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Fréchet differentiability of the solution to the acoustic Neumann scattering problem with respect to the domain

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Abstract — Using integral equation methods to solve the time harmonic acoustic scattering problem with Neumann boundary condition it is possible to reduce the solution of the scattering problem to the solution of a boundary integral equation of the second kind. We show the Fréchet differentiability of the boundary integral operators which occur. They are considered in dependence of the boundary as integral operators in the spaces of continuous functions. Then we use this to prove the Fréchet differentiability of the scattered fields. Finally we characterize the Fréchet derivatives of the scattered fields by a suitable boundary value problem.

1. INTRODUCTION

In this paper we deal with time-harmonic acoustic scattering problem by a bounded obstacle with Neumann boundary condition. The problem is one of the standard problems of mathematical physics. Different methods have been developed to solve the problem. Here we refer to the integral equation approach wich can be found in [3].

Especially in the framework of *inverse problems* it is interesting to study the solutions to the scattering problems in dependence of the domain of the scatterer. By Γ we denote the boundary of a suitable domain $D \subset \mathbb{R}^3$. We consider the solution to the scattering problem on a set $M \subset \mathbb{R}^3 \setminus \overline{D}$. Let the operator R map the boundary Γ onto the solution $u^s|_M$ of the direct scattering problem for a fixed entire incident field u^i, i.e. we have

$$u^s|_M = R(\Gamma). \tag{1.1}$$

The *inverse scattering problem* consists of looking for a solution Γ of (1.1) given u^s on the exterior set M or looking for Γ given the farfield u^∞ of u^s, respectively. R is nonlinear and equation (1.1) is ill-posed, which makes it in general difficult to solve.

Using boundary integral equation methods to solve the scattering problem following Colton and Kress [3] it is possible to derive a representation of R consisting of acoustic single and double layer potentials and weakly singular boundary integral operators. We briefly recall this method in Section 2. Section 3 we use to establish some facts about the Fréchet derivative of integral operators. In Section 4 we prove the Fréchet differentiability in dependence of the domain and calculate the Fréchet derivative of the integral operators

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used in 2, considered as operators in the spaces of continuous functions on Γ. Using the well known properties of the Fréchet derivative it is now possible to obtain the Fréchet differentiability and the Fréchet derivative of the scattering operator R. In Section 5 we give a characterization of the derivative of u^s as a solution to a boundary value problem with Neumann boundary condition.

The differentiability properties of the integral operators considered as operators in the spaces of continuous functions have been studied for the case of the obstacle scattering problem with *Dirichlet* boundary condition in [12]. For the scattering problem with the Dirichlet boundary condition the characterization of the derivative of the scattered field has been done by Kirsch [4], [5], [6] using variational methods and by Potthast [12] within the framework of integral equation methods.

The great potential of the integral equation approach to study the dependence of the domain has also been shown in [13] – where the author studied the Fréchet differentiability of the integral operators in the spaces of höldercontinuously differentiable functions, extended the results to higher derivatives and applied the method also to the electromagnetic scattering problem for a perfect conductor. Kress applied the approach in [8] to the inverse scattering problem from an open arc.

The results open the possibility to use Newton's method or Newton-type methods to solve the inverse scattering problem. For the application of Newton's method to solve inverse scattering problems we refer to the standard literature concerning Newton's method in Banach spaces and to [14], [8], [9], [6], [10], [11], [16] and [17].

2. THE SCATTERING MAPS R AND THE INVERSE SCATTERING PROBLEM

By Ω_L we denote the Ball with radius L in $\mathbb{R}^3$. Let $D \subset \Omega_L \subset \mathbb{R}^3$ be a bounded domain with boundary ∂D of class C^2, $B \supset \overline{\Omega_L}$ an open set. A function $w \in C^1(\mathbb{R}^3 \setminus \overline{\Omega_L})$ satisfies the *Sommerfeld radiation condition* if there holds

$$\frac{x}{|x|} \cdot (\text{grad } w)(x) - ikw(x) = o\left(\frac{1}{|x|}\right), \qquad |x| \to \infty \tag{2.1}$$

uniformly on $\Omega = \left\{\hat{x} := x/|x|,\ x \in \mathbb{R}^3 \setminus \{0\}\right\}$. By

$$\Phi(x,y) = \frac{1}{4\pi}\frac{e^{i\kappa|x-y|}}{|x-y|}, \qquad x \neq y$$

– where we assume $\kappa \geq 0$ for the wave number κ – we denote the *fundamental solution* of the *Helmholtz equation*

$$\Delta u + \kappa^2 u = 0. \tag{2.2}$$

$\Phi(\cdot, y)$ solves the Helmholtz equation in $\mathbb{R}^3 \setminus \{y\}$ and satisfies the Sommerfeld radiation condition uniformly for $y \in \Omega_L \subset \mathbb{R}^3$. For $\varphi \in C(\partial D)$ the *acoustic single layer potential*

$$u(x) := \int_{\partial D} \Phi(x,y)\varphi(y)\, ds(y), \qquad x \in \mathbb{R}^3 \setminus \partial D \tag{2.3}$$

and the *acoustic double layer potential*

$$v(x) := \int_{\partial D} \frac{\partial \Phi(x,y)}{\partial \nu(y)}\varphi(y)\, ds(y), \qquad x \in \mathbb{R}^3 \setminus \partial D \tag{2.4}$$

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are solutions to the Helmholtz equation in $\mathbb{R}^3 \setminus \partial D$ and satisfy the Sommerfeld radiation condition. Consider the *Neumann obstacle scattering problem*: For a given solution $u^i \in C^1(B)$ to the Helmholtz equation find a function $u^s \in C^2(\mathbb{R}^3 \setminus \overline{D}) \cap C(\mathbb{R}^3 \setminus D)$, which satisfies the Helmholtz equation in $\mathbb{R}^3 \setminus \overline{D}$ and the Sommerfeld radiation condition, such that for the normal derivative of the total field we have $\partial u/\partial \nu = \partial\{u^i + u^s\}/\partial \nu = 0$ on ∂D. Following Colton and Kress [3] we look for a solution to the Neumann obstacle scattering problem using a *modified single and double layer potential*

$$u^s(x) = \int_{\partial D} \left\{ \Phi(x,y)\varphi(y) + \left\{ \mathrm{i}\eta \frac{\partial \Phi(x,y)}{\partial \nu(y)} \right\} (S_0^2\varphi)(y) \right\} \mathrm{d}s(y), \quad x \in \mathbb{R}^3 \setminus \partial D \tag{2.5}$$

$\eta \in \mathbb{R}$, where S_0 denotes the operator S defined by (2.7) in the potential theoretic case $\kappa = 0$. Using the classical jump relations for the single and double layer potential [2] the potential (2.5) solves the Neumann scattering problem if the density $\varphi \in C(\partial D)$ is a solution to the boundary integral equation

$$(I - K^* - \mathrm{i}\eta T S_0^2)\varphi = 2\frac{\partial u^i}{\partial \nu}. \tag{2.6}$$

Here the operators

$$(S\varphi)(x) := 2\int_{\partial D} \Phi(x,y)\varphi(y)\,\mathrm{d}s(y), \qquad x \in \partial D \tag{2.7}$$

and

$$(K^*\varphi)(x) := 2\int_{\partial D} \frac{\partial \Phi(x,y)}{\partial \nu(x)}\varphi(y)\,\mathrm{d}s(y), \qquad x \in \partial D \tag{2.8}$$

are linear, they have weakly singular kernels and are therefore compact operators $C(\partial D) \to C(\partial D)$. S is bounded from $C(\partial D)$ into $C^{0,\alpha}(\partial D)$ and from $C^{0,\alpha}(\partial D)$ into $C^{1,\alpha}(\partial D)$. The operator

$$(T\varphi)(x) := 2\frac{\partial}{\partial \nu(x)} \int_{\partial D} \frac{\partial \Phi(x,y)}{\partial \nu(y)}\varphi(y)\,\mathrm{d}s(y), \qquad x \in \partial D \tag{2.9}$$

is bounded from $C^{1,\alpha}(\partial D)$ into $C^{0,\alpha}(\partial D)$. Existence and boundedness of the inverse of the operator $I + K - \mathrm{i}\eta T S_0^2$ can be obtained using the Riesz-Fredholm theory for equations of the second kind with compact operators [7].

To keep our analysis as simple as possible we want to use another form of equation (2.6). With the help of the jump relations and Green's theorem Colton and Kress derive in [3], equation (3.13), the relation

$$TS = K^{*2} - I. \tag{2.10}$$

With (2.10) and $\tilde{T} := T - T_0$, where T_0 denotes the operator T in the potential theoretic case, the equation (2.6) takes the form

$$\left(I - K^* - \mathrm{i}\eta(K_0^{*2} - I + \tilde{T}S_0)S_0\right)\varphi = 2\frac{\partial u^i}{\partial \nu}. \tag{2.11}$$

Proceeding in this way we can avoid the examination of the strongly singular operator T and we can restrict our analysis to weakly singular operators in the spaces of continuous functions. The analysis for the operators T and S in Hölder spaces – their Fréchet differentiability in dependence of the domain – can be found in [13].

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We are interested in the values of the scattered field on a set $M \subset \mathbb{R}^3 \setminus \overline{D}$. Therefore we combine the potential (2.5) with the restriction $u^s \mapsto u^s|_M$ to a linear bounded mapping $P : C(\partial D) \to C(M)$. Using the operator

$$N : C^1(B) \to C(\partial D), \qquad Nu^i(x) := \frac{\partial}{\partial \nu(x)} u^i(x), \qquad x \in \partial D$$

we can write the solution of the Neumann scattering problem in the form

$$u^s = 2\, P\Big(I - K^* - i\eta(K_0^{*2} - I + \tilde{T} S_0) S_0\Big)^{-1} N u^i. \tag{2.12}$$

The *inverse Neumann scattering problem* consists of looking for a domain D, which for a given number of incident fields u^i with corresponding scattered fields u^s satisfies (2.12). In order to use *Newton-type methods* to solve the inverse scattering problem we have to study the differentiability properties of the mapping defined by equation (2.12).

First we want to transform the operators onto a fixed reference boundary. Similar to [11], [4], [5], [6], [12], [13] or [15] we use the mapping $\phi_r : \partial D \to \partial D_r : x \mapsto x + r(x)$ where $r \in C^2(\partial D)$ is a two times continuously differentiable vector field and ∂D_r is defined by $\partial D_r := \{x + r(x), x \in \partial D\}$. For sufficiently small $\rho > 0$ depending on ∂D each ∂D_r with $\|r\|_{C^2(\partial D)} \le \rho$ is again a boundary of class C^2 of a domain D_r. We use $\mathcal{C}^2_\rho := \{r \in C^2(\partial D), \|r\| < \rho\}$. By $\nu(r, x)$ we denote the exterior unit normal vector to the boundary ∂D_r at the point $x_r := x + r(x)$. We use $\nu(x) := \nu(0, x)$.

By $B(X, Y)$ we denote the space of all bounded linear operators mapping a normed space X into a normed space Y. Now for each $r \in \mathcal{C}^2_\rho$ we transform functions $\varphi \in C(\partial D_r)$ into functions $\tilde{\varphi} \in C(\partial D)$ using $\tilde{\varphi}(x) := \varphi(x_r)$. Analogously we transform operators $I : C(\partial D_r) \to C(\partial D_r)$ into operators $\tilde{I} : C(\partial D) \to C(\partial D)$. Since in this way the space $C(\partial D)$ is isomorphic to $C(\partial D_r)$ and $B(C(\partial D), C(\partial D))$ is isomorphic to $B(C(\partial D_r), C(\partial D_r))$ usually we just write $\tilde{\varphi} = \varphi$ and $\tilde{I} = I$.

3. SOME REMARKS TO FRÉCHET DIFFERENTIABILITY OF INTEGRAL OPERATORS

For the well-known properties of the Fréchet derivative of a nonlinear mapping we refer to [1]; here we just give a summary of our notation.

Let Y be a normed space, X be a Banach space and let $U \subset Y$ be an open set. A mapping $A : U \to X$ is called *Fréchet differentiable in* $r_0 \in U$, if there is a bounded linear mapping $\partial A/\partial r \in B(Y, X)$, a neighborhood V of 0 in Y and a mapping $A_1 : V \to X$ for which holds

$$A(r_0 + h) = A(r_0) + \frac{\partial A}{\partial r}(h) + A_1(h), \qquad h \in V, \tag{3.1}$$

$$A_1(h) = o(\|h\|).$$

The mapping $\partial A/\partial r$ is called the *Fréchet derivative* of A in r_0. If A is Fréchet differentiable in U the Fréchet derivative can be considered as a mapping $U \to B(Y, X), r \to \partial A(r,)/\partial r$. If this mapping is again Fréchet differentiable, we speak of the second derivative of A. We have $\partial^2 A/\partial r^2 \in B(Y, B(Y, X))$ and we use $\partial^2 A(r, h)/\partial r^2 := \partial^2 A(r, h, h)/\partial r^2$. The *chain rule* and the *product rule* are valid analogously to the finite dimensional case. As a consequence of *Taylor's theorem* for twice continuously Fréchet differentiable functions we obtain:

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Theorem 3.1. *Let Y be a normed space, let X be a Banach space and let $U \subset Y$ be an open set. Assume that $f : U \to X$ is a twice continuously differentiable function on U and let the second derivative be bounded, i.e. there exists $c > 0$ such that $\|\partial^2 f(r;\cdot)/\partial r^2\| \le c$ on U. If $r + th \in U$ for all $t \in [0,1]$ we have the equality*

$$f(r+h) = f(r) + \frac{\partial f}{\partial r}(r,h) + f_1(r,h) \tag{3.2}$$

with some function f_1 satisfying

$$\|f_1(r,h)\| \le \sup_{r\in U} \left\| \frac{\partial^2 f}{\partial r^2}(r,\cdot) \right\| \|h\|^2 . \tag{3.3}$$

Proof. An application of Taylor's theorem [1] yields

$$f(r+h) = f(r) + \frac{\partial f}{\partial r}(r,h) + \int_0^1 (1-t)\frac{\partial^2 f}{\partial r^2}(r+th,h)\,\mathrm{d}t. \tag{3.4}$$

Since we have $\|\partial^2 f(\cdot)/\partial r^2\| \le c$ on U the statement of the theorem is a direct consequence of the inequality

$$\left\| \int_0^1 (1-t)\frac{\partial^2 f}{\partial r^2}(r+th;h)\,\mathrm{d}t \right\| \le \sup_{r\in U} \left\| \frac{\partial^2 f}{\partial r^2}(r;\cdot) \right\| \|h\|^2 . \tag{3.5}$$

□

In order to show the Fréchet differentiability of $\left(I + K^* - \mathrm{i}\eta(K_0^{*2} - I + \tilde{T}S_0)S_0\right)^{-1}$ we need the following theorem.

Theorem 3.2. *Let Y be a normed space, $U \subset Y$ an open set and X a Banach algebra with neutral element e. Let $A : U \to X$ be Fréchet differentiable in $y_0 \in U$. Assume there is a neighborhood W of y_0 such that for all $y \in W$ the element $A(y)$ is invertible in X and the mapping $y \mapsto (A(y))^{-1}$ is continuous in y_0. Then $A^{-1}(y)$ is Fréchet differentiable in y_0 with Fréchet derivative*

$$(A^{-1})'(y_0,h) = -A^{-1}(y_0)\left(\frac{\partial A}{\partial r}(y_0,h)\right)A^{-1}(y_0) \tag{3.6}$$

Proof. We follow [3]. Define

$$z(y_0,h) := A^{-1}(y_0+h) - A^{-1}(y_0) + A^{-1}(y_0)\frac{\partial A}{\partial r}(y_0,h)A^{-1}(y_0).$$

We have to show $z(y_0,h) = o(\|h\|)$. For this we multiply from the left and from the right by $A(y_0)$ and use the continuous invertibility and the Fréchet differentiability of A. We obtain $A(y_0)z(y_0,h)A(y_0) = o(\|h\|)$ and therefore the statement of the theorem. □

We want to show the Fréchet differentiability of integral operators of the form

$$(A(r)\varphi)(x) := \int_{\partial D} f(r,x,y)\varphi(y)\,\mathrm{d}s(y), \qquad x \in G, \quad r \in V. \tag{3.7}$$

Here $D \subset \mathbb{R}^3$ is a bounded domain with boundary of class C^2, G is an arbitrary subset of $\mathbb{R}^3$ and $V \subset Y$ is a subset of a normed space Y. We want to treat integral operators

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which have for fixed r weakly singular kernels $f(x,y,r)$. Then for fixed $r \in V$ the operator A is a bounded linear operator $C(\partial D) \to C(G)$. We consider A as a mapping

$$V \to B\big(C(\partial D), C(G)\big).$$

In the next theorem we will show that, for suitable properties of the kernel f, the differentiation of (3.7) can be reduced to the differentiation of the kernel f and that the derivative of A is given by the operator

$$\big(\tilde{A}(r,h)\varphi\big)(x) := \int_{\partial D} \frac{\partial f}{\partial r}(r,h,x,y)\varphi(y)\,\mathrm{d}s(y), \qquad x \in G, \quad r \in V, \quad h \in Y. \tag{3.8}$$

This includes the classical theorem concerning the differentiation of an integral depending on a parameter. We use $\Delta_G := \big\{(x,y),\ x = y,\ x \in G,\ y \in \partial D\big\}$.

Theorem 3.3. *Consider a bounded domain $D \subset \mathbb{R}^3$ with boundary of class C^2, an arbitrary subset G of $\mathbb{R}^3$ and an open convex subset $V \subset Y$ of a Banach space Y. Let $f : V \times ((G \times \partial D) \setminus \Delta_G) \to \mathbb{C}$ be a continuous function with the following properties:*

- *for all fixed $x \in G, y \in \partial D, x \neq y$ the function $f(\cdot, x, y) : V \to \mathbb{C}$ is two times continuously Fréchet differentiable,*
- *there is a weakly singular function $g : (G \times \partial D) \setminus \Delta_G \to \mathbb{R}$ such that for $j = 0,1,2$ and for all $x \in G, y \in \partial D, x \neq y$ we have the estimate*

$$\left|\left(\frac{\partial^j f}{\partial r^j}\right)(r,h,x,y)\right| \le g(x,y)$$

uniformly for all $r \in V, h \in Y$ with $\|h\| \le 1$.

Then, considered as a mapping $V \to B\big(C(\partial D), C(G)\big)$, $r \mapsto A(r)$ the operator A is Fréchet differentiable and the derivative of A is given by $(\partial A/\partial r)(r,h) = \tilde{A}(r,h)$, where $\tilde{A}$ is given by (3.8).

Remark. The theorem covers the case $G = \partial D$ and weakly singular f as well as $G \cap \partial D = \emptyset$ and continuous f . Therefore it can be applied to the operators S, K^* and P. The theorem in this form can be used to prove the existence of higher derivatives by induction (see [13]). □

Proof. We consider a point r_0 in V. For all sufficiently small h we have $r_0 + h \in V$ and the convexity of V yields $r_0 + th \in V$ for all $t \in [0,1]$. Then, as in Theorem 3.1, there holds the decomposition

$$f(r_0 + h, x, y) = f(r_0, x, y) + \frac{\partial f}{\partial r}(r_0, h, x, y) + f_1(r_0, h, x, y) \tag{3.9}$$

and we have

$$|f_1(r_0,h,x,y)| \le \sup_{r \in V} \left\|\frac{\partial^2 f}{\partial r^2}(r,\cdot,x,y)\right\| \|h\|^2, \qquad h \in Y, \quad (x,y) \in (G \times \partial D) \setminus \Delta_G.$$

Because of

$$\left|\frac{\partial^2 f}{\partial r^2}(r,h,x,y)\right| \le g(x,y), \qquad r \in V, \quad \|h\| \le 1$$

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we find

$$\left\| \frac{\partial^2 f}{\partial r^2}(r,\cdot,x,y) \right\| \leq g(x,y), \qquad r \in V.$$

Therefore we obtain integrability of f_1 and the inequality

$$\begin{aligned}\int_{\partial D} |f_1(r_0,h,x,y)|\, \mathrm{d}s(y) &\leq \int_{\partial D} \sup_{r\in U} \left\| \frac{\partial^2 f}{\partial r^2}(r,\cdot,x,y) \right\| \|h\|^2\, \mathrm{d}s(y) \\ &\leq \left(\int_{\partial D} g(x,y)\, \mathrm{d}s(y) \right) \|h\|^2 .\end{aligned}$$

We now know that all terms in equation (3.9) are integrable on ∂D and can use the linearity of the integral to obtain

$$\begin{aligned}(A(r_0+h)\varphi)(x) &= \int_{\partial D} f(r_0+h,x,y)\varphi(y)\, \mathrm{d}s(y) \\ &= \int_{\partial D} f(r_0,x,y)\varphi(y)\, \mathrm{d}s(y) + \int_{\partial D} \frac{\partial f}{\partial r}(r_0,h,x,y)\varphi(y)\, \mathrm{d}s(y) \\ &\quad + \int_{\partial D} f_1(r_0,h,x,y)\varphi(y)\, \mathrm{d}s(y) \\ &= (A(r_0)\varphi)(x) + (\tilde{A}(r_0,h)\varphi)(x) + (A_1(r_0,h)\varphi)(x)\end{aligned}$$

where the operator A_1 satisfies

$$|(A_1(r_0,h)\varphi)(x)| \leq c\, \|\varphi\|_\infty\, \|h\|^2$$

with some constant c. Therefore A is Fréchet differentiable in r_0 considered as a mapping $V \to B\big(\mathrm{C}(\partial D), \mathrm{C}(G)\big)$ with the derivative given by $\partial A/\partial r = \tilde{A}$. $\square$

4. FRÉCHET DIFFERENTIABILITY OF SPECIAL OPERATORS

As an application of Theorem 3.3 we want to show the Fréchet differentiability of the operators occuring in Section 2. First we deal with S and K^*. Using the transformations described in Section 2 the operators can be brought into the form

$$(S(r)\varphi)(x) = \int_{\partial D} \frac{h_1(|x_r - y_r|)}{|x_r - y_r|} J_r(y)\varphi(y)\, \mathrm{d}s(y) \tag{4.1}$$

$$\begin{aligned}(K^*(r)\varphi)(x) &= \int_{\partial D} \langle \nu(r,x), y_r - x_r \rangle \left\{ \frac{h_2(|x_r-y_r|)}{|x_r-y_r|^3} + \frac{h_3(|x_r-y_r|)}{|x_r-y_r|^2} \right\} \\ &\qquad \times J_T(r,y)\varphi(y)\, \mathrm{d}s(y).\end{aligned} \tag{4.2}$$

where the functions h_1, h_2 and h_3 are analytic complex valued functions, and where $J_T(r,y)$ denotes the Jacobian of the transformation ϕ_r in $y \in \partial D$. The operator $\tilde{T}$ can be handled in the same manner. For a suitable decomposition of its kernel we refer to [2], equation (2.57).

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Theorem 4.4. *The integral operators S, K^* and $\tilde{T} = T - T_0$ are Fréchet differentiable in $\mathcal{C}^2_\rho$, considered as mappings*

$$\mathcal{C}^2_\rho \to B\Big(\mathrm{C}(\partial D), \mathrm{C}(\partial D)\Big).$$

The Fréchet derivative is obtained by differentiation of the kernels according to Theorem 3.3.

We base the proof of the theorem on the following lemma.

Lemma 4.1. *The kernels of the integral operators given by (4.1), (4.2) and the kernel of $\tilde{T} = T - T_0$ ([2], equation (2.57)) are for all fixed $x, y \in \partial D$ with $x \neq y$ two times continuously Fréchet differentiable as mappings $\mathcal{C}^2_\rho \to \mathbb{C}$. The kernels and their first two derivatives are bounded by*

$$g(x,y) = \frac{C}{|x-y|} \qquad x,\ y \in \partial D \tag{4.3}$$

with some constant $C > 0$ uniformly for $(r, h) \in \mathcal{C}^2_\rho \times \mathcal{C}^2_1$.

Proof of Theorem 4.4. We establish the assumptions made in Theorem 3.3. Lemma 4.1 states the Fréchet differentiability of the kernels of S and K^* and also gives estimates for their singularity and those of their derivatives: there is a weakly singular majorante g and therefore they are weakly singular. Now by standard arguments S and K^* and the operators which are built by integration of the derivatives of the kernels are well defined bounded linear operators $\mathrm{C}(\partial D) \to \mathrm{C}(\partial D)$. Thus we apply Theorem 3.3 to obtain Theorem 4.4. □

Proof of Lemma 4.1. We verify the Fréchet differentiability of the kernels by four elementary steps. We will use the letter c to denote a generic constant.

I. The mapping $g_{x,y} : \mathcal{C}^2_\rho \to \mathbb{R}^3$ defined by

$$g_{x,y}(r) := x_r - y_r = (x + r(x)) - (y + r(y))$$

is the sum of a constant and a linear mapping and therefore, for all fixed $x, y \in \partial D$, it is Fréchet differentiable with derivative

$$\frac{\partial g_{x,y}}{\partial r}(r; h) = h(x) - h(y), \qquad h \in \mathrm{C}^2(\partial D).$$

The derivative does not depend on $r \in \mathcal{C}^2_\rho$ and therefore it is continuous. Since for $x \neq y$ we have $x_r - y_r \neq 0$ for all $r \in \mathcal{C}^2_\rho$, using the chain rule, we obtain the Fréchet differentiability of the mapping

$$g_{1,x,y} : \mathcal{C}^2_\rho \to \mathbb{R}, \qquad r \mapsto |x_r - y_r|$$

for all $r \in \mathcal{C}^2_\rho, x \neq y,\ x, y \in \partial D$. The Fréchet derivative is given by

$$\frac{\partial g_{1,x,y}}{\partial r}(r; h) = \frac{1}{|x_r - y_r|} \langle x_r - y_r, h(x) - h(y) \rangle, \qquad h \in \mathrm{C}^2(\partial D). \tag{4.4}$$

We use the mean value theorem for the differentiable vector fields $r \in \mathcal{C}^2_\rho$ on the manifold ∂D to obtain the estimates

$$\gamma_1 |x - y| \leq |x_r - y_r| \tag{4.5}$$

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$$|x_r - y_r| \leq \gamma_2 \, |x - y| \tag{4.6}$$

uniformly on $\mathcal{C}^2_\rho$, where γ_1 and γ_2 are constants depending on ρ and ∂D. Again with the help of the mean value theorem – this time applied to h – we derive from (4.5) and (4.6) the inequalities

$$\left|\frac{\partial g_{1,x,y}}{\partial r}(r;h)\right| \leq c \, \|h\|_{\mathrm{C}^2(\partial D)} \, |x-y| \qquad \forall r \in \mathcal{C}^2_\rho, \quad h \in \mathrm{C}^2(\partial D) \tag{4.7}$$

with some constant c. Proceeding as for $g_{1,x,y}$ we obtain the Fréchet differentiability of the mapping

$$g_{2,x,y} : \mathcal{C}^2_\rho \to \mathbb{R}, \qquad r \mapsto \frac{1}{|x_r - y_r|^n} \tag{4.8}$$

the derivative

$$\frac{\partial g_{2,x,y}}{\partial r}(r;h) = (-n) \frac{1}{|x_r - y_r|^{n+2}} \langle x_r - y_r, h(x) - h(y) \rangle \tag{4.9}$$

and the estimate

$$\left|\frac{\partial g_{2,x,y}}{\partial r}(r;h)\right| \leq c \frac{1}{|x_r - y_r|^n} \, \|h\|_{\mathrm{C}^2(\partial D)} \tag{4.10}$$

with some constant c. We also want to compute the second derivatives of the terms and give similar estimates. To do so we have to consider the first derivatives as mappings $\mathcal{C}^2_\rho \to B(\mathrm{C}^2(\partial D), \mathbb{R})$. Using the same arguments as above we obtain

$$\begin{aligned} \frac{\partial^2 g_{1,x,y}}{\partial r^2}(r;h) &= \frac{(-1)}{|x_r - y_r|^3} \langle x_r - y_r, h(x) - h(y) \rangle^2 \\ &\quad + \frac{1}{|x_r - y_r|} \langle h(x) - h(y), h(x) - h(y) \rangle \end{aligned} \tag{4.11}$$

$$\begin{aligned} \frac{\partial^2 g_{2,x,y}}{\partial r}(r;h) &= n(n+1) \frac{1}{|x_r - y_r|^{n+2}} \left(\frac{\partial g_{1,x,y}}{\partial r}(r;h)\right)^2 \\ &\quad - n \frac{1}{|x_r - y_r|^{n+1}} \; \frac{\partial^2 g_{1,x,y}}{\partial r^2}(r;h) \end{aligned} \tag{4.12}$$

and the estimates

$$\left|\frac{\partial^2 g_{1,x,y}}{\partial r^2}(r;h)\right| \leq c \, \|h\|^2_{\mathrm{C}^2(\partial D)} \, |x-y| \,, \qquad r \in \mathcal{C}^2_\rho, \quad h \in \mathrm{C}^2(\partial D) \tag{4.13}$$

and

$$\left|\frac{\partial^2 g_{2,x,y}}{\partial r^2}(r;h)\right| \leq c \, \|h\|^2_{\mathrm{C}^2(\partial D)} \, \frac{1}{|x_r - y_r|^n}, \qquad r \in \mathcal{C}^2_\rho, \quad h \in \mathrm{C}^2(\partial D). \tag{4.14}$$

The estimates show that the degree of the singularity in $|x-y|$ of the functions under consideration does not increase when we differentiate. We also want to prove this for the other components of the kernels.

II. Consider the term $\langle \nu(r,x), x_r - y_r \rangle$ and use local coordinates (u,v). With $x = x(u_1, v_1)$ and $y = y(u_2, v_2)$ we have the estimate $\tilde{\gamma}_1 \, |(u_1, v_1) - (u_2, v_2)| \leq |x - y| \leq$

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$\tilde{\gamma}_2 |(u_1, v_1) - (u_2, v_2)|$ for $x \in U(y)$, where $U(y)$ is a neighborhood of y and $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ are constants [2]. In $U(y)$ we can write

$$\langle \nu(r,x), x_r - y_r \rangle = \frac{1}{g_{3,y}(r)} \left\langle \left(\frac{\partial y}{\partial u_2} + \frac{\partial r(y)}{\partial u_2} \right) \times \left(\frac{\partial y}{\partial v_2} + \frac{\partial r(y)}{\partial v_2} \right), x + r(x) - y - r(y) \right\rangle \tag{4.15}$$

with

$$g_{3,y}(r) := \left| \left(\frac{\partial y}{\partial u_2} + \frac{\partial r(y)}{\partial u_2} \right) \times \left(\frac{\partial y}{\partial v_2} + \frac{\partial r(y)}{\partial v_2} \right) \right|.$$

The function $g_{3,y}$ is Fréchet differentiable in $\mathcal{C}^2_\rho$ and there exist constants c_1 and c_2 with $0 < c_1 \leq g_{3,y} \leq c_2$ and $0 < c_1 \leq \partial g_{3,y}/\partial r \leq c_2$ $\forall r \in \mathcal{C}^2_\rho$. Therefore $1/g_{3,y}$ is also Fréchet differentiable in $\mathcal{C}^2_\rho$ and the derivative is bounded. Using the chain rule, clearly the other terms of (4.15) are Fréchet differentiable. For the derivative

$$f(u_1, v_1) := \frac{\partial}{\partial r} \left(\left\langle \left(\frac{\partial y}{\partial u_2} + \frac{\partial r(y)}{\partial u_2} \right) \times \left(\frac{\partial y}{\partial v_2} + \frac{\partial r(y)}{\partial v_2} \right), x + r(x) - (y - r(y)) \right\rangle \right) (r, h) \tag{4.16}$$

we want to show that

$$|f(u_1, v_1)| \leq L \, |(u_1, v_1) - (u_2, v_2)|^2 \tag{4.17}$$

uniformly for $r \in \mathcal{C}^2_\rho$ and $h \in K_1 \subset \mathrm{C}^2(\partial D)$. The estimate (4.17) is a direct consequence of Taylor's theorem applied to the twice continuously differentiable function $f : \mathbb{R} \to \mathbb{R}$, if we are able to show that $\operatorname{grad}_{(u_1,v_1)} f|_{u_1=u_2, v_1=v_2} = 0$. This can be verified by a straightforward but lengthy calculation. Now collecting all terms and using the product rule for the differentiation of (4.15) we obtain the estimate

$$\left| \frac{\partial}{\partial r} \{ \nu_r(x) \cdot (x_r - y_r) \} (r; h) \right| \leq c \, \|h\|_{\mathrm{C}^2(\partial D)} \, |x - y|^2 \tag{4.18}$$

for all $r \in \mathcal{C}^2_\rho$. For the second derivative we obtain the analogous result

$$\left| \frac{\partial^2}{\partial r^2} \{ \nu_r(x) \cdot (x_r - y_r) \} (r; h) \right| \leq c \, \|h\|^2_{\mathrm{C}^2(\partial D)} \, |x - y|^2 \tag{4.19}$$

for all $r \in \mathcal{C}^2_\rho$.

III. We obtain the differentiability of $J_T(r, y)$ using the representation

$$J_T(r, y) = \left| \frac{\partial}{\partial u_1}(y + r(y)) \times \frac{\partial}{\partial u_2}(y + r(y)) \right| \left| \frac{\partial}{\partial u_2} y \times \frac{\partial}{\partial v_2} y \right|^{-1}$$

which is valid in local coordinates $y = y(u_1, u_2)$. The derivatives of J_T are uniformly bounded for $r \in \mathcal{C}^2_\rho, y \in \partial D$.

IV. The statement of Lemma 4.1 can now be verified using the estimates of I., II. and III., the chain and product rule. □

Corollary 4.1. *The operator $(I - K^* - i\eta(K_0^{*2} - I + \tilde{T} S_0) S_0)^{-1}$ is Fréchet differentiable considered as a mapping $\mathcal{C}^2_\rho \to B\big(\mathrm{C}(\partial D), \mathrm{C}(\partial D)\big)$ and the Fréchet derivative is given by*

$$\frac{\partial \big((I - K^* - i\eta(K_0^{*2} - I + \tilde{T} S_0) S_0)^{-1} \big)}{\partial r} = \Big(I - K^* - i\eta(K_0^{*2} - I + \tilde{T} S_0) S_0 \Big)^{-1} \tag{4.20}$$

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$$\times \frac{\partial\Big(K^* + \mathrm{i}\eta(K_0^{*2} - I + \tilde{T}S_0)S_0\Big)}{\partial r}\Big(I - K^* - \mathrm{i}\eta(K_0^{*2} - I + \tilde{T}S_0)S_0\Big)^{-1}.$$

Proof. The statement follows by combining Theorems 3.2 and 4.4. □

We transform the operator P onto the reference surface ∂D:

$$\begin{aligned}(P(r)\varphi)(x) \;=\; & \int_{\partial D}\left\{\frac{h_1(|x-y_r|)}{|x-y_r|}\varphi(y) - \mathrm{i}\eta\;\langle\nu(r,y), x-y_r\rangle\right. \qquad (4.21)\\ & \times\left.\left[\frac{h_2(|x-y_r|)}{|x-y_r|^3} + \frac{h_3(|x-y_r|)}{|x-y_r|^2}\right] J_T(r,y)(S_0^2(r)\varphi)(y)\right\}\mathrm{d}s(y), \quad x\in M\end{aligned}$$

and establish the following result.

Theorem 4.5. *The integral operator $P : C_\rho^2 \to B\Big(\mathrm{C}(\partial D), \mathrm{C}(M)\Big)$ is Fréchet differentiable and the derivative can be computed by differentiation of the kernel of P.*

Analogously to the proof of Theorem 4.4 we base the proof on the following lemma which can be shown analogously to Lemma 4.1. It is actually more simple since the kernels have no singularities.

Lemma 4.2. *The kernel of the operator P given by* (4.21) *is two times continuously Fréchet differentiable as a mapping $C_\rho^2 \to \mathbb{C}$ for fixed $x \in M, y \in \partial D$. The derivatives are continuous on $C_\rho^2 \times M \times \partial D$ and bounded by a constant $C \in \mathbb{R}$.*

Proof of Theorem 4.5. We verify the assumptions of Theorem 3.3. The differentiability of the kernels and their continuity is stated in Lemma 4.2. Therefore P and the operators which are built by integration of the derivatives of the kernel are well defined bounded linear operators $\mathrm{C}(\partial D) \to \mathrm{C}(M)$. Now Theorem 3.3 can be applied to obtain the statement of Theorem 4.5.

Now consider the operator N. We can write

$$(N(r)u^i)(x) = \Big\langle \nu(r,x), \mathrm{grad}\; u^i(x_r)\Big\rangle, \qquad x \in \partial D.$$

Theorem 4.6. *The operator $N : C_\rho^2 \to B\Big(\mathrm{C}^2(B), \mathrm{C}(\partial D)\Big)$ is Fréchet differentiable with derivative*

$$\Big(\frac{\partial N}{\partial r}(r,h)u^i\Big)(x) = \left\langle \frac{\partial \nu}{\partial r}(r,h), (\mathrm{grad}\;_x u^i)(x_r)\right\rangle + \sum_{k,j} h_k(x)\frac{\partial^2 u^i}{\partial x_k \partial x_j}(x)\nu_j(x), \quad x \in \partial D.$$

Proof. The proof is a simple application of the chain and the product rule. □

We now obtain the central statement as a corollary.

Corollary 4.2. *The nonlinear mapping $R : C_\rho^2 \to \mathrm{C}(M), r \mapsto u^s|_M$ is Fréchet differentiable and the derivative is given by*

$$\frac{\partial R}{\partial r} \;=\; 2\frac{\partial P}{\partial r}(I - K^* - \mathrm{i}\eta(K_0^{*2} - I + \tilde{T}S_0)S_0)^{-1} N u^i$$

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$$+2P(I-K^*-\mathrm{i}\eta(K_0^{*2}-I+\tilde{T}S_0)S_0)^{-1}\frac{\partial(K^*+\mathrm{i}\eta(K_0^{*2}-I+\tilde{T}S_0)S_0)}{\partial r}$$
$$\times(I-K^*-\mathrm{i}\eta(K_0^{*2}-I+\tilde{T}S_0)S_0)^{-1}Nu^i$$
$$+2P(I-K^*-\mathrm{i}\eta(K_0^{*2}-I+\tilde{T}S_0)S_0)^{-1}\frac{\partial N}{\partial r}u^i. \tag{4.22}$$

5. CHARACTERIZATION OF THE DERIVATIVES

The actual numerical evaluation of $\partial R/\partial r$ using corollary 4.2 is rather lengthy. Therefore we characterize the derivative of R as the solution of a Neumann boundary value problem analogous to the characterization in the Dirichlet case ([4], [5], [12], [13]). In the case of the Neumann boundary condition we need more regularity of the boundary ∂D of the domain D than in the Dirichlet case, since we have to compute the second spatial derivatives of the scattered fields on the boundary.

Theorem 5.7. *Let D be a bounded domain with boundary of class $C^{2,\alpha}$. The Fréchet derivative $\partial R/\partial r(0,h)$ of the operator R at the point $r\equiv 0$ is given by the solution to the exterior Neumann problem for the domain D with boundary values*

$$-\left\langle \frac{\partial\nu(r,x)}{\partial r}(0,h),(\operatorname{grad}\ u)(x)\right\rangle-\sum_{k,j}h_k(x)\frac{\partial^2u(x)}{\partial x_k\partial x_j}\nu_j(x),\qquad x\in\partial D \tag{5.1}$$

where u denotes the solution to the original scattering problem.

Proof. We show that $\partial R(0,h)/\partial r$ given by Corollary 4.2 is a solution to the exterior Neumann problem with boundary values given by (5.1). $\partial R(0,h)/\partial r$ solves the Helmholtz equation in $\mathbb{R}^3\setminus\overline{D_r}$ and satisfies the Sommerfeld radiation condition which can be seen from (4.22). We have to compute the normal derivative on the boundary.

We want to apply the techniques which are used in [2] to compute the boundary values of the single and double layer potential. There the singularity of the kernel of the integrals is reduced by adding a suitable term, such that the sum of the original and the added term is continuous on the boundary. Since we have to substract the added term it should be chosen in such a way that its boundary values exist. In the case of the Potential $\partial P(0,h)/\partial r\varphi$ one possible term is given by the Fréchet derivative of $(P(r)\varphi)(r,h,x_q^\tau)$ with respect to the function $q\in C^2(\partial D,\mathbb{R}^3)$ in the point $q\equiv 0$, where x_q^τ is given by

$$x_q^\tau := x+q(x)+\nu(q,x)\cdot\tau. \tag{5.2}$$

We write x^τ for x_0^τ. Differentiating instead of $P(r)$ the whole expression

$$(P(r)\varphi)(r,x_r^\tau) \tag{5.3}$$

with respect to r using the chain rule exactly the right term is added. These observations are used in equation (5.6). In addition we will observe: the limit

$$\lim_{\tau\to 0}\frac{\partial}{\partial\tau}\frac{\partial}{\partial r}\left\{(P(r)\varphi)(r,x_r^\tau)\right\}$$

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of the Fréchet derivative of (5.3) vor $\tau \to 0$ is given by

$$\frac{1}{2}\frac{\partial\Big(K^* + i\eta(K^{*2} - I + \tilde{T}S_0)S_0\Big)}{\partial r}\varphi.$$

Choosing the density $\varphi = 2\Big(I - K^* - i\eta(K^{*2} - I + \tilde{T}S_0)S_0\Big)^{-1} Nu^i$ these are exactly the boundary values of the second term of (4.22). Since the signs differ this terms compensate each other and in the following proof the boundary values of the first two terms are given by the boundary values of the Fréchet derivative of $(P(r)\varphi)(r, h, x_q^\tau)$ with respect to q.

We consider a small strip of parallel surfaces to the boundary ∂D of the domain D. For $\tau < \tau_0$, τ_0 sufficiently small, the strip $D^{\tau_0} := \Big\{x \in \mathbb{R}^3, \min_{y\in\partial D} |x - y| < \tau_0\Big\}$ is bijectively mapped onto the set $\{(x, \tau), x \in \partial D, -\tau_0 < \tau < \tau_0\}$. For brevity we again use equation (2.10) and write TS_0^2 instead of $(K^{*2} - I + \tilde{T}S_0)S_0$.

I. We compute the normal derivative of $P(I - K^* - i\eta TS_0^2)^{-1}\partial N/\partial r u^i$, i.e. the last term of (4.22). The jump relations yield

$$2NP = -(I - K^* - i\eta TS_0^2). \tag{5.4}$$

Therefore we obtain

$$\begin{aligned}
&\left(2NP(I - K^* - i\eta TS_0^2)^{-1}\frac{\partial N}{\partial r}(0, h)u^i\right)(x)\\
&\quad = \quad -\left(\frac{\partial N}{\partial r}(0, h)u^i\right)(x)\\
&\quad = \quad -\left\langle\frac{\partial \nu}{\partial r}(0, h, x), (\text{grad } u^i)(x)\right\rangle - \sum_{k,j} h_k(x)\frac{\partial^2 u^i(x)}{\partial x_k \partial x_j}\nu_j(x)
\end{aligned}$$

II. We want to show that for the normal derivative of the first two terms in (4.22) at the point $r \equiv 0$ there holds

$$\begin{aligned}
&2\left(N\frac{\partial P}{\partial r}(0, h)(I - K^* - i\eta TS_0^2)^{-1}Nu^i\right)(x)\\
&+2\left(NP(I - K^* - i\eta TS_0^2)^{-1}\left(\frac{\partial(K^* + i\eta TS_0^2)}{\partial r}\right)(0, h)\right.\\
&\qquad \left.\times\,(I - K^* - i\eta TS_0^2)^{-1}Nu^i\right)(x)\\
= \quad &-\left\langle\frac{\partial \nu}{\partial r}(0, h, x), (\text{grad } \ u^s)(x)\right\rangle - \sum_{k,j} h_k(x)\frac{\partial^2 u^s(x_r)}{\partial x_k \partial x_j}\nu_j(x), \qquad x \in \partial D.
\end{aligned} \tag{5.5}$$

For a function $w \in C^1(\mathbb{R}^3 \setminus D)$ we have

$$(Nw)(x) = \lim_{\tau\to 0}\{\langle \nu(x), (\text{grad } w)(x^\tau)\rangle\} = \lim_{\tau\to 0}\frac{\partial w(x^\tau)}{\partial \tau}.$$

We are interested in the function $w = \partial P(0, h)/\partial r(I - K^* i\eta TS_0^2)^{-1}Nu^i$. Define $\varphi := (I - K^* - i\eta TS_0^2)^{-1}Nu^i$ where the operators are considered at the point $r \equiv 0$. Thus they

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are independent of r. Using the chain rule as mentioned at the beginning of the proof we derive

$$2\frac{\partial}{\partial r}\big\{P\varphi\big\}(r,h)(x_r^\tau) = 2\frac{\partial}{\partial r}\big\{(P\varphi)(x_r^\tau)\big\}(0,h) - \left\langle \text{grad } (2P\varphi)(x^\tau), \frac{\partial x_r^\tau}{\partial r}(0,h)\right\rangle. \tag{5.6}$$

Since we have a boundary ∂D of class $C^{2,\alpha}$ we have $u^s \in C^{2,\alpha}(\mathbb{R}^3 \setminus D)$ for the solution to the scattering problem (see [13]). Now for the last term of (5.6) we obtain

$$\begin{aligned}
&\frac{\partial}{\partial\tau}\left\langle \text{grad } (2P\varphi)(x^\tau), \frac{\partial x_r^\tau}{\partial r}(0,h)\right\rangle \\
&= \frac{\partial}{\partial\tau}\left\langle \text{grad}\,(2P\varphi)(x^\tau), h(x) + \frac{\partial\nu}{\partial r}(0,h,x)\tau\right\rangle \\
&= \left\langle \frac{\partial}{\partial\tau}\big\{\text{grad}\,(2P\varphi)(x^\tau)\big\}, h(x) + \frac{\partial\nu}{\partial r}(0,h,x)\tau\right\rangle \\
&\quad + \left\langle \text{grad}\,(2P\varphi)(x^\tau), \frac{\partial\nu}{\partial r}(0,h,x)\right\rangle \\
&\to \sum_{k,j} h_k(x)\frac{\partial^2 u^s(x)}{\partial x_k \partial x_j}\nu_j(x) + \left\langle (\text{grad } u^s)(x), \frac{\partial\nu}{\partial r}(0,h,x)\right\rangle
\end{aligned} \tag{5.7}$$

for $\tau \to 0$, where we have used $u^s = 2P\varphi$. Because of (5.6) and (5.7) in order to show (5.5) it remains to verify

$$\lim_{\tau\to 0}\left\{2\frac{\partial}{\partial\tau}\frac{\partial}{\partial r}\big\{(P\varphi)(x_r^\tau)\big\}(r,h) - \left(\frac{\partial(K^* + i\eta T S_0^2)}{\partial r}(0,h)\varphi\right)(x)\right\} = 0. \tag{5.8}$$

For the sake of simplicity we will write this down only for the potential theoretic case $k = 0$. The case $k \neq 0$ can be handled in the same manner. We split the potential P into two parts: the single layer potential P_1 with density φ and $i\eta$ times the double layer potential P_2 with density $S_0^2\varphi$.

III. First we show

$$\lim_{\tau\to 0}\left\{\frac{\partial}{\partial\tau}\frac{\partial}{\partial r}\big\{2(P_1(r)\varphi)(x_r^\tau)\big\}(0,h) - \left(\frac{\partial K^*}{\partial r}(0,h)\varphi\right)(x)\right\} = 0. \tag{5.9}$$

We derive

$$\begin{aligned}
&\frac{\partial}{\partial\tau}\frac{\partial}{\partial r}\big\{2(P_1(r)\varphi)(x_r^\tau)\big\}(0,h) \\
&= \frac{\partial}{\partial r}\frac{\partial}{\partial\tau}\left\{\int_{\partial D} 2\Phi(x_r^\tau, y_r)J_T(r,y)\varphi(y)\,\mathrm{d}s(y)\right\}(0,h) \\
&= 2\frac{\partial}{\partial r}\left\langle \nu(r,x), \int_{\partial D}(\text{grad }_x\Phi)(x_r^\tau, y_r)\varphi(y)J_T(r,y)\,\mathrm{d}s(y)\right\rangle(0,h)
\end{aligned} \tag{5.10}$$

and

$$\left(\frac{\partial K^*}{\partial r}(0,h)\varphi\right)(x) = 2\frac{\partial}{\partial r}\left\{\int_{\partial D}\langle \nu(r,x), (\text{grad }_x\Phi)(x_r, y_r)\rangle\, \varphi(y)J_T(r,y)\,\mathrm{d}s(y)\right\}(0,h) \tag{5.11}$$

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$$x \in \partial D.$$

In order to show (5.9) we have to verify the continuity of (5.10) at $\tau = 0$. We use the decomposition

$$\begin{aligned}
&\int_{\partial D} \langle \nu(r,x), (\operatorname{grad}_x \Phi)(x,y_r)\rangle \, \varphi(y) J_T(r,y)\, ds(y) \\
&\quad = \varphi(x) \int_{\partial D} \langle \nu(r,x), (\operatorname{grad}_x \Phi)(x,y_r)\rangle \, J_T(r,y)\, ds(y) \\
&\qquad + \int_{\partial D} \langle \nu(r,x), (\operatorname{grad}_x \Phi)(x,y_r)\rangle \, [\varphi(y) - \varphi(x)] J_T(r,y)\, ds(y). \qquad (5.12)
\end{aligned}$$

We define

$$u_r(x) := \int_{\partial D} \langle \nu(r,x), (\operatorname{grad}_x \Phi)(x,y_r)\rangle \, [\varphi(y) - \varphi(x)] J_T(r,y)\, ds(y), \qquad x \in \mathbb{R}^3.$$

The continuity of $\partial u_r(x_r^\tau)(0,h)/\partial r$ for $\tau \to 0$ can be shown analogously to Lemma 3 of [12]. The proof is literally the same. For the first term of the right-hand side of (5.12) we use ([2], page 52)

$$\int_{\partial D} \langle \nu(r,x), (\operatorname{grad}_x \Phi)(x,y_r)\rangle \, J_T(r,y)\, ds(y) = \langle \nu(r,x), U_r(x)\rangle + \langle \nu(r,x), V_r(x)\rangle\,, \quad x \in \mathbb{R}^3$$

where

$$\begin{aligned}
U_r(x) : &= -\int_{\partial D} (\operatorname{Grad}_y \Phi)(x,y_r) J_T(r,y)\, ds(y) \\
&= 2\int_{\partial D} H(r,y)\nu_r(y)\Phi(x,y_r) J_T(r,y)\, ds(y), \qquad x \in \mathbb{R}^3
\end{aligned}$$

and

$$V_r(x) = -\int_{\partial D} \nu(r,y) \frac{\partial \Phi(x,y_r)}{\partial \nu(r,y)} J_T(r,y)\, ds(y), \qquad x \in \mathbb{R}^3. \qquad (5.13)$$

By $H(r,y)$ we denote the mean curvature of the surface ∂D_r which is defined in [2]. The continuity of $\frac{\partial}{\partial r}\{\langle \nu(r,x), U_r(x_r^\tau)\rangle\}(r,h)$ for $\tau \to 0$ can be proved analogously to the continuity of the Fréchet derivative of the single layer potential (Theorem 7 of [12]). In order to prove the continuity of $\frac{\partial}{\partial r}\{\langle \nu(r,x), V_r(x_r^\tau)\rangle\}(r,h)$ for $\tau \to 0$ we use the decomposition

$$\begin{aligned}
\langle \nu(r,x), V_r(x)\rangle &= \int_{\partial D} \Big(\langle \nu(r,x), \nu(r,x) - \nu(r,y)\rangle \Big) \frac{\partial \Phi(x,y_r)}{\partial \nu(r,y)} J_T(r,y)\, ds(y) \\
&\quad - \int_{\partial D} \frac{\partial \Phi(x,y_r)}{\partial \nu(r,y)} J_T(r,y)\, ds(y), \qquad x \in \mathbb{R}^3. \qquad (5.14)
\end{aligned}$$

The continuity of the Fréchet derivative of the first term of (5.14) for $\tau \to 0$ can be shown by an application of Lemma 3 of [12]. For the second term we use

$$\int_{\partial D} \frac{\partial \Phi(x,y_r)}{\partial \nu(r,y)} J_T(r,y)\, ds(y) = \begin{cases} 1, & x \notin \partial D_r \\ 1/2, & x \in \partial D_r \end{cases}$$

to conclude that the derivative vanishes identically. Now collecting all terms we obtain the equation (5.9).

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IV. Now we show

$$\lim_{\tau\to 0}\left\{\frac{\partial}{\partial\tau}\frac{\partial}{\partial r}\Big\{2(P_2S_0^2\varphi)(x_r^\tau)\Big\}(0,h)-\frac{\partial}{\partial r}\Big\{(TS_0^2\varphi)(x)\Big\}(0,h)\right\}=0. \tag{5.15}$$

We derive

$$\begin{aligned}\frac{\partial}{\partial\tau}\frac{\partial}{\partial r}\Big\{(P_2S_0^2\varphi)(x_r^\tau)\Big\}(0,h) &= \frac{\partial}{\partial r}\frac{\partial}{\partial\tau}\Big\{(P_2S_0^2\varphi)(x_r^\tau)\Big\}(0,h)\\ &= \frac{\partial}{\partial r}\left\{\langle\nu(r,x),(\text{grad } v_r)(x_r^\tau)\rangle\right\}(0,h) \qquad (5.16)\end{aligned}$$

where we use

$$v_r(x):=\int_{\partial D}\langle\nu(r,y),(\text{grad }_x\Phi)(x,y_r)\rangle\, J_T(r,y)(S_0^2\varphi)(y)\,\mathrm{d}s(y),\qquad x\in\mathbb{R}^3.$$

According to Theorem 2.23 of [2] we have

$$\begin{aligned}(\text{grad } v_r)(x_r^\tau) &= k^2\int_{\partial D}\Phi(x_r^\tau,y_r)\nu(r,y)J_T(r,y)(S_0^2\varphi)(y)\,\mathrm{d}s(y) \qquad (5.17)\\ &\quad+\int_{\partial D}(\text{grad }_x\Phi)(x_r^\tau,y_r)\times\Big(\nu(r,y)\times(\text{Grad}\,(S_0^2\varphi))(y)\Big)J_T(y,,r)\,\mathrm{d}s(y)\end{aligned}$$

where Grad ψ denotes the surface gradient of ψ with respect to the surface ∂D_r. For the second term of (5.15) we obtain with the help of the same theorem

$$\begin{aligned}(TS_0^2\varphi)(x) &= \Big\langle\nu(r,x),k^2\int_{\partial D}\Phi(x_r,y_r)\nu(r,y)(S_0^2\varphi)(y)J_T(r,y)\,\mathrm{d}s(h)\Big\rangle\\ &\quad-\Big\langle\nu(r,x),\int_{\partial D}(\text{grad }_x\Phi)(x_r,y_r)\Big(\nu(r,y)\times(\text{Grad}\,(S_0^2\varphi))(yy)\Big)\\ &\qquad\times J_T(y,r)\,\mathrm{d}s(y)\Big\rangle,\qquad x\in\partial D.\end{aligned}$$

We define

$$w_r(x_r^\tau):=\begin{cases}\langle\nu(r,x),2(\text{grad } v_r)(x_r^\tau)\rangle\,, & \tau_0>\tau>0\\ (TS_0^2\varphi)(x), & \tau=0\end{cases}$$

We have to verify the continuity of $\partial/\partial r\,\{w_r(x_r^\tau)\}\,(r,h)$ for $\tau\to 0$. For the firsst integral of w_r we proceed as in the case of the single layer potential. For the second inntegral we use the relations

$$\begin{aligned}&-\Big\langle\nu(r,x),\int_{\partial D}\Big(\text{grad }_x\Phi\Big)(x_r^\tau,y_r)\times b(y)J_T(r,y)\,\mathrm{d}s(y)\Big\rangle\\ &\quad=\nu_i(r,x)\epsilon_{i,j,k}\int_{\partial D}\Big(\text{grad }_x\Phi(x_r^\tau,y_r)\Big)_j b_k(y)J_T(r,y)\,\mathrm{d}s(y)\\ &\quad=\nu_i(r,x)\epsilon_{i,j,k}\int_{\partial D}\Big(\text{grad }_x\Phi(x_r^\tau,y_r)\Big)_j\Big[b_k(y)-b_k(x)\Big]J_T(r,y)\,\mathrm{dd}s(y)\\ &\qquad+\nu_i(r,x)\epsilon_{i,j,k}b_k(x)\int_{\partial D}\Big(\text{grad }_x\Phi(x_r^\tau,y_r)\Big)_j\,J_T(r,y)\,\mathrm{d}s(y)\end{aligned}$$

where $\epsilon_{i,j,k}$ denotes the total antisymmetric tensor of rank 3,

$$\nu_i(r,x)\epsilon_{i,j,k}b_k(x)\int_{\partial D}\Big(\text{grad }_x\Phi(x_r^\tau,y_r)\Big)_j J_T(x,y)\,\mathrm{d}s(x)$$

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$$= \nu_i(r,x)\epsilon_{i,j,k}b_k(x)\int_{\partial D}\Big\{2H(r,y)\nu_j(r,y)\Phi(x_r^\tau,y_r) - \nu_j(r,y)\frac{\partial\Phi}{\partial\nu(y)}(x_r^\tau,y_r)\Big\}J_T(r,y)\,\mathrm{d}s(y)$$

and

$$\nu_i(r,x)\epsilon_{i,j,k}b_k(x)\int_{\partial D}\nu_j(r,y)\frac{\partial\Phi}{\partial\nu(y)}(x_r^\tau,y_r)J_T(r,y)\,\mathrm{d}s(y)$$

$$= \nu_i(r,x)\epsilon_{i,j,k}b_k(x)\int_{\partial D}\Big(\nu_j(r,y)-\nu_j(r,x)\Big)\frac{\partial\Phi}{\partial\nu(y)}(x_r^\tau,y_r)J_T(r,y)\,\mathrm{d}s(y)$$

$$+ \nu_i(r,x)\nu_j(r,x)\epsilon_{i,j,k}b_k(x)\int_{\partial D}\frac{\partial\Phi}{\partial\nu(y)}(x_r^\tau,y_r)J_T(r,y)\,\mathrm{d}s(y)$$

and the antisymmetry of $\epsilon_{i,j,k}$ to derive the continuity. This can be done analogously to III. □

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