Network Modelling of Physical Systems: a Geometric Approach*

Arjan van der Schaft[1], Bernhard Maschke[2], and Romeo Ortega[3]

[1] Fac. of Mathematical Sciences, University of Twente, Enschede, The Netherlands
[2] Laboratoire d'Automatique et de Genie des Procedes, Université Claude Bernard - Lyon 1, Villeurbanne Cedex, France
[3] Lab. des Signaux et Systèmes, CNRS–Supélec, Gif-sur-Yvette, France

Abstract. It is discussed how *network modeling* of lumped-parameter physical systems naturally leads to a geometrically defined class of systems, called *port-controlled Hamiltonian systems (with dissipation)*. The structural properties of these systems are investigated, in particular the existence of Casimir functions and their implications for stability. It is shown how the power-conserving interconnection with a controller system which is also a port-controlled Hamiltonian system defines a closed-loop port-controlled Hamiltonian system; and how this may be used for control by shaping the internal energy. Finally, extensions to implicit system descriptions (constraints, no a priori input-output structure) are discussed.

1 Introduction

Nonlinear systems and control theory has witnessed tremendous developments over the last three decades, see for example the textbooks [12,25]. Especially the introduction of geometric tools like Lie brackets of vector fields on manifolds has greatly advanced the theory, and has enabled the proper generalization of many fundamental concepts known for *linear control systems* to the nonlinear world. While the emphasis in the eighties has been primarily on the *structural* analysis of smooth nonlinear dynamical control systems, in the nineties this has been combined with analytic techniques for stability, stabilization and robust control, leading e.g. to backstepping techniques and nonlinear H_∞- control. Moreover, in the last decade the theory of *passive* systems, and its implications for regulation and tracking, has undergone a remarkable revival. This last development was also spurred by work in robotics on the possibilities of *shaping* by feedback the *physical energy* in such a way that it can be used as a suitable Lyapunov function for the control purpose at hand, see e.g. the influential paper [42]. This has led to what is called *passivity-based control*, see e.g. [26,32,13].

In this lecture we want to stress the importance of *modelling* for nonlinear control. Of course, this is well-known for (nonlinear) control applications, but

* This paper is an adapted and expanded version of [33]. Part of this material can be also found in [32].

in our opinion also the development of nonlinear control *theory* for physical systems should be integrated with a theoretical framework for modelling. We discuss how *network modelling* of (lumped-parameter) physical systems naturally leads to a geometrically defined class of systems, called *port-controlled Hamiltonian systems with dissipation* (PCHD systems). This provides a unified mathematical framework for the description of physical systems stemming from different physical domains, such as mechanical, electrical, thermal, as well as mixtures of them.

Historically, the Hamiltonian approach has its roots in analytical mechanics and starts from the principle of least action, via the Euler-Lagrange equations and the Legendre transform, towards the Hamiltonian equations of motion. On the other hand, the network approach stems from electrical engineering, and constitutes a cornerstone of systems theory. While most of the *analysis* of physical systems has been performed within the Lagrangian and Hamiltonian framework, the network modelling point of view is prevailing in *modelling* and *simulation* of (complex) physical systems. The framework of PCHD systems *combines* both points of view, by associating with the interconnection structure ("generalized junction structure" in bond graph terminology) of the network model a *geometric structure* given by a *Poisson structure*, or more generally a *Dirac structure*. The Hamiltonian dynamics is then defined with respect to this Poisson (or Dirac) structure *and* the Hamiltonian given by the total stored energy, as well as the energy-dissipating elements and the ports of the system.

Dirac structures encompass the "canonical" structures which are classically being used in the geometrization of mechanics, since they also allow to describe the geometric structure of systems with *constraints* as arising from the interconnection of sub-systems. Furthermore, Dirac structures allow to extend the Hamiltonian description of *distributed parameter systems* to include variable boundary conditions, leading to port-controlled distributed parameter Hamiltonian systems with boundary ports, see [17].

The structural properties of PCHD systems can be investigated through geometric tools stemming from the theory of Hamiltonian systems.We shall indicate how the *interconnection* of PCHD systems leads to another PCHD system, and how this may be exploited for control and design. In particular, we investigate the existence of Casimir functions for the feedback interconnection of a plant PCHD system and a controller PCHD system, leading to a reduced PCHD system on invariant manifolds with *shaped* energy. We thus provide an interpretation of *passivity-based control* from an *interconnection* point of view. This point of view can be further extended to what has been recently called Interconnection-Damping Assignment Passivity-Based Control (IDA-PBC).

2 Port-controlled Hamiltonian systems

2.1 From the Euler-Lagrange and Hamiltonian equations to port-controlled Hamiltonian systems

Let us briefly recall the standard Euler-Lagrange and Hamiltonian equations of motion. The standard *Euler-Lagrange equations* are given as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}(q,\dot{q})\right) - \frac{\partial L}{\partial q}(q,\dot{q}) = \tau, \tag{1}$$

where $q = (q_1, \ldots, q_k)^T$ are generalized configuration coordinates for the system with k degrees of freedom, the Lagrangian L equals the difference $K - P$ between kinetic energy K and potential energy P, and $\tau = (\tau_1, \ldots, \tau_k)^T$ is the vector of generalized forces acting on the system. Furthermore, $\frac{\partial L}{\partial \dot{q}}$ denotes the column-vector of partial derivatives of $L(q,\dot{q})$ with respect to the generalized velocities $\dot{q}_1, \ldots, \dot{q}_k$, and similarly for $\frac{\partial L}{\partial q}$. In standard mechanical systems the kinetic energy K is of the form

$$K(q,\dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q} \tag{2}$$

where the $k \times k$ inertia (generalized mass) matrix $M(q)$ is symmetric and positive definite for all q. In this case the vector of generalized *momenta* $p = (p_1, \ldots, p_k)^T$, defined for any Lagrangian L as $p = \frac{\partial L}{\partial \dot{q}}$, is simply given by

$$p = M(q)\dot{q}, \tag{3}$$

and by defining the state vector $(q_1, \ldots, q_k, p_1, \ldots, p_k)^T$ the k second-order equations (1) transform into $2k$ first-order equations

$$\begin{aligned} \dot{q} &= \tfrac{\partial H}{\partial p}(q,p) \qquad (= M^{-1}(q)p) \\ \dot{p} &= -\tfrac{\partial H}{\partial q}(q,p) + \tau \end{aligned} \tag{4}$$

where

$$H(q,p) = \frac{1}{2}p^T M^{-1}(q)p + P(q) \quad \left(= \frac{1}{2}\dot{q}^T M(q)\dot{q} + P(q)\right) \tag{5}$$

is the total energy of the system. The equations (4) are called the *Hamiltonian equations* of motion, and H is called the *Hamiltonian*. The following *energy balance* immediately follows from (4):

$$\frac{d}{dt}H = \frac{\partial^T H}{\partial q}(q,p)\dot{q} + \frac{\partial^T H}{\partial p}(q,p)\dot{p} = \frac{\partial^T H}{\partial p}(q,p)\tau \;=\; \dot{q}^T\tau, \tag{6}$$

expressing that the increase in energy of the system is equal to the supplied work (*conservation of energy*).

If the potential energy is *bounded from below*, that is $\exists C > -\infty$ such that $P(q) \geq C$, then it follows that (4) with inputs $u = \tau$ and outputs $y = \dot{q}$ is a *passive* (in fact, a *lossless*) state space system with storage function $H(q,p) - C \geq 0$ (see e.g. [43,11,32] for the general theory of passive and dissipative systems). Since the energy is only defined up to a constant, we may as well take as potential energy the function $P(q) - C \geq 0$, in which case the total energy $H(q,p)$ becomes nonnegative and thus itself is the storage function.

System (4) is an example of a *Hamiltonian system* with collocated inputs and outputs, which more generally is given in the following form

$$\begin{aligned} \dot{q} &= \frac{\partial H}{\partial p}(q,p), \qquad (q,p) = (q_1, \dots, q_k, p_1, \dots, p_k) \\ \dot{p} &= -\frac{\partial H}{\partial q}(q,p) + B(q)u, \qquad u \in \mathbb{R}^m, \\ y &= B^T(q)\frac{\partial H}{\partial p}(q,p) \quad (= B^T(q)\dot{q}), \qquad y \in \mathbb{R}^m, \end{aligned} \tag{7}$$

Here $B(q)$ is the input force matrix, with $B(q)u$ denoting the generalized forces resulting from the control inputs $u \in \mathbb{R}^m$. The state space of (7) with local coordinates (q,p) is usually called the *phase space*. Normally $m < k$, in which case we speak of an *underactuated* system.

Because of the form of the output equations $y = B^T(q)\dot{q}$ we again obtain the energy balance

$$\frac{dH}{dt}(q(t),p(t)) = u^T(t)y(t) \tag{8}$$

and if H is bounded from below, any Hamiltonian system (7) is a lossless state space system. For a system-theoretic treatment of Hamiltonian systems (7), we refer to e.g. [4,29,30,6,25].

A major generalization of the class of Hamiltonian systems (7) is to consider systems which are described in local coordinates as

$$\begin{aligned} \dot{x} &= J(x)\tfrac{\partial H}{\partial x}(x) + g(x)u, \qquad x \in \mathcal{X}, u \in \mathbb{R}^m \\ y &= g^T(x)\tfrac{\partial H}{\partial x}(x), \qquad y \in \mathbb{R}^m \end{aligned} \tag{9}$$

Here $J(x)$ is an $n \times n$ matrix with entries depending smoothly on x, which is assumed to be *skew-symmetric*

$$J(x) = -J^T(x), \tag{10}$$

and $x = (x_1, \dots, x_n)$ are local coordinates for an n-dimensional state space manifold $\mathcal{X}$. Because of (10) we easily recover the energy-balance $\frac{dH}{dt}(x(t)) =$

$u^T(t)y(t)$, showing that (9) is lossless if $H \geq 0$. We call (9) with J satisfying (10) a *port-controlled Hamiltonian (PCH) system* with *structure matrix* $J(x)$ and *Hamiltonian* H ([21,16,15]).

As an important mathematical note, we remark that in many examples the structure matrix J will satisfy the "*integrability*" conditions

$$\sum_{l=1}^{n} \left[J_{lj}(x)\frac{\partial J_{ik}}{\partial x_l}(x) + J_{li}(x)\frac{\partial J_{kj}}{\partial x_l}(x) + J_{lk}(x)\frac{\partial J_{ji}}{\partial x_l}(x) \right] = 0$$

$$i, j, k = 1, \ldots, n \tag{11}$$

In this case we may find, by Darboux's theorem (see e.g. [14]) around any point x_0 where the rank of the matrix $J(x)$ is constant, local coordinates $\tilde{x} = (q, p, s) = (q_1, \ldots, q_k, p_1, \ldots, p_k, s_1, \ldots s_l)$, with $2k$ the rank of J and $n = 2k + l$, such that J in these coordinates takes the form

$$J = \begin{bmatrix} 0 & I_k & 0 \\ -I_k & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{12}$$

The coordinates (q, p, s) are called *canonical* coordinates, and J satisfying (10) and (11) is called a *Poisson structure matrix*. In such canonical coordinates the equations (9) are very close to the standard Hamiltonian form (7).

PCH systems arise systematically from *network-type models* of physical systems as formalized within the (generalized) bond graph language ([28,3]). Indeed, the structure matrix $J(x)$ and the input matrix $g(x)$ may be directly associated with the network interconnection structure given by the bond graph, while the Hamiltonian H is just the sum of the energies of all the energy-storing elements; see our papers [16,21,18,22,35,36,23,31]. This is most easily exemplified by electrical circuits.

Example 1 (LCTG circuits) *Consider a controlled LC-circuit consisting of two parallel inductors with magnetic energies $H_1(\varphi_1), H_2(\varphi_2)$ (φ_1 and φ_2 being the magnetic flux linkages), in parallel with a capacitor with electric energy $H_3(Q)$ (Q being the charge). If the elements are linear then $H_1(\varphi_1) = \frac{1}{2L_1}\varphi_1^2$, $H_2(\varphi_2) = \frac{1}{2L_2}\varphi_2^2$ and $H_3(Q) = \frac{1}{2C}Q^2$. Furthermore let $V = u$ denote a voltage source in series with the first inductor. Using Kirchhoff's laws one immediately arrives at the dynamical equations*

$$\begin{bmatrix} \dot{Q} \\ \dot{\varphi}_1 \\ \dot{\varphi}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}}_{J} \begin{bmatrix} \frac{\partial H}{\partial Q} \\ \frac{\partial H}{\partial \varphi_1} \\ \frac{\partial H}{\partial \varphi_2} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u \tag{13}$$

$$y = \frac{\partial H}{\partial \varphi_1} \qquad (= \textit{current through first inductor})$$

with $H(Q, \varphi_1, \varphi_2) := H_1(\varphi_1) + H_2(\varphi_2) + H_3(Q)$ *the total energy. Clearly the matrix* J *is skew-symmetric, and since* J *is constant it trivially satisfies (11). In [22] it has been shown that in this way every LC-circuit with independent elements can be modelled as a port-controlled Hamiltonian system, with the constant skew-symmetric matrix* J *being solely determined by the network topology (i.e., Kirchhoff's laws). Furthermore, also any LCTG-circuit with independent elements can be modelled as a PCH system, with* J *determined by Kirchhoff's laws and the constitutive relations of the transformers* T *and gyrators* G*.* □

Another important class of PCH systems are mechanical systems as arising from *reduction by a symmetry group*, such as Euler's equations for a rigid body.

2.2 Basic properties of port-controlled Hamiltonian systems

Recall that a port-controlled Hamiltonian system is defined by a state space manifold $\mathcal{X}$ endowed with a *triple* (J, g, H). The pair $(J(x), g(x)), x \in \mathcal{X}$, captures the *interconnection structure* of the system, with $g(x)$ modeling in particular the *ports* of the system. Independently from the interconnection structure, the function $H : \mathcal{X} \to \mathbb{R}$ defines the total stored *energy* of the system.

PCH systems are intrinsically *modular* in the sense that any power-conserving interconnection of a number of PCH systems again defines a PCH system, with its overall interconnection structure determined by the interconnection structures of the composing individual PCH systems together with their power-conserving interconnection, and the Hamiltonian just the sum of the individual Hamiltonians (see [36,31,7]). The only thing which needs to be taken into account is the fact that a general power-conserving interconnection of PCH systems not always leads to a PCH system with respect to a Poisson structure $J(x)$ and input matrix $g(x)$ as above, since the interconnection may introduce *algebraic constraints* between the state variables of the individual sub-systems. Nevertheless, also in this case the resulting system still can be seen as a PCH system, which now, however, is defined with respect to a *Dirac structure*, generalizing the notion of a Poisson structure. The resulting class of *implicit* PCH systems, see e.g. [36,31,7], will be discussed in Section 4.

From the structure matrix $J(x)$ of a port-controlled Hamiltonian system one can directly extract useful information about the dynamical properties of the system. Since the structure matrix is directly related to the modeling of the system (capturing the interconnection structure) this information usually has a direct physical interpretation. A very important property is the possible existence of dynamical invariants *independent* of the Hamiltonian H. Consider the set of p.d.e.'s

$$\frac{\partial^T C}{\partial x}(x) J(x) = 0, \qquad x \in \mathcal{X}, \tag{14}$$

in the unknown (smooth) function $C : \mathcal{X} \to \mathbb{R}$. If (14) has a solution C then it follows that the time-derivative of C along the port-controlled Hamiltonian system (9) satisfies

$$\begin{aligned} \frac{dC}{dt} &= \frac{\partial^T C}{\partial x}(x) J(x) \frac{\partial H}{\partial x}(x) + \frac{\partial^T C}{\partial x}(x) g(x) u \\ &= \frac{\partial^T C}{\partial x}(x) g(x) u \end{aligned} \tag{15}$$

Hence, for the input $u = 0$, or for *arbitrary* input functions if additionally $\frac{\partial^T C}{\partial x}(x) g(x) = 0$, the function $C(x)$ *remains constant* along the trajectories of the port-controlled Hamiltonian system, *irrespective* of the precise form of the Hamiltonian H. A function $C : \mathcal{X} \to \mathbb{R}$ satisfying (14) is called a *Casimir function* (of the structure matrix $J(x)$).

It follows that the level sets $L_C := \{x \in \mathcal{X} | C(x) = c\}, c \in \mathbb{R}$, of a Casimir function C are *invariant* sets for the autonomous Hamiltonian system $\dot{x} = J(x)\frac{\partial H}{\partial x}(x)$, while the dynamics *restricted* to any level set L_C is given as the *reduced* Hamiltonian dynamics

$$\dot{x}_C = J_C(x_C) \frac{\partial H_C}{\partial x}(x_C) \tag{16}$$

with H_C and J_C the *restriction* of H, respectively J, to L_C. The existence of Casimir functions has immediate consequences for stability analysis of (9) for $u = 0$. Indeed, if $C_1, \cdots, C_r$ are Casimirs, then by (14) not only $\frac{dH}{dt} = 0$ for $u = 0$, but

$$\frac{d}{dt}\left(H + H_a(C_1, \cdots, C_r)\right)(x(t)) = 0 \tag{17}$$

for *any* function $H_a : \mathbb{R}^r \to \mathbb{R}$. Hence, if H is not positive definite at an equilibrium $x^* \in \mathcal{X}$, then $H + H_a(C_1, \cdots, C_r)$ may be rendered positive definite at x^* by a proper choice of H_a, and thus may serve as a Lyapunov function. This method for stability analysis is called the *Energy-Casimir method*, see e.g. [14].

Example 2 (Example 1 continued) *The quantity $\phi_1 + \phi_2$ is a Casimir function.*

2.3 Port-controlled Hamiltonian systems with dissipation

Energy-dissipation is included in the framework of port-controlled Hamiltonian systems (9) by terminating some of the ports by resistive elements. In the sequel we concentrate on PCH systems with *linear* resistive elements $u_R = -Sy_R$ for some positive semi-definite symmetric matric $S = S^T \geq 0$, where u_R and y_R are the power variables at the resistive ports. This leads to models of the form

$$\begin{aligned} \dot{x} &= [J(x) - R(x)] \frac{\partial H}{\partial x}(x) + g(x) u \\ y &= g^T(x) \frac{\partial H}{\partial x}(x) \end{aligned} \tag{18}$$

where $R(x)$ is a positive semi-definite symmetric matrix, depending smoothly on x. In this case the energy-balancing property (7) takes the form

$$\begin{aligned}\frac{dH}{dt}(x(t)) &= u^T(t)y(t) - \frac{\partial^T H}{\partial x}(x(t))R(x(t))\frac{\partial H}{\partial x}(x(t)) \\ &\leq u^T(t)y(t). \end{aligned} \tag{19}$$

showing passivity if the Hamiltonian H is bounded from below. We call (18) a *port-controlled Hamiltonian system with dissipation* (PCHD system). Note that in this case *two* geometric structures play a role: the internal power-conserving interconnection structure given by $J(x)$, and an additional resistive structure given by $R(x)$.

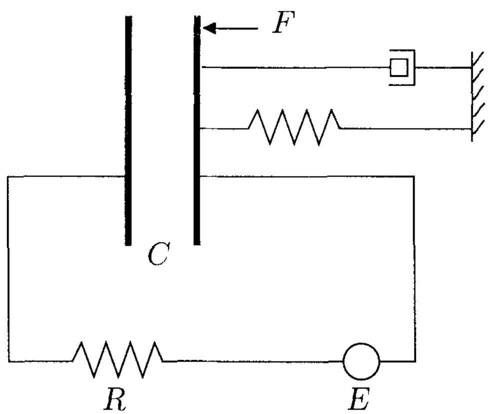

Fig. 1. Capacitor microphone

Example 3 *([24]) Consider the capacitor microphone depicted in Figure 1. Here the capacitance $C(q)$ of the capacitor is varying as a function of the displacement q of the right plate (with mass m), which is attached to a spring (with spring constant $k > 0$) and a damper (with constant $c > 0$), and affected by a mechanical force F (air pressure arising from sound). Furthermore, E is a voltage source. The dynamical equations of motion can be written* *as the PCHD system*

$$\begin{bmatrix} \dot{q} \\ \dot{p} \\ \dot{Q} \end{bmatrix} = \left(\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & \frac{1}{R} \end{bmatrix} \right) \begin{bmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \\ \frac{\partial H}{\partial Q} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} F + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{R} \end{bmatrix} E$$

$$y_1 = \frac{\partial H}{\partial p} = \dot{q} \tag{20}$$

$$y_2 = \frac{1}{R}\frac{\partial H}{\partial Q} = I$$

with p the momentum, R the resistance of the resistor, I the current through the voltage source, and the Hamiltonian H being the total energy

$$H(q,p,Q) = \frac{1}{2m}p^2 + \frac{1}{2}k(q-\bar{q})^2 + \frac{1}{2C(q)}Q^2, \tag{21}$$

with $\bar{q}$ denoting the equilibrium position of the spring. Note that $F\dot{q}$ is the mechanical power, and EI the electrical power applied to the system. In the application as a microphone the voltage over the resistor will be used (after amplification) as a measure for the mechanical force F.

A rich class of examples of PCHD systems is provided by electro-mechanical systems such as induction motors, see e.g. [27]. In some examples the interconnection structure $J(x)$ is actually *varying*, depending on the mode of operation of the system, as is the case for power converters (see e.g. [9]) or for mechanical systems with variable constraints.

3 Control of port-controlled Hamiltonian systems with dissipation

The aim of this section is to discuss a general methodology for controlling PCH or PCHD systems which exploits their Hamiltonian properties in an intrinsic way. Since this exposition is based on ongoing recent research (see e.g. [19,39,20,27,32]) we only try to indicate its potential. An expected benefit of such a methodology is that it leads to physically interpretable controllers, which possess inherent robustness properties. Future research is aimed at corroborating these claims.

We have already seen that PCH or PCHD systems are *passive* if the Hamiltonian H is bounded from below. Hence in this case we can use all the results from the theory of passive systems, such as asymptotic stabilization by the insertion of *damping* by negative output feedback, see e.g. [32]. The emphasis in this section is however on the somewhat complementary aspect of *shaping the energy* of the system, which directly involves the Hamiltonian structure of the system, as opposed to the more general passivity structure.

3.1 Control by interconnection

Consider a port-controlled Hamiltonian system with dissipation (18) regarded as a plant system to be controlled. Recall the well-known result that the standard feedback interconnection of two passive systems again is a passive system; a basic fact which can be used for various stability and control purposes ([11,26,32]). In the same vein we consider the interconnection of the plant (18) with *another* port-controlled Hamiltonian system with dissipation

$$C: \begin{array}{l} \dot{\xi} = [J_C(\xi) - R_C(\xi)]\frac{\partial H_C}{\partial \xi}(\xi) + g_C(\xi)u_C \\ \qquad\qquad\qquad\qquad\qquad\qquad \xi \in \mathcal{X}_C \\ y_C = g_C^T(\xi)\frac{\partial H_C}{\partial \xi}(\xi) \end{array} \tag{22}$$

regarded as the *controller* system, via the standard feedback interconnection

$$\begin{aligned} u &= -y_C + e \\ u_C &= y + e_C \end{aligned} \tag{23}$$

with e, e_C external signals inserted in the feedback loop. The closed-loop system takes the form

$$\begin{aligned} \begin{bmatrix} \dot{x} \\ \dot{\xi} \end{bmatrix} &= \Big(\underbrace{\begin{bmatrix} J(x) & -g(x)g_C^T(\xi) \\ g_C(\xi)g^T(x) & J_C(\xi) \end{bmatrix}}_{J_{cl}(x,\xi)} - \underbrace{\begin{bmatrix} R(x) & 0 \\ 0 & R_C(\xi) \end{bmatrix}}_{R_{cl}(x,\xi)} \Big) \begin{bmatrix} \frac{\partial H}{\partial x}(x) \\ \frac{\partial H_C}{\partial \xi}(\xi) \end{bmatrix} \\ &\quad + \begin{bmatrix} g(x) & 0 \\ 0 & g_C(\xi) \end{bmatrix} \begin{bmatrix} e \\ e_C \end{bmatrix} \\ \begin{bmatrix} y \\ y_C \end{bmatrix} &= \begin{bmatrix} g(x) & 0 \\ 0 & g_C(\xi) \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial x}(x) \\ \frac{\partial H_C}{\partial \xi}(\xi) \end{bmatrix} \end{aligned} \tag{24}$$

which again is a port-controlled Hamiltonian system with dissipation, with state space given by the product space $\mathcal{X} \times \mathcal{X}_C$, total Hamiltonian $H(x) + H_C(\xi)$, inputs (e, e_C) and outputs (y, y_C). Hence the feedback interconnection of any two PCHD systems results in another PCHD system; just as in the case of passivity. This is a special case of a theorem ([32]), which says that any regular power-conserving interconnection of PCHD systems defines another PCHD system.

It is of interest to investigate the *Casimir functions* of the closed-loop system, especially those relating the state variables ξ of the controller system to the state variables x of the plant system. Indeed, from a control point of view the Hamiltonian H is *given* while H_C can be *assigned.* Thus if we can find Casimir functions $C_i(\xi, x), i = 1, \cdots, r$, relating ξ to x then by the Energy-Casimir method the Hamiltonian $H + H_C$ of the closed-loop system may be replaced by the Hamiltonian $H + H_C + H_a(C_1, \cdots, C_r)$, thus creating the possibility of obtaining a suitable Lyapunov function for the closed-loop system.

Example 4 *[38] Consider the "plant" system*

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \end{bmatrix} \tag{25}$$

with q the position and p being the momentum of the mass m, in feedback interconnection ($u = -y_C + e, u_C = y$) with the controller system (see Figure 2)

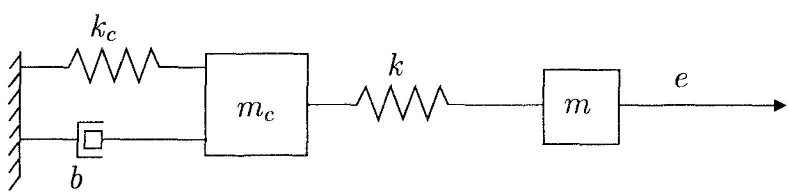

Fig. 2. Controlled mass

$$\begin{bmatrix} \dot{\Delta q_c} \\ \dot{p}_c \\ \dot{\Delta q} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & -b & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H_C}{\partial \Delta q_c} \\ \frac{\partial H_C}{\partial p_c} \\ \frac{\partial H_C}{\partial \Delta q} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u_C$$
$$y_C = \frac{\partial H_C}{\partial \Delta q}$$

where Δq_c is the displacement of the spring k_c, Δq is the displacement of the spring k, and p_c is the momentum of the mass m_c. The plant Hamiltonian is $H(p) = \frac{1}{2m}p^2$, and the controller Hamiltonian is given as $H_C(\Delta q_c, p_c, \Delta q) = \frac{1}{2}(\frac{p_c^2}{m_c} + k(\Delta q)^2 + k_c(\Delta q_c)^2)$. The variable $b > 0$ is the damping constant, and e is an external force. The closed-loop system possesses the Casimir function

$$C(q, \Delta q_c, \Delta q) = \Delta q - (q - \Delta q_c), \tag{26}$$

implying that along the solutions of the closed-loop system

$$\Delta q = q - \Delta q_c + c \tag{27}$$

with c a constant depending on the initial conditions. With the help of LaSalle's Invariance principle it can be shown that restricted to the invariant manifolds (27) the system is asymptotically stable for the equilibria $q = \Delta q_c = p = p_c = 0$. □

As a special case (see [32] for a more general discussion) let us consider Casimir functions for (24) of the form

$$\xi_i - G_i(x) \qquad , \quad i = 1, \dots, \dim \mathcal{X}_C = n_C \tag{28}$$

That means that we are looking for solutions of the p.d.e.'s (with e_i denoting the i-th basis vector)

$$\begin{bmatrix} -\frac{\partial^T G_i}{\partial x}(x) \; e_i^T \end{bmatrix} \begin{bmatrix} J(x) - R(x) & -g(x)g_C^T(\xi) \\ g_C(\xi)g^T(x) & J_C(\xi) - R_C(\xi) \end{bmatrix} = 0$$

for $i = 1, \dots, n_C$, relating *all* the controller state variables $\xi_1, \dots, \xi_{n_C}$ to the plant state variables x. Denoting $G = (G_1, \dots, G_{n_C})^T$ this means ([32]) that G should satisfy

$$\begin{aligned} &\tfrac{\partial^T G}{\partial x}(x) J(x) \tfrac{\partial G}{\partial x}(x) = J_C(\xi) \\ &R(x) \tfrac{\partial G}{\partial x}(x) = 0 = R_C(\xi) \\ &\tfrac{\partial^T G}{\partial x}(x) J(x) = g_C(\xi)\, g^T(x) \end{aligned} \tag{29}$$

In this case the reduced dynamics on any multi-level set

$$L_C = \{(x, \xi) | \xi_i = G_i(x) + c_i, i = 1, \dots n_C\} \tag{30}$$

can be immediately recognized ([32]) as the PCHD system

$$\dot{x} = [J(x) - R(x)] \frac{\partial H_s}{\partial x}(x), \tag{31}$$

with the same interconnection and dissipation structure as before, but with *shaped* Hamiltonian H_s given by

$$H_s(x) = H(x) + H_C(G(x) + c). \tag{32}$$

In the context of actuated mechanical systems this amounts to the shaping of the *potential energy* as in the classical paper [42], see [32].

A direct interpretation of the shaped Hamiltonian H_s in terms of *energy-balancing* is obtained as follows. Since $R_C(\xi) = 0$ by (29) the controller Hamiltonian H_C satisfies $\frac{dH_C}{dt} = u_C^T y_C$. Hence along any multi-level set L_C given by (30) $\frac{dH_s}{dt} = \frac{dH}{dt} + \frac{dH_C}{dt} = \frac{dH}{dt} - u^T y$, since $u = -y_C$ and $u_C = y$. Therefore, up to a constant,

$$H_s(x(t)) = H(x(t)) - \int_0^t u^T(\tau) y(\tau) d\tau, \tag{33}$$

and the shaped Hamiltonian H_s is the original Hamiltonian H *minus the energy* supplied to the plant system (18) by the controller system (22). From a stability analysis point of view (33) can be regarded as an effective way of *generating* candidate Lyapunov functions H_s from the Hamiltonian H.

3.2 Passivity-based control of port-controlled Hamiltonian systems with dissipation

In the previous section we have seen how under certain conditions the feedback interconnection of a PCHD system having Hamiltonian H (the "plant") with another PCHD system with Hamiltonian H_C (the "controller") leads to a reduced dynamics given by (31) for the shaped Hamiltonian H_s. From a *state feedback* point of view the dynamics (31) could have been directly obtained by a state feedback $u = \alpha(x)$ such that

$$g(x)\alpha(x) = [J(x) - R(x)] \frac{\partial H_C(G(x)+c)}{\partial x} \tag{34}$$

Indeed, such an $\alpha(x)$ is given in explicit form as

$$\alpha(x) = -g_C^T(G(x)+c)\frac{\partial H_C}{\partial \xi}(G(x)+c) \tag{35}$$

The state feedback $u = \alpha(x)$ is customarily called a *passivity-based control law*, since it is based on the passivity properties of the original plant system (18) and transforms (18) into *another* passive system with *shaped* storage function (in this case H_s).
Seen from this perspective we have shown in the previous section that the passivity-based state feedback $u = \alpha(x)$ satisfying (34) *can be derived* from the interconnection of the PCHD plant system (18) with a PCHD controller system (22). This fact has some favorable consequences. Indeed, it implies that the passivity-based control law defined by (34) can be equivalently *generated* as the feedback interconnection of the passive system (18) with another passive system (22). In particular, this implies an inherent *invariance* property of the controlled system: the plant system (18), the controller system (32), as well as any other passive system interconnected to (18) in a power-conserving fashion, may change in any way as long as they remain passive, and for any perturbation of this kind the controlled system will remain stable. For a further discussion of passivity-based control from this point of view we refer to [27].

3.3 Interconnection and damping assignment passivity-based control

A further generalization of the previous subsection is to use state feedback in order to *change* the interconnection structure and the resistive structure of the plant system, and thereby to create more flexibility to shape the storage function for the (modified) port-controlled Hamiltonian system to a desired form. This methodology has been called Interconnection-Damping Assignment Passivity-Based Control (IDA-PBC) in [27], and has been succesfully applied to a number of applications. The method is especially attractive if the newly assigned interconnection and resistive structures are judiciously chosen on the basis of physical considerations, and represent some "ideal" interconnection and resistive structures for the physical plant. For an extensive treatment of IDA-PBC we refer to [27].

4 Physical systems with algebraic constraints

From a general modeling point of view physical systems are, at least in first instance, often described by DAE's, that is, a mixed set of differential and *algebraic* equations. This stems from the fact that in many modelling approaches the system under consideration is naturally regarded as obtained from interconnecting simpler sub-systems. These interconnections in general, give rise to algebraic constraints between the state space variables of the sub-systems; thus leading to implicit systems. While in the linear case one may argue that it is often relatively straightforward to eliminate the algebraic constraints, and thus to reduce the system to an *explicit* form without constraints, in the nonlinear case such a conversion from implicit to explicit form is usually fraught with difficulties. Indeed, if the algebraic constraints are nonlinear then they need not be analytically solvable (locally or globally). More importantly perhaps, even if they are analytically solvable, then often one would prefer *not* to eliminate the algebraic constraints, because of the complicated and physically not easily interpretable expressions for the reduced system which may arise.

4.1 Power-conserving interconnections

In order to geometrically describe network models of physical systems we first consider the notion of a *Dirac structure*, formalizing the concept of a power-conserving interconnection. Let $\mathcal{F}$ be an ℓ-dimensional linear space, and denote its dual (the space of linear functions on $\mathcal{F}$) by $\mathcal{F}^*$. The product space $\mathcal{F} \times \mathcal{F}^*$ is considered to be the space of power variables, with *power* intrinsically defined by

$$P = < f^* | f >, \quad (f, f^*) \in \mathcal{F} \times \mathcal{F}^*, \tag{36}$$

where $< f^* | f >$ denotes the duality product, that is, the linear function $f^* \in \mathcal{F}^*$ acting on $f \in \mathcal{F}$. Often we call $\mathcal{F}$ the space of *flows* f, and $\mathcal{F}^*$ the space of *efforts* e, with the power of an element $(f, e) \in \mathcal{F} \times \mathcal{F}^*$ denoted as $< e | f >$.

Remark 1 *If $\mathcal{F}$ is endowed with an inner product structure $<,>$, then $\mathcal{F}^*$ can be naturally identified with $\mathcal{F}$ in such a way that $< e | f > = < e, f >$, $f \in \mathcal{F}$, $e \in \mathcal{F}^* \simeq \mathcal{F}$.*

Example 5 *Let $\mathcal{F}$ be the space of generalized velocities, and $\mathcal{F}^*$ be the space of generalized forces, then $< e|f >$ is mechanical power. Similarly, let $\mathcal{F}$ be the space of currents, and $\mathcal{F}^*$ be the space of voltages, then $< e|f >$ is electrical power.*

There exists on $\mathcal{F} \times \mathcal{F}^*$ a canonically defined symmetric bilinear form

$$< (f_1, e_1), (f_2, e_2) >_{\mathcal{F}\times\mathcal{F}^*} := < e_1|f_2 > + < e_2|f_1 > \tag{37}$$

for $f_i \in \mathcal{F}$, $e_i \in \mathcal{F}^*, i = 1, 2$. Now consider a linear subspace $S \subset \mathcal{F} \times \mathcal{F}^*$, and its orthogonal complement with respect to the bilinear form $<,>_{\mathcal{F}\times\mathcal{F}^*}$ on $\mathcal{F} \times \mathcal{F}^*$, denoted as $S^\perp \subset \mathcal{F} \times \mathcal{F}^*$. Clearly, if S has dimension d, then the subspace $S^\perp$ has dimension $2\ell - d$. (Since dim $(\mathcal{F} \times \mathcal{F}^*) = 2\ell$, and $<,>_{\mathcal{F}\times\mathcal{F}^*}$ is a non-degenerate form.)

Definition 1 *[5,8,7] A constant Dirac structure on $\mathcal{F}$ is a linear subspace $\mathcal{D} \subset \mathcal{F} \times \mathcal{F}^*$ such that*

$$\mathcal{D} = \mathcal{D}^\perp \tag{38}$$

It immediately follows that the dimension of any Dirac structure $\mathcal{D}$ on an ℓ-dimensional linear space is equal to ℓ. Furthermore, let $(f, e) \in \mathcal{D} = \mathcal{D}^\perp$. Then by (37)

$$0 = < (f, e), (f, e) >_{\mathcal{F}\times\mathcal{F}^*} = 2 < e|f > . \tag{39}$$

Thus for all $(f, e) \in \mathcal{D}$ we obtain $< e|f > = 0$; and hence any Dirac structure $\mathcal{D}$ on $\mathcal{F}$ defines a power-conserving relation between the power variables $(f, e) \in \mathcal{F} \times \mathcal{F}^*$.

Remark 2 *The property dim $\mathcal{D} =$ dim $\mathcal{F}$ is intimately related to the usually expressed statement that a physical interconnection can not determine at the same time both the flow and effort (e.g. current and voltage, or velocity and force).*

Constant Dirac structures admit different *matrix representations*. Here we just list three of them, without giving proofs and algorithms to convert one representation into another, see e.g. [7].
Let $\mathcal{D} \subset \mathcal{F} \times \mathcal{F}^*$, with dim $\mathcal{F} = \ell$, be a constant Dirac structure. Then $\mathcal{D}$ can be represented as

1. (*Kernel and Image representation*, [7,35]).

$$\mathcal{D} = \{(f, e) \in \mathcal{F} \times \mathcal{F}^* | Ff + Ee = 0\} \tag{40}$$

for $\ell \times \ell$ matrices F and E satisfying

$$\begin{aligned} &(i) \;\; EF^T + FE^T = 0 \\ &(ii) \;\; \text{rank}\, [F \vdots E] = \ell \end{aligned} \tag{41}$$

Equivalently,

$$\mathcal{D} = \{(f,e) \in \mathcal{F} \times \mathcal{F}^* | f = E^T\lambda,\ e = F^T\lambda,\ \lambda \in \mathbb{R}^\ell\} \tag{42}$$

2. (*Constrained input-output representation*, [7]).

$$\mathcal{D} = \{(f,e) \in \mathcal{F} \times \mathcal{F}^* | f = -Je + G\lambda,\ G^T e = 0\} \tag{43}$$

for an $\ell \times \ell$ skew-symmetric matrix J, and a matrix G such that $\mathrm{Im}G = \{f|(f,0) \in \mathcal{D}\}$. Furthermore, $\mathrm{Ker}J = \{e|(0,e) \in \mathcal{D}\}$.

3. (*Canonical coordinate representation*, [5]).
There exist linear coordinates (q,p,r,s) for $\mathcal{F}$ such that in these coordinates and dual coordinates for $\mathcal{F}^*$, $(f,e) = (f_q, f_p, f_r, f_s, e_q, e_p, e_r, e_s) \in \mathcal{D}$ if and only if

$$\begin{cases} f_q = e_p,\ f_p = -e_q \\ f_r = 0,\quad e_s = \ 0 \end{cases} \tag{44}$$

Example 6 *Kirchhoff's laws are a special case of (40). By taking $\mathcal{F}$ the space of currents and $\mathcal{F}^*$ the space of voltages, Kirchhoff's current laws determine a subspace $\mathcal{V}$ of $\mathcal{F}$, while Kirchhoff's voltage laws determine the orthogonal subspace $\mathcal{V}^{orth}$ of $\mathcal{F}^*$. Hence, the Dirac structure determined by Kirchhoff's laws is given as $\mathcal{V} \times \mathcal{V}^{orth} \subset \mathcal{F} \times \mathcal{F}^*$, with kernel representation of the form*

$$\mathcal{D} = \{(f,e) \in \mathcal{F} \times \mathcal{F}^* | Ff = 0, Ee = 0\}, \tag{45}$$

for suitable matrices F and E (consisting only of elements $+1$, -1 and 0), such that Ker $F = \mathcal{V}$ and Ker $E = \mathcal{V}^{orth}$. In this case the defining property $\mathcal{D} = \mathcal{D}^\perp$ of the Dirac structure amounts to Tellegen's theorem.

Example 7 *Any skew-symmetric map $J : \mathcal{F}^* \to \mathcal{F}$ defines the Dirac structure*

$$\mathcal{D} = \{(f,e) \in \mathcal{F} \times \mathcal{F}^* | f = -Je\}, \tag{46}$$

as a special case of (43). Furthermore, any interconnection structure (J,g) with J skew-symmetric defines a Dirac structure given in hybrid input-output representation as

$$\begin{bmatrix} f_S \\ e_P \end{bmatrix} = \begin{bmatrix} -J & -g \\ g^T & 0 \end{bmatrix} \begin{bmatrix} e_S \\ f_P \end{bmatrix} \tag{47}$$

Given a Dirac structure $\mathcal{D}$ on $\mathcal{F}$, the following subspaces of $\mathcal{F}$, respectively $\mathcal{F}^*$, will shown to be of importance in the next section

$$\begin{aligned} G_1 &:= \{f \in \mathcal{F} \mid \exists e \in \mathcal{F}^* \text{ s.t. } (f,e) \in \mathcal{D}\} \\ P_1 &:= \{e \in \mathcal{F}^* \mid \exists f \in \mathcal{F} \text{ s.t. } (f,e) \in \mathcal{D}\} \end{aligned} \tag{48}$$

The subspace G_1 expresses the set of admissible flows, and P_1 the set of admissible efforts. In the image representation (42) they are given as

$$G_1 = \ \mathrm{Im}\ E^T, \qquad P_1 = \ \mathrm{Im}\ F^T. \tag{49}$$

4.2 Implicit port-controlled Hamiltonian systems

From a network modeling perspective, see e.g. [28,3], a (lumped-parameter) physical system is directly described by a set of (possibly multi-dimensional) *energy-storing* elements, a set of *energy-dissipating* or *resistive* elements, and a set of *ports* (by which interaction with the environment can take place), interconnected to each other by a *power-conserving interconnection*, see Figure 3. Associated with the energy-storing elements are energy-variables

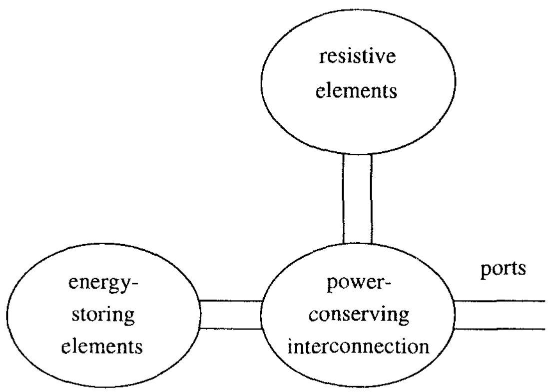

Fig. 3. Network model of physical systems

$x_1, \cdots, x_n$, being coordinates for some n-dimensional state space manifold $\mathcal{X}$, and a total energy $H : \mathcal{X} \to \mathbb{R}$. The power-conserving interconnection also includes power-conserving elements like (in the electrical domain) transformers, gyrators, or (in the mechanical domain) transformers, kinematic pairs and kinematic constraints. In first instance (see later on for the non-constant case) the power-conserving interconnection will be formalized by a constant Dirac structure on a finite-dimensional linear space $\mathcal{F} := \mathcal{F}_S \times \mathcal{F}_R \times \mathcal{F}_P$, with $\mathcal{F}_S$ denoting the space of flows f_S connected to the energy-storing elements, $\mathcal{F}_R$ denoting the space of flows f_R connected to the dissipative (resistive) elements, and $\mathcal{F}_P$ the space of external flows f_P which can be connected to the environment. Dually, we write $\mathcal{F}^* = \mathcal{F}_S^* \times \mathcal{F}_R^* \times \mathcal{F}_P^*$, with $e_S \in \mathcal{F}_S^*$ the efforts connected to the energy-storing elements, $e_R \in \mathcal{F}_R^*$ the efforts connected to the resistive elements, and $e_P \in \mathcal{F}_P^*$ the efforts to be connected to the environment of the system.

In kernel representation, the Dirac structure on $\mathcal{F} = \mathcal{F}_S \times \mathcal{F}_R \times \mathcal{F}_P$ is given as

$$\begin{aligned} \mathcal{D} = \{ & (f_S, f_R, f_P, e_S, e_R, e_P) \mid \\ & F_S f_S + E_S e_S + F_R f_R + E_R e_R + F_P f_P + E_P e_P = 0 \} \end{aligned} \tag{50}$$

for certain matrices $F_S, E_S, F_R, E_R, F_P, E_P$ satisfying

$$\begin{aligned} &(i) \;\; E_S F_S^T + F_S E_S^T + E_R F_R^T + F_R E_R^T + E_P F_P^T + F_P E_P^T = 0 \\ &(ii) \;\; \text{rank} \left[F_S \vdots F_R \vdots F_P \vdots E_S \vdots E_R \vdots E_P \right] = \dim \mathcal{F} \end{aligned} \tag{51}$$

The flow variables of the energy-storing elements are given as $\dot{x}(t) = \frac{dx}{dt}(t), t \in \mathbb{R}$, and the effort variables of the energy-storing elements as $\frac{\partial H}{\partial x}(x(t))$ (implying that $< \frac{\partial H}{\partial x}(x(t)) | \dot{x}(t) >= \frac{dH}{dt}(x(t))$ is the increase in energy). In order to have a consistent sign convention for energy flow we put

$$\begin{aligned} f_S &= -\dot{x} \\ e_S &= \tfrac{\partial H}{\partial x}(x) \end{aligned} \tag{52}$$

Restricting to *linear* resistive elements, the flow and effort variables connected to the resistive elements are related as

$$f_R = -S e_R \tag{53}$$

for some matrix $S = S^T \geq 0$. Substitution of (52) and (53) into (50) yields

$$-F_S \dot{x}(t) + E_S \frac{\partial H}{\partial x}(x(t)) - F_R S e_R + E_R e_R + F_P f_P + E_P e_P = 0 \tag{54}$$

with $F_S, E_S, F_R, E_R, F_P, E_P$ satisfying (51). We call (54) an *implicit port-controlled Hamiltonian system with dissipation*, defined with respect to the constant Dirac structure $\mathcal{D}$, the Hamiltonian H, and the resistive structure S.

Actually, for many purposes this definition of an implicit PCHD system is not general enough, since often the Dirac structure is not constant, but *modulated* by the state variables x. In this case the matrices $F_S, E_S, F_R, E_R, F_P, E_P$ depend (smoothly) on x, leading to the implicit PCHD system

$$\begin{aligned} &-F_S(x(t))\dot{x}(t) + E_S(x(t))\tfrac{\partial H}{\partial x}(x(t)) - F_R(x(t)) S e_R(t) \\ &+ E_R(x(t)) e_R(t) + F_P(x(t)) f_P(t) + E_P(x(t)) e_P(t) = 0, \;\; t \in \mathbb{R} \end{aligned} \tag{55}$$

with

$$\begin{aligned} &E_S(x)F_S^T(x) + F_S(x)E_S^T(x) + E_R(x)F_R^T(x) + F_R(x)E_R^T(x) \\ &+ E_P(x)F_P^T(x) + F_P(x)E_P^T(x) = 0, \quad \forall x \in \mathcal{X} \\ &\text{rank} \left[F_S(x) \vdots F_R(x) \vdots F_P(x) \vdots E_S(x) \vdots E_R(x) \vdots E_P(x) \right] = \dim \mathcal{F} \end{aligned} \tag{56}$$

Remark 3 *Strictly speaking the flow and effort variables $\dot{x}(t) = -f_S(t)$, respectively $\frac{\partial H}{\partial x}(x(t)) = e_S(t)$, are not living in the constant linear space $\mathcal{F}_S$, respectively $\mathcal{F}_S^*$, but instead in the tangent spaces $T_{x(t)}\mathcal{X}$, respectively co-tangent spaces $T^*_{x(t)}\mathcal{X}$, to the state space manifold $\mathcal{X}$. This is formalized in the definition of a non-constant Dirac structure on a manifold; see [5,8,7,32].*

By the power-conservation property of a Dirac structure (cf. (39)) it follows directly that any implicit PCHD system satisfies the energy-inequality

$$\begin{aligned} \tfrac{dH}{dt}(x(t)) &= < \tfrac{\partial H}{\partial x}(x(t))|\dot{x}(t) >= \\ &= -e_R^T(t)Se_R(t) + e_P^T(t)f_P(t) \leq e_P^T(t)f_P(t), \end{aligned} \tag{57}$$

showing passivity if $H \geq 0$. The *algebraic constraints* that are present in the implicit system (55) are expressed by the subspace P_1, and the Hamiltonian H. In fact, since the Dirac structure $\mathcal{D}$ is modulated by the x-variables, also the subspace P_1 is modulated by the x-variables, and thus the effort variables e_S, e_R and e_P necessarily satisfy $(e_S, e_R, e_P) \in P_1(x),\ x \in \mathcal{X}$, and thus, because of (49),

$$e_S \in \text{ Im } F_S^T(x), e_R \in \text{ Im } F_R^T(x), e_P \in \text{ Im } F_P^T(x). \tag{58}$$

The second and third inclusions entail the expression of e_R and e_P in terms of the other variables, while the first inclusion determines, since $e_S = \frac{\partial H}{\partial x}(x)$, the following algebraic constraints on the state variables

$$\frac{\partial H}{\partial x}(x) \in \text{ Im } F_S^T(x). \tag{59}$$

The *Casimir functions* $C : \mathcal{X} \to \mathbb{R}$ of the implicit system (55) are determined by the subspace $G_1(x)$. Indeed, necessarily $(f_S, f_R, f_P) \in G_1(x)$, and thus by (49)

$$f_S \in \text{ Im } E_S^T(x), f_R \in \text{ Im } E_R^T(x), f_P \in \text{ Im } E_P^T(x). \tag{60}$$

Since $f_S = -\dot{x}(t)$, the first inclusion yields the *flow constraints* $\dot{x}(t) \in \text{ Im } E_S^T(x(t)),\quad t \in \mathbb{R}$. Thus $C : \mathcal{X} \to \mathbb{R}$ is a Casimir function if $\frac{dC}{dt}(x(t)) = \frac{\partial^T C}{\partial x}(x(t))\dot{x}(t) = 0$ for all $\dot{x}(t) \in \text{ Im } E_S^T(x(t))$. Hence $C : \mathcal{X} \to \mathbb{R}$ is a Casimir of the implicit PCHD system (54) if it satisfies the set of p.d.e.'s

$$\frac{\partial C}{\partial x}(x) \in \text{ Ker } E_S(x) \tag{61}$$

Remark 4 *Note that $C : \mathcal{X} \to \mathbb{R}$ satisfying (61) is a Casimir function of (54) in a strong sense: it is a dynamical invariant $\left(\frac{dC}{dt}(x(t)) = 0\right)$ for every port behavior and every resistive relation (53).*

Example 8 *[7,36,35] Consider a mechanical system with k degrees of freedom, locally described by k configuration variables* $q = (q_1, \ldots, q_k)$*. Suppose that there are constraints on the generalized velocities* $\dot{q}$*, described as* $A^T(q)\dot{q} = 0$*, with* $A(q)$ *a* $r \times k$ *matrix of rank* r *everywhere (that is, there are* r *independent kinematic constraints). This leads to the following constrained Hamiltonian equations*

$$\begin{aligned} \dot{q} &= \frac{\partial H}{\partial p}(q,p) \\ \dot{p} &= -\frac{\partial H}{\partial q}(q,p) + A(q)\lambda + B(q)u \\ y &= B^T(q)\frac{\partial H}{\partial p}(q,p) \\ 0 &= A^T(q)\frac{\partial H}{\partial p}(q,p) \end{aligned} \tag{62}$$

where $B(q)u$ *are the external forces (controls) applied to the system, for some* $k \times m$ *matrix* $B(q)$*, while* $A(q)\lambda$ *are the constraint forces. The Lagrange multipliers* $\lambda(t)$ *are uniquely determined by the requirement that the constraints* $A^T(q(t))\dot{q}(t) = 0$ *have to be satisfied for all* t*. One way of proceeding with these equations is to eliminate the constraint forces, and to reduce the equations of motion to the constrained state space* $\mathcal{X}_c = \{(q,p) \mid A^T(q)\frac{\partial H}{\partial p}(q,p) = 0\}$*, thereby obtaining an (explicit) port-controlled Hamiltonian system; see [34]. An alternative, and more direct, approach is to view the constrained Hamiltonian equations (62) as an implicit port-controlled Hamiltonian system with respect to the Dirac structure* $\mathcal{D}$*, given in constrained input-output representation (43) by*

$$\begin{aligned} \mathcal{D} = \{(f_S, f_P, e_S, e_P) | 0 = A^T(q)e_S,\ e_P = B^T(q)e_S, \\ -f_S = \begin{bmatrix} 0 & I_k \\ -I_k & 0 \end{bmatrix} e_S + \begin{bmatrix} 0 \\ A(q) \end{bmatrix} \lambda + \begin{bmatrix} 0 \\ B(q) \end{bmatrix} f_P,\ \lambda \in \mathbb{R}^r \} \end{aligned} \tag{63}$$

In this case, the algebraic constraints on the state variables (q,p) *are given as* $A^T(q)\frac{\partial H}{\partial p}(q,p) = 0$*, while the Casimir functions* C *are determined by the equations*

$$\frac{\partial^T C}{\partial q}(q)\dot{q} = 0, \qquad \text{for all } \dot{q} \text{ satisfying } A^T(q)\dot{q} = 0. \tag{64}$$

Hence, finding Casimir functions amounts to integrating the kinematic constraints $A^T(q)\dot{q} = 0$*.*

Remark 5 *For a proper notion of integrability of non-constant Dirac structures, generalizing the Jacobi identity for the structure matrix* $J(x)$*, we refer e.g. to [7]. For example, the Dirac structure (63) is integrable if and only if the kinematic constraints are* holonomic.

In principle, the theory presented in Section 3 for stabilization of *explicit* port-controlled Hamiltonian systems can be directly extended, mutatis mutandis, to *implicit* port-controlled Hamiltonian system. In particular, the standard feedback interconnection of an implicit port-controlled Hamiltonian system P with port variables f_P, e_P (the "plant") with another implicit port-controlled Hamiltonian system with port variables f_P^C, e_P^C (the "controller"), via the interconnection relations

$$\begin{aligned} f_P &= -e_P^C + f^{\text{ext}} \\ f_P^C &= e_P + e^{\text{ext}} \end{aligned} \tag{65}$$

is readily seen to result in a closed-loop implicit port-controlled Hamiltonian system with port variables $f^{\text{ext}}, e^{\text{ext}}$. Furthermore, as in the explicit case, the Hamiltonian of this closed-loop system is just the *sum* of the Hamiltonian of the plant PCHD system and the Hamiltonian of the controller PCHD system. Finally, the Casimir analysis for the closed-loop system can be performed along the same lines as before.

5 Conclusions and future research

We have shown how network modelling of (lumped-parameter) physical systems, e.g. using bond graphs, leads to a mathematically well-defined class of open dynamical systems, which are called port-controlled Hamiltonian systems (with dissipation). Furthermore, we have tried to emphasize that this definition is completely *modular*, in the sense that any power-conserving interconnection of these systems defines a system in the same class, with overall interconnection structure defined by the individual interconnection structures, together with the power-conserving interconnection.

Clearly, the theory presented in this paper opens up the way for many other control and design problems than the stabilization problem as briefly discussed in the present paper. Its potential for set-point regulation has already received some attention (see [19,20,27,32]), while the extension to *tracking problems* is wide open. In this context we also like to refer to some recent work concerned with the shaping of the *Lagrangian*, see e.g. [2]. Also, the control of mechanical systems with nonholonomic kinematic constraints can be fruitfully approached from this point of view, see e.g. [10], as well as the modelling and control of multi-body systems, see [18,23,40]. The framework of PCHD systems seems perfectly suited to theoretical investigations on the topic of *impedance control*; see already [38] for some initial results in this direction. Also the connection with multi-modal (*hybrid*) systems, corresponding to PCHD systems with varying interconnection structure [9], needs further investigations. Finally, our current research is concerned with the formulation of *distributed parameter systems* as port-controlled Hamiltonian systems, see [17], and applications in tele-manipulation [41] and smart structures [37].

References

1. A.M. Bloch & P.E. Crouch, "Representations of Dirac structures on vector spaces and nonlinear *LC* circuits", Proc. Symposia in Pure Mathematics, Differential Geometry and Control Theory, G. Ferreyra, R. Gardner, H. Hermes, H. Sussmann, eds., Vol. 64, pp. 103-117, AMS, 1999.
2. A. Bloch, N. Leonard & J.E. Marsden, "Matching and stabilization by the method of controlled Lagrangians", in Proc. 37th IEEE Conf. on Decision and Control, Tampa, FL, pp. 1446-1451, 1998.
3. P.C. Breedveld, *Physical systems theory in terms of bond graphs*, PhD thesis, University of Twente, Faculty of Electrical Engineering, 1984
4. R.W. Brockett, "Control theory and analytical mechanics", in *Geometric Control Theory*, (eds. C. Martin, R. Hermann), Vol. VII of Lie Groups: History, Frontiers and Applications, Math. Sci. Press, Brookline, pp. 1-46, 1977.
5. T.J. Courant, "Dirac manifolds", *Trans. American Math. Soc.*, 319, pp. 631-661, 1990.
6. P.E. Crouch & A.J. van der Schaft, *Variational and Hamiltonian Control Systems*, Lect. Notes in Control and Inf. Sciences 101, Springer-Verlag, Berlin, 1987.
7. M. Dalsmo & A.J. van der Schaft, "On representations and integrability of mathematical structures in energy-conserving physical systems", *SIAM J. Control and Optimization*, 37, pp. 54-91, 1999.
8. I. Dorfman, *Dirac Structures and Integrability of Nonlinear Evolution Equations*, John Wiley, Chichester, 1993.
9. G. Escobar, A.J. van der Schaft & R. Ortega, "A Hamiltonian viewpoint in the modelling of switching power converters", *Automatica*, Special Issue on Hybrid Systems, 35, pp. 445-452, 1999.
10. K. Fujimoto, T. Sugie, "Stabilization of a class of Hamiltonian systems with nonholonomic constraints via canonical transformations", Proc. European Control Conference '99, Karlsruhe, 31 August - 3 September 1999.
11. D.J. Hill & P.J. Moylan, "Stability of nonlinear dissipative systems," *IEEE Trans. Aut. Contr.*, AC-21, pp. 708-711, 1976.
12. A. Isidori, *Nonlinear Control Systems* (2nd Edition), Communications and Control Engineering Series, Springer-Verlag, London, 1989, 3rd Edition, 1995.
13. R. Lozano, B. Brogliato, O. Egeland and B. Maschke, *Dissipative systems*, Communication and Control Engineering series, Springer, London, March 2000.
14. J.E. Marsden & T.S. Ratiu, *Introduction to Mechanics and Symmetry*, Texts in Applied Mathematics 17, Springer-Verlag, New York, 1994.
15. B.M. Maschke, *Interconnection and structure of controlled Hamiltonian systems: a network approach*, (in French), Habilitation Thesis, No.345, Dec. 10, 1998, University of Paris-Sud , Orsay, France.
16. B.M. Maschke, A.J. van der Schaft, "An intrinsic Hamiltonian formulation of network dynamics: non-standard Poisson structures and gyrators", J. Franklin Institute, vol. 329, no.5, pp. 923-966, 1992.
17. B.M. Maschke, A.J. van der Schaft, "Port controlled Hamiltonian representation of distributed parameter systems", Proc. IFAC Workshop on Lagrangian and Hamiltonian methods for nonlinear control, Princeton University, March 16-18, pp. 28-38, 2000.

18. B.M. Maschke, C. Bidard & A.J. van der Schaft, "Screw-vector bond graphs for the kinestatic and dynamic modeling of multibody systems", in Proc. ASME Int. Mech. Engg. Congress, 55-2, Chicago, U.S.A., pp. 637-644, 1994.
19. B.M. Maschke, R. Ortega & A.J. van der Schaft, "Energy-based Lyapunov functions for forced Hamiltonian systems with dissipation", in Proc. 37th IEEE Conference on Decision and Control, Tampa, FL, pp. 3599-3604, 1998.
20. B.M. Maschke, R. Ortega, A.J. van der Schaft & G. Escobar, "An energy-based derivation of Lyapunov functions for forced systems with application to stabilizing control", in Proc. 14th IFAC World Congress, Beijing, Vol. E, pp. 409-414, 1999.
21. B.M. Maschke & A.J. van der Schaft, "Port-controlled Hamiltonian systems: Modelling origins and system-theoretic properties", in Proc. 2nd IFAC NOLCOS, Bordeaux, pp. 282-288, 1992.
22. B.M. Maschke, A.J. van der Schaft & P.C. Breedveld, "An intrinsic Hamiltonian formulation of the dynamics of LC-circuits, *IEEE Trans. Circ. and Syst.*, CAS-42, pp. 73-82, 1995.
23. B.M. Maschke & A.J. van der Schaft, "Interconnected Mechanical Systems, Part II: The Dynamics of Spatial Mechanical Networks", in *Modelling and Control of Mechanical Systems*, (eds. A. Astolfi, D.J.N. Limebeer, C. Melchiorri, A. Tornambe, R.B. Vinter), pp. 17-30, Imperial College Press, London, 1997.
24. J.I. Neimark & N.A. Fufaev, *Dynamics of Nonholonomic Systems*, Vol. 33 of Translations of Mathematical Monographs, American Mathematical Society, Providence, Rhode Island, 1972.
25. H. Nijmeijer & A.J. van der Schaft, *Nonlinear Dynamical Control Systems*, Springer-Verlag, New York, 1990.
26. R. Ortega, A. Loria, P.J. Nicklasson & H. Sira-Ramirez, *Passivity-based Control of Euler-Lagrange Systems*, Springer-Verlag, London, 1998.
27. R. Ortega, A.J. van der Schaft, B.M. Maschke & G. Escobar, "Interconnection and damping assignment passivity-based control of port-controlled Hamiltonian systems", 1999, submitted for publication.
28. H. M. Paynter, *Analysis and design of engineering systems*, M.I.T. Press, MA, 1960.
29. A.J. van der Schaft, *System theoretic properties of physical systems*, CWI Tract 3, CWI, Amsterdam, 1984.
30. A.J. van der Schaft, "Stabilization of Hamiltonian systems", *Nonl. An. Th. Math. Appl.*, 10, pp. 1021-1035, 1986.
31. A.J. van der Schaft, "Interconnection and geometry", in *The Mathematics of Systems and Control, From Intelligent Control to Behavioral Systems* (eds. J.W. Polderman, H.L. Trentelman), Groningen, 1999.
32. A.J. van der Schaft, *L_2-Gain and Passivity Techniques in Nonlinear Control*, 2nd revised and enlarged edition, Springer-Verlag, Springer Communications and Control Engineering series, p. xvi+249, London, 2000 (first edition Lect. Notes in Control and Inf. Sciences, vol. 218, Springer-Verlag, Berlin, 1996).
33. A.J. van der Schaft, "Port-controlled Hamiltonian systems: Towards a theory for control and design of nonlinear physical systems", J. of the Society of Instrument and Control Engineers of Japan (SICE), vol. 39, no.2, pp. 91-98, 2000.
34. A.J. van der Schaft & B.M. Maschke, "On the Hamiltonian formulation of nonholonomic mechanical systems", *Rep. Math. Phys.*, 34, pp. 225-233, 1994.

35. A.J. van der Schaft & B.M. Maschke, "The Hamiltonian formulation of energy conserving physical systems with external ports", *Archiv für Elektronik und Übertragungstechnik*, 49, pp. 362-371, 1995.
36. A.J. van der Schaft & B.M. Maschke, "Interconnected Mechanical Systems, Part I: Geometry of Interconnection and implicit Hamiltonian Systems", in *Modelling and Control of Mechanical Systems*, (eds. A. Astolfi, D.J.N. Limebeer, C. Melchiorri, A. Tornambe, R.B. Vinter), pp. 1-15, Imperial College Press, London, 1997.
37. K. Schlacher, A. Kugi, "Control of mechanical structures by piezoelectric actuators and sensors". In *Stability and Stabilization of Nonlinear Systems*, eds. D. Aeyels, F. Lamnabhi-Lagarrigue, A.J. van der Schaft, Lecture Notes in Control and Information Sciences, vol. 246, pp. 275-292, Springer-Verlag, London, 1999.
38. S. Stramigioli, *From Differentiable Manifolds to Interactive Robot Control*, PhD Dissertation, University of Delft, Dec. 1998.
39. S. Stramigioli, B.M. Maschke & A.J. van der Schaft, "Passive output feedback and port interconnection", in Proc. 4th IFAC NOLCOS, Enschede, pp. 613-618, 1998.
40. S. Stramigioli, B.M. Maschke, C. Bidard, "A Hamiltonian formulation of the dynamics of spatial mechanism using Lie groups and screw theory", to appear in Proc. Symposium Commemorating the Legacy, Work and Life of Sir R.S. Ball, J. Duffy and H. Lipkin organizers, July 9-11, 2000, University of Cambridge, Trinity College, Cambridge, U.K..
41. S. Stramigioli, A.J. van der Schaft, B. Maschke, S. Andreotti, C. Melchiorri, "Geometric scattering in tele-manipulation of port controlled Hamiltonian systems", 39th IEEE Conf. Decision & Control, Sydney, 2000.
42. M. Takegaki & S. Arimoto, "A new feedback method for dynamic control of manipulators", *Trans. ASME, J. Dyn. Systems, Meas. Control*, 103, pp. 119-125, 1981.
43. J.C. Willems, "Dissipative dynamical systems - Part I: General Theory", *Archive for Rational Mechanics and Analysis*, 45, pp. 321-351, 1972.