Master Thesis

H(div)-based approximation of the Stokes problem with Navier slip boundary conditions

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Contents

1 Introduction

In the low Renolds number regime, the stationary velocity field $\mathbf{u}$ and pressure field p of an incompressible fluid on a domain $\Omega \subset \mathbb{R}^N$, $N = 2, 3$ with a Lipschitz boundary Γ can be described by the Stokes problem

$$-2\,\nabla\cdot\boldsymbol{\epsilon}(\mathbf{u}) + \nabla p = \mathbf{f}, \quad \text{in } \Omega, \tag{1}$$

$$\nabla\cdot\mathbf{u} = 0, \quad \text{in } \Omega, \tag{2}$$

where $\boldsymbol{\epsilon}(\mathbf{u})$ denotes symmetric gradient of $\mathbf{u}$, namely

$$\boldsymbol{\epsilon}(\mathbf{u}) \coloneqq \frac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^T),$$

and $\mathbf{f}$ is a given forcing term. The first equation expresses the conservation of momentum and the second the incompressibility condition. The system can be supplemented with various boundary conditions, including Dirichlet boundary conditions on $\mathbf{u}$ or Neumann boundary conditions prescribing the normal force on Γ. In this project, we are interested in imposing Navier slip boundary conditions, namely

$$\mathbf{u}\cdot\mathbf{n} = 0, \quad \text{on } \Gamma, \tag{3}$$

$$\gamma_{\|}(\boldsymbol{\epsilon}(\mathbf{u})\mathbf{n}) = 0, \quad \text{on } \Gamma, \tag{4}$$

where $\mathbf{n}$ denotes the outward oriented unit normal vector on Γ and $\gamma_{\|}(\boldsymbol{\zeta})$ denotes the tangential part of a vector field $\boldsymbol{\zeta}$ on Γ, i.e.

$$\gamma_{\|}(\boldsymbol{\zeta}) \coloneqq \boldsymbol{\zeta} - \boldsymbol{\zeta}\cdot\mathbf{n} = \mathbf{n}\times\boldsymbol{\zeta}\times\mathbf{n}.$$

A conventional weak formulation of the Stokes problem can be derived by using the function spaces $\mathbf{H}^1(\Omega)$ and $\mathrm{L}^2(\Omega)$ for $\mathbf{u}$ and p, respectively, as is described in [9, Chapter 53]. For some structure-preserving numerical schemes for the Navier-Stokes equation [18] or MHD [11], however, it is beneficial to use a mixed formulation seeking the vorticity $\mathbf{w} = \nabla\times\mathbf{u}$, the velocity $\mathbf{u}$ and the pressure p in (subspaces of) the function spaces $\mathbf{H}(\mathrm{curl};\Omega)$, $\mathbf{H}(\mathrm{div};\Omega)$ and $\mathrm{L}^2(\Omega)$, respectively. This motivates our use of the Vorticity-Velocity-Pressure (VVP) formulation proposed by Nédélec [15] for the discretization of the Stokes problem with Navier slip boundary conditions. Although the VVP formulation has been extensively analyzed by Dubois and his collaborators [6, 5, 7], it is *a priori* not clear how to incorporate Navier slip boundary conditions. Following the approach adopted by Wouter and his collaborators in [3], we derive the variational formulation with the help of an equivalent formulation of Navier slip boundary conditions proposed by Mitrea and Monniaux [14, Eq. 2.9], which can be viewed as a Robin type boundary condition for the velocity $\mathbf{u}$. The resulting variational problem is shown to be well-posed on the continuous level, which is then discretized using appropriate finite element spaces. The convergence of the numerical scheme is confirmed by the results of numerical experiments conducted.

The remaining of the thesis proceeds as follows. In Section 2, we introduce the notations and derive the VVP formulation for the Stokes problem with pure natural boundary conditions and Navier slip boundary conditions. In Section 3, we recall important results on establishing the well-posedness of variational problems and prove the well-posedness of the variational problems obtained in Section 2. After discretizing the variational problems in Section 4, Section 5 presents the results from numerical experiments and Section 6 draws conclusions.

2 VVP formulations

Let us first introduce some notations. Throughout the thesis, the domain of the function spaces is Ω unless otherwise stated and will be omitted in notation. We use the standard Lebesgue and Sobolev spaces L^p and H^p, and the vector valued version of these $\mathbf{L}^p$ and $\mathbf{H}^p$. For sufficiently smooth scaler field σ and vector field $\mathbf{u} = (u_1, \dots, u_N)$ defined on $\Omega \subset \mathbb{R}^N$, we define the following calculus operators

$$\begin{aligned}
\nabla\sigma &\coloneqq \left(\frac{\partial\sigma}{\partial x_1}, \dots, \frac{\partial\sigma}{\partial x_N}\right), && N = 2,3,\\
\nabla\times\sigma &\coloneqq \left(\frac{\partial\sigma}{\partial x_2}, -\frac{\partial\sigma}{\partial x_1}\right), && N = 2,\\
\nabla\cdot\mathbf{u} &\coloneqq \sum_{i=1}^{i=N} \frac{\partial u_i}{\partial x_i}, && N = 2,3,\\
\nabla\times\mathbf{u} &\coloneqq \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}, && N = 2,\\
\nabla\times\mathbf{u} &\coloneqq \left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}, \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}, \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right), && N = 3.
\end{aligned}$$

and the spaces $\mathrm{H}(\mathrm{curl})$, $\mathbf{H}(\mathrm{div})$, and $\mathbf{H}(\mathrm{curl})$ are defined as

$$\begin{aligned}
\mathrm{H}(\mathrm{curl}) &\coloneqq \{\, \sigma \in \mathrm{L}^2 \mid \nabla\times\sigma \in \mathrm{L}^2 \,\} = \mathrm{H}^1, && N = 2,\\
\mathbf{H}(\mathrm{div}) &\coloneqq \{\, \mathbf{u} \in \mathbf{L}^2 \mid \nabla\cdot\mathbf{u} \in \mathbf{L}^2 \,\}, && N = 2,3,\\
\mathbf{H}(\mathrm{curl}) &\coloneqq \{\, \mathbf{u} \in \mathbf{L}^2 \mid \nabla\times\mathbf{u} \in \mathbf{L}^2 \,\}, && N = 2,3.
\end{aligned}$$

We will also need a larger space than $\mathbf{H}(\mathrm{curl})$

$$\mathbf{\Sigma} \coloneqq \{\, \mathbf{w} \in \mathbf{L}^2 \mid \nabla\times\mathbf{w} \in \mathring{\mathbf{H}}(\mathrm{div})' \,\},$$

which is equipped with the norm

$$\|\cdot\|_{\mathbf{\Sigma}} = \|\cdot\|_{\mathbf{L}^2} + \|\nabla\times\cdot\|_{\mathring{\mathbf{H}}(\mathrm{div})'}.$$

The 2-dimensional counterpart of $\mathbf{\Sigma}$ is defined as

$$\Sigma \coloneqq \{\, \sigma \in \mathrm{L}^2 \mid \nabla\times\sigma \in \mathring{\mathbf{H}}(\mathrm{div})' \,\},$$

the norm on which is defined in the same way as $\mathbf{\Sigma}$. These spaces have been introduced in [7] and [2] for the studies of the VVP formulation of the Stokes problem with no slip boundary conditions.

The standard L^2 and $\mathbf{L}^2$ inner product will be denoted as $(\cdot,\cdot)$, the duality pair between $\mathring{\mathbf{H}}(\mathrm{div})'$ and $\mathring{\mathbf{H}}(\mathrm{div})$ as $\langle\cdot,\cdot\rangle$, and the inner product on $\mathrm{L}^2(\Gamma)$ and $\mathbf{L}^2(\Gamma)$ as $(\cdot,\cdot)_\Gamma$. $\mathbf{H}(\mathrm{div})$ and $\mathbf{H}(\mathrm{curl})$ are equipped with the following inner products

$$\begin{aligned}\langle \mathbf{u},\mathbf{v}\rangle_{\mathbf{H}(\mathrm{div})} &= (\mathbf{u},\mathbf{v}) + (\nabla\cdot\mathbf{u},\nabla\cdot\mathbf{v}), \qquad \forall \mathbf{u},\mathbf{v}\in\mathbf{H}(\mathrm{div}),\\ \langle \mathbf{w},\mathbf{z}\rangle_{\mathbf{H}(\mathrm{curl})} &= (\mathbf{w},\mathbf{z}) + (\nabla\times\mathbf{w},\nabla\times\mathbf{z}), \quad \forall \mathbf{w},\mathbf{z}\in\mathbf{H}(\mathrm{curl}),\end{aligned}$$

with respect to which they are Hilbert spaces. The closure of C_c^∞ in H^p, $\mathbf{H}^p$, $\mathbf{H}(\mathrm{div})$ and $\mathbf{H}(\mathrm{curl})$ with respect to the corresponding norms are denoted by $\mathring{\mathrm{H}}^p$, $\mathring{\mathbf{H}}^p$, $\mathring{\mathbf{H}}(\mathrm{div})$ and $\mathring{\mathbf{H}}(\mathrm{curl})$, respectively.

We will denote the subspace of divergence-free functions in $\mathbf{H}(\mathrm{div})$ as $\mathbf{H}(\mathrm{div}_0)$ and the subspace of curl-free functions in $\mathbf{H}(\mathrm{curl})$ as $\mathbf{H}(\mathrm{curl}_0)$. The orthogonal complement of $\mathbf{H}(\mathrm{div}_0)$ in $\mathbf{H}(\mathrm{div})$ and of $\mathbf{H}(\mathrm{curl}_0)$ in $\mathbf{H}(\mathrm{curl})$ with respect to the inner products defined above will be denoted as $\mathbf{H}(\mathrm{div}_0)^\perp$ and $\mathbf{H}(\mathrm{curl}_0)^\perp$, respectively. Similarly, $\mathring{\mathbf{H}}(\mathrm{div}_0)$ and $\mathring{\mathbf{H}}(\mathrm{curl}_0)$ will denote the corresponding subspaces in $\mathring{\mathbf{H}}(\mathrm{div})$ and $\mathring{\mathbf{H}}(\mathrm{curl})$, respectively, and $\mathring{\mathbf{H}}(\mathrm{div}_0)^\perp$ and $\mathring{\mathbf{H}}(\mathrm{curl}_0)^\perp$ their orthogonal complement. The subspace of L^2 functions with zero mean will be denoted as

$$\hat{\mathrm{L}}^2 \coloneqq \{\, \sigma\in\mathrm{L}^2 \mid \int_\Omega \sigma\,dx = 0\,\}.$$

Finally, let us denote the trace map of $\mathbf{H}^1$ functions as

$$\begin{aligned}\gamma : \mathbf{H}^1 &\to \mathbf{H}^{\frac{1}{2}}(\Gamma)\\ \boldsymbol{\xi} &\mapsto \boldsymbol{\xi}|_\Gamma\end{aligned}$$

and introduce the space of tangential vector fields in $\mathbf{H}^{\frac{1}{2}}(\Gamma)$

$$\mathrm{T}\mathbf{H}^{\frac{1}{2}}(\Gamma) \coloneqq \{\, \gamma(\boldsymbol{\xi}) \mid \boldsymbol{\xi}\in\mathbf{H}^1,\ \gamma(\boldsymbol{\xi})\cdot\mathbf{n} = 0 \text{ on } \Gamma\,\}.$$

It has been shown in [7, Prop. 4.5] that there exists a continuous trace map from $\boldsymbol{\Sigma}$ to $(\mathrm{T}\mathbf{H}^{\frac{1}{2}}(\Gamma))'$

$$\begin{aligned}\gamma_t : \boldsymbol{\Sigma} &\to (\mathrm{T}\mathbf{H}^{\frac{1}{2}}(\Gamma))'\\ \boldsymbol{\varphi} &\mapsto \boldsymbol{\varphi}\times\mathbf{n}.\end{aligned}$$

Denoting the duality pair between $(\mathrm{T}\mathbf{H}^{\frac{1}{2}}(\Gamma))'$ and $\mathrm{T}\mathbf{H}^{\frac{1}{2}}(\Gamma)$ as $\langle\cdot,\cdot\rangle_\Gamma$, $\boldsymbol{\varphi}\times\mathbf{n}$ is defined by

$$\langle \boldsymbol{\varphi}\times\mathbf{n}, \gamma(\boldsymbol{\xi})\rangle_\Gamma = (\boldsymbol{\varphi}, \nabla\times\boldsymbol{\xi}) - \langle \nabla\times\boldsymbol{\varphi}, \boldsymbol{\xi}\rangle, \quad \forall\boldsymbol{\varphi}\in\boldsymbol{\Sigma},\ \forall\gamma\boldsymbol{\xi}\in\mathrm{T}\mathbf{H}^{\frac{1}{2}}(\Gamma).$$

The definitions for $N = 2$ are analogous, where Σ contains scalar-valued functions. For 2-dimensional vector-valued functions, we define the tangential trace on $\mathbf{H}(\text{curl})$ in a similar way

$$\begin{aligned} \gamma_\perp : \mathbf{H}(\text{curl}) &\to (\mathrm{H}^{\frac{1}{2}}(\Gamma))' \\ \boldsymbol{\xi} &\mapsto \boldsymbol{\xi} \times \mathbf{n}, \end{aligned}$$

where $\boldsymbol{\xi} \times \mathbf{n}$ is defined by

$$\langle \boldsymbol{\xi} \times \mathbf{n}, \gamma(\psi) \rangle_\Gamma = (\boldsymbol{\xi}, \nabla\times \psi) - (\nabla\times \boldsymbol{\xi}, \psi), \quad \forall \boldsymbol{\xi} \in \mathbf{H}(\text{curl}), \forall \psi \in \mathrm{H}^1.$$

If $\boldsymbol{\xi}$ is restricted to $\mathbf{H}^1$, then $\gamma(\boldsymbol{\xi}) \in \mathbf{H}^{\frac{1}{2}}(\Gamma) \subset \mathbf{L}^2(\Gamma)$ and $\boldsymbol{\xi} \times \mathbf{n}$ is given by a tangential vector field in $\mathrm{L}^2(\Gamma)$.

2.1 The Stokes problem: reformulated

Following [3], we reformulate the Stokes problem with Navier slip boundary conditions in this subsection.

For sufficiently smooth $\mathbf{u}$ with $\nabla\cdot \mathbf{u} = 0$, we have

$$\begin{aligned} -2\,\nabla\cdot \boldsymbol{\epsilon}(\mathbf{u}) &= -\Delta \mathbf{u} - \nabla(\nabla\cdot \mathbf{u}) \\ &= \nabla\times \nabla\times \mathbf{u} - 2\nabla(\nabla\cdot \mathbf{u}) \\ &= \nabla\times \nabla\times \mathbf{u}. \end{aligned}$$

Substituting into (1) and introducing a new variable $\mathbf{w} = \nabla\times \mathbf{u}$ for the vorticity, we obtain

$$\nabla\times \mathbf{w} + \nabla p = \mathbf{f}, \quad \text{in } \Omega. \tag{5}$$

For Navier slip boundary conditions, the following conditions have been proven by Mitrea and Monniaux [14, Eq. 2.9] to be an equivalent formulation if Γ is C^2:

$$\mathbf{u} \cdot \mathbf{n} = 0, \quad \text{on } \Gamma, \tag{6}$$

$$-\mathbf{n} \times \mathbf{w} + 2\mathcal{W}(\gamma_\parallel(\mathbf{u})) = 0, \quad \text{on } \Gamma, \tag{7}$$

where $\mathcal{W} : T\Gamma \to T\Gamma$ denotes the Weingarten map on the space $T\Gamma$ of tangential vector fields on Γ.

Using (5),(6) and (7) and adding the definition of $\mathbf{w}$, we reformulate the Stokes problem with Navier slip boundary conditions as

$$\mathbf{w} - \nabla\times \mathbf{u} = \mathbf{0} \quad \text{in } \Omega, \tag{8}$$

$$\nabla\times \mathbf{w} + \nabla p = \mathbf{f} \quad \text{in } \Omega, \tag{9}$$

$$\nabla\cdot \mathbf{u} = 0 \quad \text{in } \Omega, \tag{10}$$

$$\mathbf{u}\cdot\mathbf{n} = 0, \quad \text{on } \Gamma, \tag{11}$$

$$-\mathbf{n}\times\mathbf{w} + \alpha(\gamma_{\|}(\mathbf{u})) = 0, \quad \text{on } \Gamma, \tag{12}$$

where $\alpha = 2\mathcal{W}$. This will be the starting point for our derivation of the VVP formulation.

Remark 2.1. *In the case of $N = 2$, $\mathbf{n}\times w$ and $\gamma_{\|}(\mathbf{u})$ are not defined since w and $\mathbf{n}\times\mathbf{u}$ are scalar-valued. However, we can circumvent the problem by interpreting the vorticity as a vector perpendicular to the plane and the velocity as a vector parallel to the plane, and the calculus operators for $N = 3$ are then compatible with those for $N = 2$. In the following, we will tacitly omit the case for $N = 2$ if such an interpretation makes both cases compatible with each other. Moreover, we denote the scalar-valued vorticity and the corresponding 3-dimensional vector perpendicular to Ω as w and $\mathbf{w}$ (z and $\mathbf{z}$ for test functions), respectively.*

2.2 Derivation of VVP formulations

We first derive the VVP formulation for the Stokes problem with pure natural boundary conditions, i.e.

$$p = 0, \quad \text{on } \Gamma, \tag{13}$$

$$\mathbf{u}\times\mathbf{n} = \mathbf{0}, \quad \text{on } \Gamma, \tag{14}$$

whose well-posedness will be shown in the next section to illustrate the standard technique. Then we will give a VVP formulation for the Stokes problem with Navier slip boundary conditions from a naive approach, which poses significant constraint on Γ. Finally, we introduce a modified variational formulation circumventing the constraint on Γ, which will be the focus of analysis in the next section.

Let us assume $\mathbf{f} \in \mathbf{L}^2$. Multiplying (8), (9) and (10) by test functions $\mathbf{z}$, $\mathbf{u}$ and q, respectively, we obtain by formally integrating by parts

$$(\mathbf{w}, \mathbf{z}) - (\nabla\times \mathbf{z}, \mathbf{u}) - (\mathbf{n} \times \mathbf{u}, \mathbf{z})_\Gamma = 0, \tag{15}$$
$$(\nabla\times \mathbf{w}, \mathbf{v}) - (p, \nabla\cdot \mathbf{v}) + (\mathbf{v} \cdot \mathbf{n}, p)_\Gamma = (\mathbf{f}, \mathbf{v}), \tag{16}$$
$$(q, \nabla\cdot \mathbf{u}) = 0. \tag{17}$$

For pure natural boundary conditions and $N = 3$, we set the trial and test space for $\mathbf{w}$, $\mathbf{u}$ and p to be $\mathbf{H}(\text{curl})$, $\mathbf{H}(\text{div})$ and L^2, respectively, and the following variational problem is derived after incorporating (13) and (14)

$$\begin{cases} \text{Seek } \mathbf{w} \in \mathbf{H}(\text{curl}),\, \mathbf{u} \in \mathbf{H}(\text{div}), p \in \mathrm{L}^2 : \\ \qquad (\mathbf{w}, \mathbf{z}) - (\nabla\times \mathbf{z}, \mathbf{u}) = 0, & \forall \mathbf{z} \in \mathbf{H}(\text{curl}), \\ \qquad (\nabla\times \mathbf{w}, \mathbf{v}) - (p, \nabla\cdot \mathbf{v}) = (\mathbf{f}, \mathbf{v}), & \forall \mathbf{v} \in \mathbf{H}(\text{div}), \\ \qquad (q, \nabla\cdot \mathbf{u}) = 0, & \forall q \in \mathrm{L}^2. \end{cases} \tag{18}$$

For $N = 2$, we obtain a similar variational problem by replacing $\mathbf{H}(\text{curl})$ by H^1.

For Navier slip boundary conditions and $N = 2$, (11) can be imposed strongly if $\mathbf{u} \in \mathbf{H}(\text{div})$ and the boundary term in (15) can be reformulated using (12) as

$$\begin{aligned} (\mathbf{n} \times \mathbf{u}, \mathbf{z})_\Gamma = &\quad ((\mathbf{n} \times \mathbf{u}) \times \mathbf{n}, \mathbf{z} \times \mathbf{n})_\Gamma = (\gamma_\| (\mathbf{u}), \mathbf{z} \times \mathbf{n})_\Gamma \\ = &\quad (\alpha^{-1}(\mathbf{n} \times \mathbf{w}), \mathbf{z} \times \mathbf{n})_\Gamma \\ = &\quad (\alpha^{-1}(\mathbf{n} \times \mathbf{w}) \times \mathbf{n}, (\mathbf{z} \times \mathbf{n}) \times \mathbf{n})_\Gamma = -(\alpha^{-1}\gamma_\|(\mathbf{w}), \gamma_\|(\mathbf{z}))_\Gamma \\ = &-(\alpha^{-1} w, z)_\Gamma, \end{aligned} \tag{19}$$

where the α^{-1} and $\times\mathbf{n}$ commute since the Weingarten map is a scalar multiplication in 2D. If we follow the approach in the case of pure natural boundary conditions, we arrive at the following variational problem by restricting the trial and test space for $\mathbf{u}$ to be $\mathring{\mathbf{H}}(\text{div}_0)$

$$\begin{cases} \text{Seek } w \in \mathrm{H}^1,\, \mathbf{u} \in \mathring{\mathbf{H}}(\text{div}),\, p \in \hat{\mathrm{L}}^2 : \\ \quad (w, z) - (\nabla\times z, \mathbf{u}) + (\alpha^{-1} w, z)_\Gamma = 0, & \forall z \in \mathrm{H}^1, \\ \qquad (\nabla\times w, \mathbf{v}) - (p, \nabla\cdot \mathbf{v}) = (\mathbf{f}, \mathbf{v}), & \forall \mathbf{v} \in \mathring{\mathbf{H}}(\text{div}), \\ \qquad (q, \nabla\cdot \mathbf{u}) = 0, & \forall q \in \hat{\mathrm{L}}^2. \end{cases} \tag{20}$$

However, there are two problems with this formulation. First, the use of α^{-1} requires the absolute value of the Weingarten map to be uniformly lower bounded, which may be violated even for a convex domain. Second, it is not clear how to generalize the formulation to the case of $N = 3$ since $\gamma_{\|}(\mathbf{w})$ is in general not in $\mathbf{L}^2(\Gamma)$ for $\mathbf{w} \in \mathbf{H}(\mathrm{curl})$. To address these issues, we first observe that without the boundary term, (20) corresponds to the VVP formulation for the Stokes problem with no slip boundary conditions. Hence, inspired by [7] and [2], we choose the trial and test space for $\mathbf{w}$ to be $\boldsymbol{\Sigma}$ and the terms $(\nabla\times \mathbf{z}, \mathbf{u})$ and $(\nabla\times \mathbf{w}, \mathbf{v})$ in (15) and (16) are given a mathematical sense by the duality pairs $\langle\nabla\times \mathbf{z}, \mathbf{u}\rangle$ and $\langle\nabla\times \mathbf{w}, \mathbf{v}\rangle$, respectively. Furthermore, we introduce a new variable for the tangential trace of $\mathbf{u}$

$$\boldsymbol{\psi} := \gamma_{\|}(\mathbf{u}),$$

and the boundary term $(\mathbf{n} \times \mathbf{u}, \mathbf{z})_\Gamma$ in (15) is interpreted as $\langle \mathbf{z} \times \mathbf{n}, \boldsymbol{\psi}\rangle_\Gamma$ by the first two equalities in (19). Using $\mathrm{T}\mathbf{H}^{\frac{1}{2}}(\Gamma)$ as the trial and test space for $\boldsymbol{\psi}$, testing (12) by $\boldsymbol{\eta} \in \mathrm{T}\mathbf{H}^{\frac{1}{2}}(\Gamma)$ yields

$$\langle \mathbf{w} \times \mathbf{n}, \boldsymbol{\eta}\rangle_\Gamma + (\alpha\boldsymbol{\psi}, \boldsymbol{\eta})_\Gamma = 0, \tag{21}$$

which, together with the weak formulation for the other equations of the Stokes problem, constitutes the following variational problem

$$\begin{cases} \text{Seek } \mathbf{w} \in \boldsymbol{\Sigma},\ \mathbf{u} \in \mathring{\mathbf{H}}(\mathrm{div}),\ p \in \hat{\mathrm{L}}^2, \boldsymbol{\psi} \in \mathrm{T}\mathbf{H}^{\frac{1}{2}}(\Gamma) : & \\ \qquad \langle \mathbf{w} \times \mathbf{n}, \boldsymbol{\eta}\rangle_\Gamma + (\alpha\boldsymbol{\psi}, \boldsymbol{\eta})_\Gamma = 0, & \forall \boldsymbol{\eta} \in \mathrm{T}\mathbf{H}^{\frac{1}{2}}(\Gamma), \\ (\mathbf{w}, \mathbf{z}) - \langle\nabla\times \mathbf{z}, \mathbf{u}\rangle - \langle \mathbf{z} \times \mathbf{n}, \boldsymbol{\psi}\rangle_\Gamma = 0, & \forall \mathbf{z} \in \boldsymbol{\Sigma}, \\ \qquad \langle\nabla\times \mathbf{w}, \mathbf{v}\rangle - (p, \nabla\cdot \mathbf{v}) = (\mathbf{f}, \mathbf{v}), & \forall \mathbf{v} \in \mathring{\mathbf{H}}(\mathrm{div}), \\ \qquad\qquad (q, \nabla\cdot \mathbf{u}) = 0, & \forall q \in \hat{\mathrm{L}}^2. \end{cases} \tag{22}$$

Remark 2.2. *Without the term* $(\alpha\boldsymbol{\psi}, \boldsymbol{\eta})_\Gamma$, $\boldsymbol{\psi}$ *can be viewed as a Lagrange multiplier for the enforcement of vanishing tangential trace of* $\mathbf{w}$.

Although (22) is more general than (20) as mentioned above, we will see in the following sections that these two formulations are related on the continuous and discrete level. The close relation between the two formulations is also supported by the results from the numerical experiments in the case of uniformly positive α, as will be shown in Section 5.

3 The continuous problem

In this section, we establish the well-posedness of the weak formulations of the Stokes problem derived in the previous section on the continuous level.

3.1 Preliminaries

Throughout the section, we assume that Ω is bounded and contractible. In particular, Ω is simply-connected and Γ is connected. We then have de Rahm complexes [8, Section 16.3] [1, Section 4.3]

$$\mathbf{H}(\mathrm{curl}) \xrightarrow{\nabla\times} \mathbf{H}(\mathrm{div}) \xrightarrow{\nabla\cdot} \mathrm{L}^2, \tag{23}$$

$$\mathring{\mathbf{H}}(\mathrm{curl}) \xrightarrow{\nabla\times} \mathring{\mathbf{H}}(\mathrm{div}) \xrightarrow{\nabla\cdot} \hat{\mathrm{L}}^2, \tag{24}$$

which will be used in the following without being explicitly mentioned.

Let us recall two forms of the Poincaré–Friedrichs inequality:

$$\|\boldsymbol{\psi}\|_{\mathbf{H}(\mathrm{div})} \leq C_{p_1}\|\nabla\cdot\boldsymbol{\psi}\|_{\mathrm{L}^2}, \quad \forall\boldsymbol{\psi} \in \mathbf{H}(\mathrm{div}_0)^{\perp} \text{ or } \forall\boldsymbol{\psi} \in \mathring{\mathbf{H}}(\mathrm{div}_0)^{\perp}, \tag{25}$$

$$\|\boldsymbol{\tau}\|_{\mathbf{H}(\mathrm{curl})} \leq C_{p_2}\|\nabla\times\boldsymbol{\tau}\|_{\mathbf{L}^2}, \quad \forall\boldsymbol{\tau} \in \mathbf{H}(\mathrm{curl}_0)^{\perp} \text{ or } \forall\boldsymbol{\psi} \in \mathring{\mathbf{H}}(\mathrm{curl}_0)^{\perp}, \tag{26}$$

which can be proved using de Rahm complexes and open mapping theorem [1, Thm. 4.6].

Next we recall the Babuška-Brezzi theorem, which can be found in [9, Thm. 49.13]. We have the following abstract setting: V and M are real reflexive Banach spaces, $Q := M'$, a and b are bounded bilinear forms defined on $V \times V$ and $V \times Q$, respectively, and $f \in V'$, $g \in Q'$. Let us set

$$\|a\| := \sup_{v\in V}\sup_{w\in W} \frac{|a(v,w)|}{\|v\|_V\|w\|_W},$$

and define $N \subseteq V$ as

$$N := \{\, v \in V \mid b(v,q) = 0 \;\forall q \in Q \,\},$$

which is a closed subspace of V. $Q' = M''$ is identified with M by reflexivity of M. The following abstract problem is considered:

$$\begin{cases} \text{Seek } u \in V \text{ and } p \in Q: \\ \qquad a(u,w) + b(w,p) = f(w), \qquad \forall w \in V, \\ \qquad\qquad\qquad\;\; b(u,q) = g(q), \qquad \forall q \in Q. \end{cases} \tag{27}$$

Theorem 3.1 (Babuška-Brezzi). (27) *is well-posed if and only if*

$$\begin{cases} \inf_{v\in N}\sup_{w\in N}\frac{|a(v,w)|}{\|v\|_V\|w\|_W} =: \alpha > 0, \\ \forall w \in N : [\forall v \in N, a(v,w) = 0] \implies [w = 0], \end{cases} \tag{28}$$

and the following inequality, usually called Babuška-Brezzi condition*, holds:*

$$\inf_{q\in Q}\sup_{v\in V}\frac{|b(v,q)|}{\|v\|_V\|q\|_Q} =: \beta > 0. \tag{29}$$

Furthermore, we have the following a priori estimates:

$$\|u\|_V \le c_1\|f\|_{V'} + c_2\|g\|_{Q'}, \tag{30}$$

$$\|p\|_Q \le c_3\|f\|_{V'} + c_4\|g\|_{Q'}, \tag{31}$$

where c_1, c_2, c_3, c_4 are generic constants that only depend on α, β and $\|a\|$.

Since b is bounded, $N \subseteq V$ is closed and hence reflexive. In view of the Banach-Nečas-Babuška (BNB) theorem [9, Thm. 25.9], (28) in Theorem 3.1 is equivalent to the well-posedness of the following problem

$$\text{Seek } \mathbf{u} \in N : \quad a(\mathbf{u}, \mathbf{w}) = f(\mathbf{w}), \quad \forall \mathbf{w} \in N,$$

where $f \in N'$. If this variational problem is also in mixed form, we can invoke Theorem 3.1 again to verify (28). This approach will be adopted in the following since we will have nested mixed variational formulations.

In the case of Navier slip boundary conditions, we have introduced the spaces $\boldsymbol{\Sigma}$ and Σ for the vorticity in Section 2. Now we derive some properties of these spaces, which will be useful in the proof of the well-posedness of (22).

In 2D, it has been shown in [2] that

$$\|\tau\|_\Sigma \sim \|\tau\|_{\mathrm{L}^2} + \|\nabla\times \tau_0\|_{\mathrm{L}^2}, \quad \forall \tau \in \Sigma,$$

where $\sim$ denotes norm equivalence and $\tau_0 \in \mathring{\mathrm{H}}^1$ is defined by

$$\langle \nabla\times \tau, \nabla\times \psi\rangle = (\nabla\times \tau_0, \nabla\times \psi) \quad \forall \psi \in \mathring{\mathrm{H}}^1.$$

As stated in the following theorem, this can be generalized to 3D.

Theorem 3.2. *For $\boldsymbol{\tau} \in \boldsymbol{\Sigma}$, let $\boldsymbol{\tau}_0 \in \mathring{\mathbf{H}}(\mathrm{curl}_0)^\perp$ be defined by*

$$\langle \nabla\times \boldsymbol{\tau}, \nabla\times \boldsymbol{\psi}\rangle = (\nabla\times \boldsymbol{\tau}_0, \nabla\times \boldsymbol{\psi}), \quad \forall \boldsymbol{\psi} \in \mathring{\mathbf{H}}(\mathrm{curl}). \tag{32}$$

Then

$$\|\boldsymbol{\tau}\|_{\boldsymbol{\Sigma}} \sim \|\boldsymbol{\tau}\|_{\mathbf{L}^2} + \|\nabla\times \boldsymbol{\tau}_0\|_{\mathbf{L}^2}. \tag{33}$$

Proof. First we show that τ_0 is well-defined. This follows from Riesz representation theorem since (26) implies that the Hilbert space structure on $\mathring{\mathbf{H}}(\mathrm{curl}_0)^\perp$ can be generated by the inner product $(\nabla\times \cdot, \nabla\times \cdot)$.

Then we define $\phi \in \hat{\mathrm{L}}^2$ by

$$(\phi, \nabla\cdot \mathbf{v}) = \langle \nabla\times \boldsymbol{\tau}, \mathbf{v}\rangle - (\nabla\times \boldsymbol{\tau}_0, \mathbf{v}), \quad \forall \mathbf{v} \in \mathring{\mathbf{H}}(\mathrm{div}). \tag{34}$$

If $\nabla\cdot \mathbf{v} = 0$, then there exists a $\boldsymbol{\psi} \in \mathring{\mathbf{H}}(\mathrm{curl}_0)^\perp$ such that $\nabla\times \boldsymbol{\psi} = \mathbf{v}$ and the right-hand side vanishes by definition of $\boldsymbol{\tau}_0$. Since $\nabla\cdot$ is surjective onto $\hat{\mathrm{L}}^2$ and $\hat{\mathrm{L}}^2$ is closed in L^2, ϕ is well-defined by Riesz representation theorem.

Now we want to find some generic constants $C_1 > 0$ and $C_2 > 0$ such that

$$C_1 (\|\boldsymbol{\tau}\|_{\mathbf{L}^2} + \|\nabla\times \boldsymbol{\tau}_0\|_{\mathbf{L}^2}) \le \|\boldsymbol{\tau}\|_{\boldsymbol{\Sigma}} \le C_2 (\|\boldsymbol{\tau}\|_{\mathbf{L}^2} + \|\nabla\times \boldsymbol{\tau}_0\|_{\mathbf{L}^2}). \tag{35}$$

By tesing (32) with $\boldsymbol{\tau}_0$, we obtain that

$$\|\nabla\times \boldsymbol{\tau}_0\|_{\mathbf{L}^2} \le \|\nabla\times \boldsymbol{\tau}\|_{\mathring{\mathbf{H}}(\mathrm{div})'}.$$

Hence

$$\|\boldsymbol{\tau}\|_{\mathbf{L}^2} + \|\nabla\times \boldsymbol{\tau}_0\|_{\mathbf{L}^2} \le \|\boldsymbol{\tau}\|_{\boldsymbol{\Sigma}}.$$

On the other hand, from (34) it holds that

$$\langle \nabla\times \boldsymbol{\tau}, \mathbf{v}\rangle \le (\|\phi\|_{\mathrm{L}^2} + \|\nabla\times \boldsymbol{\tau}_0\|_{\mathbf{L}^2}) \|\mathbf{v}\|_{\mathbf{H}(\mathrm{div})},$$

which implies that

$$\|\nabla\times \boldsymbol{\tau}\|_{\mathring{\mathbf{H}}(\mathrm{div})'} \le (\|\phi\|_{\mathrm{L}^2} + \|\nabla\times \boldsymbol{\tau}_0\|_{\mathbf{L}^2}).$$

From [12, Cor. 2.4], we can find $\tilde{\mathbf{v}} \in \mathring{\mathbf{H}}^1$ such that

$$\nabla\cdot \tilde{\mathbf{v}} = \phi, \quad \|\tilde{\mathbf{v}}\|_{\mathbf{H}^1} \le C\|\phi\|_{\mathrm{L}^2},$$

where C is a generic constant. Testing (34) with $\tilde{\mathbf{v}}$ we obtain that

$$\begin{aligned}\|\phi\|_{\mathrm{L}^2}^2 &= (\boldsymbol{\tau}, \nabla\times \tilde{\mathbf{v}}) - (\nabla\times \boldsymbol{\tau}_0, \tilde{\mathbf{v}}) \\ &\leq (\|\boldsymbol{\tau}\|_{\mathbf{L}^2} + \|\nabla\times \boldsymbol{\tau}_0\|_{\mathbf{L}^2}) \|\tilde{\mathbf{v}}\|_{\mathbf{H}^1} \\ &\leq C\,(\|\boldsymbol{\tau}\|_{\mathbf{L}^2} + \|\nabla\times \boldsymbol{\tau}_0\|_{\mathbf{L}^2}) \|\phi\|_{\mathrm{L}^2}.\end{aligned}$$

It follows that

$$\|\phi\|_{\mathrm{L}^2} \leq C\,(\|\boldsymbol{\tau}\|_{\mathbf{L}^2} + \|\nabla\times \boldsymbol{\tau}_0\|_{\mathbf{L}^2})$$

and hence

$$\|\boldsymbol{\tau}\|_{\boldsymbol{\Sigma}} \leq (C+1)\,(\|\boldsymbol{\tau}\|_{\mathbf{L}^2} + \|\nabla\times \boldsymbol{\tau}_0\|_{\mathbf{L}^2})\,.$$

□

Let us introduce the space $\boldsymbol{\mathcal{H}}$ as

$$\boldsymbol{\mathcal{H}} \coloneqq \{\, \boldsymbol{\tau} \in \boldsymbol{\Sigma} \mid \langle \nabla\times \boldsymbol{\tau}, \nabla\times \boldsymbol{\psi}\rangle = 0,\ \forall \boldsymbol{\psi} \in \mathring{\mathbf{H}}(\mathrm{curl})\,\}.$$

By Theorem 3.2, $\boldsymbol{\Sigma}$ can be decomposed as

$$\boldsymbol{\Sigma} = \mathring{\mathbf{H}}(\mathrm{curl}_0)^{\perp} \oplus \boldsymbol{\mathcal{H}}, \tag{36}$$

and

$$\|\boldsymbol{\tau}\|_{\boldsymbol{\Sigma}} \sim \|\boldsymbol{\tau}\|_{\mathbf{L}^2} \text{ on } \boldsymbol{\mathcal{H}}. \tag{37}$$

We now give some equivalent characterizations of $\boldsymbol{\mathcal{H}}$:

Proposition 3.3. *For $\boldsymbol{\tau} \in \mathbf{L}^2$, the following statements are equivalent:*

(1) $\boldsymbol{\tau} \in \boldsymbol{\mathcal{H}}$,

(2) $\boldsymbol{\tau} \in \boldsymbol{\Sigma}$ *and* $\forall \mathbf{v} \in \mathring{\mathbf{H}}(\mathrm{div}_0) : \langle \nabla\times \boldsymbol{\tau}, \mathbf{v}\rangle = 0$,

(3) $\forall \mathbf{v} \in \mathring{\mathbf{H}}(\mathrm{div}_0) \cap \mathring{\mathbf{H}}^1 : \langle \nabla\times \boldsymbol{\tau}, \mathbf{v}\rangle_{(\mathring{\mathbf{H}}^1)', \mathring{\mathbf{H}}^1} = 0$,

(4) $\nabla\times \boldsymbol{\tau} = \nabla p$ *for some* $p \in \mathrm{L}^2$.

The same holds for $\Omega \subset \mathbb{R}^2$.

Before giving the proof, we recall a classical result [12, Lemma 2.1]:

Lemma 3.4. *If* $\mathbf{f} \in (\mathring{\mathbf{H}}^1)'$ *satisfies*

$$\langle \mathbf{f}, \mathbf{v} \rangle_{(\mathring{\mathbf{H}}^1)', \mathring{\mathbf{H}}^1} = 0, \quad \forall \mathbf{v} \in \mathring{\mathbf{H}}^1 \cap \mathring{\mathbf{H}}(\mathrm{div}_0),$$

then there exists $p \in \mathrm{L}^2$ *such that*

$$\mathbf{f} = \nabla p,$$

where the equality holds in distributional sense.

Proof of Proposition 3.3. The proof for 2D and 3D are the same. We prove for $\Omega \subset \mathbb{R}^3$ that $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4 \Rightarrow 1$.

$1 \Rightarrow 2$ $\boldsymbol{\tau} \in \boldsymbol{\Sigma}$ by definition of $\boldsymbol{\mathcal{H}}$. For all $\mathbf{v} \in \mathring{\mathbf{H}}(\mathrm{div}_0)$, there exists a $\boldsymbol{\psi} \in \mathring{\mathbf{H}}(\mathrm{curl})$ such that $\nabla\times \boldsymbol{\psi} = \mathbf{v}$. Again by definition of $\boldsymbol{\mathcal{H}}$, we have that

$$\langle \nabla\times \boldsymbol{\tau}, \mathbf{v} \rangle = \langle \nabla\times \boldsymbol{\tau}, \nabla\times \boldsymbol{\psi} \rangle = 0.$$

$2 \Rightarrow 3$ For all $\mathbf{v} \in \mathring{\mathbf{H}}^1 \cap \mathring{\mathbf{H}}(\mathrm{div}_0)$, $\langle \nabla\times \boldsymbol{\tau}, \mathbf{v} \rangle = 0$ by assumption. Let $(\mathbf{v}_n)_{n\in\mathbb{N}}$ be a sequence in C_c^∞ converging to $\mathbf{v}$ in $\mathbf{H}^1$-norm. Since $\mathbf{H}(\mathrm{div})$-norm is upper bounded by $\mathbf{H}^1$-norm up to a constant, $(\mathbf{v}_n)_{n\in\mathbb{N}}$ converges to $\mathbf{v}$ also in $\mathbf{H}(\mathrm{div})$-norm. Hence we have

$$\begin{aligned} \langle \nabla\times \boldsymbol{\tau}, \mathbf{v} \rangle_{(\mathring{\mathbf{H}}^1)', \mathring{\mathbf{H}}^1} &= \lim_{n\to\infty} \langle \nabla\times \boldsymbol{\tau}, \mathbf{v}_n \rangle_{(\mathring{\mathbf{H}}^1)', \mathring{\mathbf{H}}^1} \\ &= \lim_{n\to\infty} \langle \nabla\times \boldsymbol{\tau}, \mathbf{v}_n \rangle \\ &= \langle \nabla\times \boldsymbol{\tau}, \mathbf{v} \rangle = 0. \end{aligned}$$

$3 \Rightarrow 4$ Follows directly from Lemma 3.4.

$4 \Rightarrow 1$ Let $\boldsymbol{\varphi} \in C_c^\infty$, it holds that

$$|\langle \nabla\times \boldsymbol{\tau}, \boldsymbol{\varphi} \rangle| = |(p, \nabla\cdot \boldsymbol{\varphi})| \leq \|p\|_{\mathrm{L}^2} \|\boldsymbol{\varphi}\|_{\mathbf{H}(\mathrm{div})},$$

which implies that $\nabla\times \boldsymbol{\tau} \in \mathring{\mathbf{H}}(\mathrm{div})'$ and hence $\boldsymbol{\tau} \in \boldsymbol{\Sigma}$. Moreover,

$$\langle \nabla\times \boldsymbol{\tau}, \nabla\times \boldsymbol{\psi} \rangle = -(p, \nabla\cdot(\nabla\times \boldsymbol{\psi})) = 0, \quad \forall \boldsymbol{\psi} \in \mathring{\mathbf{H}}(\mathrm{curl}).$$

This shows that $\boldsymbol{\tau} \in \boldsymbol{\mathcal{H}}$.

□

Finally, we want to establish a connection between $\mathrm{T}\mathbf{H}^{\frac{1}{2}}(\Gamma)$ and $\mathrm{H}^{\frac{1}{2}}(\Gamma)$ in the case of $N = 2$. We first note that for $\boldsymbol{\xi} \in \mathbf{H}^1$, $\gamma_\perp(\boldsymbol{\xi})$ only depends on $\gamma(\boldsymbol{\xi})$. Hence, we define the following function by a slight abuse of notation

$$\begin{aligned}\gamma_\perp : \mathbf{H}^{\frac{1}{2}}(\Gamma) &\to \mathrm{L}^2(\Gamma)\\ \gamma(\boldsymbol{\xi}) &\mapsto \gamma(\boldsymbol{\xi}) \times \mathbf{n}.\end{aligned}$$

Now we have the following result

Theorem 3.5. *If $N = 2$ and Γ is $C^{1,1}$, we have*

$$\gamma_\perp(\mathrm{T}\mathbf{H}^{\frac{1}{2}}(\Gamma)) = \mathrm{H}^{\frac{1}{2}}(\Gamma)$$

and $\gamma_\perp$ is continuous as a function from $\mathrm{T}\mathbf{H}^{\frac{1}{2}}(\Gamma)$ to $\mathrm{H}^{\frac{1}{2}}(\Gamma)$.

Proof. We first show $\gamma_\perp(\mathrm{T}\mathbf{H}^{\frac{1}{2}}(\Gamma)) \subset \mathrm{H}^{\frac{1}{2}}(\Gamma)$. Let $\boldsymbol{\psi} \in \mathrm{T}\mathbf{H}^{\frac{1}{2}}(\Gamma)$. By [12, Lemma 2.2], we can find an extension $\tilde{\boldsymbol{\psi}} \in \mathbf{H}^1$ of $\boldsymbol{\psi}$ such that

$$\nabla\cdot \tilde{\boldsymbol{\psi}} = 0,$$

i.e. $\tilde{\boldsymbol{\psi}} \in \mathring{\mathbf{H}}(\mathrm{div}_0) \cap \mathbf{H}^1$. Now [12, Thm. 3.1] asserts that there is a stream function $u \in \mathrm{H}^1$ such that

$$\tilde{\boldsymbol{\psi}} = \nabla\times u.$$

Since $\tilde{\boldsymbol{\psi}} \in \mathbf{H}^1$, $u \in \mathrm{H}^2$. We observe that $(\nabla\times u) \times \mathbf{n} = \nabla u \cdot \mathbf{n}$, since

$$\begin{aligned}\forall \xi \in \mathrm{H}^1 :\ (\nabla u \cdot \mathbf{n}, \gamma(\xi))_\Gamma &= (\nabla u, \nabla \xi) + (\nabla\cdot(\nabla u), \xi)\\ &= (\nabla\times u, \nabla\times \xi) - (\nabla\times(\nabla\times u), \xi)\\ &= ((\nabla\times u) \times \mathbf{n}, \gamma(\xi))_\Gamma.\end{aligned}$$

Hence $(\nabla\times u) \times \mathbf{n} \in \mathrm{H}^{\frac{1}{2}}(\Gamma)$ by [12, Thm. 1.6].

Now we show $\mathrm{H}^{\frac{1}{2}}(\Gamma) \subset \gamma_\perp(\mathrm{T}\mathbf{H}^{\frac{1}{2}}(\Gamma))$. Let $g \in \mathrm{H}^{\frac{1}{2}}(\Gamma)$. Again by [12, Thm. 1.6], we can find $u \in \mathrm{H}^2$ such that

$$\gamma(u) = 0, \quad \nabla u \cdot \mathbf{n} = g.$$

Now we claim that $\nabla\times u \in \mathbf{H}^1$ is the candidate we seek. Indeed, using the same argument as above we have $(\nabla\times u) \times \mathbf{n} = \nabla u \cdot \mathbf{n} = g$ and

$$\begin{aligned}\forall \xi \in \mathrm{H}^1 :\ ((\nabla\times u) \cdot \mathbf{n}, \gamma(\xi))_\Gamma &= (\nabla\cdot(\nabla\times u), \xi) + (\nabla\times u, \nabla \xi)\\ &= (\nabla\times u, \nabla \xi) - (u, \nabla\times(\nabla \xi))\\ &= (\nabla \xi \times \mathbf{n}, \gamma(u))_\Gamma\\ &= 0,\end{aligned}$$

which shows that $(\nabla\times u)\cdot\mathbf{n}=0$ and hence $\gamma(\nabla\times u)\in \mathrm{T}\mathbf{H}^{\frac{1}{2}}(\Gamma)$.

Finally, we verify the continuity of $\gamma_\perp$ as a function from $\mathrm{T}\mathbf{H}^{\frac{1}{2}}(\Gamma)$ to $\mathrm{H}^{\frac{1}{2}}(\Gamma)$. Let $g=\boldsymbol{\psi}\times\mathbf{n}\in\mathrm{H}^{\frac{1}{2}}(\Gamma)$. Since $\times\mathbf{n}$ does not alter the $\mathrm{L}^2(\Gamma)$ norm, $\gamma_\perp$ is injective. By the previous argument, there exists $u\in\mathrm{H}^2$ such that

$$\gamma(u)=0,\quad \nabla u\cdot\mathbf{n}=(\nabla\times u)\times\mathbf{n}=g,\quad (\nabla\times u)\cdot\mathbf{n}=0,$$

which implies that $\gamma(\nabla\times u)=\boldsymbol{\psi}$ by the injectivity of $\gamma_\perp$. By [12, Thm. 1.6], the normal trace of the gradient is a continuous map from H^2 to $\mathrm{H}^{\frac{1}{2}}(\Gamma)$. Hence we have

$$\|g\|_{\mathrm{H}^{\frac{1}{2}}(\Gamma)}\le C_1\|u\|_{\mathrm{H}^2}=C_1(\|u\|_{\mathrm{H}^1}+\|\mathrm{grad}(\nabla\times u)\|_{\mathbf{L}^2})$$

where grad is the gradient operator for vector fields and $C_1>0$ is a generic constant. Moreover, since $\gamma(u)=0$, the Poincaré inequality [12, Thm. 1.1] gives

$$\|u\|_{\mathrm{H}^1}\le C_2\|\nabla\times u\|_{\mathbf{L}^2}$$

for some generic constant $C_2>0$. Combining these inequalities we get

$$\|g\|_{\mathrm{H}^{\frac{1}{2}}(\Gamma)}\le C_1C_2\|\nabla\times u\|_{\mathbf{H}^1}.$$

Since the left-hand side only depends on $\gamma(\nabla\times u)$ and

$$\forall\boldsymbol{\xi}\in\mathbf{H}(\mathrm{div}_0)\cap\mathring{\mathbf{H}}^1:\ \gamma(\nabla\times u+\boldsymbol{\xi})=\gamma(\nabla\times u),$$

taking the infimum over all such $\boldsymbol{\xi}\in\mathbf{H}(\mathrm{div}_0)\cap\mathring{\mathbf{H}}^1$ gives

$$\|g\|_{\mathrm{H}^{\frac{1}{2}}(\Gamma)}\le C\|\gamma(\nabla\times u)\|_{\mathrm{T}\mathbf{H}^{\frac{1}{2}}(\Gamma)}=C\|\boldsymbol{\psi}\|_{\mathrm{T}\mathbf{H}^{\frac{1}{2}}(\Gamma)}$$

by [12, Lemma 2.2], where $C>0$ is a generic constant. □

Remark 3.6. *By [12, Remark 1.1], the first part of Theorem 3.5 can be extended to bounded polygons with the sides* $\Gamma_j, j=1\dots J$ *by replacing* $\mathrm{H}^{\frac{1}{2}}(\Gamma)$ *with* $\prod_{j=1}^J\mathrm{H}^{\frac{1}{2}}(\Gamma_j)$*, i.e.*

$$\gamma_\perp(\mathrm{T}\mathbf{H}^{\frac{1}{2}}(\Gamma))=\prod_{j=1}^J\mathrm{H}^{\frac{1}{2}}(\Gamma_j).$$

The proof is analogous to that of Theorem 3.5.

Remark 3.7. *This proof may be simplified using a more tractable definition of* $\mathrm{T}\mathbf{H}^{\frac{1}{2}}(\Gamma)$ *and* $\mathrm{H}^{\frac{1}{2}}(\Gamma)$*, e.g. the definition using the Sobolev-Slobodeckij norm as in [3].*

3.2 Pure natural boundary conditions

In this subsection, we establish the well-posedness of the VVP formulation of the Stokes problem with pure natural boundary conditions in the case of $N=3$ to illustrate the iterated use of 3.1. The adaptation of the arguments to the case of $N=2$ amounts to replacing $\mathbf{H}(\text{curl})$ by H^1. We consider the variational problem (18) with general source functions $\mathbf{f}=(\mathbf{f}_1,\mathbf{f}_2)\in(\mathbf{H}(\text{curl})\times\mathbf{H}(\text{div}))'$ and $g\in\mathrm{L}^2$ on the right-hand side

$$\begin{cases}\text{Seek } \mathbf{w}\in\mathbf{H}(\text{curl}),\ \mathbf{u}\in\mathbf{H}(\text{div}), p\in\mathrm{L}^2: \\ \qquad (\mathbf{w},\mathbf{z})-(\nabla\times\mathbf{z},\mathbf{u})=\langle\mathbf{f}_1,\mathbf{z}\rangle_{\mathbf{H}(\text{curl})',\mathbf{H}(\text{curl})}, & \forall\mathbf{z}\in\mathbf{H}(\text{curl}),\\ \qquad (\nabla\times\mathbf{w},\mathbf{v})-(p,\nabla\cdot\mathbf{v})=\langle\mathbf{f}_2,\mathbf{v}\rangle_{\mathbf{H}(\text{div})',\mathbf{H}(\text{div})}, & \forall\mathbf{v}\in\mathbf{H}(\text{div}),\\ \qquad (q,\nabla\cdot\mathbf{u})=(g,q), & \forall q\in\mathrm{L}^2,\end{cases} \tag{38}$$

and present the following result:

Theorem 3.8. (38) *is well-posed.*

Proof. We first invoke 3.1 with the following setting:

$$\begin{aligned}V&=\mathbf{H}(\text{curl})\times\mathbf{H}(\text{div}),\ Q=\mathrm{L}^2,\\ a((\mathbf{w},\mathbf{u}),(\mathbf{z},\mathbf{v}))&=(\mathbf{w},\mathbf{z})-(\nabla\times\mathbf{z},\mathbf{u})-(\nabla\times\mathbf{w},\mathbf{v}),\\ b((\mathbf{z},\mathbf{v}),p)&=(\nabla\cdot\mathbf{v},p),\end{aligned}$$

where V is endowed with the norm $\|(\mathbf{a},\mathbf{b})\|_V=\|\mathbf{a}\|_{\mathbf{H}(\text{curl})}+\|\mathbf{b}\|_{\mathbf{H}(\text{div})}$.

For (29), let $q\in\mathrm{L}^2$ and $\boldsymbol{\varphi}\in\mathbf{H}(\text{div}_0)^\perp$ be such that

$$\nabla\cdot\boldsymbol{\varphi}=q.$$

We then have the following inequality:

$$\begin{aligned}\sup_{(\mathbf{z},\mathbf{v})\in V}\frac{|b((\mathbf{z},\mathbf{v}),q)|}{\|(\mathbf{z},\mathbf{v})\|_V}&=\sup_{(\mathbf{0},\mathbf{v})\in V}\frac{|b((\mathbf{0},\mathbf{v}),q)|}{\|(\mathbf{0},\mathbf{v})\|_V}=\sup_{(\mathbf{0},\mathbf{v})\in V}\frac{|b((\mathbf{0},\mathbf{v}),q)|}{\|\mathbf{v}\|_{\mathbf{H}(\text{div})}}\\ &\geq\frac{|b((\mathbf{0},\boldsymbol{\varphi}),q)|}{\|\boldsymbol{\varphi}\|_{\mathbf{H}(\text{div})}}=\frac{\|q\|_{\mathrm{L}^2}^2}{\|\boldsymbol{\varphi}\|_{\mathbf{H}(\text{div})}}\\ &\geq\frac{\|q\|_{\mathrm{L}^2}^2}{C_{p_1}\|q\|_{\mathrm{L}^2}}=\frac{1}{C_{p_1}}\|q\|_{\mathrm{L}^2},\end{aligned}$$

where in the last step the Poincaré–Friedrichs inequality (25) is used. This establishes (29).

For (28), we note that

$$(\forall q \in \mathrm{L}^2 : \, b((\mathbf{z},\mathbf{v}),q) = 0) \iff \nabla\cdot \mathbf{v} = 0,$$

i.e.

$$N = \mathbf{H}(\mathrm{curl}) \times \mathbf{H}(\mathrm{div}_0).$$

Observing that a on $N \times N$ is a bilinear form in mixed form, we invoke Theorem 3.1 again with the following setting:

$$\begin{aligned} \tilde{V} = \mathbf{H}(\mathrm{curl})\,,\ \tilde{Q} &= \mathbf{H}(\mathrm{div}_0), \\ \tilde{a}(\mathbf{w},\mathbf{z}) &= (\mathbf{w},\mathbf{z}), \\ \tilde{b}(\mathbf{z},\mathbf{u}) \ &= -(\nabla\times \mathbf{z},\mathbf{u}). \end{aligned}$$

For (29), let $\mathbf{v} \in \mathbf{H}(\mathrm{div}_0)$ and $\boldsymbol{\rho} \in \mathbf{H}(\mathrm{curl}_0)^\perp$ be such that

$$\nabla\times \boldsymbol{\rho} = \mathbf{v}.$$

It follows that

$$\begin{aligned} \sup_{\mathbf{z}\in V} \frac{\left|\tilde{b}(\mathbf{z},\mathbf{v})\right|}{\|\mathbf{z}\|_{\tilde{V}}} &\geq \frac{\left|\tilde{b}(\boldsymbol{\rho},\mathbf{v})\right|}{\|\boldsymbol{\rho}\|_{\mathbf{H}(\mathrm{curl})}} = \frac{\|\mathbf{v}\|_{\mathbf{L}^2}^2}{\|\boldsymbol{\rho}\|_{\mathbf{H}(\mathrm{curl})}} \\ &\geq \frac{\|\mathbf{v}\|_{\mathbf{L}^2}^2}{C_{p_2}\|\mathbf{v}\|_{\mathbf{L}^2}} = \frac{1}{C_{p_2}}\|\mathbf{v}\|_{\mathbf{L}^2} = \frac{1}{C_{p_2}}\|\mathbf{v}\|_{\tilde{Q}}, \end{aligned}$$

by the Poincaré–Friedrichs inequality (26), which establishes (29).

Observing that

$$\tilde{N} = \mathbf{H}(\mathrm{curl}_0),$$

(28) can be easily verified since $\tilde{a}$ is coercive with respect to $\|\cdot\|_{\mathbf{H}(\mathrm{curl})}$ on $\tilde{N}$. □

Remark 3.9. *We can see that the de Rahm complex* (23) *is used every time we invoke Theorem 3.1. Similarly, we can use the de Rahm complex* (24) *to show the well-posedness of the VVP formulation for essential boundary conditions, namely*

$$\begin{aligned} \mathbf{w} \times \mathbf{n} = \mathbf{0} \ \ &\text{on } \Gamma, \\ \mathbf{u} \cdot \mathbf{n} = 0 \ \ &\text{on } \Gamma. \end{aligned}$$

The importance of the Hilbert complex, in particular for the optimal convergence rates of numerical methods, has already been indicated in [2].

3.3 Navier slip boundary conditions

Recall that we have the following two variational formulations for $N = 2$ and α uniformly positive:

$$\begin{cases} \text{Seek } w \in \mathrm{H}^1,\ \mathbf{u} \in \mathring{\mathbf{H}}(\mathrm{div}),\ p \in \hat{\mathrm{L}}^2 : \\ \quad (w, z) - (\nabla\times z, \mathbf{u}) + (\alpha^{-1} w, z)_\Gamma = 0, & \forall z \in \mathrm{H}^1, \\ \quad (\nabla\times w, \mathbf{v}) - (p, \nabla\cdot \mathbf{v}) = (\mathbf{f}, \mathbf{v}), & \forall \mathbf{v} \in \mathring{\mathbf{H}}(\mathrm{div}), \\ \quad (q, \nabla\cdot \mathbf{u}) = 0, & \forall q \in \hat{\mathrm{L}}^2. \end{cases} \tag{39}$$

$$\begin{cases} \text{Seek } w \in \Sigma,\ \mathbf{u} \in \mathring{\mathbf{H}}(\mathrm{div}),\ p \in \hat{\mathrm{L}}^2, \boldsymbol{\psi} \in \mathrm{T}\mathbf{H}^{\frac{1}{2}}(\Gamma) : \\ \quad \langle w \times \mathbf{n}, \boldsymbol{\eta}\rangle_\Gamma + (\alpha\boldsymbol{\psi}, \boldsymbol{\eta})_\Gamma = 0, & \forall \boldsymbol{\eta} \in \mathrm{T}\mathbf{H}^{\frac{1}{2}}(\Gamma), \\ \quad (w, z) - \langle\nabla\times z, \mathbf{u}\rangle - \langle z \times \mathbf{n}, \boldsymbol{\psi}\rangle_\Gamma = 0, & \forall z \in \Sigma, \\ \quad \langle\nabla\times w, \mathbf{v}\rangle - (p, \nabla \cdot \mathbf{v}) = (\mathbf{f}, \mathbf{v}), & \forall \mathbf{v} \in \mathring{\mathbf{H}}(\mathrm{div}), \\ \quad (q, \nabla \cdot \mathbf{u}) = 0, & \forall q \in \hat{\mathrm{L}}^2. \end{cases} \tag{40}$$

These formulations are related in the following sense

Theorem 3.10. *Assume Γ is $C^{1,1}$ and $\alpha \in \mathrm{L}^\infty(\Gamma)$ is bounded away from 0. Let $(w, \mathbf{u}, p) \in \mathrm{H}^1 \times \mathring{\mathbf{H}}(\mathrm{div}) \times \hat{\mathrm{L}}^2$.*
Then $(w, \mathbf{u}, p)$ is a solution to (39) if and only if there exists $\boldsymbol{\psi} \in \mathrm{T}\mathbf{H}^{\frac{1}{2}}(\Gamma)$ such that $(w, \mathbf{u}, p, \boldsymbol{\psi})$ is a solution to (40).

Proof. We first assume that $(w, \mathbf{u}, p) \in \mathrm{H}^1 \times \mathring{\mathbf{H}}(\mathrm{div}) \times \hat{\mathrm{L}}^2$ is a solution to (39). Testing the first equation of (39) with $z \in C_c^\infty$, we obtain that

$$w = \nabla\times \mathbf{u}. \tag{41}$$

Since $w \in \mathrm{H}^1$, $\mathbf{u} \in \mathbf{H}(\mathrm{curl})$. By the last equation in (39) and using $\nabla\cdot(\mathring{\mathbf{H}}(\mathrm{div})) = \hat{\mathrm{L}}^2$, we obtain that $\nabla\cdot \mathbf{u} = 0$. Hence $\mathbf{u} \in \mathbf{H}(\mathrm{curl}) \cap \mathring{\mathbf{H}}(\mathrm{div}_0)$ and the assumptions on the domain allows us to invoke [12, Prop. 3.1] to deduce that $\mathbf{u} \in \mathbf{H}^1 \cap \mathring{\mathbf{H}}(\mathrm{div}_0)$. Now testing with $z \in \mathrm{H}^1$ and using (41), the first equation in (39) yields

$$-(\mathbf{u} \times \mathbf{n}, z)_\Gamma + (\alpha^{-1} w, z)_\Gamma = 0, \quad \forall z \in \mathrm{H}^1.$$

Let $\boldsymbol{\psi} = \gamma(\mathbf{u}) \in \mathrm{T}\mathbf{H}^{\frac{1}{2}}(\Gamma)$. The assumption on Γ implies the density of $\mathrm{H}^{\frac{1}{2}}(\Gamma)$ in $\mathrm{L}^2(\Gamma)$, and we obtain that

$$\alpha^{-1}\gamma(w) = \boldsymbol{\psi} \times \mathbf{n}. \tag{42}$$

Since

$$(\boldsymbol{\psi} \times \mathbf{n}, z)_\Gamma = -\langle z \times \mathbf{n}, \boldsymbol{\psi} \rangle_\Gamma,$$

the first equation in (39) can now be reformulated as

$$(w, z) - (\nabla\times z, \mathbf{u}) - \langle z \times \mathbf{n}, \boldsymbol{\psi} \rangle_\Gamma = 0, \quad \forall z \in \mathrm{H}^1. \tag{43}$$

Let $\boldsymbol{\eta} \in \mathrm{T}\mathbf{H}^{\frac{1}{2}}(\Gamma)$. Multiplying (42) by α and testing with $\boldsymbol{\eta} \times \mathbf{n}$ yields

$$\begin{aligned} 0 &= -(w, \boldsymbol{\eta} \times \mathbf{n})_\Gamma + (\alpha\boldsymbol{\psi} \times \mathbf{n}, \boldsymbol{\eta} \times \mathbf{n})_\Gamma \\ &= \langle w \times \mathbf{n}, \boldsymbol{\eta} \rangle_\Gamma + (\alpha\boldsymbol{\psi} \times \mathbf{n}, \boldsymbol{\eta} \times \mathbf{n})_\Gamma \\ &= \langle w \times \mathbf{n}, \boldsymbol{\eta} \rangle_\Gamma + (\alpha\boldsymbol{\psi}, \boldsymbol{\eta})_\Gamma, \quad \forall \boldsymbol{\eta} \in \mathrm{T}\mathbf{H}^{\frac{1}{2}}(\Gamma). \end{aligned} \tag{44}$$

By (43) and (44), we know that $(w, \mathbf{u}, p, \boldsymbol{\psi})$ satisfies the weak formulation (40) after restricting z to be in H^1. The continuity of the left-hand side with respect to $\|\cdot\|_\Sigma$ and the density of H^1 in Σ [7, Prop. 4.4] then implies that $(w, \mathbf{u}, p, \boldsymbol{\psi})$ is a solution to (40).

Now let $(w, \mathbf{u}, p, \boldsymbol{\psi}) \in \mathrm{H}^1 \times \mathring{\mathbf{H}}(\mathrm{div}) \times \hat{\mathrm{L}}^2 \times \mathrm{T}\mathbf{H}^{\frac{1}{2}}(\Gamma)$ be a solution to (40). By the same argument as in (44), the first equation can be reformulated as

$$-(w, \boldsymbol{\eta} \times \mathbf{n})_\Gamma + (\alpha\boldsymbol{\psi} \times \mathbf{n}, \boldsymbol{\eta} \times \mathbf{n})_\Gamma = 0, \quad \forall \boldsymbol{\eta} \in \mathrm{T}\mathbf{H}^{\frac{1}{2}}(\Gamma),$$

which is equivalent to the following condition by Theorem 3.5

$$-(w, z)_\Gamma + (\alpha\boldsymbol{\psi} \times \mathbf{n}, z)_\Gamma = 0, \quad \forall z \in \mathrm{H}^1.$$

Again by the density of $\mathrm{H}^{\frac{1}{2}}(\Gamma)$ in $\mathrm{L}^2(\Gamma)$, we obtain that

$$\gamma(w) = \alpha\boldsymbol{\psi} \times \mathbf{n}. \tag{45}$$

Let $z \in \mathrm{H}^1$. The boundary term in the second equation in (40) can be reformulated using (45) as

$$-\langle z \times \mathbf{n}, \boldsymbol{\psi} \rangle_\Gamma = (\boldsymbol{\psi} \times \mathbf{n}, z)_\Gamma = (\alpha^{-1} w, z)_\Gamma$$

and we have

$$(w, z) - (\nabla\times z, \mathbf{u}) + (\alpha^{-1} w, z)_\Gamma = 0, \quad \forall z \in \mathrm{H}^1,$$

which implies that $(w, \mathbf{u}, p)$ is a solution to (39). □

Remark 3.11. *Note that if $\alpha = 2\mathcal{W}$, then $\alpha \in \mathrm{L}^\infty(\Gamma)$ by the assumption on Γ.*

Now we focus on establishing the well-posedness of (40) since it is more general than (39) as mentioned in Section 2. It should be noted that the uniqueness of the solution of the Stokes problem with Navier slip boundary conditions is in general not guaranteed. For example, the rotation field $\mathbf{u} = [-y, x]^T$ is a solution to the system on the domain $\Omega = \{ (x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1 \}$ with $\mathbf{f} = \mathbf{0}$. Thus, the goal of the remainder of this subsection is to show that the kernel of (40) has finite dimension, and the strategy is to view the term $(\alpha\boldsymbol{\psi}, \boldsymbol{\eta})_\Gamma$ as a compact perturbation of the following variational problem

$$\begin{cases} \text{Seek } \mathbf{w} \in \boldsymbol{\Sigma},\ \mathbf{u} \in \mathring{\mathbf{H}}(\mathrm{div}),\ p \in \hat{\mathrm{L}}^2, \boldsymbol{\psi} \in \mathrm{T}\mathbf{H}^{\frac{1}{2}}(\Gamma): \\ \qquad \langle \mathbf{w} \times \mathbf{n}, \boldsymbol{\eta} \rangle_\Gamma = \langle \mathbf{f}_1, \boldsymbol{\eta} \rangle_\Gamma \quad \forall \boldsymbol{\eta} \in \mathrm{T}\mathbf{H}^{\frac{1}{2}}(\Gamma), \\ (\mathbf{w}, \mathbf{z}) - \langle \nabla\times \mathbf{z}, \mathbf{u} \rangle - \langle \mathbf{z} \times \mathbf{n}, \boldsymbol{\psi} \rangle_\Gamma = \langle \mathbf{f}_2, \mathbf{z} \rangle_{\boldsymbol{\Sigma}', \boldsymbol{\Sigma}} \quad \forall \mathbf{z} \in \boldsymbol{\Sigma}, \\ \qquad \langle \nabla\times \mathbf{w}, \mathbf{v} \rangle - (p, \nabla \cdot \mathbf{v}) = \langle \mathbf{f}_3, \mathbf{v} \rangle \quad \forall \mathbf{v} \in \mathring{\mathbf{H}}(\mathrm{div}), \\ \qquad (q, \nabla \cdot \mathbf{u}) = (f_4, q) \quad \forall q \in \hat{\mathrm{L}}^2. \end{cases} \tag{46}$$

where $\mathbf{f}_1 \in (\mathrm{T}\mathbf{H}^{\frac{1}{2}}(\Gamma))', \mathbf{f}_2 \in \boldsymbol{\Sigma}', \mathbf{f}_3 \in \mathring{\mathbf{H}}(\mathrm{div})', f_4 \in \mathrm{L}^2$ are general source terms.

We now present the following result

Theorem 3.12. (46) *is well-posed iff the following condition is satisfied*

$$\inf_{\boldsymbol{\psi} \in \mathrm{T}\mathbf{H}^{\frac{1}{2}}(\Gamma)} \sup_{\boldsymbol{\tau} \in \boldsymbol{\mathcal{H}}} \frac{|\langle \boldsymbol{\tau} \times \mathbf{n}, \boldsymbol{\psi} \rangle|}{\|\boldsymbol{\tau}\|_{\boldsymbol{\Sigma}} \|\boldsymbol{\psi}\|_{\mathrm{T}\mathbf{H}^{\frac{1}{2}}(\Gamma)}} > 0, \tag{47}$$

which is equivalent to the surjectivity of $\gamma_t : \boldsymbol{\Sigma} \to (\mathrm{T}\mathbf{H}^{\frac{1}{2}}(\Gamma))'$.

Proof. Using the same argument as in Subsection 3.2, we can reduce the system to

$$\begin{cases} \text{Seek } \mathbf{w} \in \boldsymbol{\Sigma},\ \mathbf{u} \in \mathring{\mathbf{H}}(\mathrm{div}_0),\ \boldsymbol{\psi} \in \mathrm{T}\mathbf{H}^{\frac{1}{2}}(\Gamma): \\ \qquad \langle \mathbf{w} \times \mathbf{n}, \boldsymbol{\eta} \rangle_\Gamma = \langle \mathbf{f}_1, \boldsymbol{\eta} \rangle_\Gamma \quad \forall \boldsymbol{\eta} \in \mathrm{T}\mathbf{H}^{\frac{1}{2}}(\Gamma), \\ (\mathbf{w}, \mathbf{z}) - \langle \nabla\times \mathbf{z}, \mathbf{u} \rangle - \langle \mathbf{z} \times \mathbf{n}, \boldsymbol{\psi} \rangle_\Gamma = \langle \mathbf{f}_2, \mathbf{z} \rangle_{\boldsymbol{\Sigma}', \boldsymbol{\Sigma}} \quad \forall \mathbf{z} \in \boldsymbol{\Sigma}, \\ \qquad \langle \nabla\times \mathbf{w}, \mathbf{v} \rangle = \langle \mathbf{f}_3, \mathbf{v} \rangle \quad \forall \mathbf{v} \in \mathring{\mathbf{H}}(\mathrm{div}_0), \end{cases} \tag{48}$$

by invoking Theorem 3.1 and using the de Rahm complex (24). Now we invoke Theorem 3.1 again with the following setting:

$$\begin{aligned} V = \boldsymbol{\Sigma} \times \mathrm{T}\mathbf{H}^{\frac{1}{2}}(\Gamma),\ Q = \mathring{\mathbf{H}}(\mathrm{div}_0), \\ a((\mathbf{w}, \boldsymbol{\psi}), (\mathbf{z}, \boldsymbol{\eta})) = (\mathbf{w}, \mathbf{z}) - \langle \mathbf{z} \times \mathbf{n}, \boldsymbol{\psi} \rangle_\Gamma - \langle \mathbf{w} \times \mathbf{n}, \boldsymbol{\eta} \rangle_\Gamma, \\ b((\mathbf{z}, \boldsymbol{\eta}), \mathbf{u}) = -\langle \nabla\times \mathbf{z}, \mathbf{u} \rangle, \end{aligned}$$

where V is equipped with the norm

$$\|(\mathbf{a},\mathbf{b})\|_V = \|\mathbf{a}\|_{\mathbf{\Sigma}} + \|\mathbf{b}\|_{\mathrm{T}\mathbf{H}^{\frac{1}{2}}(\Gamma)}.$$

Observing that for $\mathbf{z} \in \mathbf{H}(\mathrm{curl}) \subseteq \mathbf{\Sigma}$ it holds that

$$|\langle \nabla\times \mathbf{z}, \mathbf{u}\rangle| = |(\nabla\times \mathbf{z}, \mathbf{u})| \leq \|\nabla\times \mathbf{z}\|_{\mathbf{L}^2}\|\mathbf{u}\|_{\mathbf{H}(\mathrm{div})}, \quad \forall \mathbf{u} \in \mathring{\mathbf{H}}(\mathrm{div}),$$

we have

$$\|\nabla\times \mathbf{z}\|'_{\mathring{\mathbf{H}}(\mathrm{div})} \leq \|\nabla\times \mathbf{z}\|_{\mathbf{L}^2}, \quad \forall \mathbf{z} \in \mathbf{H}(\mathrm{curl}),$$

and hence

$$\|\nabla\times \mathbf{z}\|_{\mathbf{\Sigma}} \leq C_{p_2}\|\nabla\times \mathbf{z}\|_{\mathbf{L}^2}, \quad \forall \mathbf{z} \in \mathring{\mathbf{H}}(\mathrm{curl}_0)^{\perp},$$

by the Poincaré–Friedrichs inequality (26). Let $\mathbf{v} \in \mathring{\mathbf{H}}(\mathrm{div}_0)$ and $\boldsymbol{\rho} \in \mathring{\mathbf{H}}(\mathrm{curl}_0)^{\perp}$ such that

$$\nabla\times \boldsymbol{\rho} = \mathbf{v}.$$

It follows that

$$\begin{aligned}\sup_{(\mathbf{z},\boldsymbol{\eta})\in V} \frac{|b((\mathbf{z},\boldsymbol{\eta}),\mathbf{v})|}{\|(\mathbf{z},\boldsymbol{\eta})\|_V} &\geq \frac{|b((\boldsymbol{\rho},\mathbf{0}),\mathbf{v})|}{\|\boldsymbol{\rho}\|_{\mathbf{\Sigma}}} = \frac{\|\mathbf{v}\|^2_{\mathbf{L}^2}}{\|\boldsymbol{\rho}\|_{\mathbf{\Sigma}}} \\ &\geq \frac{\|\mathbf{v}\|^2_{\mathbf{L}^2}}{C_{p_2}\|\mathbf{v}\|_{\mathbf{L}^2}} = \frac{1}{C_{p_2}}\|\mathbf{v}\|_{\mathbf{L}^2} = \frac{1}{C_{p_2}}\|\mathbf{v}\|_Q,\end{aligned}$$

which establishes (29) for (48).

Since for $\boldsymbol{\tau} \in \mathbf{\Sigma}$ it holds that

$$(\forall \mathbf{v} \in \mathring{\mathbf{H}}(\mathrm{div}_0) : \langle \nabla\times \boldsymbol{\tau}, \mathbf{v}\rangle = 0) \iff (\forall \boldsymbol{\xi} \in \mathring{\mathbf{H}}(\mathrm{curl}) : \langle \nabla\times \boldsymbol{\tau}, \nabla\times \boldsymbol{\xi}\rangle = 0)$$

we obtain that

$$N = \boldsymbol{\mathcal{H}} \times \mathrm{T}\mathbf{H}^{\frac{1}{2}}(\Gamma).$$

Now we invoke Theorem 3.1 again with

$$\begin{aligned}&\tilde{V} = \boldsymbol{\mathcal{H}},\ \tilde{Q} = \mathrm{T}\mathbf{H}^{\frac{1}{2}}(\Gamma), \\ &\tilde{a}(\mathbf{w},\mathbf{z}) = (\mathbf{w},\mathbf{z}), \\ &\tilde{b}(\mathbf{z},\boldsymbol{\psi}) = -\langle \mathbf{z}\times\mathbf{n}, \boldsymbol{\psi}\rangle_\Gamma,\end{aligned}$$

and $\tilde{N}$ is determined By

$$\tilde{N} = \{\, \boldsymbol{\tau} \in \boldsymbol{\mathcal{H}} \mid \langle \boldsymbol{\tau}\times\mathbf{n}, \boldsymbol{\psi}\rangle_\Gamma = 0,\ \forall \boldsymbol{\psi} \in \mathrm{T}\mathbf{H}^{\frac{1}{2}}(\Gamma) \,\} = \{\, \boldsymbol{\tau} \in \boldsymbol{\mathcal{H}} \mid \boldsymbol{\tau}\times\mathbf{n} = \mathbf{0} \,\}.$$

Since $\tilde{a}$ is coercive by (37), it is also coercive on $\tilde{N} \subseteq \boldsymbol{\mathcal{H}}$ and the condition (28) is easily verified. Hence the condition for the well-posedness of (46) is reduced to

$$\inf_{\boldsymbol{\psi} \in \mathrm{T}\mathbf{H}^{\frac{1}{2}}(\Gamma)} \sup_{\boldsymbol{\tau} \in \boldsymbol{\mathcal{H}}} \frac{\left|\tilde{b}(\boldsymbol{\tau}, \boldsymbol{\psi})\right|}{\|\boldsymbol{\tau}\|_{\boldsymbol{\Sigma}} \|\boldsymbol{\psi}\|_{\mathrm{T}\mathbf{H}^{\frac{1}{2}}(\Gamma)}} > 0,$$

which is equivalent to the surjectivity of

$$\begin{aligned} B : \boldsymbol{\mathcal{H}} &\to (\mathrm{T}\mathbf{H}^{\frac{1}{2}}(\Gamma))' \\ \boldsymbol{\tau} &\mapsto \tilde{b}(\boldsymbol{\tau}, \cdot), \end{aligned}$$

namely

$$B = \gamma_t|_{\boldsymbol{\mathcal{H}}}$$

by [9, Thm. C.40]. We conclude by noting that $\gamma_t(\boldsymbol{\Sigma}) = \gamma_t(\boldsymbol{\mathcal{H}})$ by (36). $\square$

To show (47), we first want to convert the bilinear form in the numerator into an inner product by finding a suitable extension of $\boldsymbol{\psi} \in \mathrm{T}\mathbf{H}^{\frac{1}{2}}(\Gamma)$. Thanks to [12, Lemma 2.2], we can find an extension $\tilde{\boldsymbol{\psi}} \in \mathbf{H}^1$ of $\boldsymbol{\psi} \in \mathrm{T}\mathbf{H}^{\frac{1}{2}}(\Gamma)$ such that

$$\nabla \cdot \tilde{\boldsymbol{\psi}} = 0,$$

i.e. $\tilde{\boldsymbol{\psi}} \in \mathring{\mathbf{H}}(\mathrm{div}_0) \cap \mathbf{H}^1$. For $\boldsymbol{\tau} \in \boldsymbol{\mathcal{H}}$, it then holds that

$$\langle \boldsymbol{\tau} \times \mathbf{n}, \boldsymbol{\psi} \rangle_\Gamma = (\boldsymbol{\tau}, \nabla \times \tilde{\boldsymbol{\psi}}) - \langle \nabla \times \boldsymbol{\tau}, \tilde{\boldsymbol{\psi}} \rangle = (\boldsymbol{\tau}, \nabla \times \tilde{\boldsymbol{\psi}})$$

by Proposition 3.3 (2). Now let us denote $\mathring{\mathbf{H}}(\mathrm{div}_0) \cap \mathbf{H}^1$ as $\mathbf{U}$ and define the bilinear form

$$c : \boldsymbol{\mathcal{H}} \times \mathbf{U} \to \mathbb{R}, \ c(\boldsymbol{\tau}, \boldsymbol{\xi}) = \langle \boldsymbol{\tau} \times \mathbf{n}, \gamma(\boldsymbol{\xi}) \rangle_\Gamma = (\boldsymbol{\tau}, \nabla \times \boldsymbol{\xi}), \tag{49}$$

(47) is then equivalent to

$$\inf_{\boldsymbol{\xi} \in \tilde{\mathbf{U}}} \sup_{\boldsymbol{\tau} \in \boldsymbol{\mathcal{H}}} \frac{|c(\boldsymbol{\tau}, \boldsymbol{\xi})|}{\|\boldsymbol{\tau}\|_{\boldsymbol{\Sigma}} \|\gamma(\boldsymbol{\xi})\|_{\mathrm{T}\mathbf{H}^{\frac{1}{2}}(\Gamma)}} > 0, \tag{50}$$

where $\tilde{\mathbf{U}} := \mathbf{U} \setminus \ker(\gamma) = \mathbf{U} \setminus (\mathring{\mathbf{H}}(\mathrm{div}_0) \cap \mathring{\mathbf{H}}^1)$.

To establish (50), we show that for all $\boldsymbol{\psi} \in \mathrm{T}\mathbf{H}^{\frac{1}{2}}(\Gamma)$, there is an extension $\boldsymbol{\xi} \in \mathbf{U}$ of $\boldsymbol{\psi}$ such that $\nabla \times \boldsymbol{\xi} \in \boldsymbol{\mathcal{H}}$. Then we choose $\boldsymbol{\tau} = \nabla \times \boldsymbol{\xi}$ and it suffices to show that

$$\|\boldsymbol{\tau}\|_{\mathbf{L}^2} \|\nabla \times \boldsymbol{\xi}\|_{\mathbf{L}^2} \geq C \|\boldsymbol{\tau}\|_{\boldsymbol{\Sigma}} \|\gamma(\boldsymbol{\xi})\|_{\mathrm{T}\mathbf{H}^{\frac{1}{2}}(\Gamma)}, \quad \forall \boldsymbol{\tau} \in \boldsymbol{\mathcal{H}} \ \forall \boldsymbol{\xi} \in \mathbf{U}$$

for some generic constant C. This will follow from some norm equivalence on the spaces $\boldsymbol{\Sigma}$ and $\mathbf{U}$.

Let $\mathrm{L} := \nabla\times \mathbf{U}$ and define the operators C and C^* as

$$C : \boldsymbol{\mathcal{H}} \to \mathbf{U}', \quad C(\boldsymbol{\tau}) = c(\boldsymbol{\tau}, \cdot),$$
$$C^* : \mathbf{U} \to \boldsymbol{\mathcal{H}}', \; C^*(\boldsymbol{\xi}) = c(\cdot, \boldsymbol{\xi}).$$

We note that if there exists $\boldsymbol{\xi} \in \ker(C^*)$ such that $\gamma(\boldsymbol{\xi}) \neq 0$, then $\boldsymbol{\xi} \in \tilde{\mathbf{U}}$ and (50) can not hold. The following lemma excludes this case:

Lemma 3.13.

$$\ker(C^*) = \mathring{\mathbf{H}}(\mathrm{div}_0) \cap \mathring{\mathbf{H}}^1.$$

Proof. First we note that

$$\forall \boldsymbol{\xi} \in \mathring{\mathbf{H}}(\mathrm{div}_0) \cap \mathring{\mathbf{H}}^1 : \; C^*(\boldsymbol{\xi}) = \langle \gamma_t(\cdot), \gamma(\boldsymbol{\xi}) \rangle_\Gamma = 0,$$

hence

$$\mathring{\mathbf{H}}(\mathrm{div}_0) \cap \mathring{\mathbf{H}}^1 \subset \ker(C^*).$$

Now let $\boldsymbol{\xi} \in \ker(C^*)$. It holds that

$$\forall \boldsymbol{\tau} \in \boldsymbol{\mathcal{H}} : \; c(\boldsymbol{\tau}, \boldsymbol{\xi}) = \langle \gamma_t(\boldsymbol{\tau}), \gamma(\boldsymbol{\xi}) \rangle_\Gamma = 0. \tag{51}$$

Since $\gamma_t(\boldsymbol{\Sigma}) = \gamma_t(\boldsymbol{\mathcal{H}})$ by (**??**), the condition (51) can be extended to $\boldsymbol{\Sigma}$. Since $\mathbf{H}^1 \subset \boldsymbol{\Sigma}$, we have

$$\forall \boldsymbol{\tau} \in \mathbf{H}^1 : \; 0 = (\gamma_t(\boldsymbol{\tau}), \gamma(\boldsymbol{\xi})) = (\boldsymbol{\tau}, \nabla\times \boldsymbol{\xi}) - (\nabla\times \boldsymbol{\tau}, \boldsymbol{\xi}) = -(\gamma(\boldsymbol{\tau}), \gamma_t(\boldsymbol{\xi})).$$

This implies that $\gamma_t(\boldsymbol{\xi}) = 0$, i.e. $\boldsymbol{\xi} \in \mathbf{U} \cap \mathring{\mathbf{H}}(\mathrm{curl}) = \mathring{\mathbf{H}}(\mathrm{div}_0) \cap \mathring{\mathbf{H}}(\mathrm{curl}) \cap \mathbf{H}^1$. By [12, Lemma 2.5], it holds that $\mathring{\mathbf{H}}(\mathrm{div}) \cap \mathring{\mathbf{H}}(\mathrm{curl}) = \mathring{\mathbf{H}}^1$ and hence $\boldsymbol{\xi} \in \mathring{\mathbf{H}}(\mathrm{div}_0) \cap \mathring{\mathbf{H}}^1$. This implies that

$$\ker(C^*) \subset \mathring{\mathbf{H}}(\mathrm{div}_0) \cap \mathring{\mathbf{H}}^1.$$

□

Remark 3.14. *The argument in showing* $\ker(C^*) \subset \mathring{\mathbf{H}}(\mathrm{div}_0) \cap \mathring{\mathbf{H}}^1$ *proves that*

$$\forall \boldsymbol{\psi} \in \mathrm{T}\mathbf{H}^{\frac{1}{2}}(\Gamma) : \; [\forall \boldsymbol{\tau} \in \boldsymbol{\Sigma} : \; \langle \gamma_t(\boldsymbol{\tau}), \boldsymbol{\psi} \rangle_\Gamma = 0] \implies [\boldsymbol{\psi} = 0].$$

Since $\mathrm{T}\mathbf{H}^{\frac{1}{2}}(\Gamma)$ *is reflexive, this shows that* $\gamma_t(\boldsymbol{\Sigma})$ *is dense in* $\left(\mathrm{T}\mathbf{H}^{\frac{1}{2}}(\Gamma)\right)'$.

Lemma 3.15. *The following decompositions are orthogonal with respect to* $\mathbf{L}^2$*:*

(1) $\boldsymbol{\mathcal{H}} = \ker(C) \oplus (\bar{\mathrm{L}} \cap \boldsymbol{\mathcal{H}})$,

(2) $\bar{\mathrm{L}} = \overline{\nabla\times \ker(C^*)} \oplus (\bar{\mathrm{L}} \cap \boldsymbol{\mathcal{H}})$.

Proof. Let us denote $\mathring{\mathbf{H}}(\mathrm{div}_0) \cap \mathring{\mathbf{H}}^1$ as $\mathbf{U}_0$.

(1) By definition of C,

$$\boldsymbol{\tau} \in \ker(C) \Leftrightarrow \boldsymbol{\tau} \in \boldsymbol{\mathcal{H}} \text{ and } [\,\forall \boldsymbol{\xi} \in \mathbf{U} : \, (\boldsymbol{\tau}, \nabla\times \boldsymbol{\xi}) = 0\,].$$

This implies that

$$\ker(C) = \bar{\mathrm{L}}^{\perp} \cap \boldsymbol{\mathcal{H}}$$

and the statement would follow if $\bar{\mathrm{L}}^{\perp} \cap \boldsymbol{\mathcal{H}} = \bar{\mathrm{L}}^{\perp}$. Now since

$$\forall \boldsymbol{\xi} \in \mathbf{U}_0 : \; (\boldsymbol{\tau}, \nabla\times \boldsymbol{\xi}) = \langle \nabla\times \boldsymbol{\tau}, \boldsymbol{\xi} \rangle_{(\mathring{\mathbf{H}}^1)', \mathring{\mathbf{H}}^1},$$

by Proposition 3.3 we have that

$$[\,\forall \boldsymbol{\xi} \in \mathbf{U} : \, (\boldsymbol{\tau}, \nabla\times \boldsymbol{\xi}) = 0\,] \Rightarrow [\,\forall \boldsymbol{\xi} \in \mathbf{U}_0 : \, \langle \nabla\times \boldsymbol{\tau}, \boldsymbol{\xi} \rangle_{(\mathring{\mathbf{H}}^1)', \mathring{\mathbf{H}}^1} = 0\,] \Rightarrow \boldsymbol{\tau} \in \boldsymbol{\mathcal{H}}.$$

Hence

$$\boldsymbol{\tau} \in \bar{\mathrm{L}}^{\perp} \Leftrightarrow [\,\forall \boldsymbol{\xi} \in \mathbf{U} : \, (\boldsymbol{\tau}, \nabla\times \boldsymbol{\xi}) = 0\,] \Rightarrow \boldsymbol{\tau} \in \boldsymbol{\mathcal{H}}$$

and $\bar{\mathrm{L}}^{\perp} \cap \boldsymbol{\mathcal{H}} = \bar{\mathrm{L}}^{\perp}$.

(2) We observe that

$$\begin{aligned} \boldsymbol{\xi} \in \ker(C^*) &\Rightarrow [\,(\boldsymbol{\tau}, \nabla\times \boldsymbol{\xi}) = 0 \;\; \forall \boldsymbol{\tau} \in \boldsymbol{\mathcal{H}}\,] \\ &\Rightarrow \nabla\times \boldsymbol{\xi} \in \boldsymbol{\mathcal{H}}^{\perp} \text{ in } \mathbf{L}^2 \Rightarrow \nabla\times \boldsymbol{\xi} \in (\bar{\mathrm{L}} \cap \boldsymbol{\mathcal{H}})^{\perp} \text{ in } \bar{\mathrm{L}}, \end{aligned}$$

hence

$$\overline{\nabla\times \ker(C^*)} \subset (\bar{\mathrm{L}} \cap \boldsymbol{\mathcal{H}})^{\perp} \text{ in } \bar{\mathrm{L}}. \tag{52}$$

On the other hand, let $\boldsymbol{\varphi} \in \bar{\mathrm{L}}$, it holds that

$$\begin{aligned} [\,(\boldsymbol{\varphi}, \nabla\times \boldsymbol{\xi}) = 0 \;\; \forall \boldsymbol{\xi} \in \mathbf{U}_0\,] &\Rightarrow [\,\langle \nabla\times \boldsymbol{\varphi}, \boldsymbol{\xi} \rangle_{(\mathring{\mathbf{H}}^1)', \mathring{\mathbf{H}}^1} = 0 \;\; \forall \boldsymbol{\xi} \in \mathbf{U}_0\,] \\ &\Rightarrow \boldsymbol{\varphi} \in \boldsymbol{\mathcal{H}} \Rightarrow \boldsymbol{\varphi} \in \bar{\mathrm{L}} \cap \boldsymbol{\mathcal{H}}, \end{aligned}$$

where in the second last step we used Proposition 3.3. This implies that

$$(\nabla\times \mathbf{U}_0)^{\perp} \subset \bar{\mathrm{L}} \cap \boldsymbol{\mathcal{H}} \text{ in } \bar{\mathrm{L}}.$$

and hence

$$(\bar{\mathrm{L}} \cap \boldsymbol{\mathcal{H}})^{\perp} \subset \overline{\nabla\times \mathbf{U}_0} \text{ in } \bar{\mathrm{L}}. \tag{53}$$

Combining (52)-(53) and using Lemma 3.13, we obtain that

$$(\bar{\mathrm{L}} \cap \boldsymbol{\mathcal{H}})^{\perp} \subset \overline{\nabla\times \mathbf{U}_0} = \overline{\nabla\times \ker(C^*)} \subset (\bar{\mathrm{L}} \cap \boldsymbol{\mathcal{H}})^{\perp} \text{ in } \bar{\mathrm{L}},$$

which proves the statement.

□

Now we can prove the well-posedness of (46) under some assumptions on Ω:

Theorem 3.16. *Assume that Ω satisfies one of the following*

(1) Γ is $C^{1,1}$,

(2) $\Omega \subset \mathbb{R}^2$ and Γ is piecewise smooth without reentrant corners,

(3) $\Omega \subset \mathbb{R}^3$ is a convex polyhedron.

Then (46) *is well-posed.*

Proof. We want to establish (50), namely

$$\inf_{\boldsymbol{\xi}\in\tilde{\mathbf{U}}} \sup_{\boldsymbol{\tau}\in\boldsymbol{\mathcal{H}}} \frac{|c(\boldsymbol{\tau},\boldsymbol{\xi})|}{\|\boldsymbol{\tau}\|_{\boldsymbol{\Sigma}}\|\gamma(\boldsymbol{\xi})\|_{\mathbf{TH}^{\frac{1}{2}}(\Gamma)}} > 0.$$

Let us first show that

$$\exists C > 0 : \ \forall \boldsymbol{\xi} \in \mathbf{U} : \ \|\nabla\times \boldsymbol{\xi}\|_{\mathbf{L}^2} \geq C\|\boldsymbol{\xi}\|_{\mathbf{H}^1}. \tag{54}$$

For $\Omega \subset \mathbb{R}^2$, this is established by the assumptions on Ω and by [12, Prop. 3.1, Remark 3.5]. For $\Omega \subset \mathbb{R}^3$, by the assumptions on Ω and [12, Thm. 3.8-3.9], $\mathring{\mathbf{H}}(\mathrm{div}) \cap \mathbf{H}(\mathrm{curl})$ is continuously imbedded in $\mathbf{H}^1$. This implies that

$$\exists C_1 > 0 : \ \forall \boldsymbol{\xi} \in \mathbf{U} : \ \|\boldsymbol{\xi}\|_{\mathbf{H}^1} \leq C_1 \left(\|\boldsymbol{\xi}\|_{\mathbf{L}^2} + \|\nabla\times \boldsymbol{\xi}\|_{\mathbf{L}^2}\right). \tag{55}$$

Furthermore, [12, Lemma 3.6] gives that

$$\exists C_2 > 0 : \ \forall \boldsymbol{\xi} \in \mathbf{U} : \ \|\boldsymbol{\xi}\|_{\mathbf{L}^2} + \|\nabla\times \boldsymbol{\xi}\|_{\mathbf{L}^2} \leq C_2\|\nabla\times \boldsymbol{\xi}\|_{\mathbf{L}^2}. \tag{56}$$

Combining (55) and (56) we obtain (54) for $\Omega \subset \mathbb{R}^3$.

We observe that by (54), $\|\nabla\times\cdot\|_{\mathbf{L}^2}$ is an equivalent norm on $\mathbf{U}$. Hence

$$\nabla\times : \mathbf{U} \to \mathrm{L}$$

is an isomorphism. Since $\mathbf{U}$ as a closed subspace of $\mathbf{H}^1$ is a Banach space, L is also a Banach space and thus closed in $\mathbf{L}^2$. Moreover, since $\ker(C^*)$ is closed in $\mathbf{U}$, $\nabla\times\ker(C^*) = \nabla\times(\mathring{\mathbf{H}}(\mathrm{div}_0)\cap\mathring{\mathbf{H}}^1)$ is also closed in L and hence in $\mathbf{L}^2$. Therefore, Lemma 3.15 gives the orthogonal decomposition

$$\mathrm{L} = \nabla\times\ker(C^*) \oplus (\mathrm{L}\cap\boldsymbol{\mathcal{H}}) \tag{57}$$

in $\mathbf{L}^2$. Now let $\boldsymbol{\xi}\in\tilde{\mathbf{U}}$ and let

$$\nabla\times\boldsymbol{\xi} = \nabla\times\boldsymbol{\varphi} + \boldsymbol{\eta}, \quad \boldsymbol{\varphi}\in\ker(C^*),\ \boldsymbol{\eta}\in\mathrm{L}\cap\boldsymbol{\mathcal{H}}$$

be the decomposition of $\nabla\times\boldsymbol{\xi}$ given by (57). Since $\boldsymbol{\xi}\notin\mathring{\mathbf{H}}(\mathrm{div}_0)\cap\mathring{\mathbf{H}}^1$, $\boldsymbol{\xi}\notin\ker(C^*)$ by Lemma 3.13 and hence $\boldsymbol{\eta}\neq 0$. Then it holds that

$$\begin{aligned}
(\eta, \nabla\times\boldsymbol{\xi}) &= (\eta, \nabla\times(\boldsymbol{\xi}-\boldsymbol{\varphi})) \\
&= \|\boldsymbol{\eta}\|_{\mathbf{L}^2}^2 = \|\boldsymbol{\eta}\|_{\mathbf{L}^2}\|\nabla\times(\boldsymbol{\xi}-\boldsymbol{\varphi})\|_{\mathbf{L}^2} \\
&\geq C\|\boldsymbol{\eta}\|_{\mathbf{L}^2}\|\boldsymbol{\xi}-\boldsymbol{\varphi}\|_{\mathbf{H}^1} \\
&\geq C\|\boldsymbol{\eta}\|_{\mathbf{L}^2}\|\gamma(\boldsymbol{\xi}-\boldsymbol{\varphi})\|_{\mathrm{T}\mathbf{H}^{\frac{1}{2}}(\Gamma)} = C\|\boldsymbol{\eta}\|_{\boldsymbol{\Sigma}}\|\gamma\boldsymbol{\xi}\|_{\mathrm{T}\mathbf{H}^{\frac{1}{2}}(\Gamma)},
\end{aligned}$$

where in the second last inequality we used (54) and in the last equality we used (37) and that $\gamma(\boldsymbol{\varphi}) = 0$ since $\boldsymbol{\varphi}\in\mathring{\mathbf{H}}^1$. This establishes (50) and proves the result. □

Corollary 3.16.1. *Assume Ω satisfies the same assumptions as in Theorem 3.16. Then γ_t is surjective onto $(\mathrm{T}\mathbf{H}^{\frac{1}{2}}(\Gamma))'$.*

Let us denote the bilinear form on the left-hand side of (46) as $\tilde{a}$, namely

$$\begin{aligned}
&V := \boldsymbol{\Sigma}\times\mathring{\mathbf{H}}(\mathrm{div})\times\hat{\mathrm{L}}^2\times\mathrm{T}\mathbf{H}^{\frac{1}{2}}(\Gamma), \quad \tilde{a} : V\times V\to\mathbb{R}, \\
&\tilde{a}((\mathbf{w},\mathbf{u},p,\boldsymbol{\psi}),(\mathbf{z},\mathbf{v},q,\boldsymbol{\eta})) = (\mathbf{w},\mathbf{z}) - \langle\nabla\times\mathbf{z},\mathbf{u}\rangle + \langle\nabla\times\mathbf{w},\mathbf{v}\rangle \\
&\qquad\qquad - (p,\nabla\cdot\mathbf{v}) + (q,\nabla\cdot\mathbf{u}) \\
&\qquad\qquad - \langle\mathbf{z}\times\mathbf{n},\boldsymbol{\psi}\rangle_\Gamma + \langle\mathbf{w}\times\mathbf{n},\boldsymbol{\eta}\rangle_\Gamma,
\end{aligned}$$

and define the bilinear form

$$k : V\times V\to\mathbb{R}^2, \quad k((\mathbf{w},\mathbf{u},p,\boldsymbol{\psi}),(\mathbf{z},\mathbf{v},q,\boldsymbol{\eta})) = (\alpha\boldsymbol{\psi},\boldsymbol{\eta})_\Gamma.$$

Let us also define

$$a := \tilde{a} + k$$

and the linear operators associated with a, $\tilde{a}$ and k as $A, \tilde{A}, K : V \to V'$, respectively:

$$A(\boldsymbol{\xi}) := a(\boldsymbol{\xi}, \cdot), \quad \tilde{A}(\boldsymbol{\xi}) := \tilde{a}(\boldsymbol{\xi}, \cdot), \quad K(\boldsymbol{\xi}) := k(\boldsymbol{\xi}, \cdot).$$

As mentioned at the beginning of the subsection, we view k as a compact perturbation and obtain that

Theorem 3.17. *Assume Ω satisfies the same assumptions as in Theorem 3.16. Then* $\ker(A)$ *is finite-dimensional.*

Proof. By Theorem 3.16, $\tilde{A}$ is a isomorphism and hence $\tilde{A}^{-1}$ exists and is continuous. On the other hand, the imbedding of $\mathrm{T}\mathbf{H}^{\frac{1}{2}}(\Gamma)$ in $\mathbf{L}^2(\Gamma)$ is compact by [10, Lemma 4.5], which implies that K is compact. Hence $\tilde{A}^{-1}K$ is also compact by [4, Prop. 6.3]. The Fredholm alternative [4, Thm. 6.6] asserts that $\ker(I + \tilde{A}^{-1}K)$ is finite-dimensional. We conclude by $A = \tilde{A}(I + \tilde{A}^{-1}K)$. □

4 The discrete problem

In this section, we discretize the VVP formulations obtained in Section 2 in the case of $N = 2$. The resulting discrete problems are then implemented in the next section for numerical tests.

4.1 Preliminaries

We need to introduce some notations for the finite element spaces. Throughout the section, let $\{\Omega_h\}_h$ be a quasi-uniform and uniformly shape regular family of meshes with mesh sizes $h > 0$ and $\{\Gamma_h\}_h$ be the $(N-1)$-dimensional boundary meshes obtained by restricting $\{\Omega_h\}_h$ to Γ. We will denote the Lagrange finite elements, the Ravier-Thomas finite elements and piecewise polynomial space of degree $r \geq 0$ on Ω_h as S_h^r, V_h^r and Q_h^r, respectively. The following subcomplex of the de Rahm complex (23) holds on Ω_h for $r > 0$

$$S_h^r \xrightarrow{\nabla\times} V_h^{r-1} \xrightarrow{\nabla\cdot} Q_h^{r-1}, \tag{58}$$

which will be the guidance of our choice of the degrees of finite element spaces.

We will also need the restriction of S_h^r to the boundary mesh, which will be denoted by $T_h^r = \gamma(S_h^r)$. Since γ is continuous from H^1 onto $\mathrm{H}^{\frac{1}{2}}(\Gamma)$ and $\{S_h^r\}_h$ is a dense sequence in H^1, $\{T_h^r\}_h$ is a dense sequence in $\mathrm{H}^{\frac{1}{2}}(\Gamma)$.

4.2 Discretizations

For the approximation of the Weingarten map, we note that on 2D the Weingarten map corresponds to the scalar curvature and consider the following variational problem as described in [16, Section 3]

$$\begin{cases} \text{Seek } \kappa \in \mathrm{H}^{\frac{1}{2}}(\Gamma) : \\ (\kappa, \psi)_\Gamma = \displaystyle\sum_{v \in V_\Gamma} \arccos(n_L(v) \cdot n_R(v))\psi(v) + (\mathbf{t} \cdot (\nabla \mathbf{n} \cdot \mathbf{t}), \psi)_\Gamma, \quad \forall \psi \in \mathrm{H}^{\frac{1}{2}}(\Gamma), \end{cases} \tag{59}$$

where V_Γ denotes the set of boundary vertices, κ the scalar curvature, $\mathbf{t}$ the tangential vector on the boundary edges, and $n_L(v)$ and $n_R(v)$ the normal vector of the left and right boundary edges that are connected by vertex v. The right-hand side of (59) splits the rotation angle weighted by ψ into two parts: the first term describes the change of angle across every vertex, and the second the rotation within every edge. We discretize (59) by restricting the trial and test space for κ to T_h^r and the approximation of α obtained in this way will be denoted as α_h in the following.

Using S_h^r as the finite element space for H^1, V_h^{r-1} for $\mathbf{H}(\mathrm{div})$, P_h^{r-1} for L^2 and considering general source functions on the right-hand side, we discretize (18) as

$$\begin{cases} \text{Seek } w_h \in S_h^r,\ \mathbf{u}_h \in V_h^{r-1}, p_h \in P_h^{r-1} : \\ \quad (w_h, z_h) - (\nabla\times z_h, \mathbf{u}_h) = \langle f_1, z_h\rangle_{(\mathrm{H}^1)',\mathrm{H}^1}, & \forall z_h \in S_h^r, \\ (\nabla\times w_h, \mathbf{v}_h) - (p_h, \nabla\cdot \mathbf{v}_h) = \langle \mathbf{f}_2, \mathbf{v}_h\rangle_{\mathbf{H}(\mathrm{div})',\mathbf{H}(\mathrm{div})}, & \forall \mathbf{v}_h \in V_h^{r-1}, \\ \quad (q_h, \nabla\cdot \mathbf{u}_h) = (g, q_h), & \forall q_h \in P_h^{r-1}. \end{cases} \tag{60}$$

For (20), we replace the trial and test space for $\mathbf{u}$ by $\mathring{V}_h^{r-1} := V_h^{r-1} \cap \mathring{\mathbf{H}}(\mathrm{div})$ and introduce a Lagrange multiplier to enforce the condition of vanishing mean value of p_h (see [13, Problem 2-15]). The discrete problem then reads

$$\begin{cases} \text{Seek } w_h \in S_h^r,\ \mathbf{u}_h \in \mathring{V}_h^{r-1},\ p_h \in P_h^{r-1}, \lambda \in \mathbb{R} : \\ (w_h, z_h) - (\nabla\times z_h, \mathbf{u}_h) + (\alpha_h^{-1} w_h, z_h)_\Gamma = \langle f_1, z_h\rangle_{(\mathrm{H}^1)',\mathrm{H}^1}, & \forall z_h \in S_h^r, \\ \quad (\nabla\times w_h, \mathbf{v}_h) - (p_h, \nabla\cdot \mathbf{v}_h) = \langle \mathbf{f}_2, \mathbf{v}_h\rangle, & \forall \mathbf{v}_h \in \mathring{V}_h^{r-1}, \\ \quad (q_h, \nabla\cdot \mathbf{u}_h) + (q_h, \lambda) = (g, q_h), & \forall q_h \in P_h^{r-1}, \\ \quad (p_h, \lambda') = 0, & \forall \lambda' \in \mathbb{R}. \end{cases} \tag{61}$$

For (22), we keep the finite element spaces of $\mathbf{u}$ and p unchanged since the function spaces for these variables are the same as in (20). For w, we also use the same finite element space as above since $\{S_h^r\}_h$ is a dense sequence in Σ. To see this, we note that $\{S_h^r\}_h$ is a dense sequence in H^1 also with respect to $\|\cdot\|_\Sigma$ since $\|\cdot\|_\Sigma$ is controlled by $\|\cdot\|_{\mathrm{H}^1}$. Hence, the density of $\{S_h^r\}_h$ in Σ follows from the density of H^1 in Σ. For the variable $\boldsymbol{\psi}$, we note that for w, z in H^1 the first two equations in (22) can be reformulated using the argument in the proof of Theorem 3.10 as

$$-(w, \boldsymbol{\eta} \times \mathbf{n})_\Gamma + (\alpha\boldsymbol{\psi} \times \mathbf{n}, \boldsymbol{\eta} \times \mathbf{n})_\Gamma = 0, \quad \forall \boldsymbol{\eta} \in \mathrm{T}\mathbf{H}^{\frac{1}{2}}(\Gamma), \tag{62}$$

$$(w, z) - (\nabla\times z, \mathbf{u}) + (\boldsymbol{\psi} \times \mathbf{n}, z)_\Gamma = 0, \quad \forall z \in \mathrm{H}^1, \tag{63}$$

which is equivalent to the following conditions in the case that Γ is $C^{1,1}$ by Theorem 3.5

$$-(w, \eta)_\Gamma + (\alpha\psi, \eta)_\Gamma = 0, \quad \forall \eta \in \mathrm{H}^{\frac{1}{2}}(\Gamma), \tag{64}$$

$$(w, z) - (\nabla\times z, \mathbf{u}) + (\psi, z)_\Gamma = 0, \quad \forall z \in \mathrm{H}^1, \tag{65}$$

where ψ is sought in $\mathrm{H}^{\frac{1}{2}}(\Gamma)$. Since the finite element space for w is chosen to be $S_h^r \subset \mathrm{H}^1$, the above reformulation holds on the discrete level for any choice of the finite element space R_h for $\boldsymbol{\psi}$. Moreover, if $\{R_h\}_h$ is a dense sequence of subspaces in $\mathrm{T}\mathbf{H}^{\frac{1}{2}}(\Gamma)$ and Γ is $C^{1,1}$, $\{R_h\} \times \mathbf{n}_h$ would be a dense sequence of subspaces in $\mathrm{H}^{\frac{1}{2}}(\Gamma)$ by Theorem 3.5. Therefore, the discretization of (62) and (63) would be equivalent to that of (64) and (65). We discretize the latter and choose T_h^r as the finite element space for ψ. Again considering general source functions on the right-hand side, the discrete problem is given by

$$\begin{cases} \text{Seek } w_h \in S_h^r,\, \mathbf{u}_h \in \mathring{V}_h^{r-1},\, p_h \in P_h^{r-1},\, \psi_h \in T_h^r,\, \lambda \in \mathbb{R}: \\ \qquad -(w_h, \eta_h)_\Gamma + (\alpha\psi_h, \eta_h)_\Gamma = \langle f_1, \eta_h \rangle_{\left(\mathrm{H}^{\frac{1}{2}}(\Gamma)\right)', \mathrm{H}^{\frac{1}{2}}(\Gamma)}, & \forall \eta_h \in T_h^r, \\ (w_h, z_h) - (\nabla\times z_h, \mathbf{u}_h) + (\psi_h, z_h)_\Gamma = \langle f_2, z_h \rangle_{(\mathrm{H}^1)', \mathrm{H}^1}, & \forall z_h \in S_h^r, \\ (\nabla\times w_h, \mathbf{v}_h) - (p_h, \nabla\cdot \mathbf{v}_h) = \langle \mathbf{f}_3, \mathbf{v}_h \rangle, & \forall \mathbf{v}_h \in \mathring{V}_h^{r-1}, \\ (q_h, \nabla\cdot \mathbf{u}_h) + (q_h, \lambda) = (g, q_h), & \forall q_h \in P_h^{r-1}, \\ (p_h, \lambda') = 0, & \forall \lambda' \in \mathbb{R}. \end{cases} \tag{66}$$

Comparing (66) with (61), we see that these two formulations differ only by an $\mathrm{L}^2(\Gamma)$ projection onto T_h^r, since (61) can be obtained if we require $\alpha_h\psi = w_h$ in (66) and assume α_h to be bounded away from 0. The similarity between the two formulations will be confirmed by the numerical results in the next section.

5 Numerical experiments

The numerical schemes derived in the last section are implemented using the open source finite element library `NGSolve` [17] and tested against numerical experiments. The source code is available in the git repository `https://gitlab.ethz.ch/peiyyu/hdiv-based-approximation-of-stokes`. In the following discussion, we will refer to the degree of the Lagrange finite element spaces S_h^r as the order of the finite element spaces used in a numerical scheme. For Navier slip boundary conditions, we refer to (61) and (66) as formulation A and B, respectively.

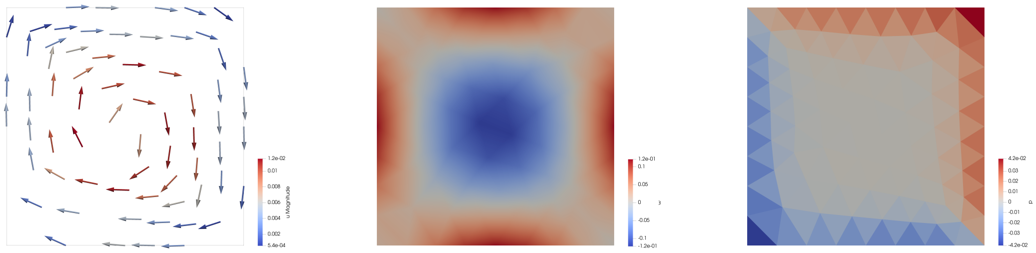
For Navier slip boundary conditions, we use a curved mesh to avoid variational crime due to the approximation of the Weingarten map. For order r finite element spaces, T_h^{r+2} is used to calculate α_h.

5.1 Pure natural boundary conditions

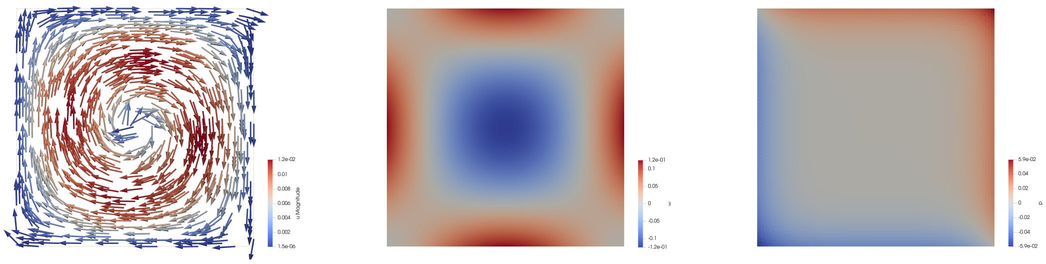
We consider $\Omega = (0,1)^2$ and choose the right-hand side f_1, $\mathbf{f}_2$ and g in (60) such that the reference solutions for $\mathbf{u} = (u_x, u_y)$, w and p are given by

$$\begin{aligned} u_x(x,y) &= -2x^2(x-1)^2 y(y-1)(2y-1), \\ u_y(x,y) &= -u_x(y,x), \\ \nabla\cdot \mathbf{u} &= 0, \\ w(x,y) &= \nabla\times \mathbf{u} = 2(y^2(y-1)^2(6x^2-6x+1) + x^2(x-1)^2(6y^2-6y+1)), \\ p(x,y) &= \left(x - \frac{1}{2}\right)^5 + \left(y - \frac{1}{2}\right)^5. \end{aligned}$$

The solutions from (60) of order 1 for $h = 2^{-3}$ and $h = 2^{-6}$ are in good accordance with the reference solution, as is shown in Fig. 1. The convergence of the scheme for orders 1 to 3 is demonstrated in Fig. 2. For the $\mathbf{H}(\mathrm{div})$ norm of $\mathbf{u}$, we see that the discrete solution preserves the divergence-free property of $\mathbf{u}$, and the $\mathbf{H}(\mathrm{div})$ error converges with rate r for order r finite element spaces. The L^2 error of p and H^1 error of w also converge with the same order as the order of finite element spaces, and the L^2 error of w enjoys an elevation of convergence order by 1. These optimal convergence rates are possibly related to the subcomplex of de Rahm complex (58), as has been pointed out in [2] for the vector Laplacian problem.

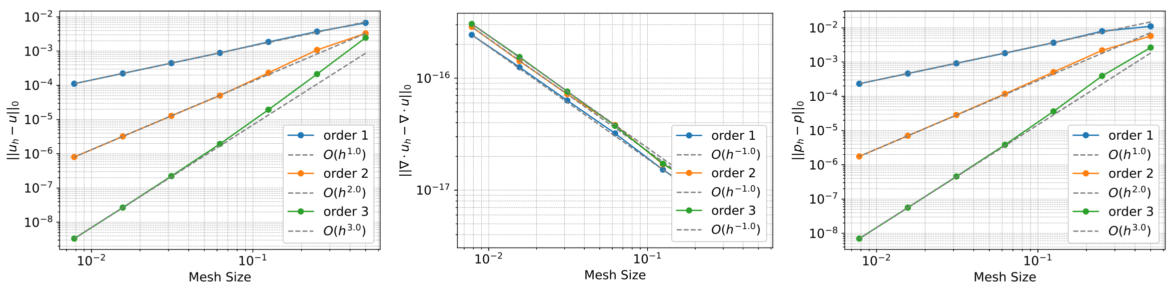


(a) $\mathbf{u}$, $h = 2^{-3}$. (b) w, $h = 2^{-3}$. (c) p, $h = 2^{-3}$.

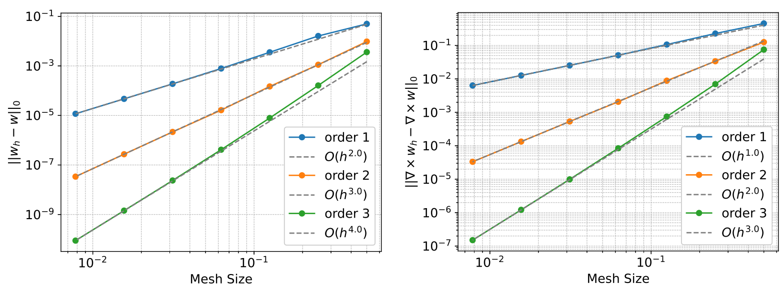


(d) $\mathbf{u}$, $h = 2^{-6}$. (e) w, $h = 2^{-6}$. (f) p, $h = 2^{-6}$.

Figure 1: Pure natural boundary conditions: color plots of $\mathbf{u}$, w and p for order 1, mesh size 2^{-3} and 2^{-6}.



(a) $\mathbf{L}^2$ error of $\mathbf{u}$. (b) L^2 error of $\nabla\cdot\mathbf{u}$. (c) L^2 error of p.



(d) L^2 error of w. (e) L^2 error of $\nabla\times w$.

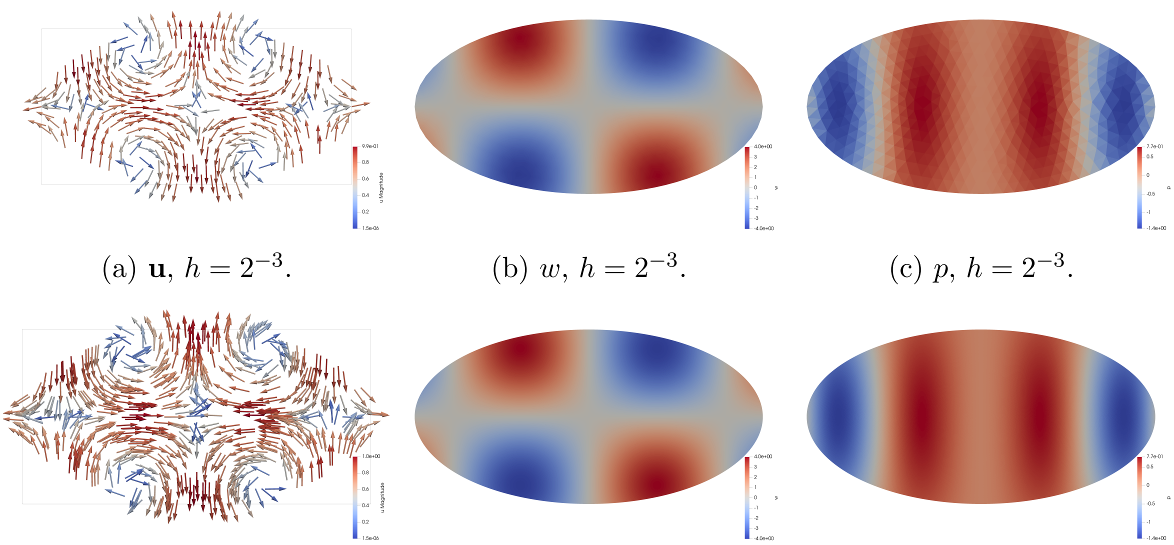
Figure 2: Pure natural boundary conditions: error plots of $\mathbf{u}$, w and p.

5.2 Navier slip boundary conditions

For Navier slip boundary conditions, we consider Ω to be an ellipse with semi-major axis of length 2 and semi-minor axis of length 1. The manufactured solutions are now defined by

$$\begin{aligned} u_x(x,y) &= -\sin(2x)\cos(2y), \\ u_y(x,y) &= -u_x(y,x), \\ w(x,y) &= \nabla\times\mathbf{u} = -4\sin(2x)\sin(2y), \\ p(x,y) &= x\sin(3x)\cos(y). \end{aligned}$$

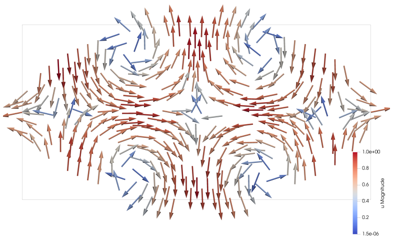
As is shown in Fig. 3 and Fig. 4, the solutions from the two formulations of order 1 show no significant difference and have no singularities. From Fig. 5 and Fig. 6 we see that the two formulations also share similar convergence behavior, and the convergence rates are summarized in Table 1. Compared to the optimal convergence rates obtained for pure natural boundary conditions, we get optimal convergence rates in the first order case, while for higher order finite element spaces the convergence rates are suboptimal by $\frac{1}{2}$ for the L^2 and H^1 error of w, which is a minor indication of the use of certain inverse inequality in the error analysis. We also note that the divergence



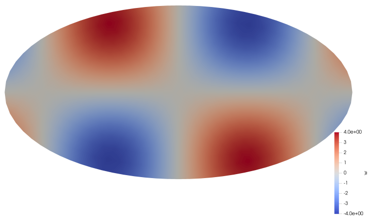
(a) $\mathbf{u}$, $h = 2^{-3}$. (b) w, $h = 2^{-3}$. (c) p, $h = 2^{-3}$.

(d) $\mathbf{u}$, $h = 2^{-6}$. (e) w, $h = 2^{-6}$. (f) p, $h = 2^{-6}$.

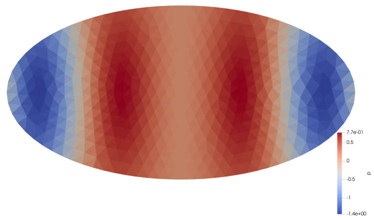
Figure 3: Navier slip boundary conditions: color plots of $\mathbf{u}$, w and p from formulation A for order 1, mesh size 2^{-3} and 2^{-6}.



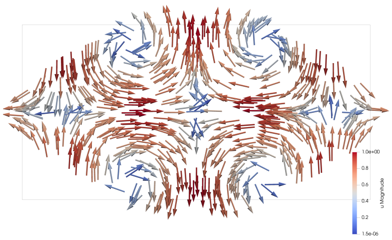
(a) $\mathbf{u}$, $h = 2^{-3}$.



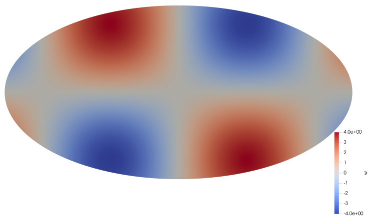
(b) w, $h = 2^{-3}$.



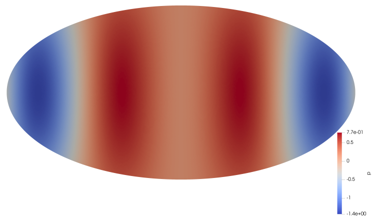
(c) p, $h = 2^{-3}$.



(d) $\mathbf{u}$, $h = 2^{-6}$.

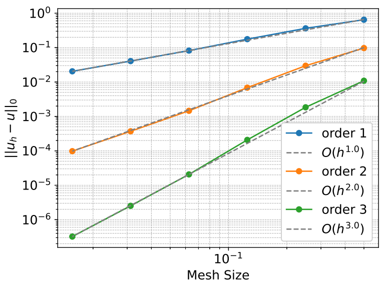


(e) w, $h = 2^{-6}$.

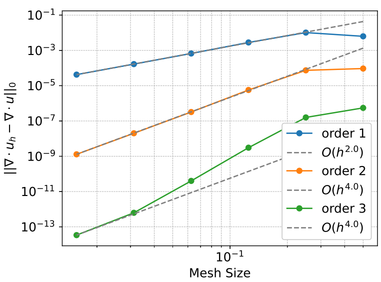


(f) p, $h = 2^{-6}$.

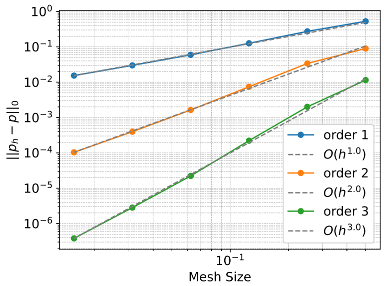
Figure 4: Navier slip boundary conditions: color plots of $\mathbf{u}$, w and p from formulation B for order 1, mesh size 2^{-3} and 2^{-6}.



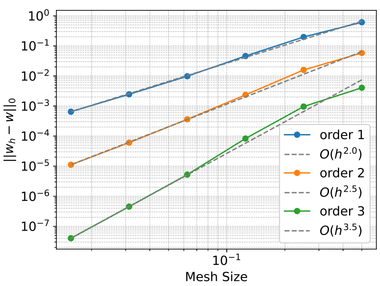
(a) $\mathbf{L}^2$ error of $\mathbf{u}$.



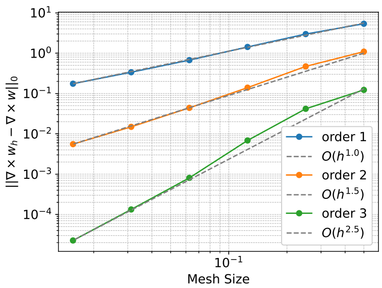
(b) L^2 error of $\nabla\cdot \mathbf{u}$.



(c) L^2 error of p.

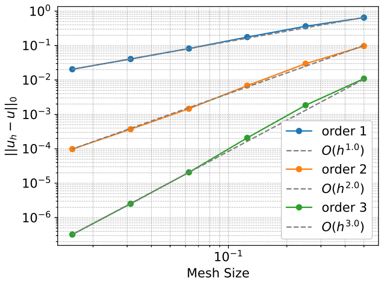


(d) L^2 error of w.

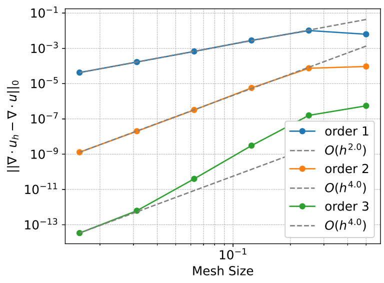


(e) L^2 error of $\nabla\times w$.

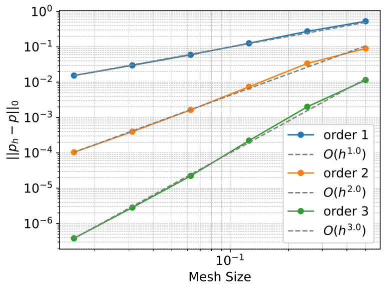
Figure 5: Navier slip boundary conditions: error plots of $\mathbf{u}$, w and p from formulation A.



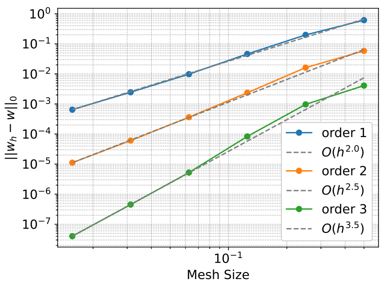
(a) $\mathbf{L}^2$ error of $\mathbf{u}$.



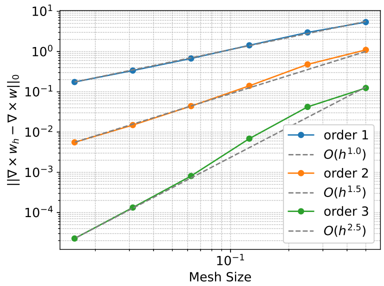
(b) L^2 error of $\nabla\cdot\mathbf{u}$.



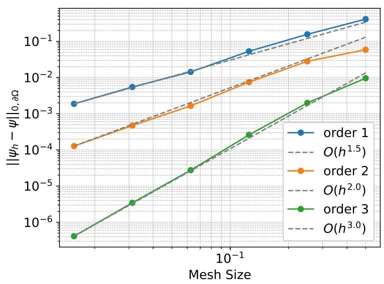
(c) L^2 error of p.



(d) L^2 error of w.



(e) L^2 error of $\nabla\times w$.



(f) $\mathrm{L}^2(\Gamma)$ error of ψ.

Figure 6: Navier slip boundary conditions: error plots of $\mathbf{u}$, w, p and ψ from formulation B.

free property of $\mathbf{u}$ is not exactly preserved in this case, which should be attributed to the curved mesh resulting in a non-linear map from the reference element to the finite elements near the boundary.

	$\|\mathbf{u}-\mathbf{u}_h\|_{\mathbf{H}(\mathrm{div})}$	$\|p-p_h\|_{\mathrm{L}^2}$	$\|w-w_h\|_{\mathrm{L}^2}$	$\|w-w_h\|_{\mathrm{H}^1}$	$\|\psi-\psi_h\|_{\mathrm{L}^2(\Gamma)}$
A	r	r	$r+1/r+0.5$	$r/r-0.5$	-
B	r	r	$r+1/r+0.5$	$r/r-0.5$	$r+0.5/r$

Table 1: Navier slip boundary conditions: convergence rates from formulation A and B using finite element spaces of order r. A slash means that the convergence rates for $r=1$ and $r>1$ are different, in which case the rate for $r=1$ is before the slash, the rate for $r>1$ after.

6 Conclusion

Motivated by the structure-preserving numerical schemes for the Navier-Stokes equation [18] and MHD [11], we try to derive VVP formulations for the Stokes problem with Navier slip boundary conditions using an equivalent formulation of the boundary condition proposed by Mitrea and Monniaux [14, Eq. 2.9]. A first variational formulation was derived by following the approach for pure natural boundary conditions, which poses significant constraint on the domain. To alleviate the issue, we enlarged the trial and test space for the vorticity and introduced a new variable for the tangential trace of the velocity, which allows us to enforce Navier slip boundary conditions in a Lagrange multiplier fashion. The resulting formulation is more general and shown to be well-posed on the continuous level, and the two formulations are closely related under certain assumptions. We also derived and simplified the discretization of the obtained variational problems, which was implemented using `NGSolve` [17] for numerical tests. Both formulations for Navier slip boundary conditions produce similar results in the numerical experiment, where the convergence rates for the L^2 error and H^1 error of w are suboptimal by half an order compared to the rates observed from pure natural boundary conditions in the case of finite element spaces with order higher than one. For the lowest order finite element spaces, optimal convergence rates are observed.

The stability and error analysis of the numerical schemes is still open for future work. In particular, the difference in the convergence behavior between the use of first order and higher order finite element spaces needs to be clarified. Moreover, additional numerical experiments will be conducted to compare the two formulations for Navier slip boundary conditions, with a focus on domains with non-invertible Weingarten maps.

References

[1] Douglas N Arnold. *Finite element exterior calculus*. SIAM, 2018.

[2] Douglas N Arnold, Richard S Falk, and Jay Gopalakrishnan. "Mixed finite element approximation of the vector Laplacian with Dirichlet boundary conditions". In: *Mathematical Models and Methods in Applied Sciences* 22.09 (2012), p. 1250024. DOI: 10.1142/S0218202512500248. eprint: https://doi.org/10.1142/S0218202512500248. URL: https://doi.org/10.1142/S0218202512500248.

[3] Wietse M Boon, Ralf Hiptmair, Wouter Tonnon, and Enrico Zampa. "H (curl)-based approximation of the Stokes problem with slip boundary conditions". In: *arXiv preprint arXiv:2407.13353* (2024).

[4] H Brezis. *Functional Analysis, Sobolev Spaces and Partial Differential Equations*. Springer, 2011.

[5] F. Dubois, M. Salaün, and S. Salmon. "First vorticity–velocity–pressure numerical scheme for the Stokes problem". In: *Computer Methods in Applied Mechanics and Engineering* 192.44 (2003), pp. 4877–4907. ISSN: 0045-7825. DOI: https://doi.org/10.1016/S0045-7825(03)00377-3. URL: https://www.sciencedirect.com/science/article/pii/S0045782503003773.

[6] François Dubois. "Vorticity–velocity–pressure formulation for the Stokes problem". In: *Mathematical Methods in the Applied Sciences* 25.13 (2002), pp. 1091–1119. DOI: https://doi.org/10.1002/mma.328. eprint: https://onlinelibrary.wiley.com/doi/pdf/10.1002/mma.328. URL: https://onlinelibrary.wiley.com/doi/abs/10.1002/mma.328.

[7] François Dubois, Michel Salaün, and Stéphanie Salmon. "Vorticity–velocity-pressure and stream function-vorticity formulations for the Stokes problem". In: *Journal de Mathématiques Pures et Appliquées* 82.11 (2003), pp. 1395–1451. ISSN: 0021-7824. DOI: https://doi.org/10.1016/j.matpur.2003.09.002. URL: https://www.sciencedirect.com/science/article/pii/S0021782403000862.

[8] Alexandre Ern and Jean-Luc Guermond. *Finite Elements*. Springer, 2021.

[9] Alexandre Ern, Jean-Luc Guermond, et al. *Finite elements II*. Springer, 2021.

[10] Gabriel N. Gatica and Salim Meddahi. "Finite element analysis of a time harmonic Maxwell problem with an impedance boundary condition". In: *IMA Journal of Numerical Analysis* 32.2 (July 2011), pp. 534–552. ISSN: 0272-4979. DOI: 10.1093/imanum/drq041. eprint: https://academic.oup.com/imajna/article-pdf/32/2/534/2257361/drq041.pdf. URL: https://doi.org/10.1093/imanum/drq041.

[11] Evan S. Gawlik and François Gay-Balmaz. "A finite element method for MHD that preserves energy, cross-helicity, magnetic helicity, incompressibility, and $\operatorname{div} B = 0$". In: *Journal of Computational Physics* 450 (2022), p. 110847. ISSN: 0021-9991. DOI: https://doi.org/10.1016/j.jcp.2021.110847. URL: https://www.sciencedirect.com/science/article/pii/S0021999121007427.

[12] Vivette Girault and Pierre-Arnaud Raviart. *Finite element methods for Navier-Stokes equations: theory and algorithms.* Vol. 5. Springer, 2012.

[13] Ralf Hiptmair. *Problem set in numerical methods for partial differential equations.* 2023. URL: https://people.math.ethz.ch/~grsam/NUMPDEFL/HOMEWORK/NPDEFL_Problems.pdf.

[14] Marius Mitrea and Sylvie Monniaux. "The nonlinear Hodge-Navier-Stokes equations in Lipschitz domains". In: *Differential and Integral Equations* 22.3/4 (2009), pp. 339–356. DOI: 10.57262/die/1356019778. URL: https://doi.org/10.57262/die/1356019778.

[15] J. C. Nédélec. "Éléments finis mixtes incompressibles pour l'équation de Stokes dans $\mathbb{R}^3$". In: *Numer. Math.* 39.1 (Feb. 1982), pp. 97–112. ISSN: 0029-599X. DOI: 10.1007/BF01399314. URL: https://doi.org/10.1007/BF01399314.

[16] Michael Neunteufel, Joachim Schöberl, and Kevin Sturm. "Numerical shape optimization of the Canham-Helfrich-Evans bending energy". In: *Journal of Computational Physics* 488 (2023), p. 112218. ISSN: 0021-9991. DOI: https://doi.org/10.1016/j.jcp.2023.112218. URL: https://www.sciencedirect.com/science/article/pii/S0021999123003133.

[17] Joachim Schöberl. "C++ 11 implementation of finite elements in NGSolve". In: *Institute for analysis and scientific computing, Vienna University of Technology* 30 (2014).

[18] Yi Zhang, Artur Palha, Marc Gerritsma, and Qinghe Yao. “A MEEVC discretization for two-dimensional incompressible Navier-Stokes equations with general boundary conditions”. In: *Journal of Computational Physics* 510 (2024), p. 113080. ISSN: 0021-9991. DOI: https://doi.org/10.1016/j.jcp.2024.113080. URL: https://www.sciencedirect.com/science/article/pii/S0021999124003292.