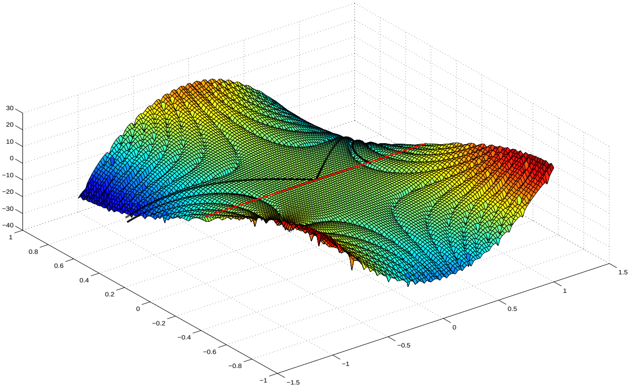
$$e^{i\omega} \int_0^\infty f(h_{-1,1}(p)) e^{-\omega p} h'_{-1,1}(p) \mathrm{d}p - \int_0^\infty f(h_{0,1}(p)) e^{-\omega p} h'_{0,1}(p) \mathrm{d}p$$
$$+ \int_0^\infty f(h_{0,2}(p)) e^{-\omega p} h'_{0,2}(p) \mathrm{d}p - e^{i\omega} \int_0^\infty f(h_{1,2}(p)) e^{-\omega p} h'_{1,2}(p) \mathrm{d}p$$

- Numerical *singularity* of the path: $h'_\xi(p) \sim p^{-1/2}$, $p \to 0$

2. The steepest descent approach

an integration path in the complex plane

Example 3: Cubic oscillator $g(x) = x^3$ **(and** $f(x) = 10 - x^2$**)**



3. The numerical steepest descent method

implementation issues and numerical results

Implementation issue 1. How to evaluate $\mathbf{F_j}$?

$$F_j(a) = e^{i\omega g(a)} \int_0^\infty f(h_a(p))h_a'(p)e^{-\omega p}dp$$
$$= \frac{e^{i\omega g(a)}}{\omega} \int_0^\infty f(h_a(\frac{q}{\omega}))h_a'(\frac{q}{\omega})e^{-q}dq$$

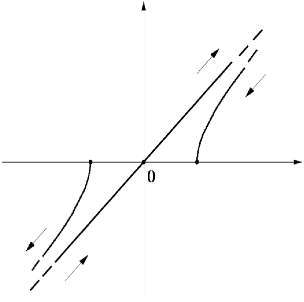
- if exponential decay: Gauss-Laguerre $(w(q) = e^{-q})$
- if singularity: generalized Gauss-Laguerre $(w(q) = q^{-\alpha}e^{-q})$
- or, with $t = \sqrt[r]{q}$, (half-line) Gauss-Hermite $(w(t) = e^{-t^r})$
- or trapezoidal rule, or Clenshaw-Curtis, ...

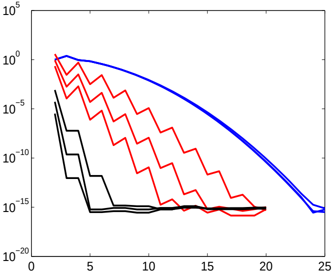
3. The numerical steepest descent method

implementation issues and numerical results

A comparison for $\int_a^b (cos(x) + sin(x)) e^{i\omega x^2} dx$.

(total contribution of the stationary point $\xi = 0$, for $\omega = 5, 20, 80$)





- Gauss-Hermite weight function e^{-q^2}, on $(-\infty, \infty)$
- Clenshaw Curtis, trapezoidal rule on $(-Q, Q)$

3. The numerical steepest descent method

implementation issues and numerical results

Example 1: $\int_0^1 \frac{1}{1+x} e^{i\omega x} dx$

$$I \approx Q_F[f, g, h_a] - Q_F[f, g, h_b] \qquad \text{using Gauss-Laguerre}$$

$\omega \setminus n$	1	2	3	4	5
10	$1.0E-3$	$3.1E-5$	$1.9E-6$	$1.7E-7$	$2.1E-8$
20	$1.2E-4$	$1.1E-6$	$2.3E-8$	$7.5E-10$	$3.2E-11$
40	$1.7E-5$	$3.9E-8$	$2.1E-10$	$2.0E-12$	$2.8E-14$
80	$2.0E-6$	$1.2E-9$	$1.7E-12$	$4.2E-15$	$1.6E-17$
rate	3.1(3)	5.0(5)	6.9(7)	8.9(9)	10.8(11)

absolute error: $O(\omega^{-(2n+1)})$ **relative error:** $O(\omega^{-2n})$

Example 2: $\int_{-1}^{1} \frac{1}{x+2} e^{i\omega x^3} dx$

$$I \approx Q[f, g, h_a] - Q_F[f, g, h_{\xi,1}] + Q_F[f, g, h_{\xi,2}] - Q_F[f, g, h_b]$$

$\omega \setminus n$	1	2	3	4	5
40	$1.5E-4$	$2.4E-6$	$1.3E-8$	$6.9E-11$	$7.6E-13$
80	$5.8E-5$	$7.1E-7$	$2.3E-9$	$6.4E-12$	$4.7E-14$
160	$2.3E-5$	$2.1E-7$	$4.2E-10$	$6.1E-13$	$3.1E-15$
320	$9.1E-6$	$6.7E-8$	$8.1E-11$	$5.8E-14$	$2.1E-16$
rate	1.3 (3/3)	1.7 (5/3)	2.4 (7/3)	3.4 (9/3)	3.9 (11/3)

For a stationary point ξ of order r, using Gauss-Hermite:

absolute error: $O(\omega^{-(2n+1)/(r+1)})$ **relative error:** $O(\omega^{-2n/(r+1)})$

3. The numerical steepest descent method

implementation issues and numerical results

Implementation issue 2. How to determine the path ?

- if inverse of g is available: $h_a(p) = g^{-1}(g(a) + pi)$
- otherwise...compute $h_a(x_i/\omega)$ and $h_a'(x_i/\omega)$ numerically

Apply *Newton iteration* to $g(h_a(p)) - g(a) - p\,i = 0$

- initial guess by truncated Taylor series of g

  e.g. $g(h_a(p)) \approx g(a) + g'(a)(h_a(p) - a)$
- only very few Newton iterations necessary

Derivative: $g(h_a(p)) - g(a) - pi = 0 \;\Rightarrow\; g'(h_a(p))h_a'(p) - i = 0$

3. The numerical steepest descent method

implementation issues and numerical results

Example: $\int_0^1 \frac{1}{1+x} e^{i\omega(x^2+x+1)^{1/3}} dx \approx Q_F[f,g,h_a] - Q_F[f,g,h_b]$

with 2^{nd} order Taylor path

$\omega \setminus n$	1	2	3	4	5
80	$3.8E-4$	$1.8E-5$	$1.7E-6$	$2.0E-7$	$2.9E-8$
160	$5.2E-5$	$1.1E-6$	$4.0E-8$	$2.1E-9$	$1.5E-10$
320	$6.7E-6$	$6.8E-8$	$7.7E-10$	$1.6E-11$	$4.4E-13$
rate	3.0	4.0	5.7	7.0	8.4

followed by 1 to 4 Newton iteration steps

$\omega \setminus n$	1	2	3	4	5
80	$3.3E-4$	$1.5E-5$	$1.2E-6$	$1.5E-7$	$2.3E-8$
160	$4.5E-5$	$6.1E-7$	$1.8E-8$	$8.7E-10$	$6.2E-11$
320	$5.9E-6$	$2.1E-8$	$1.8E-10$	$2.7E-12$	$6.2E-14$
rate	3.0	5.0	6.9	8.8	10.5

4. A Filon type quadrature rule

a localized Filon-Iserles-Nørsett method

The Generalized Filon method (Iserles and Nørsett)

Idea: *approximate f globally on* $[a,b]$ by Hermite interpolation

Assume: $f(x) \approx \sum_{i=1}^{N} c_i \phi_i(x)$

Then: $I := \int_a^b f(x) e^{i\omega g(x)}\, dx \approx \sum_{i=1}^{N} w_i c_i$

with $w_i = \int_a^b \phi_i(x) e^{i\omega g(x)} dx$

- interpolate derivatives
  - of order $0 \ldots s-1$ at boundary points
  - of order $0 \ldots (s-1)(r+1)$ at stationary points
- convergence $O(\omega^{-s-1/(r+1)})$

A localized Filon method

Idea: apply *local Hermite (or Taylor) approximation* of f for each integral contribution $\mathbf{F_j}$

Define: $F_j[f](x) := e^{i\omega g(x)} \int_0^\infty f(h_x(p))h_x'(p)e^{-\omega p}dp$

From: $f(z) \approx \sum_{i=0}^{d_j} f^{(i)}(x)\frac{(z-x)^i}{i!}$

We have: $F_j[f](x) \approx \sum_{i=0}^{d_j} w_{i,j} f^{(i)}(x)$

with: $w_{i,j} := F_j[\frac{(z-x)^i}{i!}](x)$

4. A Filon type quadrature rule

a localized Filon-Iserles-Nørsett method

A "classical" quadrature rule

$$I \approx \sum_{j=0}^{\ell} \sum_{i=0}^{d_j} w_{i,j} f^{(i)}(x_j)$$

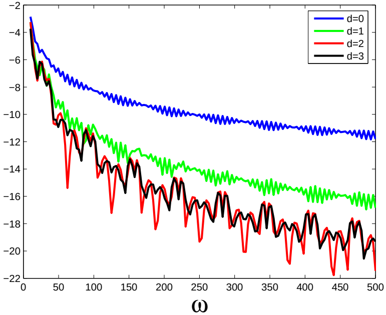
Convergence result:

Let $d_j = s - 1$ at a non-stationary point, and let $d_j = s(r+1) - 1$ at a stationary point then our rule has an absolute error of $O(\omega^{-s-1/(r+1)})$ and a relative error of $O(\omega^{-s})$, asymptotically for large ω.

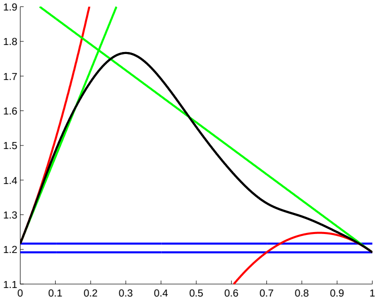
4. A Filon type quadrature rule

a localized Filon-Iserles-Nørsett method

Example 1: Fourier oscillator: $\int_0^1 f(x)e^{i\omega x}dx$



absolute error as a function of ω



function $f(x)$ and the local approximations in the endpoints

4. A Filon type quadrature rule

a localized Filon-Iserles-Nørsett method

Example 2: The Hankel oscillator

$$I_H[f] := \int_a^b f(x) H_\nu^{(1)}(\omega g_1(x)) e^{i\omega g_2(x)} \mathrm{d}x,$$

For large arguments

$$\begin{aligned} H_\nu^{(1)}(z) &\sim \sqrt{\frac{2}{\pi z}} e^{i(z - \frac{1}{2}\nu\pi - 1/4\pi)}, -\pi < arg z < \pi, \ |z| \to \infty. \\ &= u(z)\, e^{iz} \end{aligned}$$

Approximate oscillator: $g(x) = g_1(x) + g_2(x)$

Quadrature rule: $I_H[f] \approx Q_H[f] := \sum_{l=0}^{\ell} \sum_{j=0}^{d_l} w_{l,j}^H f^{(j)}(x_l).$

4. A Filon type quadrature rule

a localized Filon-Iserles-Nørsett method

Example 2: The Hankel oscillator

$$\int_0^1 \cos(x-1) H_0^{(1)}(\omega x) e^{i\omega(x^2+x^3-x)} dx \qquad (\sim \mathcal{O}(\omega^{-3/4})).$$

Two quadrature points:

$x = 0$: a singularity and a stationary point of order 1

$x = 1$: a regular endpoint

$\omega \setminus (d_0, d_1)$	$(0,0)$	$(1,0)$	$(2,0)$	$(3,1)$
100	$1.2E-3$	$2.8E-5$	$1.3E-6$	$2.6E-8$
200	$5.1E-4$	$8.6E-6$	$2.9E-7$	$4.1E-9$
400	$2.2E-4$	$2.6E-6$	$6.4E-8$	$6.2E-10$
800	$9.3E-5$	$7.8.1E-7$	$1.4E-8$	$9.7E-11$
rate	1.23 (1.25)	1.73 (1.75)	2.20 (2.25)	2.68 (2.75)

5. Multivariate oscillatory integrals

a repeated one-dimensional approach

Model form

$$I_n := \int_S f(\mathbf{x}) e^{i\omega g(\mathbf{x})}\, d\mathbf{x} \quad \text{with} \quad S \subset \mathbb{R}^n$$

Contributing points?

- corner points
- critical points: $\nabla g = 0$
- *resonance* points: $\nabla g \perp \partial S$

5. Multivariate oscillatory integrals

a repeated one-dimensional approach

Main approach

- deform onto path of steepest descent for inner variable

$$I_1(\mathbf{x}) := \int_{a(\mathbf{x})}^{b(\mathbf{x})} f(\mathbf{x}, y) e^{i\omega g(\mathbf{x},y)} \, dy = F(\mathbf{x}, a(\mathbf{x})) - F(\mathbf{x}, b(\mathbf{x}))$$

- the function F is evaluated only in points on the boundary

$$F(\mathbf{x}, a(\mathbf{x})) = e^{i\omega g(\mathbf{x},a(\mathbf{x}))} \int_0^\infty f(\mathbf{x}, u(\mathbf{x}, p)) \frac{\partial u(\mathbf{x}, p)}{\partial p} e^{-\omega p} dp$$

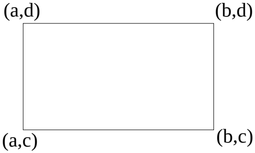
- oscillator of $F(\mathbf{x}, a(\mathbf{x}))$ is exactly known: $g(\mathbf{x}, a(\mathbf{x}))$!

5. Multivariate oscillatory integrals

a repeated one-dimensional approach

Example 1: Rectangular domain in two dimensions

$$I := \int_a^b \int_c^d f(x,y) e^{i\omega(x+y)}\, dy\, dx$$



- Step 1: deform onto path of steepest descent for y

$$I := \int_a^b G(x,c) e^{i\omega(x+c)} - G(x,d) e^{i\omega(x+d)} dx$$

- Smooth function G is given by

$$G(x,y) = \int_0^\infty f(x, y+ip) i e^{-\omega p}\, dp$$

5. Multivariate oscillatory integrals

a repeated one-dimensional approach

Example 1: Rectangular domain in two dimensions

$$\int_a^b G(x,c)e^{i\omega(x+c)} = \tilde{G}(a,c)e^{i\omega(a+c)} - \tilde{G}(b,c)e^{i\omega(b+c)}$$

- Total decomposition

$$I := F(a,c) - F(b,c) - F(a,d) + F(b,d)$$

- Contributions are given by non-oscillatory double integrals with exponential decay in both variables

$$F(x,y) = e^{i\omega(x+y)} \int_0^\infty \int_0^\infty f(x+ip, y+iq) i^2 e^{-\omega(p+q)} \, dq \, dp$$

5. Multivariate oscillatory integrals

a repeated one-dimensional approach

More general boundaries

$$I := \int_a^b \int_{c(x)}^{d(x)} f(x,y) e^{i\omega(x+y)}\, dy\, dx$$

$$= \int_a^b G(x, c(x)) e^{i\omega(x+c(x))} - G(x, d(x)) e^{i\omega(x+d(x))} dx$$

- new oscillator $x + c(x)$ may have stationary points!
- *resonance points*: stationary point of oscillator $g(x, c(x))$ evaluated along the boundary. This happens when $\nabla g \perp \partial S$

$$0 = \frac{d}{dx} g(x, c(x)) = \frac{\partial g}{\partial x} + \frac{\partial g}{\partial y} \cdot \frac{dc}{dx} = \left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}\right) \cdot \left(1, \frac{dc}{dx}\right)^T$$

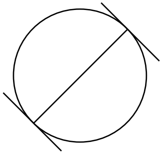
5. Multivariate oscillatory integrals

a repeated one-dimensional approach

Example 2: Fourier integral on a circle

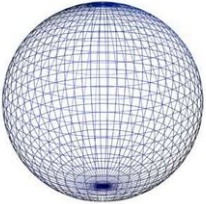
$$I := \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} f(x,y) e^{i\omega(x+y)}\, dy\, dx$$

$$= F\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right) - F\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$$



5. Multivariate oscillatory integrals

a repeated one-dimensional approach



Example 3:

Sphere with Fourier oscillator

$$I_3 = \int_{-1}^{1} \int_{-\sqrt{1-x_1^2}}^{\sqrt{1-x_1^2}} \int_{-\sqrt{1-x_1^2-x_2^2}}^{\sqrt{1-x_1^2-x_2^2}} e^{x_1+x_2^2 x_3} (3x_3 + \cos(x_2)) e^{i\omega(x_1+x_2+x_3)} \, \mathrm{d}x_3 \, \mathrm{d}x_2 \, \mathrm{d}x_1.$$

Two contributing resonance points: $\nabla g \perp \partial S$

$(-\sqrt{3}/3, -\sqrt{3}/3, -\sqrt{3}/3)$ and $(\sqrt{3}/3, \sqrt{3}/3, \sqrt{3}/3)$

5. Multivariate oscillatory integrals

a repeated one-dimensional approach

Cubature rule

$$I_3 \approx \sum_{i=1}^{2} \sum_{j} \sum_{k} \sum_{l} w_{i,j,k,l} \frac{\partial^{j+k+l} f}{\partial x_1^j \partial x_2^k \partial x_3^l}(\mathbf{x}_i)$$

Convergence

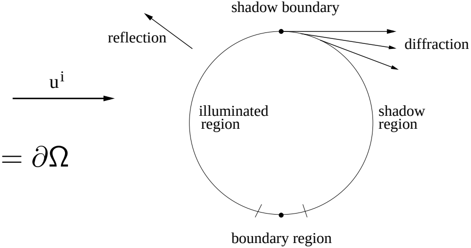
$\omega \setminus d$	0	1	2
100	$2.6e-5$	$2.4e-6$	$1.2e-7$
200	$3.2e-6$	$3.6e-7$	$8.0e-9$
400	$3.9e-7$	$5.2e-8$	$5.5e-10$
800	$5.0e-8$	$3.8e-9$	$2.7e-11$
1600	$6.3e-9$	$5.2e-10$	$1.7e-12$
rate	3.0 (2.5)	2.9 (3.0)	4.0 (3.5)

(*with classical rule:* $\omega = 10$, error 10^{-7}: need 100000 function evaluations)

6. An oscillatory integral equation

Scattering by a smooth, convex 2D obstacle

$\Delta u + k^2 u = 0$, with $u = 0$ on $\Gamma := \partial\Omega$



Total field $u = u^i + u^s$ with $u^s = -Sq$ where

$$(Sq)(x) = \int_\Gamma \frac{i}{4} H_0^{(1)}(k|x-y|)q(y)\,\mathrm{d}s_y$$

The density q satisfies the **combined potential integral equation**

$$\frac{q(x)}{2} + \int_\Gamma \left(\frac{\partial K}{\partial n_x}(x,y) + i\eta K(x,y)\right) q(y)\,\mathrm{d}s_y = r.h.s. \quad x \in \Gamma$$

6. An oscillatory integral equation

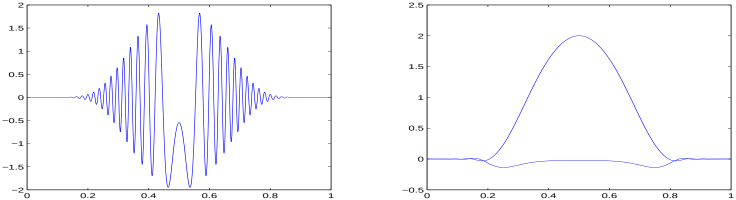
Ansatz for the solution (Bruno, Geuzaine, Monro, Reitich, 2004)

Assume incoming wave is given by $u^i(x) = u^i_s(x)e^{ikg^i(x)}$, then

$$q(\kappa(t)) = q_s(t)e^{ikg^i(\kappa(t))} \quad (\text{with } \Gamma : x = \kappa(t), t \in [0,1])$$

Function $q_s(t)$ is much less oscillatory than $q(\kappa(t))$. It satisfies

$$\frac{1}{2}q_s(t)e^{ikg^i(\kappa(t))} + \int_0^1 G(t,\tau)q_s(\kappa(\tau))e^{ikg^i(\kappa(\tau))}\,\mathrm{d}\tau = r.h.s.$$



6. An oscillatory integral equation

Collocation approach for the discretisation

Look for a solution $q_c(t) = \sum_{m=1}^{N} c_m \phi_m(t)$, with ϕ_m having local support.

Choose N distinct collocation points $x_n = \kappa(t_n)$, $t_n \in [0,1]$

For each collocation point t_n, we have one equation of the form

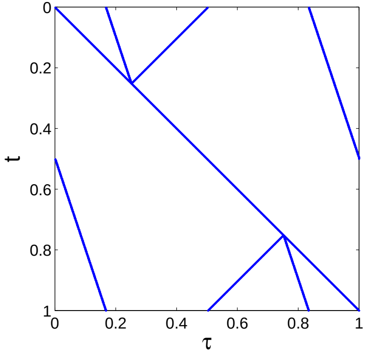
$$\frac{1}{2} q_c(t_n) e^{ikg^i(\kappa(t_n))} + \int_0^1 G(t_n,\tau) q_c(\tau) e^{ikg^i(\kappa(\tau))}\, \mathrm{d}\tau = r.h.s.$$

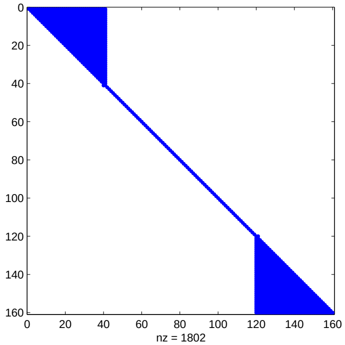
Compute integral with localized Filon

$$\begin{aligned} I[q_c;t_n] \approx Q[q_c;t_n] \quad &:= \quad \sum_{l=0}^{\ell_n} \sum_{j=0}^{d_{n,l}} w_{n,l,j} q_c^{(j)}(\tau_{n,l}) \\ &= \quad \sum_{m=1}^{N} \left(\sum_{l=0}^{\ell_n} \sum_{j=0}^{d_{n,l}} w_{n,l,j} \phi_m^{(j)}(\tau_{n,l}) \right) c_m \end{aligned}$$

6. An oscillatory integral equation

Location of quadrature points and discretization matrix.





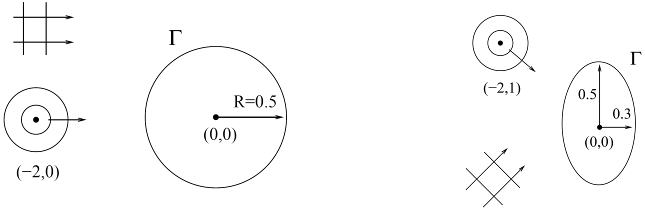
Quadrature rule fails in shadow boundary region... replace part of integral by classical rule

Discretization matrix: small and sparse, with structure independent of k.

6. An oscillatory integral equation

Example: Two smooth convex scattering obstacles.

The boundary conditions are plane waves,
or circular waves originating from a point source



6. An oscillatory integral equation

Total solution time in seconds, as a function of wavenumber k.
We used $d = 2$ derivatives in the quadrature rules.
(N_1, N_2, N_3): number of unknowns in region 1, 2, and 3.

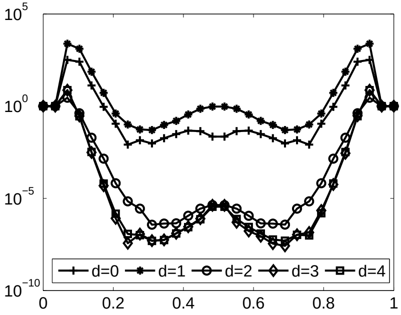
	Circle		Ellipse	
	Plane wave	Point source	Plane wave	Point source
(N_1, N_2, N_3)	$(30, 30, 30)$	$(30, 30, 30)$	$(60, 30, 60)$	$(60, 30, 60)$
$k = 200$	$287s$	$312s$	$463s$	$512s$
$k = 400$	$283s$	$308s$	$450s$	$496s$
$k = 800$	$281s$	$306s$	$443s$	$491s$
$k = 1600$	$279s$	$302s$	$438s$	$484s$
$k = 10000$	$273s$	$294s$	$422s$	$470s$
$k = 100000$	$269s$	$289s$	$416s$	$459s$

6. An oscillatory integral equation

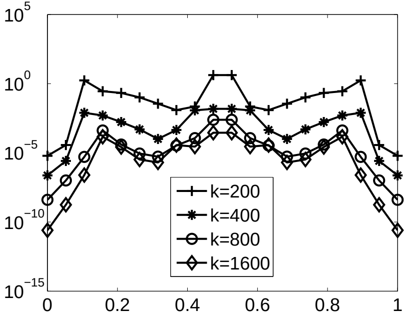
Scattering of a plane wave by a circle

Relative error: $\dfrac{|q_s(t) - q_c(t)|}{|q_s(t)|}$

Absolute error: $\dfrac{|q_s(t) - q_c(t)|}{k}$



$k = 200,\ d = 0, 1, 2, 3, 4$



$d = 1,\ k = 200, 400, 800, 1600$

7. Concluding remarks

1. **One-dimensional integrals**
   - two types of points: stationary points and endpoints
   - integration along the path of steepest descent
   - construction of Filon-type quadrature rules
2. **Multi-dimensional integrals**
   - three types of points: corners, critical points, resonance points
   - integration on a manifold of steepest descent
3. **Oscillatory integral equations**
   - hybrid method: basis functions incorporate asymptotic behaviour
   - small and sparse discrete system

References

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