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The domain derivative and two applications in inverse scattering theory

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Abstract. The first part of this paper recalls the notion of the domain derivative of a functional. The existence and a characterization of the domain derivative of the far-field pattern for a scattering problem with Dirichlet boundary condition is proved. The second part uses this characterization for an efficient implementation of a Gauss–Newton method for solving the inverse obstacle scattering problem. The third part deals with the sensitivity of the linearized obstacle problem.

1. Introduction

The inverse scattering problem for acoustic waves which we shall consider in this paper is the problem of recovering the shape of a scatterer from the far-field pattern of the scattered wave. Inverse problems of this type occur in various applications such as remote sensing, ultrasound tomography and seismic imaging. They are difficult to solve since they are ill-posed and nonlinear when formulated as an equation of the kind $F(\Gamma) = f$. Here, f denotes the (measured) far-field pattern and F the far-field operator which assigns to every suitable boundary Γ the corresponding far-field pattern of the scattered wave (see section 2 for the exact definition).

This paper consists of three parts. In section 2 we rigorously derive a formulation for the domain derivative of the far-field operator which is independent of the parametrization of the boundary. The notion of a domain derivative is explained in [24]. This section contains the main theoretical result of the paper.

Section 3 uses this result for the implementation of a Gauss–Newton method for (a discretized version of) the equation $F(\Gamma) = f$. This method requires, at every step, the solution of a direct scattering problem (for the evaluation of $F(\Gamma)$) and several boundary value problems on the same region for the same Helmholtz equation (for the evaluation of the derivative of F). Because the domain derivative is known explicitly we could reduce the computation time considerably. Compared with the methods proposed by Colton and Monk [3–5] or Kirsch and Kress [13, 14] which also use Gauss–Newton methods in their second steps we observed a more critical dependence on the regularization parameter. Also, the initial curve for starting the optimization procedure had to be chosen closer to the true one. On the other hand, we observed fast convergence as one would expect from a Newton-type method. Also, the numerical results are not very sensitive to noise in the data.

In section 4 we want, for the first time, study the 'worst case error' of the linearized inverse scattering problem. The basic question is: given a known error ϵ in the far-field pattern and some *a priori* information about the scatterer how large could the error in the boundary be? In a recent paper Isakov [10] proves a stability estimate for the full nonlinear inverse problem, giving (to the author's opinion optimal but) rather pessimistic estimates. In this paper we study the numerical aspects of this question for a special class of boundaries, which arises as parametrizations in various inverse scattering solvers. The numerical results are much better and reflect the observed quality of the recovered objects.

2. The domain derivative

Let $k > 0$ be the (given and fixed) wavenumber, $\hat{d} \in \mathbb{R}^N$ ($N = 2$ or 3) with $|\hat{d}| = 1$ the given direction of the incident plane wave, and

$X \subset \{\Gamma \in C^2 : \exists \Omega \subset \mathbb{R}^N \text{ open, bounded and connected such that } \Gamma = \partial\Omega\}$

some set of admissible boundaries. For $\Gamma \in X$ and $\Gamma = \partial\Omega$, set $\Omega^\epsilon := \mathbb{R}^N \setminus \bar{\Omega}$ consider the obstacle scattering problem: Find $u \in C^2(\Omega^\epsilon) \cap C(\bar{\Omega}^\epsilon)$ with

$$\Delta u + k^2 u = 0 \text{ in } \Omega^\epsilon \quad u = 0 \text{ on } \Gamma \quad (2.1)$$

and $u^s(x) := u(x) - \exp(ik\hat{d}^\top x)$ satisfies the radiation condition†

$$\begin{aligned} \frac{\partial u^s(x)}{\partial r} - ik u^s(x) &= o(r^{(1-N)/2}) \\ r = |x| \rightarrow \infty &\quad \text{uniformly in } \frac{x}{|x|}. \end{aligned}$$

It is well known that a unique solution of this problem exists (cf Colton and Kress [2]). Certainly, u depends on Γ . The radiation condition implies the following behaviour at infinity:

$$\begin{aligned} u^s(x) &= \frac{\exp(ikr)}{r^{(N-1)/2}} u_\infty(\hat{x}) + o(r^{(1-N)/2}) \\ r \rightarrow \infty &\quad \text{uniformly in } \hat{x} := \frac{x}{|x|}. \end{aligned} \quad (2.2)$$

u_∞ is uniquely determined by u^s and is called the far-field pattern (or scattering amplitude) of u^s . It is an analytic function on the unit sphere S^{N-1} and admits the following representation

$$\begin{aligned} u_\infty(\hat{x}) &= \int_{\partial K_R} \left(u^s(y) \frac{\partial}{\partial n(y)} e^{-ik\hat{x}^\top y} - e^{-ik\hat{x}^\top y} \frac{\partial u^s(y)}{\partial n} \right) ds(y) \\ \hat{x} &\in S^{N-1} \end{aligned} \quad (2.3)$$

† $\hat{d}^\top x$ means the inner product of the vectors $\hat{d}, x \in \mathbb{R}^N$.

where $R > 0$ such that $\bar{\Omega} \subset K_R := \{y \in \mathbb{R}^N : |y| < R\}$. Now we are able to introduce the far-field operator $F : X \rightarrow C(S^{N-1})$ which assigns to every $\Gamma \in X$ the far-field pattern u_∞ of the corresponding solution of the scattering problem in Ω^ε . Recall that we fix the wavenumber k and the incident wave u^i . Then F is well defined by the preceding remarks.

A derivative of F at Γ can be defined as follows (cf Pironeau [24]): For any real vector field $a \in C^2(\Gamma; \mathbb{R}^N)$ we denote by Γ_a the set

$$\Gamma_a := \{x + a(x) : x \in \Gamma\}$$

which is the C^2 boundary of a domain Ω_a provided $\|a\|_\infty := \max_{x \in \Gamma} |a(x)|$ is small enough. Now we define the domain derivative of F at Γ 'in the direction' a by

$$F'(\Gamma; a) := \lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon} [F(\Gamma_{\varepsilon a}) - F(\Gamma)] \tag{2.4}$$

where the limit should exist uniformly on S^{N-1} . Then $F'(\Gamma; \cdot) : C^2(\Gamma; \mathbb{R}^N) \rightarrow C(S^{N-1})$ is called *domain derivative* of F at Γ . The main theoretical result of this paper is the following

Theorem 2.1. Let be $\Gamma \in C^2$, $a \in C^2(\Gamma; \mathbb{R}^N)$ and $u^0 \in H_{loc}^2(\Omega^\varepsilon)$ be the solution of the scattering problem (2.1). Then the domain derivative $F'(\Gamma; a)$ exists and is given by the far-field pattern u'_∞ of u' , where $u' \in C^2(\Omega^\varepsilon) \cap C(\bar{\Omega}^\varepsilon)$ solves the exterior boundary value problem

$$\begin{aligned} \Delta u' + k^2 u' &= 0 \text{ in } \Omega^\varepsilon \\ u' &= -a^\top n \frac{\partial u^0}{\partial n} \text{ on } \Gamma \\ \frac{\partial u'(x)}{\partial r} - i k u'(x) &= o(r^{(1-N)/2}) \quad r \rightarrow \infty \text{ uniformly in } \frac{x}{|x|} \end{aligned} \tag{2.5}$$

where $n(x)$ denotes the outer unit normal vector at $x \in \Gamma$.

Remark. Since $\partial u^0 / \partial n \in C(\Gamma)$ (cf [2, theorem 3.21]) this is a classical exterior boundary value problem for the Helmholtz equation.

Proof. We prove the assertion only for the case $N = 3$. The plan of the proof is as follows: First we derive a variational equation in a region $\Omega_R := K_R \cap \Omega^\varepsilon$ where $R > 0$ is chosen such that $\bar{\Omega} \subset K_{R/2}$. This variational formulation uses a non-local boundary condition on the outer boundary ∂K_R and simplifies arguments of the paper of Kress and Zinn [18]. We do this for all the regions outside Γ_ε . By a change of variables we transform them into variational equations on a fixed reference region and study the limit of the difference quotients for $\varepsilon \rightarrow 0$.

By Green's theorem and the Helmholtz equation we conclude that the total field u^0 satisfies

$$\iint_{\Omega_R} (\nabla \bar{v}^\top \nabla u^0 - k^2 u^0 \bar{v}) \, dx = \int_{\partial K_R} \frac{\partial u^0}{\partial n} \bar{v} \, ds \tag{2.6}$$

for all $v \in \tilde{H}_0^1(\Omega_R) := \{v \in H^1(\Omega_R) : v|_\Gamma = 0\}$.

By $L : H^{1/2}(\partial K_R) \rightarrow H^{-1/2}(\partial K_R)$ we denote the Dirichlet-to-Neumann map

$$L : g \mapsto \partial w / \partial n$$

where w solves the exterior Dirichlet problem for the Helmholtz equation in the region $\{x \in \mathbb{R}^3 : |x| > R\}$ with $w|_{\partial K_R} = g$. Denoting by Y_n^m , $|m| \leq n$, $n \in \mathbb{N} \cup \{0\}$, the spherical harmonics of order n , and by $h_n^{(1)}$ the spherical Hankel functions of the first kind and order n (cf [21]) we can solve the exterior Dirichlet problem in $\{x \in \mathbb{R}^3 : |x| > R\}$ explicitly and, thus, derive the following expression for L :

$$(Lg)(R, \hat{x}) = \sum_{n=0}^{\infty} \sum_{|m| \leq n} g_n^m \frac{k h_n^{(1)'}(kR)}{h_n^{(1)}(kR)} Y_n^m(\hat{x}) \quad (2.7)$$

where $g(R, \hat{x}) = \sum_{n=0}^{\infty} \sum_{|m| \leq n} g_n^m Y_n^m(\hat{x})$.

Let L_0 be the corresponding operator for wavenumber $k = 0$ which is given by $(L_0g)(R, \hat{x}) = -\frac{1}{R} \sum_{n=0}^{\infty} \sum_{|m| \leq n} (n+1) g_n^m Y_n^m(\hat{x})$ for g as in (2.7). We see directly from the orthogonality of Y_n^m that $-L_0$ is strictly coercive, i.e.

$$-\langle L_0g, g \rangle \geq c \|g\|_{H^{1/2}(\partial K_R)}^2 \quad \text{for all } g \in H^{1/2}(\partial K_R)$$

where $\langle \cdot, \cdot \rangle$ denotes the dual bracket in $\langle H^{-1/2}(\partial K_R), H^{1/2}(\partial K_R) \rangle$. Furthermore, from $kR h_n^{(1)'}(kR)/h_n^{(1)}(kR) = -(n+1) + \mathcal{O}(1/n)$ for $n \rightarrow \infty$ (cf [21]) we conclude that the difference operator $L - L_0$ is compact from $H^{1/2}(\partial K_R)$ into $H^{-1/2}(\partial K_R)$.

From (2.6) above we conclude that the solution u^0 of the scattering problem satisfies the sesquilinear equation (with $u^i(x) := \exp(ik\hat{d}^\top x)$ being the incident field)

$$S(u^0; v) := \iint_{\Omega_R} (\nabla \bar{v}^\top \nabla u^0 - k^2 u^0 \bar{v}) dx - \langle L u^0, v \rangle = \int_{\partial K_R} \bar{v} \left(\frac{\partial u^i}{\partial n} - L u^i \right) ds \quad (2.8)$$

for all $v \in \tilde{H}_0^1(\Omega_R)$. One splits S into $S_0 + S_1$ with strictly coercive

$$S_0(u; v) := \iint_{\Omega_R} (\nabla \bar{v}^\top \nabla u + \bar{v} u) dx - \langle L_0 u, v \rangle$$

and

$$S_1(u; v) := -(k^2 + 1) \iint_{\Omega_R} \bar{v} u dx + \int_{\partial K_R} \bar{v} (L_0 - L) u ds.$$

By the theorem of Lax and Milgram there exists a unique isomorphism T_0 from $\tilde{H}_0^1(\Omega_R)$ onto itself with

$$S_0(u; v) = (T_0 u, v)_{H^1} \quad \text{for all } u, v \in \tilde{H}_0^1(\Omega_R).$$

By the Riesz representation theorem there exists a bounded operator T_1 from $\tilde{H}_0^1(\Omega_R)$ into itself and an element $r \in \tilde{H}_0^1(\Omega_R)$ with

$$S_1(u; v) = (T_1 u, v)_{H^1} \quad \int_{\partial K_R} \bar{v} \left(\frac{\partial u^i}{\partial n} - L u^i \right) ds = (r, v)_{H^1}$$

for all $u, v \in \tilde{H}_0^1(\Omega_R)$. Then equation (2.8) is equivalent to

$$u^0 + T_0^{-1} T_1 u^0 = T_0^{-1} r \quad \text{in } \tilde{H}_0^1(\Omega_R). \quad (2.9)$$

From the compactness of $L - L_0$ and the imbedding $\tilde{H}_0^1(\Omega_R) \subset L^2(\Omega_R)$ it is easily shown that T_1 is compact in $\tilde{H}_0^1(\Omega_R)$. Therefore, the Fredholm alternative is applicable to (2.9). By the uniqueness of the scattering problem $u^0 \in \tilde{H}_0^1(\Omega_R)$ is the unique solution (2.9) and thus also of (2.8), and it depends continuously on the right-hand side. Once the solution u^0 of equation (2.8) is found the far-field pattern u_∞ can be expressed by equation (2.3), i.e.

$$\begin{aligned} u_\infty(\hat{x}) &= \int_{\partial K_R} u^s \frac{\partial}{\partial n} e^{-ik\hat{x} \cdot} ds - \langle L u^s, e^{ik\hat{x} \cdot} \rangle \\ &= \int_{\partial K_R} (u^0 - u^i) \frac{\partial}{\partial n} e^{-ik\hat{x} \cdot} ds - \langle L(u^0 - u^i), e^{ik\hat{x} \cdot} \rangle \end{aligned} \quad (2.10)$$

for $\hat{x} \in S^{N-1}$.

This formulation as a variational problem with a non-local boundary condition can be done for each of the regions Ω_ϵ^e exterior to $\Gamma_{\epsilon a}$. Then the sesquilinear forms S depend on ϵ through its domain of definition $\tilde{H}_0^1(K_R \cap \Omega_\epsilon^e)$. Let u^ϵ be the solutions of the scattering problems corresponding to the regions Ω_ϵ^e . Then u^ϵ are the unique solutions of the analogous variational equation on $K_R \cap \Omega_\epsilon^e$, i.e.

$$\int_{K_R \cap \Omega_\epsilon^e} (\nabla \bar{v}^\top \nabla u^\epsilon - k^2 u^\epsilon \bar{v}) dx - \langle L u^\epsilon, v \rangle = \int_{\partial K_R} \bar{v} \left(\frac{\partial u^i}{\partial n} - L u^i \right) ds. \quad (2.11)$$

Now, for given $a \in C^2(\Gamma; \mathbb{R}^3)$ we choose an extension $\bar{a} \in C^2(\Omega^e, \mathbb{R}^3)$ with $\bar{a}(x) = 0$ for $|x| \geq R/2$. Denote by $\phi^\epsilon : \Omega^e \rightarrow \Omega_\epsilon^e$ the mapping $\phi^\epsilon(y) := y + \epsilon a(y)$, $y \in \Omega^e$. For small $\epsilon > 0$ ϕ^ϵ is a diffeomorphism which maps Ω_R onto $K_R \cap \Omega_\epsilon^e$. Let $\psi^\epsilon : \Omega_\epsilon^e \rightarrow \Omega^e$ be the inverse of ϕ^ϵ . Making the change of variables $x = \phi^\epsilon(y)$ and $\bar{u}^\epsilon = u^\epsilon \circ \phi^\epsilon$ we get

$$\begin{aligned} &\int_{K_R \cap \Omega_\epsilon^e} (\nabla \bar{v}^\top \nabla u^\epsilon - k^2 \bar{v} u^\epsilon) dx \\ &= \iint_{\Omega_R} \left(\sum_{i,j=1}^3 b_{ij}^\epsilon \frac{\partial \bar{u}^\epsilon}{\partial y_i} \frac{\partial (\bar{v} \circ \phi^\epsilon)}{\partial y_j} - k^2 \bar{u}^\epsilon (\bar{v} \circ \phi^\epsilon) \right) \det J_{\phi^\epsilon} dy \end{aligned}$$

where J_{ϕ^ϵ} is the Jacobian matrix of ϕ^ϵ and

$$b_{ij}^\epsilon = \sum_{l=1}^3 \frac{\partial \psi_i^\epsilon}{\partial x_l} \frac{\partial \psi_j^\epsilon}{\partial x_l} \quad i, j = 1, 2, 3.$$

We define $S^\epsilon : \tilde{H}_0^1(\Omega_R) \times \tilde{H}_0^1(\Omega_R) \rightarrow \mathbb{C}$ by

$$S^\epsilon(u; v) := \iint_{\Omega_R} \left(\sum_{i,j=1}^3 b_{ij}^\epsilon \frac{\partial u}{\partial y_i} \frac{\partial \bar{v}}{\partial y_j} - k^2 u \bar{v} \right) \det J_{\phi^\epsilon} dy - \langle Lu, v \rangle$$

and see from (2.11) that $\tilde{u}^\epsilon \in \tilde{H}_0^1(\Omega_R)$ is the unique solution of

$$S^\epsilon(u; v) = \int_{\partial K_R} \bar{v} \left(\frac{\partial u^i}{\partial n} - Lu^i \right) ds \quad \text{for all } v \in \tilde{H}_0^1(\Omega_R).$$

Since, by (2.10)

$$\frac{1}{\epsilon} (u_\infty^\epsilon(\hat{x}) - u_\infty(\hat{x})) = \int_{\partial K_R} \frac{1}{\epsilon} (\tilde{u}^\epsilon - u^0) \frac{\partial}{\partial n} e^{-ik\hat{x}} ds - \left\langle \frac{1}{\epsilon} L(\tilde{u}^\epsilon - u^0), e^{ik\hat{x}} \right\rangle$$

for $|\hat{x}| = 1$, we have to compute the derivative $\lim_{\epsilon \rightarrow 0+} \frac{1}{\epsilon} (\tilde{u}^\epsilon - u^0)$ in $\tilde{H}_0^1(\Omega_R)$.

For any $v \in \tilde{H}_0^1(\Omega_R)$ we have

$$S \left(\frac{1}{\epsilon} (\tilde{u}^\epsilon - u^0); v \right) = -\frac{1}{\epsilon} [S^\epsilon(\tilde{u}^\epsilon; v) - S(\tilde{u}^\epsilon; v)] \quad (2.12)$$

and we have to show that the limit at the right-hand side of (2.9) exists for $\epsilon \rightarrow 0+$ and to compute its value. The right-hand side of (2.12) is

$$-\frac{1}{\epsilon} \iint_{\Omega_R} \left(\sum_{i,j=1}^3 (b_{ij}^\epsilon \det J_{\phi^\epsilon} - \delta_{ij}) \frac{\partial \tilde{u}^\epsilon}{\partial y_i} \frac{\partial \bar{v}}{\partial y_j} - k^2 (\det J_{\phi^\epsilon} - 1) \tilde{u}^\epsilon \bar{v} \right) dy$$

with Kronecker symbol δ_{ij} . Since $\det J_{\phi^\epsilon} = \det(I + \epsilon J_a) = 1 + \epsilon \operatorname{div} a + \mathcal{O}(\epsilon^2)$ and $J_{\psi^\epsilon} = J_{\phi^\epsilon}^{-1} \circ \psi^\epsilon = I - \epsilon J_a + \mathcal{O}(\epsilon^2)$ uniformly in $y \in \bar{\Omega}_R$ we conclude that

$$\frac{1}{\epsilon} (\det J_{\phi^\epsilon} - 1) \rightarrow \operatorname{div} a \quad (\epsilon \rightarrow 0+)$$

$$\frac{1}{\epsilon} (b_{ij}^\epsilon \det J_{\phi^\epsilon} - \delta_{ij}) \rightarrow \delta_{ij} \operatorname{div} a - \left(\frac{\partial a_i}{\partial y_j} + \frac{\partial a_j}{\partial y_i} \right) \quad (\epsilon \rightarrow 0+)$$

uniformly in $\bar{\Omega}_R$ for $i, j = 1, 2, 3$.

Since \tilde{u}^ϵ converges to u^0 in $H^1(\Omega_R)$ by the continuity arguments (cf [18]) we conclude that the right-hand side of (2.12) tends to

$$\begin{aligned} & \iint_{\Omega_R} \left[\sum_{i,j=1}^3 \left(\frac{\partial a_i}{\partial y_j} + \frac{\partial a_j}{\partial y_i} - \delta_{ij} \operatorname{div} a \right) \frac{\partial u^0}{\partial y_i} \frac{\partial \bar{v}}{\partial y_j} + k^2 u^0 \bar{v} \operatorname{div} a \right] dy \\ & = \iint_{\Omega_R} [\nabla \bar{v}^\top (J_a + J_a^\top - \operatorname{div} a I) \nabla u^0 + k^2 u^0 \bar{v} \operatorname{div} a] dy. \end{aligned}$$

Thus $\frac{1}{\epsilon}(\tilde{u}^\epsilon - u^0)$ is convergent in $\tilde{H}_0^1(\Omega_R)$, and we define $\hat{u} \in \tilde{H}_0^1(\Omega_R)$ by $\hat{u} := \lim_{\epsilon \rightarrow 0^+} \frac{1}{\epsilon}(\tilde{u}^\epsilon - u^0)$ in Ω_R .

Extend \hat{u} outside Ω_R by solving the exterior Dirichlet problem with boundary data \hat{u} on ∂K_R . Then \hat{u} satisfies

$$S(\hat{u}; v) = \iint_{\Omega_R} [\nabla \bar{v}^\top (J_a + J_a^\top - \div a I) \nabla u^0 + k^2 u^0 \bar{v} \div a] \, dx$$

for all $v \in \tilde{H}_0^1(\Omega_R)$. Now we observe that, by well known regularity results (cf [7, theorem 8.12]), $u^0 \in \tilde{H}_0^1(\Omega_R) \cap H^2(\Omega_R)$ and use for $v \in \tilde{H}_0^1(\Omega_R) \cap H^2(\Omega_R)$ the formula

$$\begin{aligned} \nabla \bar{v}^\top (J_a + J_a^\top - \div a I) \nabla u^0 \\ = \operatorname{div} [(a^\top \nabla u^0) \nabla \bar{v} + (a^\top \nabla \bar{v}) \nabla u^0 - (\nabla \bar{v}^\top \nabla u^0) a] \\ - (a^\top \nabla u^0) \Delta \bar{v} - (a^\top \nabla \bar{v}) \Delta u^0 \end{aligned}$$

in Ω_R and apply Gauss' theorem to the divergence term. This yields (since $a = 0$ near ∂K_R)

$$\begin{aligned} S(\hat{u}; v) = \iint_{\Omega_R} [k^2 u^0 \bar{v} \div a - (a^\top \nabla u^0) \Delta \bar{v} - (a^\top \nabla \bar{v}) \Delta u^0] \, dx \\ - \int_{\Gamma} [(a^\top \nabla u^0) \nabla \bar{v} + (a^\top \nabla \bar{v}) \nabla u^0 - (\nabla \bar{v}^\top \nabla u^0) a]^\top n \, ds. \end{aligned}$$

Now we use the homogeneous boundary conditions for v and u^0 , the Helmholtz equation for Δu^0 and Green's theorem for Δv . This yields

$$S(\hat{u}; v) = k^2 \iint_{\Omega_R} \div (u^0 \bar{v} a) \, dx + \iint_{\Omega_R} [\nabla \bar{v}^\top \nabla (a^\top \nabla u^0) - k^2 \bar{v} (a^\top \nabla u^0)] \, dx$$

i.e. applying Gauss' theorem again

$$\iint_{\Omega_R} [\nabla \bar{v}^\top \nabla \hat{u} - k^2 \hat{u} \bar{v}] \, dx - \langle L \hat{u}, v \rangle = \iint_{\Omega_R} [\nabla \bar{v}^\top \nabla (a^\top \nabla u^0) - k^2 \bar{v} (a^\top \nabla u^0)] \, dx.$$

Since $a^\top \nabla u^0 = 0$ for $|x| \geq R/2$ this proves that $(\Delta + k^2)(\hat{u} - a^\top \nabla u^0) = 0$ in Ω_R weakly, $(\hat{u} - a^\top \nabla u^0)|_{\Gamma} = -(a^\top n) \partial u^0 / \partial n$ and also $\partial \hat{u} / \partial n|_+ = L \hat{u} = \partial \hat{u} / \partial n|_-$ on Γ_R , i.e. $\hat{u} -$ and thus $u' -$ is C^2 in Ω^e . This proves that $u' = \hat{u} - a^\top \nabla u^0$ and therefore

$$\lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} (u^\epsilon - u^0) = u' \quad \text{in } H^{1/2}(\partial K_R).$$

This, finally, proves

$$\lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} (u_\infty^\epsilon(\hat{x}) - u_\infty(\hat{x})) = u'_\infty(\hat{x}) \quad \text{uniformly for } |\hat{x}| = 1.$$

This ends the proof.

Remarks. (a) For the proof it is not essential that ∂K_R is the boundary of a sphere but can be the C^2 -boundary of any bounded domain containing Ω in its interior. From this observation and Green's representation theorem it follows that $\lim_{\epsilon \rightarrow 0^+} \frac{1}{\epsilon}(u^\epsilon - u^0) = u'$ uniformly in every compact subset of Ω^ϵ .

(b) By obvious modifications of the proof it can be shown that $F'(\Gamma; a)$ is a kind of Frechet derivative of F at Γ , i.e.

$$\lim_{\|a\|_{1,\infty} \rightarrow 0} \frac{1}{\|a\|_{1,\infty}} (F(\Gamma_a) - F(\Gamma) - F'(\Gamma; a)) = 0 \quad \text{uniformly on } S^2$$

where $\|a\|_{1,\infty} := \max_{x \in \Gamma} |a(x)| + \max_{x \in \Gamma} \sum_{j=1}^3 |\text{Grad } a_j(x)|$ with surface gradient $\text{Grad } a_j$ of the j th component of a (cf [2] for a definition of the surface gradient).

(c) From this theorem we observe that the domain derivative can be extended to a bounded operator $F'(\Gamma; \cdot) : C(\Gamma; \mathbb{R}^N) \rightarrow C(S^{N-1})$ (or even to Sobolev spaces).

In order to deal with 'non-degenerated' surfaces we assume that $a^\top n$ has only finitely many zeros on Γ . Then we can prove

Lemma 2.2. The operator $F'(\Gamma; \cdot)$ is one-to-one on the space

$$\{a \in C(\Gamma; \mathbb{R}^N) : a^\top n \text{ has only finitely many zeros on } \Gamma\}.$$

Proof. Let be $F'(\Gamma; a) = 0$ in $L^2(S^{N-1})$. Then the far-field pattern u'_∞ vanishes where u' solves the exterior boundary value problem with boundary data $u' = -(a^\top n) \partial u^0 / \partial n$ on Γ . From Rellich's theorem and analytic continuation u' vanishes outside of Ω . Since $\partial u^0 / \partial n$ cannot vanish on any $U \cap \Gamma$ for U being a disk and it is continuous $a^\top n$ has to vanish on Γ . The assumption on a implies that $a = 0$, which ends the proof.

Remark. In general it is not known if F itself is one-to-one, see [6] for a detailed discussion of the problem of uniqueness of the inverse scattering problem.

3. An application to a Gauss-Newton method

This section describes the essential step of an algorithm for the following *inverse scattering problem* which we formulate in two dimensions. Given the (measured) far-field pattern $f \in L^2(S^1)$, the direction of the incident plane wave and the wave number $k > 0$ compute a domain such that the scattered wave has the far-field pattern f . There exist a large number of algorithms for this problem. Roughly speaking, they fall into two classes of methods.

The algorithms of the first class all consist of two steps, the analytic continuation step (which is linear but highly ill-posed) and a nonlinear step of finding a 'level curve' of a certain field (cf Imbriale and Mittra [9], Colton and Monk [3-5] and Kirsch and Kress [13, 14]).

The algorithms of the second class are called 'output least squares methods' in the parameter estimation community. As typical papers in this field we mention only those by Roger [25] and Kristensson and Vogel [19]. The idea is to solve the equation $F(\Gamma) = f$ by a Newton-type method or minimize $\|F(\Gamma) - f\|$ by an optimization

method. In any case the unknown boundary Γ has to be parametrized. Without loss of generality we represent (as also done by most of the authors, cf [3-5, 13-15, 18, 19, 25]) Γ in polar coordinates by

$$x_\alpha(t) = r_\alpha(t) \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} \quad 0 \leq t \leq 2\pi$$

$$\text{where } r_\alpha(t) = \alpha_0 + \sum_{j=1}^M [\alpha_j \cos jt + \alpha_{j+M} \sin jt]$$

with $\alpha = (\alpha_0, \dots, \alpha_{2M})^T \in \mathbb{R}^{2M+1}$ for some fixed number $M \in \mathbb{N}$. Let

$$U_M := \{ \alpha \in \mathbb{R}^{2M+1} : \rho_1 \leq r_\alpha(t) \leq \rho_2, t \in [0, 2\pi] \} \quad \text{for some } 0 < \rho_1 < \rho_2.$$

We can assign to each $\alpha \in U_M$ the far-field patterns $F(\Gamma)(\hat{x}_i) \in \mathbb{C}$, $i = 1, \dots, P$, at a number P of given and fixed 'observation points' $\hat{x}_i \in S^1$ where Γ denotes the boundary parameterized by $x_\alpha(t)$. This defines a (nonlinear) mapping $\bar{F} : \mathbb{R}^{2M+1} \supset U_M \rightarrow \mathbb{C}^P$. For the following we fix M and P . A simple application of theorem 2.1 shows

Theorem 3.1. For $\alpha \in \text{int } U_M$ † the mapping \bar{F} is (totally) differentiable with $\partial \bar{F}_i(\alpha) / \partial \alpha_j = u'_{j,\infty}(\hat{x}_i)$ for $i = 1, \dots, P$ and $j = 0, \dots, 2M$. Here, $u'_{j,\infty}$ is the far-field pattern of $u'_j \in C^2(\Omega^e) \cap C(\bar{\Omega}^e)$ which is the solution of the exterior boundary value problem

$$\Delta u'_j + k^2 u'_j = 0 \text{ in } \Omega^e \quad \frac{\partial u'_j}{\partial r} - ik u'_j = o(r^{-1/2}) \quad r \rightarrow \infty$$

and

$$u'_j(x_\alpha(t)) = -\frac{r_\alpha(t)}{\sqrt{r'_\alpha(t)^2 + r_\alpha(t)^2}} \times \frac{\partial}{\partial n} u^0(x_\alpha(t)) \begin{cases} \cos jt & j = 0, \dots, M \\ \sin(j - M)t & j = M + 1, \dots, 2M \end{cases}$$

for $t \in [0, 2\pi]$.

Proof. Let be $i \in \{1, \dots, P\}$ and $j \in \{0, \dots, 2M\}$ fixed, $\varepsilon > 0$ and $e^{(j)}$ be the j th unit vector in \mathbb{R}^{2M+1} . Let Γ and Γ^ε be parametrized by

$$r_\alpha(t) \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} \\ r_{\alpha + \varepsilon e^{(j)}}(t) \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} = r_\alpha(t) \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} + \varepsilon r_{e^{(j)}}(t) \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$$

respectively. Thus $\Gamma^\varepsilon = \Gamma_{\varepsilon a_j}$ in the notation of (2.4) with $a^j(x_\alpha(t)) = r_{e^{(j)}}(t) \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$. Then

$$\frac{1}{\varepsilon} [\bar{F}_i(\alpha + \varepsilon e^{(j)}) - \bar{F}_i(\alpha)] = \frac{1}{\varepsilon} [F(\Gamma_{\varepsilon a_j})(\hat{x}_i) - F(\Gamma)(\hat{x}_i)]$$

† $\text{int } U$ denotes the interior of the set U

and this converges by theorem 2.1 to $u'_{j,\infty}(\hat{x}_i)$, where u'_j solves the exterior Dirichlet problem with boundary data $-(n^\top a^j) \partial u^0 / \partial n$. With

$$n(t)^\top a^j(x_\alpha(t)) = \frac{r_\alpha(t)}{\sqrt{r'_\alpha(t)^2 + r_\alpha(t)^2}} \begin{cases} \cos jt & j = 0, \dots, M \\ \sin(j - M)t & j = M + 1, \dots, 2M \end{cases}$$

the assertion follows. This ends the proof.

The computation of one function value $\bar{F}(\alpha)$ involves solving an exterior boundary value problem for the Helmholtz equation in a region parametrized by $x_\alpha(t)$ and to evaluate the far-field pattern at P points. The computation of the Jacobian of this mapping means solving $2M + 1$ exterior boundary value problems for the same Helmholtz equation in the same region with different boundary values. After discretization this leads to a system of linear equations with $2M + 1$ right-hand sides which is not much more work than solving one boundary value problem (at least if $2M + 1$ is small compared to the number of discretization points). Therefore, the advantage of theorem 3.1 is that one gets the Jacobian of \bar{F} almost for free.

We will now show by some numerical examples how one can solve the system of equations $\bar{F}_i(\alpha) = \bar{f}_i := f(\hat{x}_i)$, $i = 1, \dots, P$, approximately. Since the equation $F(\Gamma) = f$ is highly ill-posed (cf [1]) we expect the system $\bar{F}(\alpha) = \bar{f}$ to be ill-conditioned. Therefore, we regularize this equation and use, for convenience, Tikhonov's method (cf [8]). Thus we minimize

$$\sum_{i=1}^P |\bar{F}_i(\alpha) - \bar{f}_i|^2 + \varepsilon \|\alpha\|^2 \quad (3.1)$$

for some regularisation parameter $\varepsilon > 0$ and some (semi-)norm on α . Recalling the meaning of α as the Fourier coefficients of r_α we take $\|\alpha\|^2 = \sum_{m=1}^M m^2 (\alpha_m^2 + \alpha_{m+M}^2)$ which is the $H^2(0, 2\pi)$ semi-norm of r_α .

The minimization problem (3.1) can be written as the least squares problem

$$\text{Minimize } \|\Phi(\alpha)\|_2^2 \quad \text{where } \begin{cases} \Phi_i(\alpha) = \bar{F}_i(\alpha) - \bar{f}_i & i = 1, \dots, P, \\ \Phi_{P+i}(\alpha) = i^2 \sqrt{\varepsilon} \alpha_i & i = 1, \dots, M, \\ \Phi_{P+M+i}(\alpha) = i^2 \sqrt{\varepsilon} \alpha_{M+i} & i = 1, \dots, M. \end{cases}$$

Certainly, this least squares problem can also be rewritten as a real valued problem. We used a Levenberg–Marquardt routine from CMLIB (US National Bureau of Standards collection of public domain software) which uses selected routines from MINPACK [22]. For this routine one has to supply the function values $\Phi_i(\alpha)$ and the Jacobian of Φ at α . We computed the solution u^0 of the scattering problem by an integral equation method based on the Green's representation theorem (cf [16]). This results in a boundary integral equation for the normal derivative $\partial u^0 / \partial n$ of the total field. By theorem 2.1 $\partial u^0 / \partial n$ is also needed for the computation of the Jacobian. For the solution of the boundary value problems for u'_j we use an integral equation method based on the layer ansatz (cf [2]) to avoid computing the normal derivative of the double layer potential. We solved the resulting boundary integral equations with a Nyström method with $2N = 64$ knots. For a detailed discussion of Nyström's method we refer the reader to Kress [17].

Numerical examples. Let Ω be the 'kite' from [15], parametrized by $x_1(t) = \cos t + 0.65(\cos(2t) - 1)$, $x_2(t) = 1.5 \sin t$, $t \in [0, 2\pi]$. The wavenumber is $k = 3$, the dimension of the ansatz space U_M is $2M + 1 = 17$, the number of observation points is $P = 32$, and we used 32 equally spaced discretization points for the Nyström method to solve the boundary integral equations for u^0 and u'_j (cf [17] for detailed discussion on Nyström's method). We added 5% relative error (random noise) on the observation points.

We observed very poor reconstruction properties when we probed the obstacle by only one incident field. Therefore, and in order to compare the method with those presented in [15] we chose two incident waves with directions 30° and 210° . Both functionals are added, i.e. instead of (3.1) we minimized

$$\sum_{i=1}^P |\tilde{F}_i^1(\alpha) - \hat{f}_i^1|^2 + \sum_{i=1}^P |\tilde{F}_i^2(\alpha) - \hat{f}_i^2|^2 + \epsilon \|\alpha\|^2$$

where the superscripts 1 and 2 indicate the incident fields.

Figures 1-4 show the results where we have chosen two different initial circles (radii 1.0 and 1.2) for the Levenberg-Marquardt algorithm and three different regularization parameters ($\epsilon = 10^{-1}$, 10^{-2} , 10^{-3}). We have plot the initial circle, the best L^2 approximation of the true curve (the kite) in the finite dimensional space U_M and the reconstructed curve.

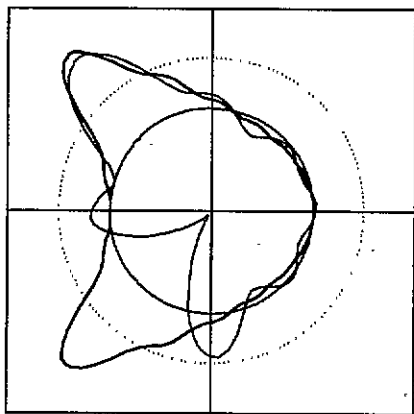


Figure 1. Results of numerical example. Radius of initial circle, 1.0; regularization parameter $\epsilon, 10^{-2}$; error in f , 0%; number of function calls, 20; number of Jacobi matrices, 11; running time, 52.

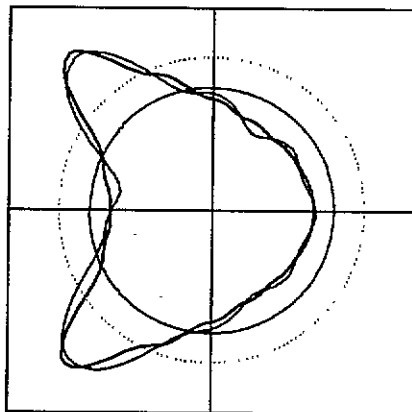


Figure 2. As figure 1, with radius of initial circle, 1.2; regularization parameter $\epsilon, 10^{-2}$; error in f , 5%; number of function calls, 18; number of Jacobi matrices, 16; running time, 65.

We clearly see the dependence on the proper choice of ϵ and the initial curve. We do not believe that the incorporation of constraints of the form $r_\alpha(t) \geq r_0$ in an Levenberg-Marquardt algorithm would improve the results in principle. The numerical tests in [11] show that the curve will always touch the constraint. A certain disadvantage, at least in principle, of these methods is the effort of computing u_∞ and $u'_{j,\infty}$ in every step. In two dimensions the Nyström method is very fast, and the computation time was not observed to be considerably higher than of other methods (cf [12]). This is certainly different in three dimensions but implementations of

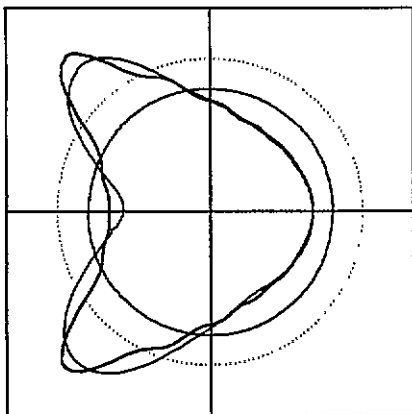


Figure 3. As figure 1, with radius of initial circle, 1.2; regularization parameter $\epsilon, 10^{-1}$; error in f , 5%; number of function calls, 20; number of Jacobi matrices, 16; running time, 82.

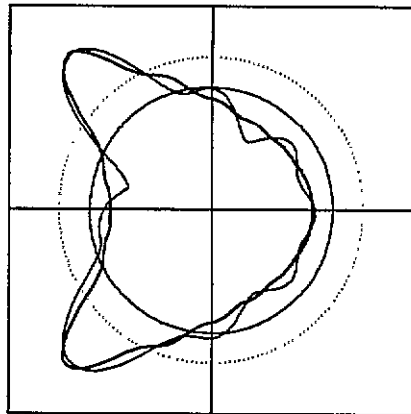


Figure 4. As figure 1, with radius of initial circle, 1.2; regularization parameter $\epsilon, 10^{-3}$; error in f , 5%; number of function calls, 20; number of Jacobi matrices, 14; running time, 74.

Newton-type methods are not known to the author. For a more detailed discussion and comparison of different numerical methods for solving the inverse scattering problem we refer the reader to [12].

4. An application to sensitivity

In this section we study the sensitivity of the linearized inverse scattering problem. As before, let, in two dimensions

$$X \subset \{\Gamma \in C^2 : \exists \Omega \subset \mathbb{R}^2 \text{ open and bounded such that } \Gamma = \partial\Omega\}$$

and $F : X \rightarrow C(S^1)$ the far-field operator. Let Γ be the true obstacle. For every (small) $\epsilon > 0$ we can define the 'worst case error' (cf Natterer [23] or Louis [20] for the linear case)

$$\sup\{d(\Gamma, \tilde{\Gamma}) : \tilde{\Gamma} \in \tilde{V}, \|F(\Gamma) - F(\tilde{\Gamma})\|_{L^2(S^1)} \leq \epsilon\}$$

where $\tilde{V} \subset X$ contains some a-priori information about Γ , and $d(\Gamma, \tilde{\Gamma})$ denotes some measure of distance between Γ and $\tilde{\Gamma}$.

We restrict $\tilde{\Gamma}$ to be of the form Γ_a for $a \in C^2(\Gamma; \mathbb{R}^2)$ with sufficiently small $\|a\|_{1,\infty}$ and define the linearized worst case error by

$$\omega := \sup\{\|a\| : a \in V, \|F'(\Gamma; a)\|_{L^2(S^1)} \leq \epsilon\}$$

where $V \subset C^1(\Gamma; \mathbb{R}^2)$ is some given subset and $\|\cdot\|$ some given semi-norm on a . Since $F'(\Gamma; \cdot)$ is linear we can assume $\epsilon = 1$ without loss of generality.

The ill-posedness of the inverse problem is expressed by the following theorem:

Theorem 4.1. Let be $V \subset C(\Gamma; \mathbb{R}^2)$ infinite dimensional, dense in $L^2(\Gamma; \mathbb{R}^2)$ and such that $a^\top n$ admits only finitely many zeros on Γ . Furthermore, let $\|\cdot\|$ be the $L^2(\Gamma)$ - or $C(\Gamma)$ -norm, or $\|a\| = |a(x)|$ for some fixed $x \in \Gamma$. Then $\omega = \infty$.

Proof. Since $F'(\Gamma; \cdot) : V \rightarrow L^2(S^1)$ is compact and one-to-one by lemma 2.2 and V is infinite dimensional a standard argument shows that $F'(\Gamma; \cdot)^{-1}$ is unbounded on the range of $F'(\Gamma; \cdot)$. This shows that $w = \infty$ in the case of $\|\cdot\|$ being the L^2 - or C -norm. For $\|a\| = |a(x)|$ simply choose a sequence $(a_j) \subset V$ with $\|a_j\|_{L^2} \leq \delta$ and $|a_j(x)| = j$, where δ is small enough such that $\|F'(\Gamma; a_j)\|_{L^2(S^1)} \leq 1$. This ends the proof.

We are especially interested in the case where Γ is parametrized by $x(t)$, $0 \leq t \leq 2\pi$, V is given by

$$V := \left\{ a_\alpha \in C(\Gamma; \mathbb{R}^2) : a_\alpha(x(t)) = r_\alpha(t) \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}, t \in [0, 2\pi], \alpha \in \mathbb{R}^{2M+1} \right\}$$

for some fixed $M \in \mathbb{N}$, where

$$r_\alpha(t) = \alpha_0 + \sum_{m=1}^M (\alpha_m \cos t + \alpha_{m+M} \sin t)$$

and $\|a\|$ is given by $|a(x)|$ for some $x \in \Gamma$ or $\|\cdot\| = \|\cdot\|_{L^2(\Gamma)}$. Since $|a_\alpha(x(t))| = r_\alpha(t)$, we define the quantities

$$\omega_1(t) := \sup\{r_\alpha(t) : \alpha \in \mathbb{R}^{2M+1}, \|F'(\Gamma; a_\alpha)\|_{L^2(S^1)} \leq 1\} \quad \text{for } t \in [0, 2\pi]$$

$$\omega_2 := \sup\{\|r_\alpha\|_{L^2(0,2\pi)} : \alpha \in \mathbb{R}^{2M+1}, \|F'(\Gamma; a_\alpha)\|_{L^2(S^1)} \leq 1\}.$$

First we note that the supremum is attained:

Lemma 4.2. There exists $\alpha^j \in \mathbb{R}^{2M+1}$ ($j = 1, 2$) with $\|F'(\Gamma; a_{\alpha^j})\|_{L^2(S^1)} \leq 1$, $r_{\alpha^1}(t) = \omega_1(t)$ and $\|r_{\alpha^2}\|_{L^2(0,2\pi)} = \omega_2$.

Proof. We note that $\alpha \mapsto F'(\Gamma; a_\alpha)$ is a bounded operator from \mathbb{R}^{2M+1} into $L^2(S^1)$ which is also one-to-one by lemma 2.2. Since \mathbb{R}^{2M+1} is finite dimensional there exist $c > 0$ with

$$1 \geq \|F'(\Gamma; a_\alpha)\|_{L^2(S^1)} \geq c\|\alpha\|_2 \quad \text{for all } \alpha \in \mathbb{R}^{2M+1} \quad \text{with } \|F'(\Gamma; a_\alpha)\|_{L^2(S^1)} \leq 1.$$

From this $r_\alpha(t) \leq c_1$ and $\|r_\alpha\|_{L^2(0,2\pi)} \leq c_2$ follows for all $t \in [0, 2\pi]$, $\alpha \in \mathbb{R}^{2M+1}$ with $\|F'(\Gamma; a_\alpha)\|_{L^2(S^1)} \leq 1$. This shows that w_j are the suprema of continuous functions on a bounded and closed set in a finite dimensional space. This proves the assertion.

Now we compute an approximation of $\omega_1(t)$ by solving for every fixed $t \in [0, 2\pi]$ the following quadratic programming problem (P_t) . Again, \hat{t}_j , $j = 1, \dots, P$, are given 'observation points', and \hat{F}' is the Jacobian of \hat{F} from theorem 3.1

(P_t) Maximize

$$r_\alpha(t) := \alpha_0 + \sum_{m=1}^M (\alpha_m \cos t + \alpha_{m+M} \sin t)$$

subject to

$$\sum_{j=1}^P |(\hat{F}'\alpha)_j|^2 \leq 1.$$

With $c_j(t) := \cos(jt)$, $j = 0, \dots, M$, $c_{j+M}(t) := \sin(jt)$, $j = 1, \dots, M$, this leads to the quadratic programming problem

(P_t) Maximize

$$c(t)^\top \alpha$$

subject to

$$\alpha^\top (\tilde{F}')^* (\tilde{F}) \alpha \leq 1.$$

This problem can easily be solved by, e.g. the Lagrange multiplier rule.

Similarly, we compute an approximation of ω_2 by solving (with the preceding notations)

P Maximize

$$\|r_\alpha\|_{L^2(0,2\pi)} = \int_0^{2\pi} |c(t)^\top \alpha|^2 dt$$

subject to

$$\alpha^\top (\tilde{F}')^* (\tilde{F}) \alpha \leq 1.$$

Using the Lagrange multiplier rule this leads to the problem of computing the smallest eigenvalue of $D^{-1}(\tilde{F}')^* (\tilde{F})$, where D is a simple diagonal matrix.

Numerical examples. We consider the same examples as in section 3. Ω is the kite, parametrized by $x_1(t) = \cos t + 0.65(\cos(2t) - 1)$, $x_2(t) = 1.5 \sin t$, $t \in [0, 2\pi]$, for k we take 1 or 3, the angle of the incoming wave is 30° . Figures 5 and 6 show the kite and $\omega_1(t)$ for these values of k . ω_1 is plotted in polar coordinates where the dotted circle denotes the zero level. In figures 7 and 8 we plot r_{α_2} against the dotted zero level.

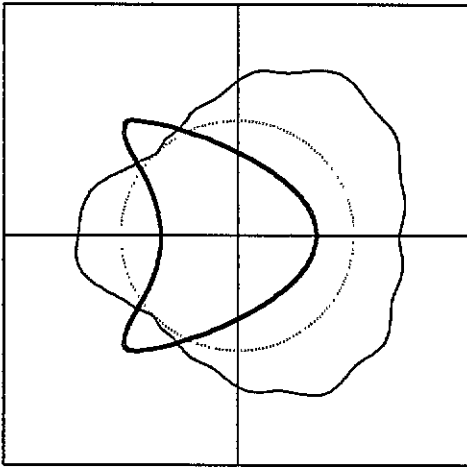


Figure 5. The function $\omega_1(t)$ for $t \in [0, 2\pi]$, with $k = 1$.

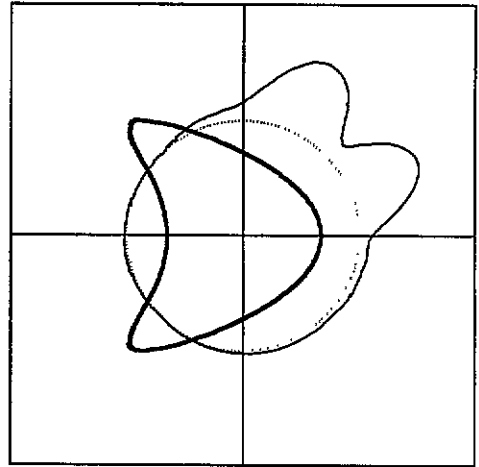


Figure 6. As figure 5, with $k = 2$.

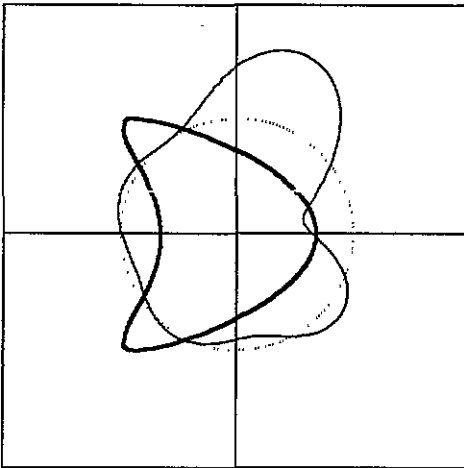


Figure 7. The function $r_{\alpha 2}$ for $t \in [0, 2\pi]$, with $k = 1$.

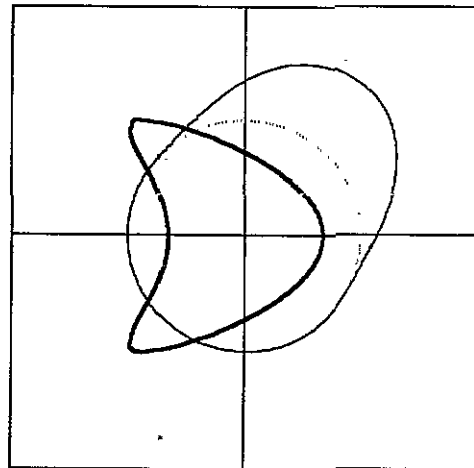


Figure 8. As figure 7, with $k = 3$.

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