JOURNAL OF APPROXIMATION THEORY **36**, 317–320 (1982)

On the Conjecture of Meinardus on Rational Approximaton of e^x

DIETRICH BRAESS

Institut für Mathematik, Ruhr-Universität Bochum, 463 Bochum, West Germany

Communicated by Carl de Boor

Received June 16, 1980

This paper is concerned with uniform approximation of e^x on the interval $[-1, +1]$ by (m, n)-degree rationals, i.e., by rational functions whose numerator and denominator have degree m and n, respectively. Several years ago, Meinardus [1, p. 168] conjectured that the norm of the error function for the best approximation is asymptotically

$$\frac{m!\, n!}{2^{m+n}(m+n)!\,(m+n+1)!} \qquad \text{as} \quad m+n \to \infty. \tag{1}$$

Recently, Newman [3] has proved that the degree of approximation is indeed better than 8 times the conjectured value. Here we will establish a lower bound by applying de la Vallée-Poussin's theorem to the rational function constructed in [3]. We will show that the error function oscillates $n + m + 1$ times by evaluating a winding number.

Let

$$p(z) = \int_0^\infty t^n (t+z)^m e^{-t}\, dt, \qquad q(z) = \int_0^\infty (t-z)^n t^m e^{-t}\, dt.$$

Then p/q is the (m, n)-degree Padé approximant to e^z. Following the evaluation in [3, p. 234] we get

$$\begin{aligned} q(z)\, e^z - p(z) &= \int_0^\infty (t-z)^n t^m e^{-t+z}\, dt - \int_0^\infty t^n (t+z)^m e^{-t}\, dt \\ &= \int_0^z (t-z)^n t^m e^{z-t}\, dt \\ &= z^{m+n+1} \int_0^1 (u-1)^n u^m e^{(1-u)z}\, du. \end{aligned}$$

0021-9045/82/120317–04$02.00/0

Hence, for $|z| \leqslant \frac{1}{2}$,

$$\begin{aligned}
|q(z)\, e^z - p(z)| &\geqslant |z|^{m+n+1} \operatorname{Re} \int_0^1 (1-u)^n\, u^m e^{(1-u)z}\, du \\
&= |z|^{m+n+1} \int_0^1 (1-u)^n\, u^m e^{(1-u)\operatorname{Re} z} \cos[(1-u) \operatorname{Im} z]\, du \\
&\geqslant |z|^{m+n+1} \int_0^1 (1-u)^n\, u^m\, du\, e^{-1/2} \cos \tfrac{1}{2} \\
&\geqslant \tfrac{7}{8} e^{-1/2} |z|^{m+n+1} \frac{m!\, n!}{(m+n+1)!}.
\end{aligned} \tag{2}$$

Observe that this is just $7/(8e)$ times the upper bound for $|qe^z - p|$ given in [3].

Next, an upper bound for $q(z)$. $|z| \leqslant \frac{1}{2}$, is derived:

$$\begin{aligned}
|q(z)| &\leqslant \int_0^\infty (t + \tfrac{1}{2})^n\, t^m e^{-t}\, dt \\
&\leqslant e^{1/2} \int_{-1/2}^\infty (t + \tfrac{1}{2})^{n+m}\, e^{-t-1/2}\, dt \\
&= e^{1/2}(m+n)!.
\end{aligned} \tag{3}$$

By combining (2) and (3) we get

$$|e^z - p(z)/q(z)| \geqslant \frac{7}{16e} \frac{2^{-m-n} m!\, n!}{(m+n)!\, (m+n+1)!}, \qquad |z| = \tfrac{1}{2}. \tag{4}$$

Given $x \in [-1, +1]$, put $z = (x+iy)/2$ with $x^2 + y^2 = 1$. Obviously, $e^x = e^{\bar{z}} e^z$. The crucial point is Newman's detection that $R(x) = p(\bar{z})\, p(z)/[q(\bar{z})\, q(z)]$ is an (m, n)-degree rational function in the variable x.

Put $a = e^z$, $b = p(z)/q(z)$. Then the error $e^x - R(x)$ is just $\bar{a}a - \bar{b}b$. It will be treated by using the formula

$$\bar{a}a - \bar{b}b = 2 \operatorname{Re} \bar{a}(a-b) - |a-b|^2, \qquad a, b \in \mathbb{C}. \tag{5}$$

From (4) we get the estimate for the first term

$$|e^{\bar{z}}[e^z - p/q]| \geqslant \frac{7}{16 e^{3/2}} \frac{2^{-m-n} m!\, n!}{(m+n)!\, (m+n+1)!}, \qquad |z| = \tfrac{1}{2}. \tag{6}$$

Denote by arg w the argument of the complex number w. Then

$$\begin{aligned}\arg\{e^{\bar{z}}[e^z - p(z)/q(z)]\} &= \arg\{e^{-z}[e^z - p(z)/q(z)]\} \\ &= \arg\left\{\frac{e^{-z}}{q(z)}[q(z)\,e^z - p(z)]\right\}. \qquad (7)\end{aligned}$$

For short, let $h(z)$ denote the function within the braces in (7).

Since p/q is the Padé approximation, $z = 0$ is a zero of $qe^z - p$ of multiplicity $n + m + 1$. Moreover, $q(z) \neq 0$ for $|z| \leqslant \frac{1}{2}$ is easily checked with the techniques in [3, p. 235]. Consequently, h has the winding number $n + m + 1$ for the circle $|z| = \frac{1}{2}$. Hence, when an entire circuit has been completed, $\arg(h(z))$ is increased by $(n + m + 1)\,2\pi$. The argument is increased by $(n + m + 1)\pi$ as z traverses the upper half of the circle, because $h(x)$ is real for x on the real line. It follows by the same arguments as in [1, pp. 38–39] that h attains real values on $n + m + 2$ points $z_k = (x_k + iy_k)/2$ with $+1 = x_1 > x_2 > \cdots > x_{n+m+2} = -1$ and that the sign changes between any pair of consecutive x's. The same is true for $e^{\bar{z}}[e^z - p/q]$. Referring to (5) we have

$$\begin{aligned}&\min_{1 \leqslant k \leqslant n+m+2} \left| e^{\bar{z}_k} e^{z_k} - \frac{p(\bar{z}_k)\,p(z_k)}{p(\bar{z}_k)\,q(z_k)} \right| \\ &\geqslant \min_{|z|=1/2} 2\,|e^{\bar{z}}[e^z - p(z)/q(z)]| - \max_{|z|=1/2} |e^z - p(z)/q(z)|^2 \\ &\geqslant \frac{7}{8e^{3/2}} \frac{2^{-m-n} m!\, n!}{(m+n)!\,(m+n+1)!} \left\{1 - \frac{\text{const}}{2^{m+n}(m+n+1)!}\right\} \qquad (8)\end{aligned}$$

From the theorem of de la Vallée-Poussin [1, p. 147] it is known that the expression in (8) is a lower bound for the distance of e^x from the (m, n)-degree rational functions. The gap between the upper bound in [2] and the lower bound is roughly a factor $e^{5/2}/[(2 - e^{1/2}) \cos \frac{1}{2}] < 40$.

If $m = n$, one gets better estimates for the constants from the result in [2].

ACKNOWLEDGMENTS

The author is indebted to Z. Ziegler and A. Pinkus for their hospitality during the Workshop on Approximation Theory at the Technion, Haifa.

REFERENCES

1. G. MEINARDUS, "Approximation of Functions: Theory and Numerical Methods." Springer Co., New York, 1967.
2. G. NÉMETH, Relative rational approximation of the function e^x. *Math. Notes* **21** (1977), 325–328.
3. D. J. NEWMAN, Rational approximation to e^x. *J. Approx. Theory* **27** (1979), 234–235.