Numerics of Hyperbolic Partial Differential Equations

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(http://www.sam.math.ethz.ch/~hiptmair/tmp/NUMHYP_07.{pdf,ps})

Contents

Reporting errors

Please report any error or dubious manipulation/assertion/reasoning by e-mail !

Examples:

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From: "MrX" <mrx@student.ethz.ch>

To: hiptmair@sam.math.ethz.ch

Subject: NAPDE05: Error

Error on page XX, Section XX, Formula (XX):
index i has to be changed to j
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From: "MrX" <mrx@student.ethz.ch>
To: hiptmair@sam.math.ethz.ch
Subject: NAPDE05: Error

Page XX, Section XX, Theorem XX:
the sign in front of the \Psi seems to be wrong
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Teaching evaluation

Course-ID: **401-3652-00L** (Numerik der hyperbolischen Differentialgleichungen)

Date: Mon, June 4, 2007

Instructor's additional questions:

D1 Do you consider the discussion of numerical examples in course useful?

(1 $\hat{=}$ not at all, 2 $\hat{=}$ hardly ever, 3 $\hat{=}$ sometimes, 4 $\hat{=}$ fairly useful, 5 $\hat{=}$ very much so)

D2 Should more numerical examples be provided in the classroom?

(1 $\hat{=}$ already way too many, 2 $\hat{=}$ less would be more, 3 $\hat{=}$ just right currently, 4 $\hat{=}$ sometimes, 5 $\hat{=}$ many more throughout)

D3 Were theoretical and practical issues properly balanced in the course?

(1 $\hat{=}$ way too much theory, 2 $\hat{=}$ slightly too much theory, 3 $\hat{=}$ well balanced, 4 $\hat{=}$ slightly too little theory, 4 $\hat{=}$ way too little theory)

D4 Do you feel bothered when personally addressed in the classroom?

(1 $\hat{=}$ not at all, 2 $\hat{=}$ hardly ever, 3 $\hat{=}$ sometimes, 4 $\hat{=}$ fairly often, 5 $\hat{=}$ extremely)

D5 Were theoretical and programming exercises well balanced?

(1 $\hat{=}$ way too much theory, 2 $\hat{=}$ slightly too much theory, 3 $\hat{=}$ well balanced, 4 $\hat{=}$ slightly too much programming, 5 $\hat{=}$ way too much programming)

1 Scalar linear second-order wave equations

Notations (see [27, Sect. 7.1]):

Ω : spatial domain, open set $\subset \mathbb{R}^d$, $d \in \mathbb{N}$,
can be unbounded !

$]0,T[$: finite time interval, $T > 0 \;\hat{=}$ final time

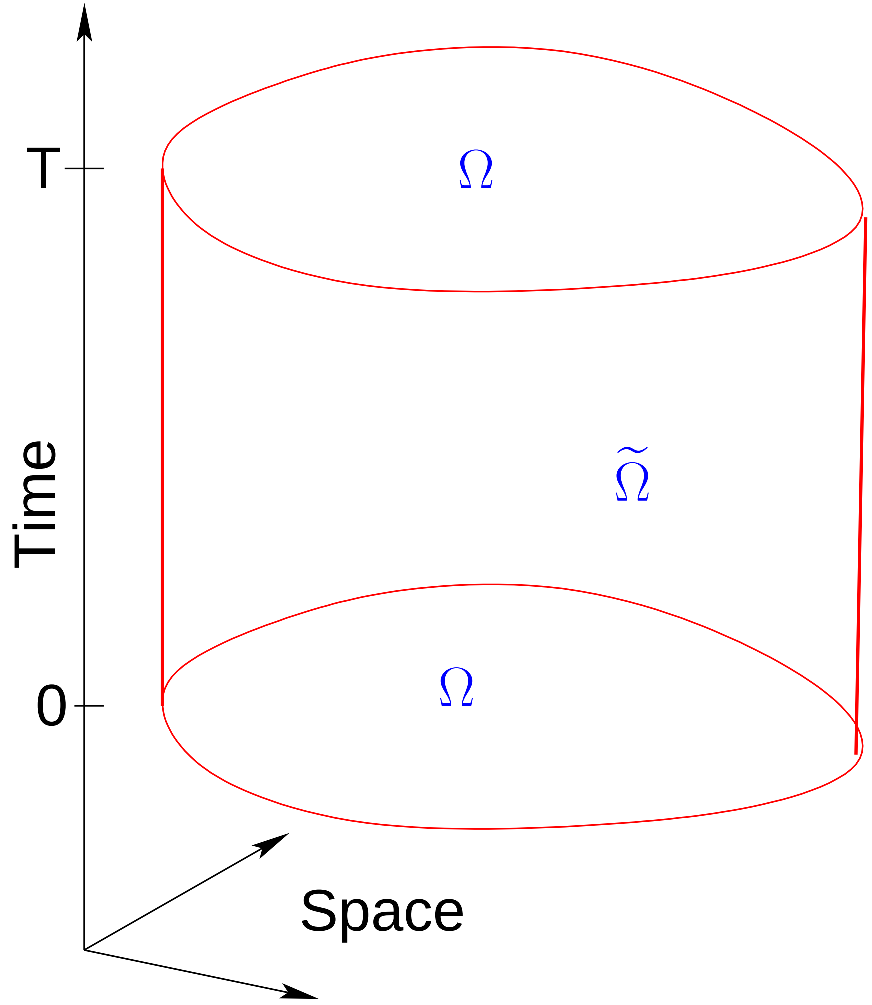
$\widetilde{\Omega}$: space-time cylinder, $\widetilde{\Omega} := \Omega \times]0,T[\subset \mathbb{R}^{d+1}$

$(\boldsymbol{x},t)$: instance in space-time

➤ function $u : \widetilde{\Omega} \mapsto \mathbb{R}$: $u = u(\boldsymbol{x},t)$
(solution of an evolution problem)

$\boldsymbol{n}$: unit normal vectorfield $\boldsymbol{n} : \partial\Omega \mapsto \mathbb{R}^d$

($\boldsymbol{x} \;\hat{=}$ spatial independent variable, $\boldsymbol{x} \in \Omega$,
$t \;\hat{=}$ temporal independent variable, $0 \le t \le T$)



1.1 Wave equations

Scalar 2nd-order spatial elliptic partial differential operator ($\rightarrow$ [27, Def. 2.3.1] & [27, (2.5.1)]):

$$\mathcal{L}_{\boldsymbol{x}}\, u := -\operatorname{div}_{\boldsymbol{x}}(\mathbf{C}\,\mathbf{grad}_{\boldsymbol{x}}\, u) + cu\ . \qquad (1.1.1)$$

differential operators act on $\boldsymbol{x}$ only !

- "conductivity tensor $\mathbf{C} \in L^\infty(\Omega, \mathbb{R}^{d,d})$ symmetric ($\mathbf{C} = \mathbf{C}^T$ a.e. in Ω) & uniformly positive definite, *cf.* [27, (2.2.3)]:

$$\exists \sigma^-, \sigma^+ > 0\colon \quad \sigma^- \left|\vec{\boldsymbol{\xi}}\right|^2 \leq \vec{\boldsymbol{\xi}}^T \mathbf{C}(\boldsymbol{x})\vec{\boldsymbol{\xi}} \leq \sigma^+ \left|\vec{\boldsymbol{\xi}}\right|^2 \quad \forall \vec{\boldsymbol{\xi}} \in \mathbb{R}^d, \quad \text{for almost all } \boldsymbol{x} \in \Omega\ . \qquad (1.1.2)$$

- "reaction coefficient" $c \in L^\infty(\Omega)$, uniformly positive : $c(\boldsymbol{x}) \geq 0$ a.e. in Ω

Terminology: (1.1.1) $\hat{=}$ divergence form

$$\mathbf{C} = \mathbf{I} \quad \rightarrow \quad \mathcal{L}_{\boldsymbol{x}} = -\Delta_{\boldsymbol{x}} = -\sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2} \quad \text{(Laplace operator, [27, Ex. 22])}$$

Definition 1.1.1 (Wave equation)**.** *Given a second-order linear scalar spatial elliptic differential operator* $\mathcal{L}_{\boldsymbol{x}}$*, a uniformly positive [27,* (2.8.2)*] "density"* $\rho \in L^\infty(\Omega)$*, and a source function* $f = f(\boldsymbol{x}, t) : \widetilde{\Omega} \mapsto \mathbb{R}$,

$$\rho \frac{\partial^2}{\partial t^2} u + \mathcal{L}_{\boldsymbol{x}} u = f(\boldsymbol{x}, t) \quad \text{in } \widetilde{\Omega} \tag{1.1.3}$$

is called a (scalar linear) wave equation for the unknown function $u = u(\boldsymbol{x}, t) : \widetilde{\Omega} \mapsto \mathbb{R}$.

☞ wave equations crucial for many mathematical models:

① *Vibrating membrane*

$\Omega \subset \mathbb{R}^2$: area occupied by flat membrane

$u = u(\boldsymbol{x}, t)$: displacement function, $[u] = 1\mathrm{m}$

➤ membrane at t:

$$M_t = \{(\boldsymbol{x}, u(\boldsymbol{x}, t)) \colon \boldsymbol{x} \in \Omega\}$$

Temporal evolution of displacement governed by

$$\rho \frac{\partial^2 u}{\partial t^2} - \operatorname{div}(\gamma \, \mathbf{grad}\, u) = f \qquad \textbf{(1.1.4)}$$

$\rho = \rho(\boldsymbol{x})$: area density, $[\rho] = \mathrm{kg\,m^{-2}}$
$\gamma = \gamma(\boldsymbol{x})$: stiffness, $[\gamma] = \mathrm{kg\,s^{-2}}$
$f = f(\boldsymbol{x}, t)$: force density, $[f] = N\mathrm{m}^{-2}$

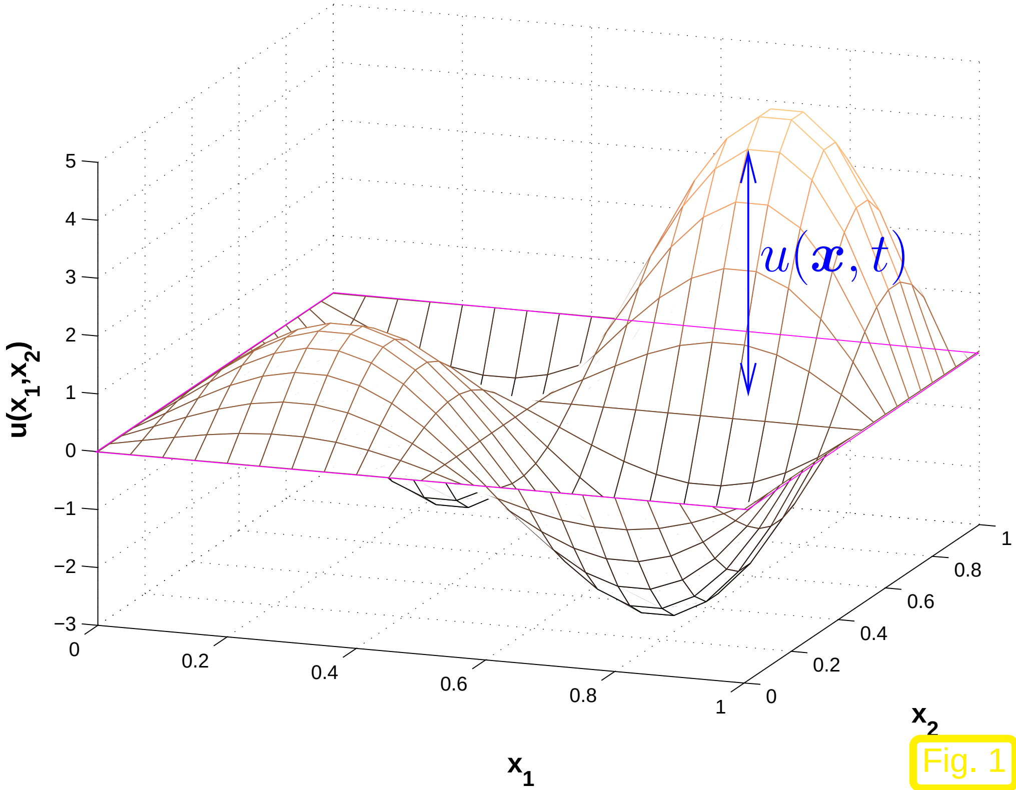


Fig. 1

② *Sound propagation*

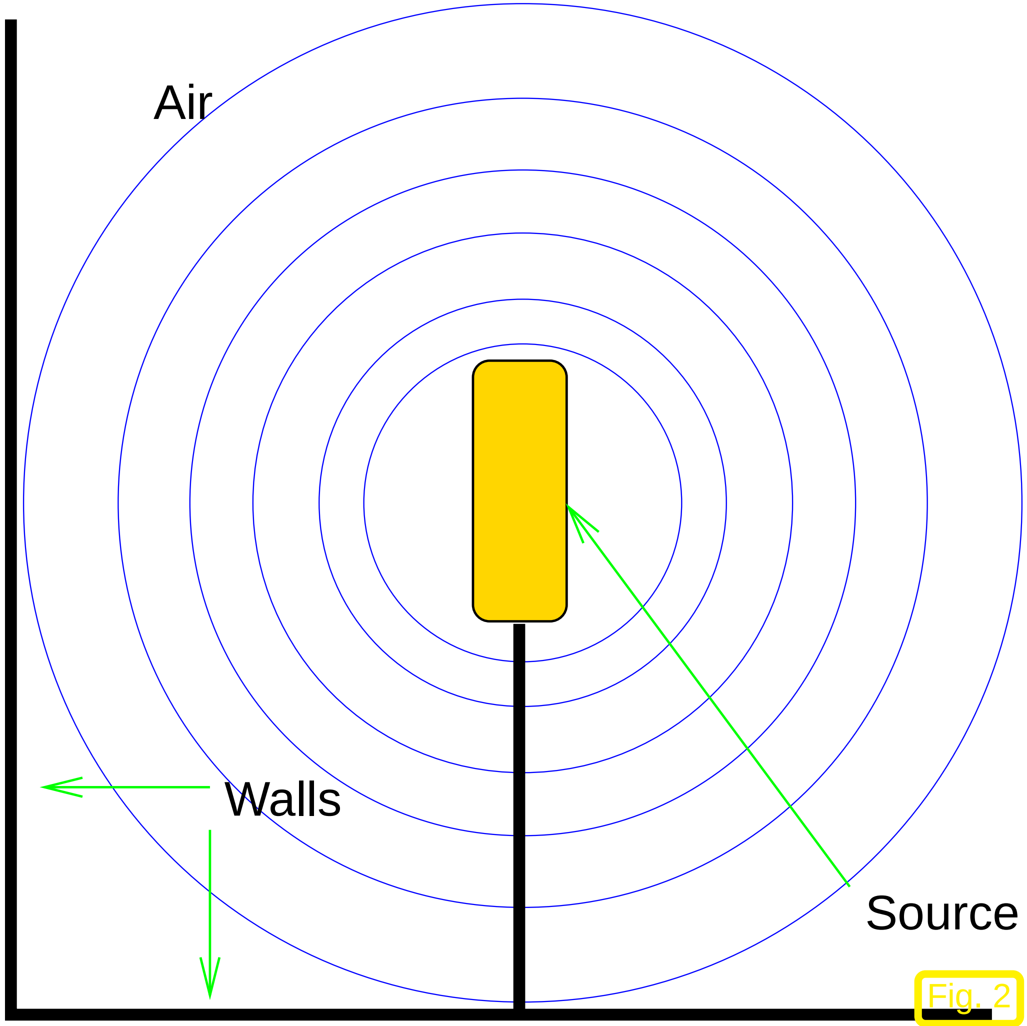


Fig. 2

$\Omega \subset \mathbb{R}^3$: (possibly unbounded) air region

Propagation of sound in Ω governed by

$$\frac{\partial \mathbf{v}}{\partial t} + \frac{1}{\rho_0}\,\mathbf{grad}\, p = 0\,, \tag{1.1.5}$$

$$\frac{\partial \rho}{\partial t} + \rho_0 \operatorname{div} \mathbf{v} = 0\,, \tag{1.1.6}$$

$$\frac{\partial \rho}{\partial t} - \frac{1}{c^2}\frac{\partial p}{\partial t} = 0\,. \tag{1.1.7}$$

(1.1.5) : linearized momentum equation,
(1.1.6) : linearized continuity equation,
(1.1.7) : linearized state equation.

$\mathbf{v} \mathrel{\hat{=}}$ velocity field ($[\mathbf{v}] = \mathrm{ms}^{-1}$), $p \mathrel{\hat{=}}$ pressure field ($[p] = \mathrm{Nm}^{-2}$), $\rho_0 = \rho_0(\boldsymbol{x}) \mathrel{\hat{=}}$ uniformly positive density ($[\rho_0] = \mathrm{kg\,m}^{-3}$), $c = c(\boldsymbol{x}) \mathrel{\hat{=}}$ uniformly positive local speed of sound ($[c] = 1\mathrm{ms}^{-1}$)

▶ Pressure wave equation: $$\frac{1}{c^2\rho_0}\frac{\partial^2 p}{\partial t^2} - \operatorname{div}(\rho_0^{-1}\,\mathbf{grad}\, p) = 0\,. \tag{1.1.8}$$

1.2 Initial and boundary conditions

☞ In the case of vibrating membrane ($\longrightarrow$ Sect. 1.1)

(Spatial) boundary conditions : $u(\boldsymbol{x},t) = 0$ for all $(\boldsymbol{x},t) \in \partial\Omega \times]0,T[$ (clamped membrane)

(Temporal) initial conditions : initial position $\leftrightarrow$ $u(\boldsymbol{x},0) = u_0$, $\boldsymbol{x} \in \Omega$,
initial velocity $\leftrightarrow$ $\dfrac{\partial u}{\partial t}(\boldsymbol{x},0) = v_0$, $\boldsymbol{x} \in \Omega$.

☞ In the case of sound propagation ($\longrightarrow$ Sect. 1.1)

(Spatial) boundary conditions : sound soft wall $\leftrightarrow$ $p(\boldsymbol{x},t) = 0$ for all $(\boldsymbol{x},t) \in \partial\Omega \times]0,T[$,
sound hard wall $\leftrightarrow$ $\rho_0^{-1}\,\mathbf{grad}\,p(\boldsymbol{x},t)\cdot\boldsymbol{n} = 0\ \forall(\boldsymbol{x},t) \in \partial\Omega \times]0,T[$.

(Temporal) initial conditions : initial pressure distribution $\leftrightarrow$ $p(\boldsymbol{x},0) = p_0$, $\boldsymbol{x} \in \Omega$,
initial compression field $\leftrightarrow$ $\dfrac{\partial p}{\partial t}(\boldsymbol{x},0) = v_0$, $\boldsymbol{x} \in \Omega$.

Suitable spatial boundary conditions for scalar linear second-order wave equations **=** meaningful boundary conditions for 2nd-order scalar elliptic BVPs [27, Sect. 2.4]

➤ spatial boundary conditions for $\rho \frac{\partial^2 u}{\partial t^2} - \mathrm{div}_{\boldsymbol{x}}(\mathbf{C}\,\mathbf{grad}_{\boldsymbol{x}}\, u) = f$:

- Spatial Dirichlet boundary conditions, *cf.* [27, (2.4.1)]:

$$u(\boldsymbol{x},t) = g(\boldsymbol{x},t) \quad \text{on } \partial\Omega\times]0,T[\ , \tag{1.2.1}$$

  with Dirichlet data $g : \partial\Omega\times]0,T[\mapsto \mathbb{R}$.

- Spatial Neumann boundary conditions, *cf.* [27, (2.4.2)]:

$$\mathbf{C}\,\mathbf{grad}\, u \cdot \boldsymbol{n} = h(\boldsymbol{x},t) \quad \text{on } \partial\Omega\times]0,T[\ , \tag{1.2.2}$$

  with Neumann data $h : \partial\Omega\times]0,T[\mapsto \mathbb{R}$.

- Spatial (nonlinear) impedance boundary conditions, *cf.* [27, (2.4.3)]

$$\mathbf{C}\,\mathbf{grad}\, u \cdot \boldsymbol{n} = \Psi(u) \quad \text{on } \partial\Omega\times]0,T[\ , \tag{1.2.3}$$

  with increasing function $\Psi : \mathbb{R} \mapsto \mathbb{R}$.

Remark 1. Sound propagation: modelling of loudspeaker

➡ prescribed velocity ↔ Inhomogeneous ($h \neq 0$) Neumann b.c (1.2.2) for pressure

△

➤ initial conditions ($\hat{=}$ temporal boundary conditions) for $\rho\frac{\partial^2 u}{\partial t^2} - \operatorname{div}_{\boldsymbol{x}}(\mathbf{C}\,\mathbf{grad}_{\boldsymbol{x}}\,u) = f$:

BOTH	initial field	$\leftrightarrow$	$u(\boldsymbol{x},0) = u_0(\boldsymbol{x})$ for all $\boldsymbol{x} \in \Omega$,	have to be specified
	initial velocity	$\leftrightarrow$	$\frac{\partial u}{\partial t}(\boldsymbol{x},0) = v_0(\boldsymbol{x})$ for all $\boldsymbol{x} \in \Omega$	

Remark 2. Remember: two initial conditions also required for 2nd-order ODE $\frac{d^2}{dt^2}y = f(y)$. △

1.3 Classical and formal solutions

Assume: smooth coefficients/sources $\mathbf{C} \in (C^1(\Omega))^{d,d}$, $\rho \in C^0(\overline{\Omega})$, $f \in C^0(\Omega)$

Definition 1.3.1 (Classical solution of wave equation, *cf.* [27, Sect. 2.5])**.**
A classical solution of the wave equation (1.1.3) *with Dirichlet boundary data* $g \in C^0(\partial\Omega)$ *is a function* $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ *that satisfies* (1.1.3) *pointwise and fulfills* $u(\boldsymbol{x}, t) = g(\boldsymbol{x})$, $\forall \boldsymbol{x} \in \partial\Omega$, $0 \leq t \leq T$.

Focus: "pure" initial value problem **=** Cauchy problem: $\Omega = \mathbb{R}^d$

1.3.1 Plane wave solutions

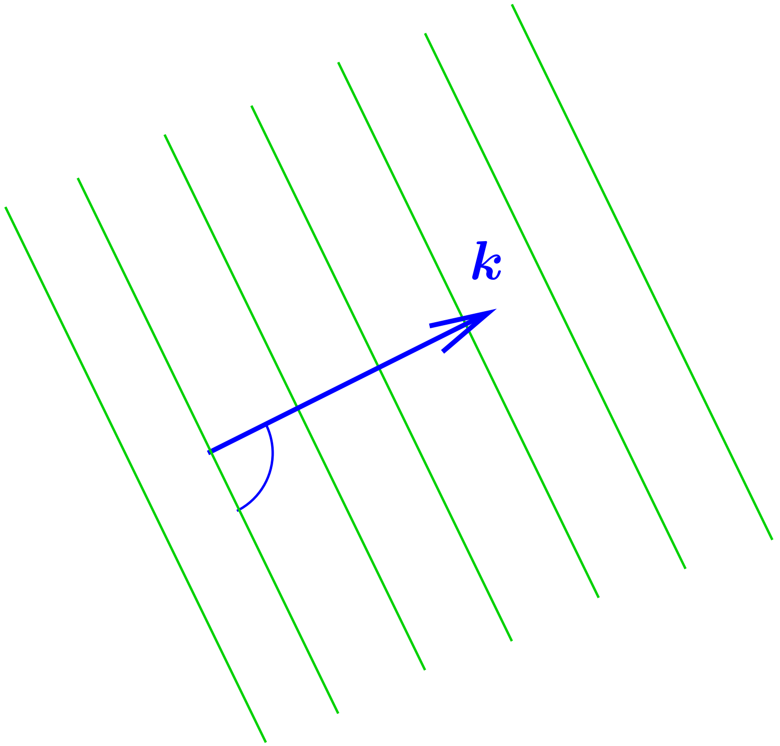
Consider Cauchy problem for (1.1.3) with $f = 0$, $\rho \equiv 1$, $\mathbf{C} = \text{const}$, $c = 0$.

Definition 1.3.2 (Plane wave)**.** *(The real part of) a complex valued function* $u(\boldsymbol{x}, t) = \exp(i(\boldsymbol{k} \cdot \boldsymbol{x} - \omega t))$, $(\boldsymbol{x}, t) \in \widetilde{\Omega}$, *is a plane wave with wave vector* $\boldsymbol{k} \in \mathbb{R}^d$ *and angular frequency* $\omega \in \mathbb{R}$.

$\boldsymbol{k}\cdot\boldsymbol{x}-\omega t \hat{=}$ wave phase

$\boldsymbol{k}$ **=** direction of propagation

phase velocity: $\boxed{\boldsymbol{c}_p = \frac{\omega}{|\boldsymbol{k}|^2}\boldsymbol{k}}$, wavelength: $\lambda = \frac{2\pi}{|\boldsymbol{k}|}$



plane wave solves (1.1.3) $\Leftrightarrow$ $|\mathbf{C}^{1/2}\boldsymbol{k}| = \pm\omega$ (1.3.1)

(1.3.1) **=** dispersion relation

Isotropic propagation: $\mathbf{C} = \gamma^2\mathbf{I}$, $\gamma > 0$ ➤ $|\boldsymbol{c}_p| = \gamma$

(1.3.1) $\Rightarrow$ $\omega = \omega(\boldsymbol{k})$: group velocity: $\boxed{\boldsymbol{c}_g = \mathbf{grad}_{\boldsymbol{k}}\,\mathrm{Re}\{\omega(\boldsymbol{k})\}}$

For wave equation (1.1.3) ($\mathbf{C} = \mathrm{const}$, $c = 0$): $\boldsymbol{c}_g(\boldsymbol{k}) = \frac{\mathbf{C}\boldsymbol{k}}{|\omega|}$

Definition 1.3.3 (Dispersionless equations). *A scalar partial differential equation (PDE) that has plane wave solutions (→ Def. 1.3.2) is dispersionless, if the group velocity $\boldsymbol{c}_g(\boldsymbol{k})$ only depends on the direction of the wave vector, but not its length.*

▶ the wave equation (→ Def. 1.1.1) is dispersionless

1.3.2 D'Alembert solution formula

Consider homogeneous Cauchy problem for $d = 1$:

$$c > 0: \quad \frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 \quad , \quad u(x,0) = u_0(x) \; , \quad \frac{\partial u}{\partial t}(x,0) = v_0(x) \; , \quad x \in \mathbb{R} \; . \tag{1.3.2}$$

Change of variables: $\xi = x + ct$, $\tau = x - ct$: $\widetilde{u}(\xi,\tau) = u(\frac{\xi+\tau}{2}, \frac{\xi-\tau}{2c})$

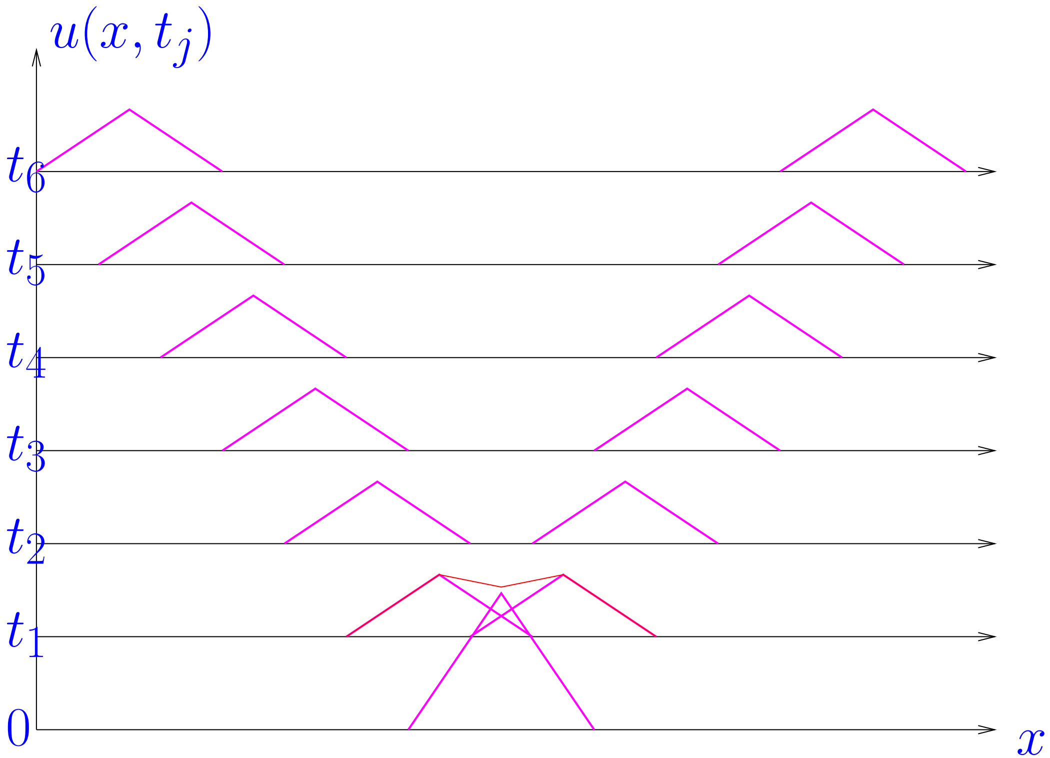
$$▶ \quad \frac{\partial^2 \widetilde{u}}{\partial \xi \partial \tau} = 0 \quad \Rightarrow \quad \widetilde{u}(\xi,\tau) = F(\xi) + G(\tau) \; ,$$

for **any** $F, G \in C^2(\mathbb{R})$!

▼ ← matching initial data

$$u(x,t) = \tfrac{1}{2}(u_0(x+ct) + u_0(x-ct)) + \tfrac{1}{2}\int_{x-ct}^{x+ct} v_0(s)\,\mathrm{d}s\;. \qquad (1.3.3)$$

(1.3.3) **=** d'Alembert solution of Cauchy problem (1.10.1).



$v_0 = 0$ ➤ initial data u_0 travel with speed c in opposite directions

> finite speed of propagation is typical feature of solutions of wave equations

Note: (1.3.3) meaningful even for discontinuous u_0, v_0 **!**

➡ "generalized solutions" **?** (*cf.* [27, Sect. 2.6])

1.3.3 Spherical mean solutions

Consider Cauchy problem for wave equation (1.1.3) with $\rho \equiv 1$, $\mathbf{C} = \mathbf{I}$, $f = 0$

- $d = 3$: Kirchhoff's formula:

$$u(\boldsymbol{x}, t) = \frac{1}{4\pi t^2} \int_{\partial B(\boldsymbol{x},t)} u_0(\boldsymbol{y}) + t v_0(\boldsymbol{y}) + \mathbf{grad}\, v_0(\boldsymbol{y}) \cdot (\boldsymbol{y} - \boldsymbol{x}) \,\mathrm{d}S(\boldsymbol{y}) \;, \quad \boldsymbol{x} \in \mathbb{R}^3, t > 0 \;. \tag{1.3.4}$$

Ball $B(\boldsymbol{x}, r) = \{\boldsymbol{y} \in \mathbb{R}^3 \colon |\boldsymbol{y} - \boldsymbol{x}| = r\}$

- $d = 2$: Poisson's formula:

$$u(\boldsymbol{x}, t) = \frac{1}{4\pi t} \int_{B(\boldsymbol{x},t)} \frac{t u_0(\boldsymbol{y}) + t^2 v_0(\boldsymbol{y}) + t\,\mathbf{grad}\, u_0(\boldsymbol{y}) \cdot (\boldsymbol{y} - \boldsymbol{x})}{\sqrt{t^2 - |\boldsymbol{y} - \boldsymbol{x}|^2}} \,\mathrm{d}S(\boldsymbol{y}) \;, \quad \boldsymbol{x} \in \mathbb{R}^2, t > 0 \;. \tag{1.3.5}$$

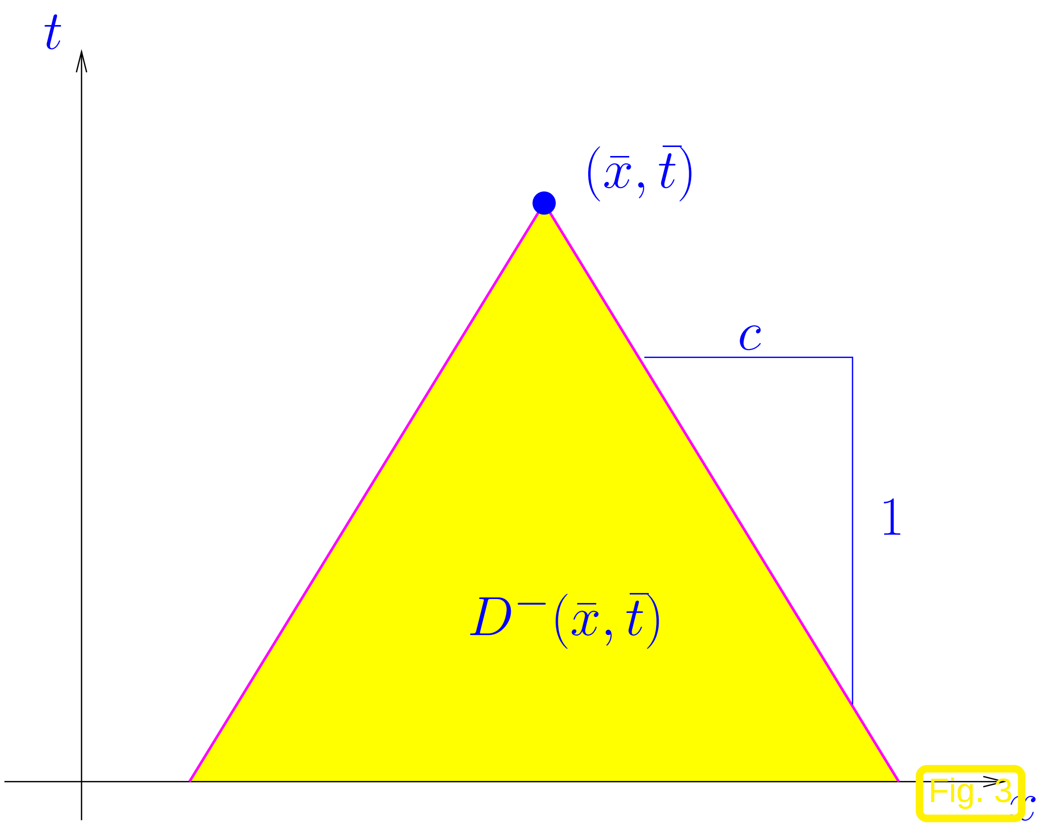
1.4 Domains of dependence and influence

finite speed of propagation ▶ "point value" $u(\bar{x}, \bar{t})$, $(\bar{x}, \bar{t}) \in \widetilde{\Omega}$, may not depend on initial values outside proper subdomain of Ω !

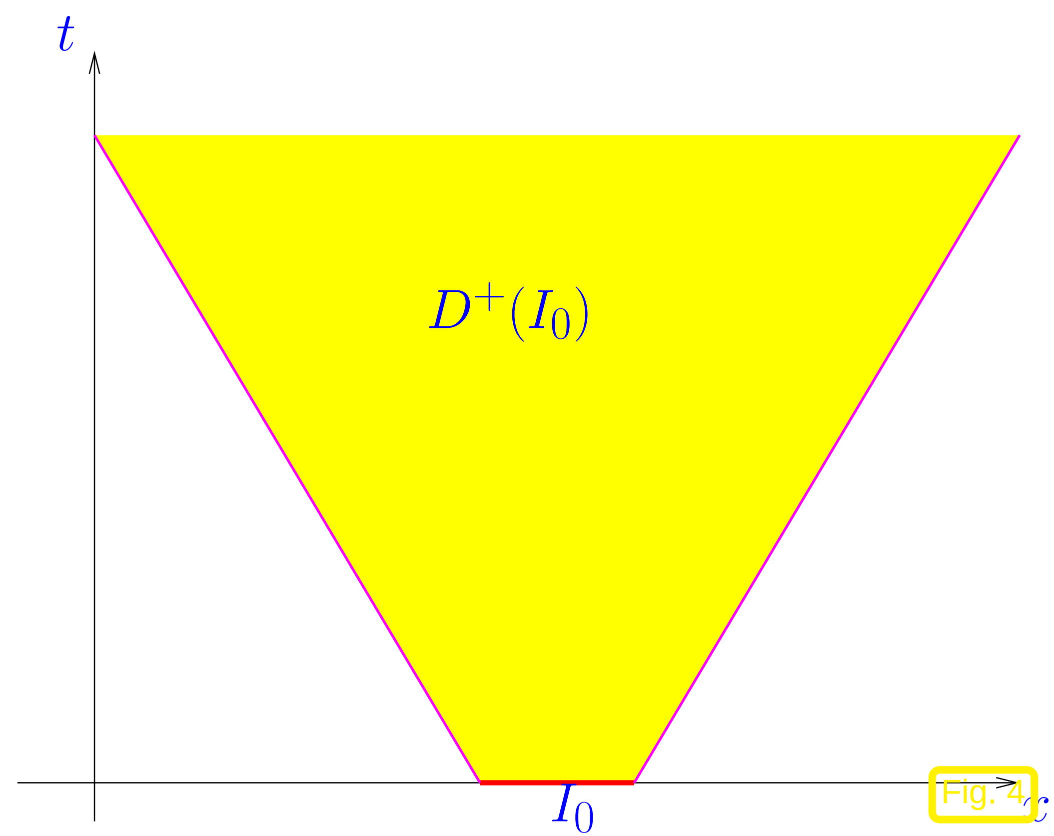
Example 3 (Domain of dependence/influence for 1D wave equation, constant coefficient case).

$d = 1$, Cauchy problem for wave equation (1.10.1): $\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$, $c > 0$:

Intuitive: from D'Alembert formula (1.3.3)



domain of dependence of $(\bar{x}, \bar{t}) \in \widetilde{\Omega}$



domain of influence of $I_0 \subset \mathbb{R}$

◇

Theorem 1.4.1 (Domain of dependence for isotropic wave equation). $\to$ *[14, 2.5, Thm. 6]*
$u : \widetilde{\Omega} \mapsto \mathbb{R} \hat{=}$ classical solution ($\to$ Def. 1.3.1) of homogeneous wave equation with $\rho = 1$, $\mathbf{C} = c^2\mathbf{I}$, $c > 0$, then

$$\Big(|\boldsymbol{x} - \boldsymbol{x}_0| \leq ct_0 \quad \Rightarrow \quad u(\boldsymbol{x}, 0) = 0\Big) \quad \Rightarrow \quad u(x, t) = 0 \quad \text{, if} \quad |\boldsymbol{x} - \boldsymbol{x}_0| \leq c(t_0 - t) \; .$$

For $\mathbf{C} = \mathbf{C}(\boldsymbol{x})$ ➤ domain of dependence is general "light cone"

Example 4 (Domain of dependence for spatially varying wave speed).

$d = 1$, $c = c(x)$ continous, uniformly positive:
$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x}\left(c^2(x)\frac{\partial u}{\partial x}\right) = 0$$

(Note: $c(x)$ provides "local" propagation speed)

▶ domain of dependence $D^-(\bar{x}, \bar{t})$:

$$D^-(\bar{x}, \bar{t}) = \{(x, t)\colon x^-(\bar{t} - t) \leq x \leq x^+(\bar{t} - t)\} \; ,$$
$$\frac{d}{dt}x^-(t) = -c(x^-(t)) \; , \quad x^-(0) = \bar{x} \quad , \quad \frac{d}{dt}x^+(t) = c(x^+(t)) \; , \quad x^+(0) = \bar{x} \; .$$

$\diamond$

Remark 5 (Infinite propagation speed for parabolic evolutions).

Consider Cauchy problem for parabolic evolution [27, Sect. 7.2]:

$$\frac{\partial u}{\partial t} - \Delta u = 0 \quad \text{on } \mathbb{R}^d \times]0, T[\quad , \quad u(0) = u_0 \in L^2(\mathbb{R}^d) \; .$$

Even if $\operatorname{supp} u_0$ bounded ➢ $\operatorname{supp} u(\cdot, t) = \mathbb{R}^d$ for all $t > 0$ **!**

$\triangle$

1.5 Weak solutions and abstract wave equations

Approach:
- consider time t as parameter in wave equation (1.1.3).
- apply standard techniques used for derivation of weak (variational) form of elliptic BVPs $\longrightarrow$ [27, Sect. 2.7]

☞ recall derivation of abstract parabolic evolution problems [27, Sect. 7.2]

STEP 1: multiply $\rho\frac{\partial^2}{\partial t^2}u - \operatorname{div}_{\boldsymbol{x}}(\mathbf{C}\,\mathbf{grad}_{\boldsymbol{x}}\,u) = f$ with *test functions* that vanish on spatial Dirichlet boundaries (*cf.* weak derivative [27, Def. 2.6.1])

STEP 2: integrate over spatial domain Ω (*cf.* weak derivative [27, Def. 2.6.1])

STEP 3: perform integration by parts using Green's formula [27, Thm. 2.7.2]

Example 6 (Formal variational formulation of wave equation with Dirchlet boundary conditions).

$$\begin{aligned}
\rho\frac{\partial^2 u}{\partial t^2} - \operatorname{div}_{\boldsymbol{x}}(\mathbf{C}\,\mathbf{grad}_{\boldsymbol{x}}\,u) &= f(\boldsymbol{x},t) \quad \text{in } \widetilde{\Omega}\,,\\
u(\boldsymbol{x},t) &= g(\boldsymbol{x},t) \quad \text{on } \partial\Omega\times]0,T[\,,\\
u(\boldsymbol{x},0) = u_0(\boldsymbol{x}) \quad , \quad \frac{\partial u}{\partial t}(\boldsymbol{x},0) &= v_0(\boldsymbol{x}) \qquad \text{in } \Omega\,.
\end{aligned}$$

seek $u :]0,T[\mapsto g(t) + V$, $V := \{v : \Omega \mapsto \mathbb{R}: v_{|\partial\Omega} = 0\}$ space of functions,

$$\int_\Omega \rho(\boldsymbol{x})\frac{\partial^2 u}{\partial t^2}(\boldsymbol{x},t)v(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} + \int_\Omega \mathbf{C}(\boldsymbol{x})\,\mathbf{grad}_{\boldsymbol{x}}\,u(\boldsymbol{x})\cdot\mathbf{grad}_{\boldsymbol{x}}\,v(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \int_\Omega f(\boldsymbol{x},t)v(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \quad \textbf{(1.5.1)}$$

for all $v \in V$.

Extension $g \to \widetilde{g} : \Omega \mapsto \mathbb{R}$, $\widetilde{g} = g$ on $\partial\Omega$ ➢ "offset function technique" [27, Sect. 2.10] incorporates Dirichlet data into source term ▷ allows to seek $\widetilde{u} :]0, T[\mapsto V$.

◇

General form of spatial variational formulation of 2nd-order hyperbolic evolution problem:

$$t \in]0, T[\mapsto u(t) \in V \quad : \quad \begin{cases} \mathsf{m}(\frac{d^2}{dt^2}u(t), v) + \mathsf{a}(u(t), v) = \langle f(t), v \rangle_V \quad \forall v \in V \,, \\ u(0) = u_0 \in V \quad , \quad \frac{du}{dt}(0) = v_0 \in H \,. \end{cases} \tag{1.5.2}$$

- V, H = Hilbert spaces [27, Def. 1.1.5]:

  ▷ $V \subset H$ with continuous [27, Def. 2.11.1] and dense [27, Def. 2.8.4] embedding $V \hookrightarrow H$

  ▷ duality pairing $\langle \cdot, \cdot \rangle_V : V' \times V \mapsto \mathbb{R}$ on $H \times V$ agrees with inner product $(\cdot, \cdot)_H$

  Terminology: $\boxed{V \subset H \subset V' = \text{evolution triple}}$

- $\mathsf{a} \in L(V \times V, \mathbb{R})$ = V-elliptic [27, Def. 1.2.3] symmetric [27, Def. 1.1.4] bilinear form [27, Def. 1.1.3] (independent of time !)

- $\mathsf{m} \in L(H \times H, \mathbb{R})$ = (an) inner product [27, Def. 1.1.4] on H (independent of time !)

- f = time-dependent continuous linear form $f(t) : V \mapsto \mathbb{R}$ [27, Def. 1.1.3], $0 < t < T$.

Convention: norms $\|\cdot\|_H$ and $\|\cdot\|_V$ ("energy norm") of V/H induced by $\mathsf{m}(\cdot,\cdot)$ and $\mathsf{a}(\cdot,\cdot)$, resp., *cf.* [27, Def. 1.1.5]: $\|v\|_V^2 = \mathsf{a}(v,v)$, $\|v\|_H^2 = \mathsf{m}(v,v)$

Operator notation: $\mathsf{A} : V \mapsto V' \overset{[27, (1.1.5)]}{\longleftrightarrow} \mathsf{a}$, $\mathsf{M} : H \mapsto H' = H \subset V' \overset{[27, (1.1.5)]}{\longleftrightarrow} \mathsf{m}$:

$$\text{(1.5.2)} \longleftrightarrow \left\{ \frac{d^2}{dt^2}\mathsf{M}u + \mathsf{A}u = f \quad \text{in } V' \right\} \quad \text{a.e. in }]0,T[\quad , \quad \begin{aligned} u(0) &= u_0 \quad \text{in } V \;, \\ \frac{du}{dt}(0) &= v_0 \quad \text{in } H \;. \end{aligned} \tag{1.5.3}$$

weak temporal derivative ! [27, Def. 2.6.1]

(1.5.3) **=** ODE in function space !

Concrete functional framework provided by Sobolev spaces [27, Sect. 2.8]

$$V = H^1(\Omega)/H_0^1(\Omega),\; H = L^2(\Omega)$$

and Bochner spaces of function space valued functions on $]0,T[$

Example 7 (Bochner spaces).

Spaces of X-valued, X = Hilbert space, functions on $]0,T[$ (Bochner spaces), e.g.,

$$H^1(]0,T[;X) := \{v :]0,T[\mapsto X \text{ measurable} : \|v\|^2_{H^1(]0,T[;X)} := \int_0^T \left\|\frac{dv}{dt}(t)\right\|^2_X + \|v(t)\|^2_X \,\mathrm{d}t < \infty\},$$

$$C^0(]0,T[;X) := \{v :]0,T[\mapsto X \text{ continuous} , \quad \|v\|_{C^0(]0,T[;X)} := \sup_{0<t<T} \|v(t)\|_X\} .$$

➤ $H^p(]0,T[;X)$, $p \in \mathbb{N}_0$ are Hilbert spaces, $C^0(]0,T[;X)$ is Banach space.

◇

Abstract hyperbolic evolution problem in weak form: [14, Sect. 7.2], [40, Sect. 10.2]

seek $u \in L^2(]0,T[;V) \cap H^1(]0,T[;H) \cap H^2(]0,T[;V')$ such that for all $v \in V$ and $w \in C_0^\infty(]0,T[;V)$

$$\int_0^T \mathsf{m}(u(t),v)\,\frac{d^2w}{dt^2}(t) + \mathsf{a}(u(t),v)\,w(t)\,\mathrm{d}t = \int_0^T \langle f(t),v\rangle_V \; w(t)\,\mathrm{d}t \;, \tag{1.5.4}$$

and $u(0) = u_0 \in V$, $\frac{du}{dt}(0) = v_0 \in H$.

Theorem 1.5.1 (Existence and uniqueness of solutions of hyperbolic evolution problems)**.**
If $f \in L^2(]0,T[;H)$*, then there exists a unique solution* u *of* (1.5.4) *that belongs to* $L^\infty(]0,T[;V) \cap W^{1,\infty}(]0,T[;H)$ *and satisfies the energy estimate*

$$\sup_{0<t<T} \left(\|u(t)\|_V^2 + \left\| \frac{du}{dt}(t) \right\|_H^2 \right) \leq C \left(\|f\|^2_{L^2(]0,T[;H)} + \|u_0\|_V^2 + \|v_0\|_H^2 \right) , \tag{1.5.5}$$

with $C = C(\mathsf{m}, \mathsf{a}) > 0$.

Proof. Thms. 2, 3, 4 & 5 in [14, Sect. 7.2] □

Under assumptions/with notations of Thm. 1.5.1: conservation of energy

$$f = 0 \quad \Rightarrow \quad E(t) = E(0) \quad \forall 0 \leq t \leq T \text{ , with “energy”} E(t) := \tfrac{1}{2} \underbrace{\|u(t)\|_V^2}_{\text{potential energy}} + \tfrac{1}{2} \underbrace{\left\| \tfrac{du}{dt}(t) \right\|_H^2}_{\text{kinetic energy}} . \tag{1.5.7}$$

Note: Energy estimates (1.5.5), (1.5.7) ➤ $\boxed{\text{stability of hyperbolic evolution problem}}$

1.5.1 Spectral decomposition

Assumption: compact embedding [27, Def. 2.11.2] $V \overset{c}{\hookrightarrow} H$

➤ operator A has pure discrete point spectrum, mutually H-orthogonal eigenspaces [27, Sect. 4.8.1]:

If $\dim V = \dim H = \infty$, $\exists$ sequence $(w_i)_{i\in\mathbb{N}} \subset V$ of eigenfunctions and a non-decreasing unbounded sequence $(\lambda_i)_{i=1}^{\infty}$ of (positive) eigenvalues such that

- $\{w_i\}_{i\in\mathbb{N}}$ is an m-orthonormal basis (ONB) of H,
- $\{w_i\}_{i\in\mathbb{N}}$ is an a-orthogonal basis of V,
- $\mathsf{a}(w_i, v) = \lambda_i \mathsf{m}(w_i, v) \quad \forall v \in V$.

Remark 8 (Compact embedding of Sobolev spaces).

Rellich's theorem [27, Thm. 2.11.3] ➤ $H^1(\Omega), H_0^1(\Omega) \overset{c}{\hookrightarrow} L^2(\Omega)$

△

Idea: "simultaneous diagonalization" of A, M

Lemma 1.5.2 (Spectral repesentation of solution of abstract wave equations)**.** *Let assumptions of Thm. 1.5.1 hold,* $\dim V = \dim H = \infty$*, and* $V \overset{c}{\hookrightarrow} H$*. Then*

$$u(t) = \sum_{l=1}^{\infty} \Big(\mathsf{m}(u_0, w_l) \cos(\sqrt{\lambda_l} t) + \mathsf{m}(v_0, w_l) \frac{1}{\sqrt{\lambda_l}} \sin(\sqrt{\lambda_l} t) + \int_0^t \frac{1}{\sqrt{\lambda_l}} \sin(\sqrt{\lambda_l}(t-s))\, \mathsf{m}(f(s), w_l)\, \mathrm{d}s \Big)\, w_l \;,$$

$0 \leq t \leq T$*, solves inhomogeneous abstract wave equations* (1.5.4)*.*

= Duhamel's principle [14, Sect. 2.3.c] ("variation of constants formula")

Rewrite representation formula using functional calculus for unbounded operators, *cf.* [40, Sect. 11.4.2]:

for operator A: $$f(\mathsf{A})v = \sum_{l=1}^{\infty} f(\lambda_l)\, \mathsf{m}(v, w_l)\, w_l \;, \quad v \in V \;. \tag{1.5.8}$$

► $$u(t) = \cos(\mathsf{A}^{1/2} t) u_0 + \mathsf{A}^{-1/2} \sin(\mathsf{A}^{1/2} t) v_0 + \int_0^t \mathsf{A}^{-1/2} \sin(\mathsf{A}^{1/2}(t-s)) f(s)\, \mathrm{d}s \;. \tag{1.5.9}$$

Example 9 (Smoothing property of hyperbolic evolution).

(1.1.3) for $d = 1$, $\mathbf{C} = \mathbf{I}$, $\rho = 1$: $$\frac{\partial^2}{\partial t^2}u(x,t) - \frac{\partial^2}{\partial x^2}u(x,t) = 0$$

$\Omega =]0,1[$, $V = H_0^1(\Omega)$, $H = L^2(\Omega)$, eigenfunctions $w_l(x) = 2\sin(\pi l x)$, $l \in \mathbb{N}$, $\lambda_l = \pi^2 l^2$

$$\begin{aligned} u_0(x) &= \sum_{l=1}^{\infty} \alpha_l \sin(\pi l x) \;, \\ v_0(x) &= \sum_{l=1}^{\infty} \beta_l \sin(\pi l x) \;, \end{aligned} \quad \Rightarrow \quad u(x,t) = \sum_{l=1}^{\infty} \underbrace{\boxed{\left(\alpha_l \cos(\pi l t) + \frac{\beta_l}{\pi l}\sin(\pi l t)\right)}}_{\text{Fourier coefficient of } u(\cdot,t)} \sin(\pi l x) \;.$$

Decay of Fourier coeffs. $\leftrightarrow$ smoothness of function $\Rightarrow$ no smoothing during hyperbolic evolution

"Rough initial data" ▶ solution "rough" for all times

(in contrast to *smoothing* parabolic evolution: $\frac{\partial}{\partial t}u - \frac{\partial^2}{\partial x^2}u = 0$, [27, Rem. 149]) $\diamond$

1.5.2 Equivalent first order system

Assume setting of abstract 2nd-order hyperbolic evolution problem (1.5.2).

Now $\mathsf{a}(u,v) = \mathsf{m}(\mathsf{B}u, \mathsf{B}v)$, $u, v \in V$, $\mathsf{B} \in L(V,H)$ & injective with closed range.

Fits (1.1.3) (with Dirichlet b.c.): here $\mathsf{B} = \mathbf{C}^{1/2}\rho^{-1/2}\,\mathbf{grad} : H_0^1(\Omega) \mapsto L^2(\Omega)$

New unknown: $$\mathbf{v}(t) := \int_0^t \mathsf{B}u(\tau)\,\mathrm{d}\tau \in H^1(]0,T[;H)$$

(apply $\int_0^t$ to (1.5.2)) ▶ (1.5.2) equivalent to

seek $u :]0,T[\mapsto V$, $v :]0,T[\mapsto H$

$$\begin{aligned} \mathsf{m}(\tfrac{\partial}{\partial t}u, w) \;+\; \mathsf{m}(\mathbf{v}, \mathsf{B}w) \;&=\; \mathsf{m}(v_0, w) + \int_0^t \langle f(\tau), w\rangle_V \,\mathrm{d}\tau \quad && \forall w \in V\;, \\ \mathsf{m}(\tfrac{\partial}{\partial t}\mathbf{v}, \mathbf{q}) \;-\; \mathsf{m}(\mathsf{B}u, \mathbf{q}) \;&=\; 0 && \forall \mathbf{q} \in H\;. \end{aligned} \tag{1.5.10}$$

$$u(0) = u_0 \quad , \quad \mathbf{v}(0) = 0\;. \tag{1.5.11}$$

1.6 Spatial semi-discretization

Assumption: spatial domain Ω **bounded** !

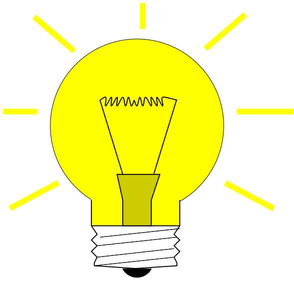
Method of lines approach:

Spatial semidiscretization of IBVP for (1.1.3) ➡ 2nd-order ODE

$$\rho\frac{\partial^2 u}{\partial t^2} + \mathcal{L}_{\boldsymbol{x}}\, u = f \quad ➡ \quad \mathbf{M}\frac{d^2}{dt^2}\vec{\mu}(t) + \mathbf{A}\vec{\mu}(t) = \vec{\varphi}(t) \tag{1.6.1}$$

($\mathbf{M}$, $\mathbf{A}$ matrices $\in \mathbb{R}^{N,N}$, $\vec{\mu}(t) \in \mathbb{R}^N$)

Insight: any method for spatial discretization of elliptic BVP for $\mathcal{L}\, u = f$ should work:

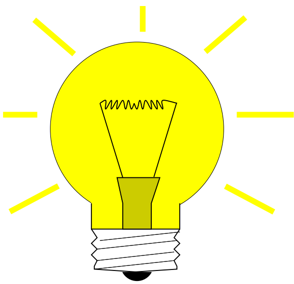


- ✔ finite difference (FD) and finite volume (FV) schemes
- ✔ various (primal/dual) finite element methods (FEM)
- ✔ discontinuous Galerkin (DG) methods, etc.

$\rightarrow$ Course “Numerics of Elliptic and Parabolic Boundary value Problems” [27]

Then apply “standard timestepping” to resulting ODE (**!** caution)

1.6.1 Finite differences (FD)



Idea: spatial "lattice model"

- deal with $\mathcal{L}_{\boldsymbol{x}}$ from (1.1.1) in *strong* (classical) form
- replace spatial derivatives with difference quotients on grid

Focus: pure Dirichlet problem: $u(\boldsymbol{x},t) = g(\boldsymbol{x},t)$, $(\boldsymbol{x},t) \in \partial\Omega\times]0,T[$, $g(t) \in C^0(\partial\Omega)$

continuous initial data: $u_0, v_0 \in C^0(\overline{\Omega})$, $u_{0|\partial\Omega} = g(0,\cdot)$

$\mathbf{C} = \gamma(\boldsymbol{x})\mathbf{I}$ with continous function $\gamma \in C^0(\overline{\Omega})$

One-dimensional case

$d = 1$ ➤ $\Omega =]0,1[$ (open interval), $\partial\Omega = \{0,1\}$, $\mathcal{L}_{\boldsymbol{x}}\, u = -\frac{\partial}{\partial x}\left(\gamma(x)\frac{\partial u}{\partial x}\right)$

grid: $\mathcal{M} := \{]x_{j-1}, x_j[: 0 = x_0 < x_1 < \cdots < x_M = 1,\ i = 1,\ldots,M\}$, $M \in \mathbb{N}$

with grid points/nodes x_j , $j = 0, \dots, M$ (node set $\mathcal{V}(\mathcal{M}) = \{x_0, x_1, \dots, x_M\}$),
(local) meshwidth $h_j := x_j - x_{j-1}$, $x_{j+1/2} := \frac{1}{2}(x_j + x_{j+1})$.

Finite difference approximation (for $f \in C^0(\overline{\Omega})$)

$$\frac{\partial}{\partial x}\left(\gamma(x)\frac{\partial f}{\partial x}\right)_{x=x_j} \approx (\mathsf{T}f)_j := \frac{\gamma(x_{j+1/2})\dfrac{f(x_{j+1}) - f(x_j)}{h_{j+1}} - \gamma(x_{j-1/2})\dfrac{f(x_j) - f(x_{j-1})}{h_j}}{1/2(h_j + h_{j+1})} . \tag{1.6.2}$$

Motivation: Taylor expansion, also shows (for sufficiently smooth γ, f)

$$\left|\frac{\partial}{\partial x}\left(\gamma(x)\frac{\partial f}{\partial x}\right)_{x=x_j} - (\mathsf{T}f)_j\right| \leq C \max\{h_j, h_{j+1}\} , \tag{1.6.3}$$

with $C > 0$ depending on (several higher) derivatives of γ, f.

$$\boxed{\text{(1.6.3)} \quad \longleftrightarrow \quad \text{(1.6.2)} \hat{=} \text{ 1st-order approximation of } \boldsymbol{\mathcal{L}_x}}$$

Note: if $h := h_j = h_{j+1}$ $\Rightarrow$ $(-\boldsymbol{\mathcal{L}_x} f)_{x=x_j} - (\mathsf{T}f)_j = O(h^2)$ (2nd-order approximation)
(equidistant grid)

Semi-discrete representation of u: $\vec{\mu} : [0,T] \mapsto \{\mathcal{V}(\mathcal{M}) \mapsto \mathbb{R}\}$

space of grid functions $\cong \mathbb{R}^{M+1}$

▶ spatial semi-discretization

$$\rho\frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x}\left(\gamma\frac{\partial u}{\partial x}\right) = f(x,t) \quad \rightarrow \quad \begin{array}{c} \rho(x_j)\dfrac{d^2}{dt^2}\vec{\mu}(x_j,t) - (\mathsf{T}\vec{\mu})_j(t) = f(x_j,t) \;, j = 1,\dots,M-1 \;, \\ \vec{\mu}(x_0,t) = g(0,t) \quad , \quad \vec{\mu}(x_M,t) = g(1,t) \;. \end{array}$$

(Linear 2nd-order ODE in $\mathbb{R}^{M-1}$)

After identiication $\vec{\mu}(t) \in \mathbb{R}^{M-1}$ ($\mu_j(t) := \vec{\mu}(x_j,t)$)

semi-discrete evolution $\longleftrightarrow$ ODE $\mathbf{M}\dfrac{d^2}{dt^2}\vec{\mu}(t) + \mathbf{A}\vec{\mu}(t) = \vec{\varphi}(t)$, **(1.6.4)**

with diagonal matrix $\mathbf{M} = \mathrm{diag}(\rho(x_1), \rho(x_2), \dots, \rho(x_{M-1})) \in \mathbb{R}^{M-1,M-1}$,

$$\mathbf{A} = (a_{ij}) \in \mathbb{R}^{M-1,M-1}: \quad a_{ij} = \frac{2}{h_j + h_{j+1}} \cdot \begin{cases} \left(\dfrac{\gamma(x_{j+1/2})}{h_{j+1}} + \dfrac{\gamma(x_{j-1/2})}{h_j}\right) & \text{, if } i = j \;, \\ -\dfrac{\gamma(x_{j+1/2})}{h_{j+1}} & \text{, if } i = j-1 \;, \\ -\dfrac{\gamma(x_{j-1/2})}{h_j} & \text{, if } i = j+1 \;, \\ 0 & \text{else.} \end{cases} \qquad \textbf{(1.6.5)}$$

Note: $\mathbf{A}$ = symmetric, positive definite tridiagonal matrix

$$\vec{\varphi}(t) \in \mathbb{R}^{M-1}: \quad \varphi_j(t) := \begin{cases} f(x_1,t) + \frac{2}{h_1+h_2}\frac{\gamma(x_{1/2})}{h_1} \cdot g(0,t) & \text{, if } j = 1 \text{ ,} \\ f(x_j,t) & \text{, if } 1 < j < M-1 \text{ ,} \\ f(x_{M-1},t) + \frac{2}{h_{M-1}+h_M}\frac{\gamma(x_{M-1/2})}{h_M} \cdot g(1,t) & \text{, if } j = M-1 \text{ .} \end{cases} \tag{1.6.6}$$

Two-dimensional case

Assumption: $\boxed{\text{Tensor product spatial domain, e.g., } \Omega =]0,1[^2}$

Tensor product grid

$$
\begin{aligned}
\mathcal{M} := \{\,]x_{i-1}, x_i[\times]y_{j-1}, y_j[, \\
& i = 1, \dots, M_x,\ j = 1, \dots, M_y\ , \\
& 0 = x_0 < x_1 < \cdots < x_{M_x} = 1, \\
& 0 = y_0 < y_1 < \cdots < y_{M_y} = 1\}
\end{aligned}
$$

(local) meshwidths $h_i^x := x_i - x_{i-1}$, $h_j^y := y_j - y_{j-1}$, nodes $(x_i, y_j) \in \overline{\Omega}$ (node set $\mathcal{V}(\mathcal{M})$)

Notation: $\boldsymbol{x}_{i,j} := (x_i, y_j)$,
$\boldsymbol{x}_{i+1/2,j} = (1/2(x_{i+1} + x_i), y_j)$, etc.

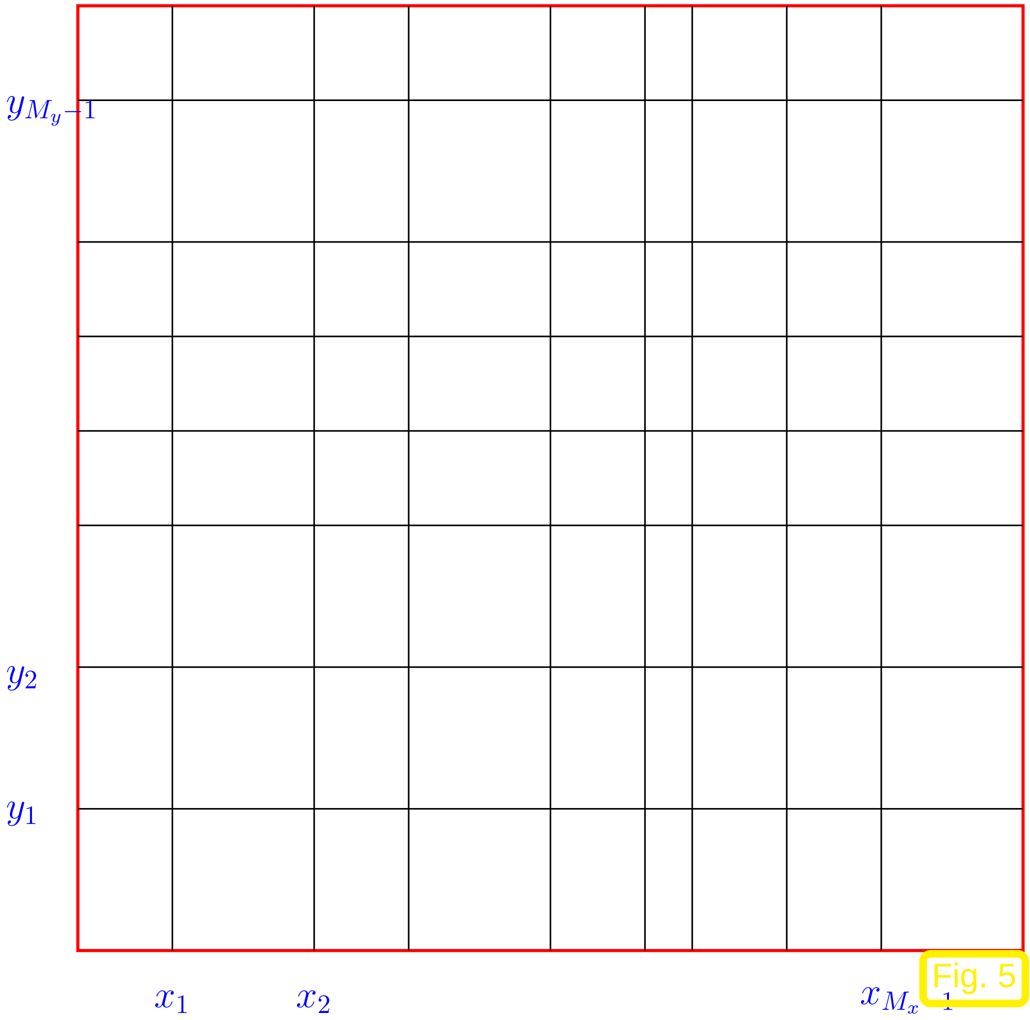


Fig. 5

(1.6.2) ➜ Two-dimensional finite difference approximation [17, Sect. 5.1.4] (for $f \in C^0(\overline{\Omega})$)

$$-\mathcal{L}_{\boldsymbol{x}} f = \operatorname{div}_{\boldsymbol{x}}(\gamma(\boldsymbol{x})\,\mathbf{grad}_{\boldsymbol{x}} f) = \frac{\partial}{\partial x}\left(\gamma(\boldsymbol{x})\frac{\partial f}{\partial x}\right) + \frac{\partial}{\partial y}\left(\gamma(\boldsymbol{x})\frac{\partial f}{\partial y}\right) \quad \text{at } (x_i, y_j)$$

$$\updownarrow$$

$$(\mathsf{T}f)_{ij} := \frac{\gamma(\boldsymbol{x}_{i+1/2,j})\dfrac{f(\boldsymbol{x}_{i+1,j}) - f(\boldsymbol{x}_{i,j})}{h^x_{i+1}} - \gamma(\boldsymbol{x}_{i-1/2,j})\dfrac{f(\boldsymbol{x}_{i,j}) - f(\boldsymbol{x}_{i-1,j})}{h^x_i}}{1/2(h^x_i + h^x_{i+1})} + \frac{\gamma(x_{i,j+1/2})\dfrac{f(\boldsymbol{x}_{i,j+1}) - f(\boldsymbol{x}_{i,j})}{h^y_{j+1}} - \gamma(\boldsymbol{x}_{i,j-1/2})\dfrac{f(\boldsymbol{x}_{i,j}) - f(\boldsymbol{x}_{i,j-1})}{h^y_j}}{1/2(h^y_j + h^y_{j+1})} \tag{1.6.7}$$

Taylor expansion (γ, f smooth) ➤ (1.6.7) $\sim$ 1st-order approximation
2nd-order approximation on equidistant grid

Semi-discrete representation of u: $\vec{\mu} : [0, T] \mapsto \{\mathcal{V}(\mathcal{M}) \mapsto \mathbb{R}\}$

space of grid functions $\cong \mathbb{R}^{(M_x+1)\cdot(M_y+1)}$

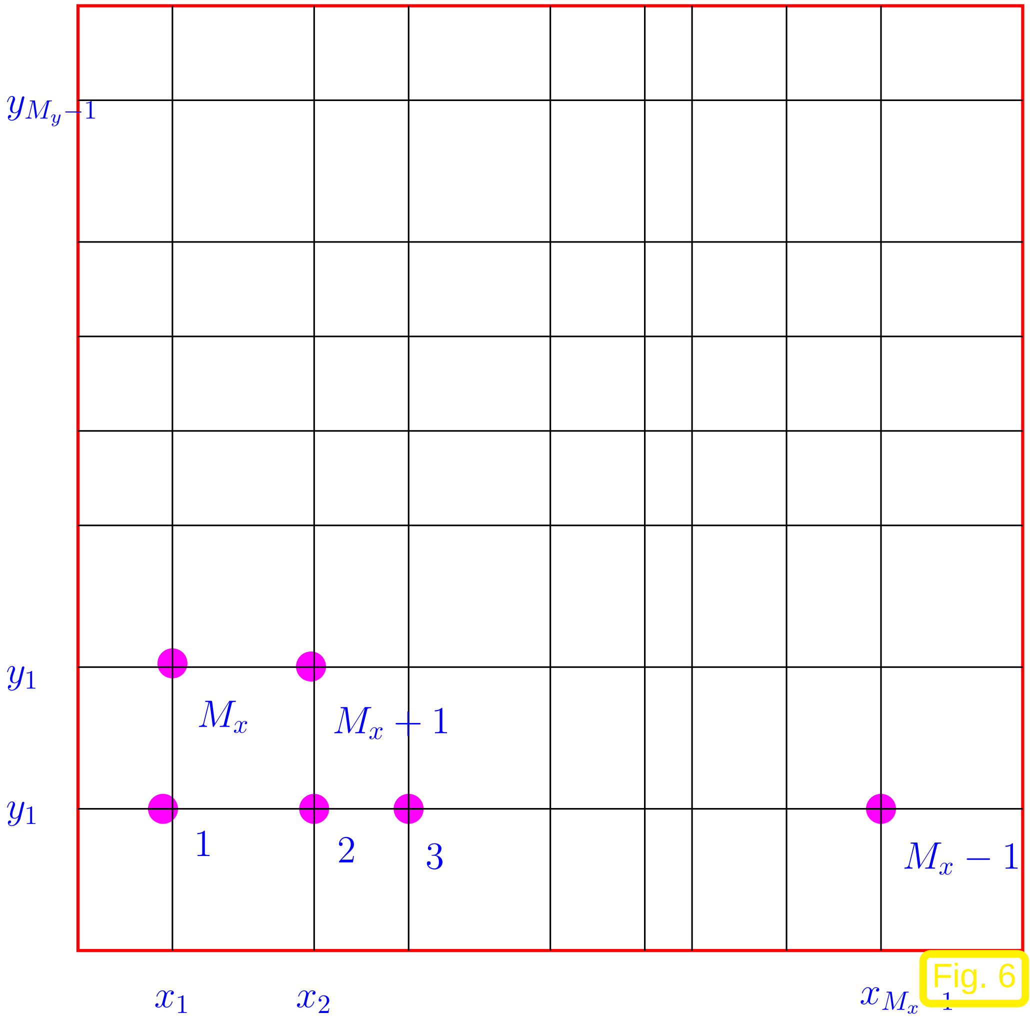


Fig. 6

lexikographic ordering of nodes:

$$\boldsymbol{x}_{1,1}, \boldsymbol{x}_{2,1}, \ldots, \boldsymbol{x}_{M_x-1,1},$$
$$\boldsymbol{x}_{1,2}, \boldsymbol{x}_{2,2}, \ldots, \boldsymbol{x}_{M_x-1,2},$$
$$\cdots$$
$$\boldsymbol{x}_{1,M_y-1}, \boldsymbol{x}_{2,M_y-1}, \ldots, \boldsymbol{x}_{M_x-1,M_y-1}$$

Identification:

(interior) grid functions on $\mathcal{M}$

$\updownarrow$

vectors $\in \mathbb{R}^{(M_x-1)\cdot(M_y-1)}$

- spatially semi-discrete problem

$$\rho(\boldsymbol{x}_{ij})\frac{d^2}{dt^2}\vec{\mu}(\boldsymbol{x}_{ij},t) - (\mathsf{T}\vec{\mu})_{ij}(t) = f(\boldsymbol{x}_{ij},t) \;, \quad \begin{array}{l} i = 1,\ldots,M_x-1 \;, \\ j = 1,\ldots,M_y-1 \;. \end{array} \tag{1.6.8}$$

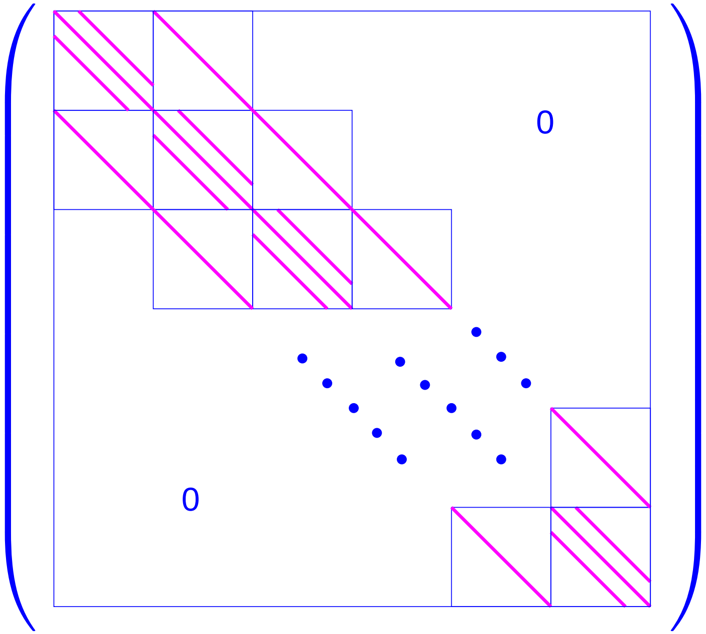
$$\vec{\mu}(\boldsymbol{x}_{ij},t) = g(\boldsymbol{x}_{ij},t) \quad \forall \boldsymbol{x}_{ij} \in \partial\Omega \;. \tag{1.6.9}$$

$\Updownarrow \leftarrow$ assuming lexikographic ordering

$$(1.6.8) \quad \longleftrightarrow \quad \text{ODE} \quad \mathbf{M}\frac{d^2}{dt^2}\vec{\mu}(t) + \mathbf{A}\vec{\mu}(t) = \vec{\varphi}(t) \ ,$$

with diagonal matrix $\mathbf{M} := \mathrm{diag}(\rho(\boldsymbol{x}_{11}, \rho(\boldsymbol{x}_{21}), \ldots, \rho(\boldsymbol{x}_{M_x-1}, M_y - 1)))$ and

$\mathbf{A} =$



$\mathbf{A}$ = $(M_x - 1)\cdot(M_y - 1) \times (M_x - 1)\cdot(M_y - 1)$ matrix: $(M_y - 1)\times(M_y - 1)$-block tridiagonal matrix with $(M_x - 1)\times(M_x - 1)$ blocks. Off-diagonal blocks are diagonal.

➤ $\mathbf{A}$ = sparse matrix [27, Def. 3.1.2]
(at most 5 nonzero entries per row)

$\mathbf{A}$ = symmetric positive definite matrix [27, Def. 1.3.9]

1.6.2 Abstract Galerkin discretization

Idea of Galerkin discretization [27, Sect. 1.3]

In (1.5.2) replace V with finite dimensional subspace V_N
(V_N = discrete trial space/test space)

▶ Abstract discrete 2nd-order hyperbolic evolution problem, *cf.* (1.5.2)

$$u_N \in C^2(]0,T[;V_N) \quad : \quad \begin{cases} \mathsf{m}(\frac{d^2}{dt^2}u_N(t), v_N) + \mathsf{a}(u_N(t), v_N) = \langle f(t), v_N\rangle_V \quad \forall v_N \in V_N\,, \\ u_N(0) = u_{N,0} \in V_N \quad , \quad \frac{du_N}{dt}(0) = v_{N,0} \in H\,. \end{cases} \tag{1.6.10}$$

$u_{N,0} \in V_N$, $v_{N,0} \in V_N$ = projection/interpolant of u_0, v_0, resp.

Note: Stability estimates, Thm. 1.5.1, also apply to (1.6.10) !

Advantage of Galerkin perspective: abstract a priori error estimates [27, Sect. 7.3]:

Tool: $\mathsf{P}_N : V \mapsto V_N$ = a-orthogonal projection onto V_N (Galerkin projection [27, Thm. 1.3.4])

Trick: split error $u - u_N = u - \mathsf{P}_N u \; + \; \mathsf{P}_N u - u_N$

spatial projection error evolution error

Assumed: extra regularity

- of initial data: $\frac{du}{dt}(0) = v_0$ in V
- of solution (in time): $u \in H^2(]0,T[;H) \cap H^1(]0,T[;V)$

$$\text{(1.5.2)} \overset{V_N \subset V}{\Longrightarrow} \begin{cases} \mathsf{m}(\frac{d^2}{dt^2}u, v_N) + \mathsf{a}(\mathsf{P}_N u, v_N) = \langle f(t), v_N \rangle_V \quad \forall v_N \in V_N \,, \\ u(0) = u_0 \quad , \quad \frac{du}{dt}(0) = v_0 \,. \end{cases} \tag{1.6.11}$$

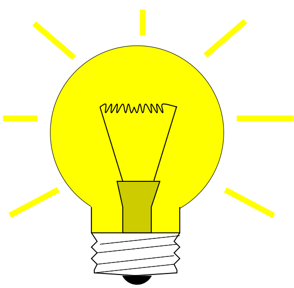
$$\blacktriangleright \quad \mathsf{m}(\mathsf{P}_N \frac{d^2}{dt^2}u, v_N) + \mathsf{a}(\mathsf{P}_N u, v_N) = f(v_N) + \mathsf{m}(\frac{d^2}{dt^2}(\mathsf{P}_N - Id)u, v_N) \quad \forall v_N \in V_N \,. \tag{1.6.12}$$

Subtract: (1.6.10) - (1.6.12)

$$\mathsf{m}\Big(\frac{d^2}{dt^2}(u_N - \mathsf{P}_N u), v_N\Big) + \mathsf{a}(u_N - \mathsf{P}_N u, v_N) = \mathsf{m}\Big(\frac{d^2}{dt^2}(\mathsf{P}_N - Id)u, v_N\Big) \quad \forall v_N \in V_N \ . \tag{1.6.13}$$

$$(u_N - \mathsf{P}_N u)(0) = u_{N,0} - \mathsf{P}_N u_0 \quad , \quad \frac{d(u_N - \mathsf{P}_N u)}{dt}(0) = v_{N,0} - \mathsf{P}_N v_0 \ .$$

$u_N - \mathsf{P}_N u$ solves a semi-discrete evolution problem like (1.6.10) with consistency error terms (residual type quantities $\longrightarrow$ “small”) on the right hand side **!**



Idea: Standard approach to

$$\mathcal{E}(\text{error term}) = \text{residual term}$$

Spatio-temporal evolution operator underlying IBVP

▷ use stability estimate, here Thm. 1.5.1

$$\left\|u_N - \mathsf{P}_N u\right\|_{L^\infty(]0,T[;V)} + \left\|\frac{du_N}{dt} - \mathsf{P}_N\Big(\frac{du}{dt}\Big)\right\|_{L^\infty(]0,T[;H)} \leq$$

$$\leq C\left(\left\|(Id - \mathsf{P}_N)\frac{d^2u}{dt^2}\right\|_{L^2(]0,T[;H)} + \left\|u_{N,0} - \mathsf{P}_N u_0\right\|_V + \left\|v_{N,0} - \mathsf{P}_N v_0\right\|_H\right) \ . \tag{1.6.14}$$

$\longleftarrow \triangle$-inequality

$$\boxed{\left\|u_N - u\right\|_{L^\infty(]0,T[;V)} + \left\|\frac{d}{dt}(u_N - u)\right\|_{L^\infty(]0,T[;H)}} \leq$$

semi-discrete " energy error"

$$\leq \ \left\|u - \mathsf{P}_N u\right\|_{L^\infty(]0,T[;V)} + \left\|\frac{du}{dt} - \mathsf{P}_N(\frac{du}{dt})\right\|_{L^\infty(]0,T[;H)} +$$
$$C\left(\left\|(Id - \mathsf{P}_N)\frac{d^2u}{dt^2}\right\|_{L^2(]0,T[;H)} + \left\|u_{N,0} - \mathsf{P}_N u_0\right\|_V + \left\|v_{N,0} - \mathsf{P}_N v_0\right\|_H\right) .$$

What can interfere with spatial/temporal smoothness of solutions of wave equation (1.1.3) **?**

- poor regularity of initial data, *cf.* Rem. 9. Also affect smoothness in time, *cf.* [27, Sect. 7.2]
- poor lifting properties of $\mathcal{L}_{\boldsymbol{x}}$ [27, Sect. 4.3]
  (due to non-smooth $\partial\Omega$, re-entrant corners, discontinuous $\mathbf{C}$)
- spatially/temporally non-smooth source function f

How to obtain final ODE (1.6.1) **?**

Choose (ordered) basis $\mathfrak{B} := \{b_N^1, \dots, b_N^N\}$, $N := \dim V_N$, of V_N, *cf.* [27, Sect. 1.3.2]:

▶ representation: $u_N(t) = \sum_{l=1}^{N} \mu_l(t) b_N^l$, $\vec{\mu}(t) := (\mu_1(t), \dots, \mu_N(t))^T \in \mathbb{R}^N$.

$$\text{(1.6.10)} \quad \Rightarrow \quad \begin{cases} \mathbf{M}\left\{\frac{d^2}{dt^2}\vec{\mu}(t)\right\} + \mathbf{A}\vec{\mu}(t) = \vec{\varphi}(t) & \text{for } 0 < t < T \ , \\ \vec{\mu}(0) = \vec{\mu}_0 \quad , \quad \frac{d\vec{\mu}}{dt}(0) = \vec{\eta}_0 \ . \end{cases} \tag{1.6.15}$$

- ▷ s.p.d. stiffness matrix $\mathbf{A} \in \mathbb{R}^{N,N}$, $(\mathbf{A})_{ij} := \mathsf{a}(b_N^j, b_N^i)$ (independent of time),
- ▷ s.p.d. mass matrix $\mathbf{M} \in \mathbb{R}^{N,N}$, $(\mathbf{M})_{ij} := \mathsf{m}(b_N^j, b_N^i)$ (independent of time),
- ▷ source (load) vector $\vec{\varphi}(t) \in \mathbb{R}^N$, $(\vec{\varphi}(t))_i := \left\langle f(t), b_N^i \right\rangle_V$ (time-dependent),
- ▷ $\vec{\mu}_0$, $\vec{\eta}_0 \;\hat{=}$ coefficient vectors of approximations $u_{N,0}$, $v_{N,0}$ of initial data u_0, v_0

Choice of basis $\mathfrak{B}$	has **no** impact on semi-discrete solution u_N of (1.6.10) crucially affects matrices $\mathbf{A}, \mathbf{M}$ (sparsity, conditioning) **!**

1.6.3 Linear Lagrangian finite elements (FE)

Finite element method [27, Ch. 3] $\leftrightarrow$ Galerkin discretization based on special trial/test spaces V_N :

☞ V_N piecewise polynomial w.r.t. partitioning (= mesh) of Ω

☞ V_N possesses basis $\mathfrak{B}$ consisting of *locally supported* functions ➤ sparse matrices

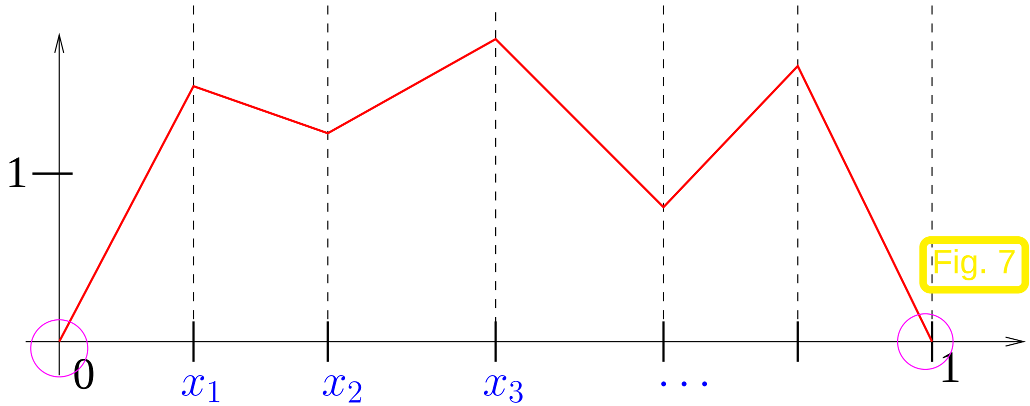
One-dimensional case

$d = 1$ ➤ (as before in Sect. 1.6.1) $\Omega =]0,1[$ (open interval), $\partial\Omega = \{0,1\}$, $\boldsymbol{\mathcal{L}_x} = -\frac{\partial}{\partial x}\left(\gamma(x)\frac{\partial f}{\partial x}\right)$

mesh: $\mathcal{M} := \{]x_{j-1}, x_j[: 0 = x_0 < x_1 < \cdots < x_M = 1,\ i = 1,\dots,M\}$, $M \in \mathbb{N}$.

x_j = nodes, $\mathcal{V}(\mathcal{M})$ = set of nodes, (local) meshwidths $h_j := x_j - x_{j-1}$, $]x_{j-1}, x_j[$ = cells.

Remember [27, Lemma 2.9.1]: $V_N \subset H^1(\Omega)$ & $\mathcal{M}$-p.w. polynomial $\Rightarrow$ $V_N \subset C^0(\overline{\Omega})$



$\Uparrow$ function $\in \mathcal{S}^0_{1,0}(\mathcal{M})$

Simplest choice (homogeneous Dirichlet b.c. !)

$$V_N = \mathcal{S}^0_{1,0}(\mathcal{M})$$
$$:= \left\{ \begin{array}{l} v \in C^0([0,1])\colon v_{|[x_{i-1},x_i]} \text{ linear}, \\ i = 1,\dots,M,\, v(0) = v(1) = 0 \end{array} \right\}$$

➢ $$V_N \subset H^1_0(\Omega)$$

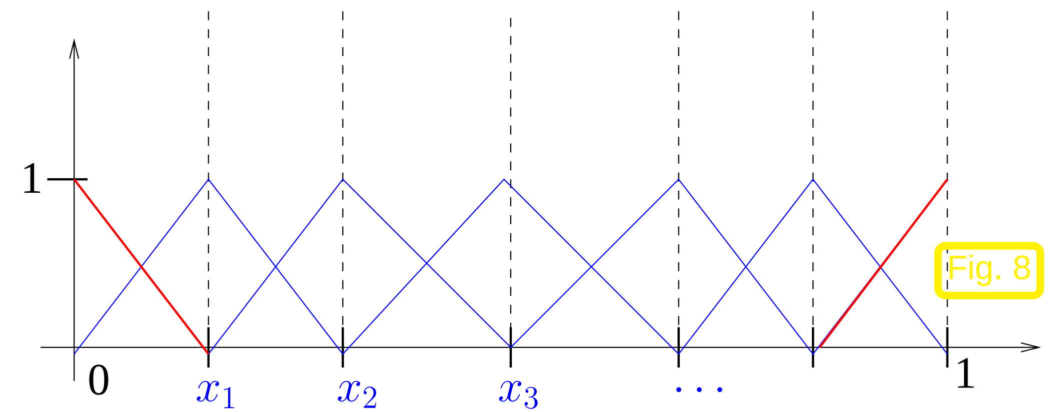
➢ $$\dim V_N = M - 1$$

Choice of ordered basis $\mathfrak{B}$ **?**

Clear: 1D "hat functions" ▷

$$\mathfrak{B} = \{b^1_N, \dots, b^{M-1}_N\} \;,$$

$$b^j_N(x_i) = \delta_{ij} := \begin{cases} 1 & \text{, if } i = j \;, \\ 0 & \text{, if } i \neq j \;, \end{cases}$$



▷ stiffness matrix $\mathbf{A} = (a_{ij}) \in \mathbb{R}^{M-1,M-1}$, $a_{ij} := \int\limits_0^1 \gamma(x) \frac{db_N^i}{dx}(x) \, \frac{db_N^j}{dx}(x) \, \mathrm{d}x$, $1 \leq i,j < M$

weak = piecewise derivatives

▷ mass matrix $\mathbf{M} = (m_{ij}) \in \mathbb{R}^{M-1,M-1}$, $m_{ij} := \int\limits_0^1 \rho(x) b_N^i(x) \, b_N^j(x) \, \mathrm{d}x$, $1 \leq i,j < M$

▷ load vector $\vec{\varphi}(t) \in \mathbb{R}^{M-1}$, $\varphi_i(t) := \int\limits_0^1 f(x,t) b_N^i(x) \, \mathrm{d}x$, $i = 1, \ldots, M-1$

(Dirichlet data contribute to $\varphi_1(t)$, $\varphi_{M-1}(t)$, see (1.6.6))

➡ Both $\mathbf{A}$ and $\mathbf{M}$ are symmetric, positive definite and tridiagonal

How to evaluate integrals **?** $\longrightarrow$ numerical quadrature

for $\mathbf{A}$: cell based midpoint rule $$\int\limits_0^1 f(x) \, \mathrm{d}x \approx \sum_{j=1}^{M} h_j f(x_{j-1/2})$$

for $\mathbf{M}$ and $\vec{\varphi}$: trapezoidal rule $$\int\limits_0^1 f(x) \, \mathrm{d}x \approx \sum_{j=1}^{M-1} 1/2(h_j + h_{j+1}) f(x_j)$$

► $\boxed{\mathbf{A}, \mathbf{M}, \text{ and } \vec{\varphi} \text{ equal to those obtained from 1D finite differences, Sect. 1.6.1 !}}$

(➢ analysis of finite differences in (perturbed) Galerkin context)

Two-dimensional case

$\Omega \subset \mathbb{R}^2$ bounded with piecewise smooth boundary ("curvilinear polygon")

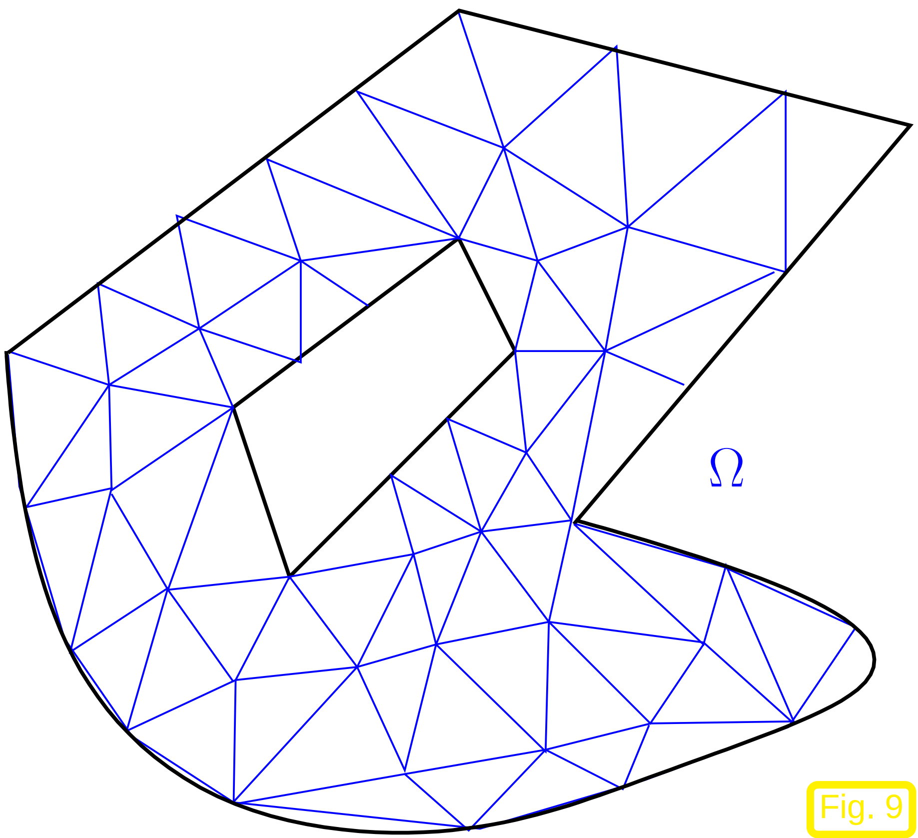


Fig. 9

Triangulation $\mathcal{M}$ of (polygonal approximation of) Ω:

- $\mathcal{M} = \{K_i\}_{i=1}^{M}$, $M \in \mathbb{N}$, $K_i \,\hat{=}$ open triangle
- disjoint interiors: $i \neq j \Rightarrow K_i \cap K_i = \emptyset$
- tiling property: $\displaystyle\bigcup_{i=1}^{M} \overline{K}_i = \overline{\Omega}$
- intersection $\overline{K}_i \cap \overline{K}_j$, $i \neq j$,
  is – either $\emptyset$
  – or an edge of both triangles
  – or a vertex of both triangles

Parlance: vertices of triangles **=** nodes of mesh (= set $\mathcal{V}(\mathcal{M})$)

Notion: meshwidth $h_{\mathcal{M}} := \max\{h_K := \operatorname{diam}(K) \colon K \in \mathcal{M}\}$ (= length of longest edge)

Important: mesh quality $\leftrightarrow$ shape regularity [27, Sect. 4.2.4]

lower bound on smallest angle of triangles (➤ limited distortion of cells)

[27, Lemma 2.9.1] ➤ $\mathcal{M}$-piecewise polynomial functions in $H^1(\Omega)$ have to be continuous

▶ simplest choice for V_N:

$$\boxed{V_N = \mathcal{S}^0_{1,0}(\mathcal{M}) := \left\{ \begin{array}{r} v \in C^0(\overline{\Omega}) \colon v_{|\partial\Omega} = 0 \;, \forall K \in \mathcal{M} \colon v_{|K}(\boldsymbol{x}) = \alpha_K + \boldsymbol{\beta}_K \cdot \boldsymbol{x}, \\ \alpha_K \in \mathbb{R}, \boldsymbol{\beta}_K \in \mathbb{R}^2, \boldsymbol{x} \in K \end{array} \right\} \subset H^1_0(\Omega)}$$

Locally supported basis functions in 2D **?**

On a triangle K with vertices $\mathbf{a}^1, \mathbf{a}^2, \mathbf{a}^3$: linear $q : K \mapsto \mathbb{R}$ uniquely determined by values $q(\mathbf{a}^i)$.

► $v_N \in \mathcal{S}^0_{1,0}(\mathcal{M})$ uniquely determined by $\{v_N(\mathbf{x}),\ \mathbf{x}$ interior node of $\mathcal{M}\}$!

► $N := \dim \mathcal{S}^0_{1,0}(\mathcal{M}) = \sharp\mathcal{V}_0(\mathcal{M})$ ($\mathcal{V}_0(\mathcal{M})$ = set of interior nodes (= vertices $\notin \partial\Omega$) of $\mathcal{M}$)

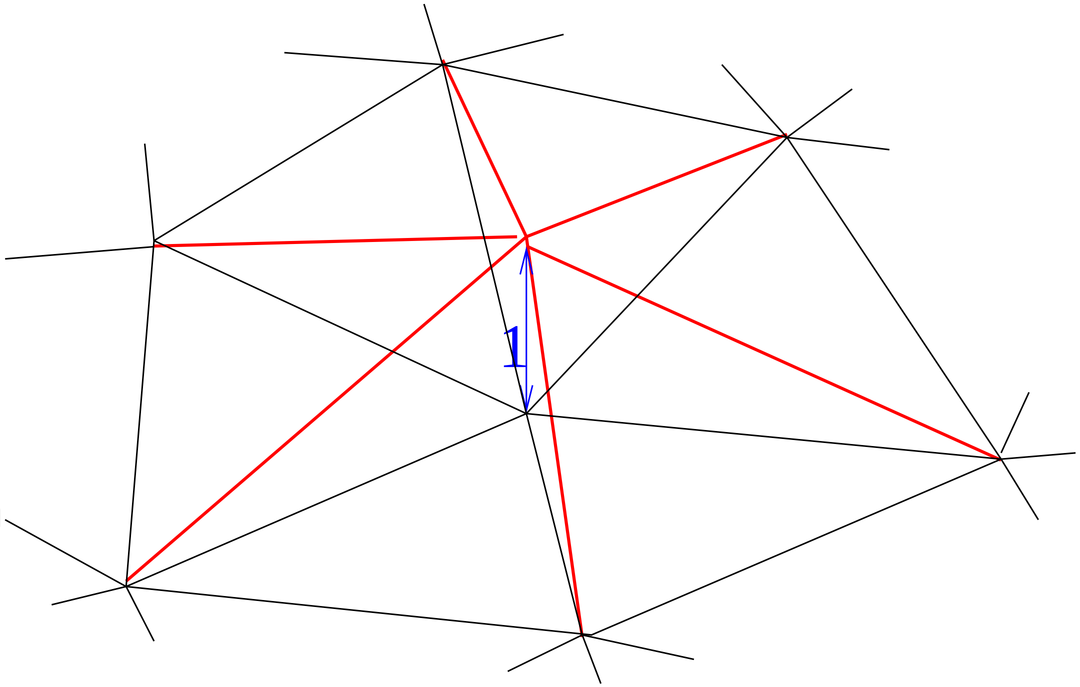
▼

$\mathcal{V}_0(\mathcal{M}) = \{\boldsymbol{x}_1, \dots, \boldsymbol{x}_N\}$: nodal basis $\mathfrak{B} := \{b^1_N, \dots, b^N_N\}$ of $\mathcal{S}^0_{1,0}(\mathcal{M})$ defined by $b^i_N(\boldsymbol{x}_j) = \delta_{ij}$.

Ordering ($\leftrightarrow$ numbering) of nodes assumed !

Piecewise linear nodal basis function

("hat function")



$$u_N = \sum_{i=1}^{N} \mu_i b^i_N \in \mathcal{S}^0_1(\mathcal{M})$$

► coefficient μ_j = "nodal value" of u_N at j-th node of $\mathcal{M}$

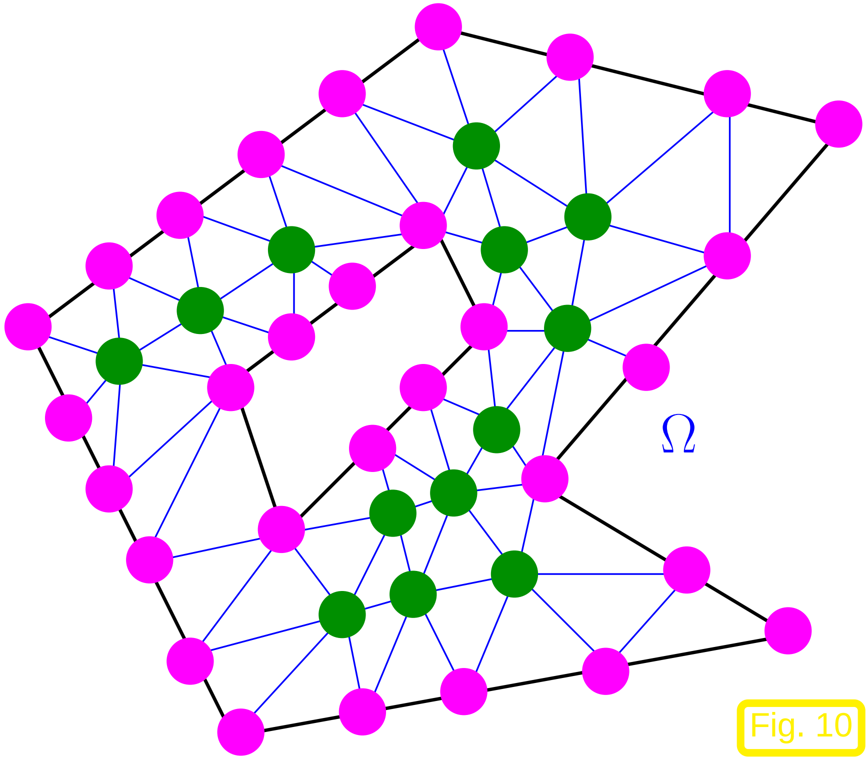


Fig. 10

◁ "Location" of nodal basis functions:

● $\longrightarrow$ nodal basis functions of $\mathcal{S}^0_{1,0}(\mathcal{M})$

● $\longrightarrow$ vertices on the boundary of Ω

▷ stiffness matrix $\mathbf{A} = (a_{ij}) \in \mathbb{R}^{N,N}$, $\quad a_{ij} := \int_\Omega \mathbf{C}(\boldsymbol{x})\,\mathbf{grad}\, b^i_N(\boldsymbol{x}) \cdot \mathbf{grad}\, b^j_N \,\mathrm{d}\boldsymbol{x}\mathrm{d}x$,
$1 \leq i,j \leq N$

▷ mass matrix $\mathbf{M} = (m_{ij}) \in \mathbb{R}^{M-1,M-1}$, $\quad m_{ij} := \int_\Omega \rho(\boldsymbol{x}) b^i_N(\boldsymbol{x})\, b^j_N \,\mathrm{d}\boldsymbol{x}$, $1 \leq i,j \leq N$

▷ load vector $\vec{\varphi}(t) \in \mathbb{R}^{M-1}$, $\quad \varphi_i(t) := \int_\Omega f(\boldsymbol{x},t) b^i_N(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$, $i = 1,\dots,N$
(Dirichlet data may contribute to $\varphi_i(t)$, when $\boldsymbol{x}_i$ shares edge with vertex on $\partial\Omega$)

$\mathbf{A}$, $\mathbf{M}$ sparse: $\quad a_{ij} \neq 0$, $m_{ij} \neq 0$ only if x_i, x_j connected by edge !

As in 1D: cell based numerical quadrature used for evaluation of integrals:

barycentric quadrature $$\int_\Omega f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \approx \sum_{K\in\mathcal{M}} |K| f(\boldsymbol{m}_K)$$ $\longrightarrow$ used for $\mathbf{A}$

($\boldsymbol{m}_K$ = barycenter of K)

vertex based quadrature $$\int_\Omega f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \approx \sum_{K\in\mathcal{M}} \tfrac{1}{3}|K| \sum_{i=1}^{3} f(\boldsymbol{a}_K^i)$$ $\longrightarrow$ used for $\mathbf{M}$, $\vec{\varphi}$

($\mathbf{a}_K^i$ = vertices of triangle K) mass lumping $\Rightarrow$ $\mathbf{M}$ diagonal

Remark 10 (FD und FEM).

Setting: $\mathbf{C} \equiv \mathbf{I}$, Ω rectangle

Galerkin FEM based on $\mathcal{S}^0_{1,0}(\mathcal{M})$

+

"structured" triangular mesh ▷

+

Numerical quadrature, see above

stiffness matrix & mass matrix agree with FD-matrices on tensor product grid

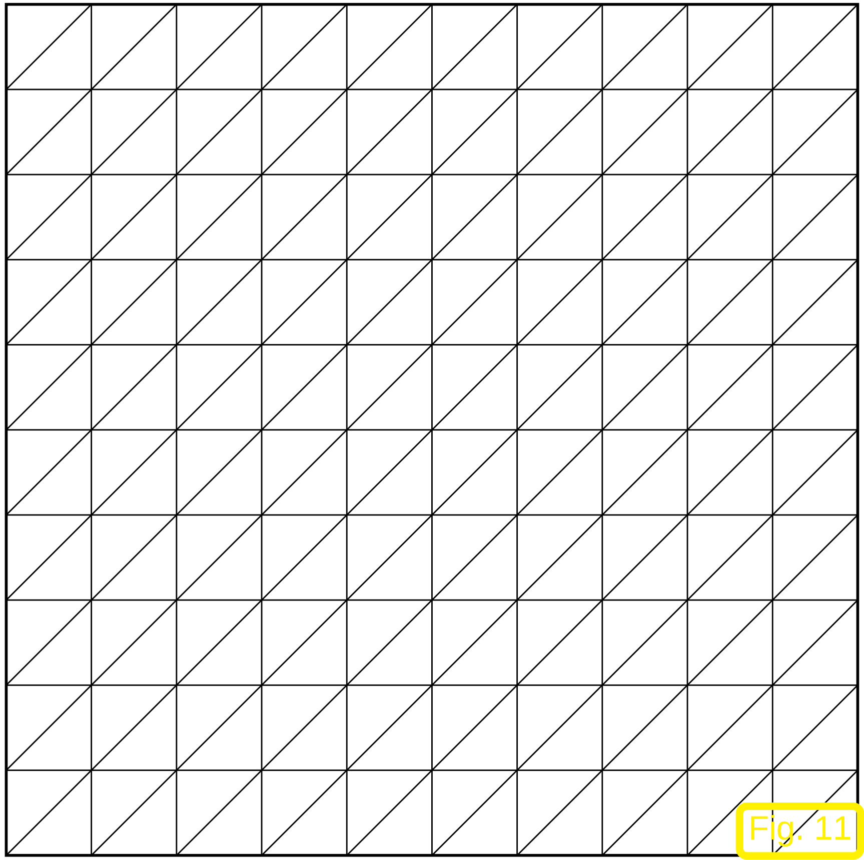


Fig. 11

△

Summary: approximation properties of Galerkin projection $\mathsf{P}_N : H^1_0(\Omega) \mapsto \mathcal{S}^0_{1,0}(\mathcal{M})$ (w.r.t. bilinear form $\mathsf{a}(u,v) = \int_\Omega \mathbf{C}\,\mathbf{grad}\,u \cdot \mathbf{grad}\,v\,\mathrm{d}\boldsymbol{x}$, $u,v \in H^1_0(\Omega)$)

Theorem 1.6.1 (Galerkin projection error for $\mathcal{S}^0_{1,0}(\mathcal{M})$). $\rightarrow$ *[27, Lemma 4.2.29]*

There is $C > 0$ only depending on $1 < s \leq 2$, Ω, $\mathbf{C}$, and the shape regularity of $\mathcal{M}$ such that

$$\|u - \mathsf{P}_N u\|_{H^1(\Omega)} \leq C h_{\mathcal{M}}^{\min\{1,s-1\}} \|u\|_{H^s(\Omega)} \quad \forall u \in H^s(\Omega) \cap H^1_0(\Omega) .$$

If the Dirichlet problem for $\mathcal{L}_{\boldsymbol{x}}$ is 2-regular [27, Sect. 4.3], then there is $C > 0$ only depending on Ω, $\mathbf{C}$, and the shape regularity of $\mathcal{M}$ such that

$$\|u - \mathsf{P}_N u\|_{L^2(\Omega)} \leq C h_{\mathcal{M}} \|u - \mathsf{P}_N u\|_{H^1(\Omega)} \quad \forall u \in H^1_0(\Omega) .$$

$\Downarrow$ $\leftarrow$ abstract convergence theory of Sect. 1.6.

Optimum for linear FE:	1st order algebraic convergence (of semi-discrete energy error) in meshwidth $h_{\mathcal{M}}$

1.7 Timestepping

Start from algebraic semi-discrete evolution (1.6.15) **=** 2nd-order ODE:

$$\mathbf{M}\left\{\frac{d^2}{dt^2}\vec{\mu}(t)\right\} + \mathbf{A}\vec{\mu}(t) = \vec{\varphi}(t) \quad , \quad \vec{\mu}(0) = \vec{\mu}_0 \; , \quad \frac{d\vec{\mu}}{dt}(0) = \vec{\eta}_0 \; . \tag{1.7.1}$$

Key features of (1.7.1) ➡ to be “approximately” respected by timestepping:

- reversibility: if $\vec{\varphi} = 0$ ➤ (1.7.1) invariant under time-reversal $t \leftarrow -t$
- energy conservation, *cf.* (1.5.7): if $\vec{\varphi} = 0$ ➤ $E_N(t) := \frac{1}{2}\frac{d\vec{\mu}}{dt} \cdot \mathbf{M}\frac{d\vec{\mu}}{dt} + \frac{1}{2}\vec{\mu} \cdot \mathbf{A}\vec{\mu} = \mathrm{const}$

Note: for Galerkin discretization of (1.5.2): $\mathbf{A}$, $\mathbf{M}$ s.p.d., *cf.* Sect. 1.6.3

1.7.1 Simple two-step methods

Definition 1.7.1 (Two-step method). *A two-step method for* (1.7.1) *with uniform timestep* $\Delta t := T/M > 0$, $M \in \mathbb{N}$, *generates sequence* $(\vec{\mu}^{(k)})_{k=0}^{M}$ *of approximations* $\vec{\mu}^{(k)} \approx \vec{\mu}(t_k)$, $t_k := k\Delta t$, $0 \leq k \leq M$, *by*

$$\vec{\mu}^{(k+1)} = \Phi(\vec{\mu}^{(k)}, \vec{\mu}^{(k-1)}; k, \Delta t) \;, \quad \Phi(\cdot,\cdot;k,\Delta t) : \mathbb{R}^N \times \mathbb{R}^N \mapsto \mathbb{R}^N \;.$$

Note: any two-step method requires special initial step ($\vec{\mu}^{(0)}$, $\vec{\mu}^{(1)}$ from $\vec{\mu}_0$, $\vec{\eta}_0$)

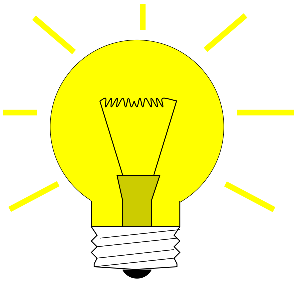
First consider (1.7.1) for $\vec{\varphi} = 0$ **&** transform

$$\vec{\nu} := \mathbf{M}^{1/2}\vec{\mu}: \qquad \frac{d^2}{dt^2}\vec{\nu} + \widetilde{\mathbf{A}}\vec{\nu} = 0 \;, \quad \widetilde{\mathbf{A}} := \mathbf{M}^{-\frac{1}{2}}\mathbf{A}\mathbf{M}^{-\frac{1}{2}} \;. \tag{1.7.2}$$

Formal solution, *cf.* (1.5.9):

$$\vec{\nu}(t) = \cos(\widetilde{\mathbf{A}}^{1/2}t)\vec{\nu}(0) + \widetilde{\mathbf{A}}^{-1/2}\sin(\widetilde{\mathbf{A}}^{1/2}t)\frac{d\vec{\nu}}{dt}(0) \;, \quad t > 0 \;. \tag{1.7.3}$$

$$\blacktriangleright \qquad \vec{\nu}(t+\Delta t) + \vec{\nu}(t-\Delta t) = 2\cos(\widetilde{\mathbf{A}}^{1/2}\Delta t)\vec{\nu}(t) \;, \quad t, \Delta t > 0 \;. \tag{1.7.4}$$



Idea: approximate $\cos(z) \approx R(z)$, R = rational function

➤ 2-step timestepping:

$$\boxed{\vec{\nu}^{(k+1)} + \vec{\nu}^{(k-1)} = 2R(\widetilde{\mathbf{A}}^{1/2}\Delta t)\vec{\nu}^{(k)}} \;, \quad k \in \mathbb{N} \,. \tag{1.7.5}$$

We expect: $\vec{\nu}^{(k-1)} \approx \vec{\nu}(t-\Delta t)$ & $\vec{\nu}^{(k)} \approx \vec{\nu}(t)$ $\Rightarrow$ $\vec{\nu}^{(k+1)} \approx \vec{\nu}(t+\Delta t)$

Obvious: if $R(z) = R(-z)$ $\Rightarrow$ (1.7.5) is time-reversible

Remark 11 (Explicit and implicit two-step methods).

$R(z)$ polynomial $\Rightarrow$ $\vec{\nu}^{(k+1)}$ only from evaluations $\mathbf{A}\times$vector (explicit)

$R(z)$ genuine rational function $\Rightarrow$ $\vec{\nu}^{(k+1)}$ by solving linear systems derived from $\widetilde{\mathbf{A}}$ (implicit)

In the case of (1.7.1): "inversion of mass matrix $\mathbf{M}$" also for explicit two-step methods

➢ importance of mass lumping !

$\triangle$

Definition 1.7.2 (Consistency of a two-step method)**.** *A two-step method* $\Phi(\cdot,\cdot;\Delta t)$ *for* (1.7.2) *(→ Def. 1.7.1) is (uniformly) consistent of order* p, $p \in \mathbb{N}_0$, *if*

$$|\Phi(\vec{\nu}(t),\vec{\nu}(t-\Delta t),\Delta t) - \vec{\nu}(t+\Delta t)| \leq C(\Delta t)^{p+2} ,$$

with $C > 0$ *independent of* $\Delta t > 0$ *(for sufficiently small* Δt*) and* $t > 0$*.*

Corollary 1.7.3. *Two-step method* (1.7.5) *for* (1.7.2) *is consistent of order* p, $p \in \mathbb{N}_0$,

$$\Leftrightarrow \qquad \exists C > 0, \delta > 0\colon \quad |R(x) - \cos x| \leq C|x|^{p+2} \quad \forall |x| \leq \delta \;.$$

1.7.1.1 Leapfrog timestepping

In (1.7.5) choose truncated Taylor series $R(z) = 1 - \frac{1}{2}z^2 \;\Rightarrow\;$ consistent of order 2

$$\blacktriangleright \qquad \frac{\vec{\nu}^{(k+1)} - 2\vec{\nu}^{(k)} + \vec{\nu}^{(k-1)}}{(\Delta t)^2} = -\widetilde{\mathbf{A}}\vec{\nu}^{(k)} \qquad (1.7.6)$$

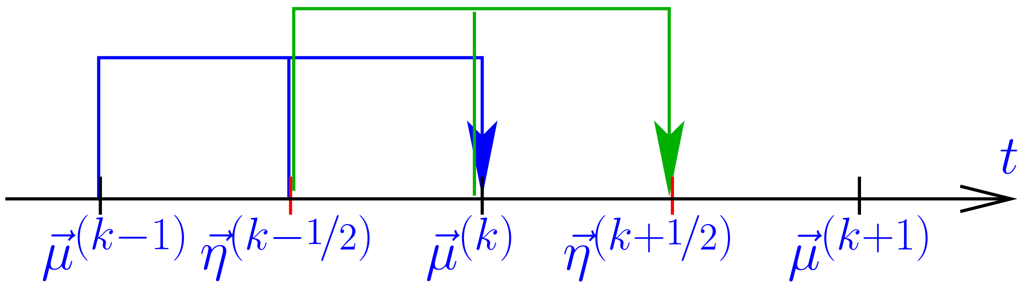
explicit trapezoidal rule/Störmer scheme for (1.7.1) (with uniform timestep $\Delta t := T/M$, $M \in \mathbb{N}$)

$$\mathbf{M}\frac{\vec{\mu}^{(k+1)} - 2\vec{\mu}^{(k)} + \vec{\mu}^{(k-1)}}{(\Delta t)^2} = -\mathbf{A}\vec{\mu}^{(k)} + \vec{\varphi}(t_k)\;, \quad k = 0, \dots, M-1\;, \tag{1.7.7}$$

$$+ \qquad \text{initial step} \quad \frac{\vec{\mu}^{(1)} - \vec{\mu}^{(-1)}}{2\Delta t} = \vec{\eta}_0\;. \tag{1.7.8}$$

Auxiliary variable: $\vec{\eta}^{(k+1/2)} := \dfrac{\vec{\mu}^{(k+1)} - \vec{\mu}^{(k)}}{\Delta t}$

$\hat{=}$ velocity approximation



$\Downarrow$

Equivalent leapfrog/Verlet-implementation of (1.7.7) (used in practice):

$$\begin{aligned} \mathbf{M}\frac{\vec{\eta}^{(k+1/2)} - \vec{\eta}^{(k-1/2)}}{\Delta t} &= -\mathbf{A}\vec{\mu}^{(k)} + \vec{\varphi}(t_k)\;, \\ \frac{\vec{\mu}^{(k+1)} - \vec{\mu}^{(k)}}{\Delta t} &= \vec{\eta}^{(k+1/2)}\;, \end{aligned} \qquad k = 0, \dots, M-1\;, \tag{1.7.9}$$

$$+ \qquad \text{initial step} \quad \vec{\eta}^{(-1/2)} + \vec{\eta}^{1/2} = 2\vec{\eta}_0\;.$$

work per step: $1\times$ evaluation $\mathbf{A}\times$vector, $1\times$ solution of linear system for $\mathbf{M}$

Remark 12 (Leap frog as variational integrator).

Discrete wave equation (1.6.15) **=** Euler-Lagrange equations for Lagrangian

$$L(\vec{\mu}, \frac{d\vec{\mu}}{dt}) := \frac{1}{2}\frac{d\vec{\mu}}{dt} \cdot \mathbf{M}\frac{d\vec{\mu}}{dt} - \frac{1}{2}\vec{\mu}\mathbf{A}\vec{\mu}$$

[28], [19, Sect. VI.6]: leap frog $\leftrightarrow$ Euler-Lagrange equations for time-discrete approximation of L

Parlance: leap frog **=** variational integration scheme

$\triangle$

Example 13 (Leap frog and energy conservation).

- $d = 1$, $\Omega =]0,1[$ 1D wave equation $c \equiv 1$, homogeneous Dirichlet b.c.
- $u(\cdot,0)$ = hat function, supported in $[\frac{1}{4},\frac{1}{2}]$, $\frac{\partial u}{\partial t}(\cdot,0) = 0$
- spatial finite difference discretization, equidistant grid, meshwidth $h > 0$
- explicit trapezoidal rule (1.7.7)

Monitored: total energy

(for $h = 1/200$)

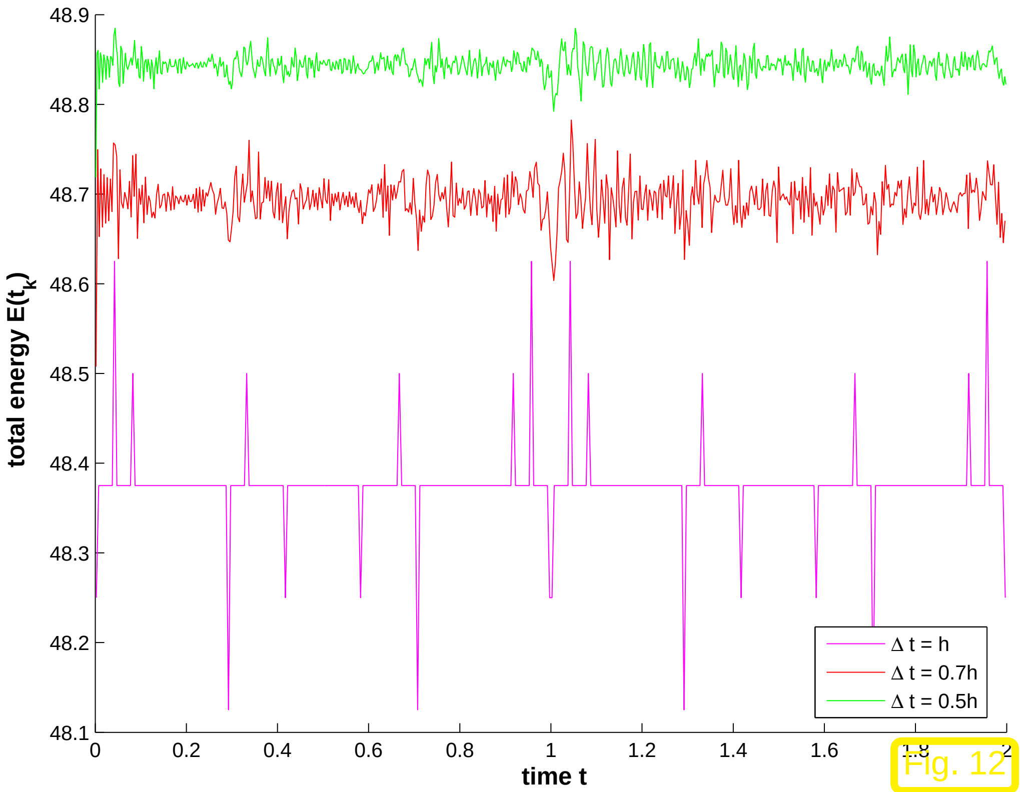


Fig. 12

$$E(t_{k+1/2}) := \frac{1}{2}\frac{\vec{\mu}^{(k+1)} - \vec{\mu}^{(k)}}{\Delta t} \cdot \mathbf{M}\frac{\vec{\mu}^{(k+1)} - \vec{\mu}^{(k)}}{\Delta t} + \frac{1}{2}\frac{\vec{\mu}^{(k+1)} + \vec{\mu}^{(k)}}{2} \cdot \mathbf{A}\frac{\vec{\mu}^{(k+1)} + \vec{\mu}^{(k)}}{2} \,.$$

► no exact energy conservation, but *no energy drift* ! $\longrightarrow$ [19, Sect. IX.3]

◇

1.7.1.2 Crank-Nicolson timestepping

In (1.7.5) choose Padé approximation of $\cos$ $R(z) = \dfrac{1 - {}^1\!/_4 z^2}{1 + {}^1\!/_4 z^2}$ $\Rightarrow$ consistent of order 2

$$\frac{\vec{\nu}^{(k+1)} - 2\vec{\nu}^{(k)} + \vec{\nu}^{(k-1)}}{(\Delta t)^2} = -\tfrac{1}{4}\widetilde{\mathbf{A}}(\vec{\nu}^{(k+1)} + 2\vec{\nu}^{(k)} + \vec{\nu}^{(k-1)})$$

implicit trapezoidal rule for (1.7.1) (with uniform timestep $\Delta t := T/M$, $M \in \mathbb{N}$)

$$\begin{aligned}\mathbf{M}\,\frac{\vec{\mu}^{(k+1)} - 2\vec{\mu}^{(k)} + \vec{\mu}^{(k-1)}}{(\Delta t)^2} = \; & -\tfrac{1}{4}\mathbf{A}(\vec{\mu}^{(k+1)} + 2\vec{\mu}^{(k)} + \vec{\mu}^{(k-1)}) \\ & + \tfrac{1}{4}(\vec{\varphi}(t_{k+1}) + 2\vec{\varphi}(t_k) + \vec{\varphi}(t_{k-1}))\,, \quad k = 0, \dots, M-1\,,\end{aligned} \tag{1.7.10}$$

$$+ \quad \text{initial step} \quad \frac{\vec{\mu}^{(1)} - \vec{\mu}^{(-1)}}{\Delta t} = \vec{\eta}_0\,.$$

Auxiliary variable: $\vec{\eta}^{(k)} := (2\mathbf{I} - \tfrac{1}{2}\Delta t\mathbf{M}^{-1}\mathbf{A})\vec{\mu}^{(k)} - (2\mathbf{I} + \tfrac{1}{2}\Delta t\mathbf{M}^{-1}\mathbf{A})\vec{\mu}^{(k+1)}$
$\hat{=}$ velocity approximation

► equivalent implementation: Crank-Nicolson timestepping:

$$\begin{aligned}\mathbf{M}\frac{\vec{\eta}^{(k+1)}-\vec{\eta}^{(k)}}{\Delta t} &= \tfrac{1}{2}\mathbf{A}(\vec{\mu}^{(k+1)}+\vec{\mu}^{(k)}) \;, \\ \frac{\vec{\mu}^{(k+1)}-\vec{\mu}^{(k)}}{\Delta t} &= -\tfrac{1}{2}(\vec{\eta}^{(k+1)}+\vec{\eta}^{(k)}) \;,\end{aligned} \qquad k=0,\dots,M-1 \;. \tag{1.7.11}$$

➤ requires solution of linear system with (non-diagonal) matrix $\mathbf{A}$ in every step ! ("implicit")

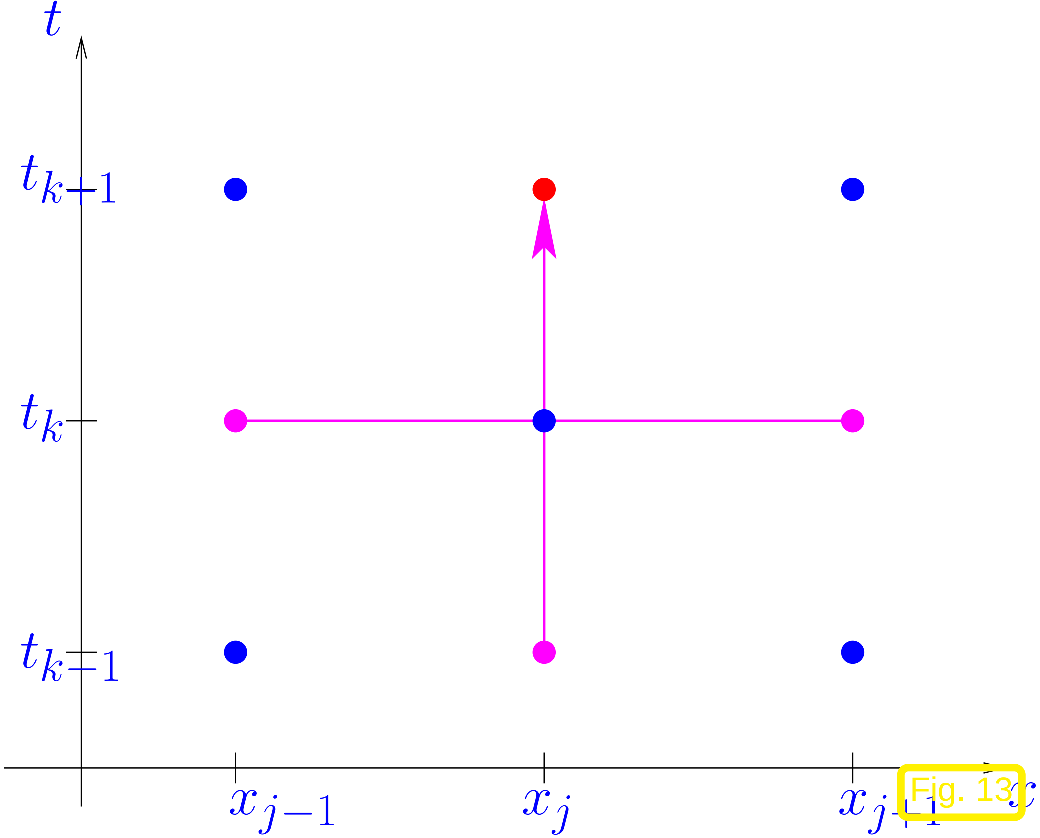
Example 14 (Space time stencils for fully discrete 1D wave equation).

- finite element (→ Sect. 1.6.3)/finite difference (→ Sect. 1.6.1) spatial discretization of 1D wave equation

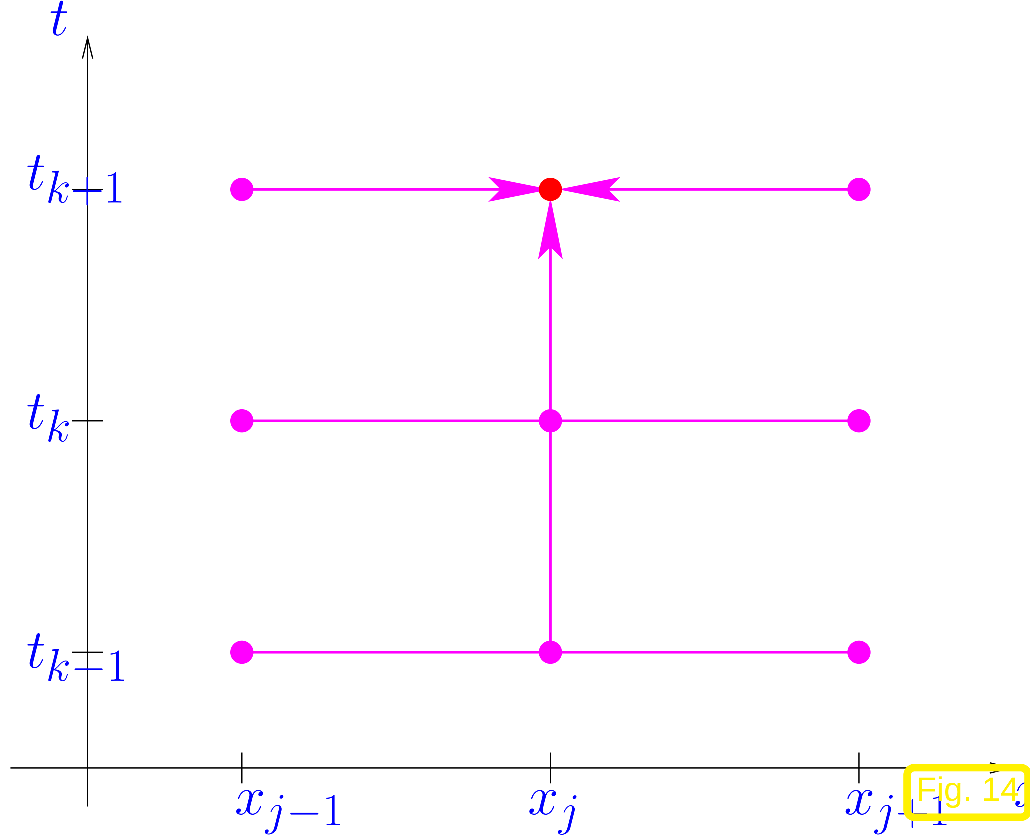
- timestepping: explicit/implicit trapezoidal rule

▼

space-time local difference formulas: representation by stencils



Explicit trapezoidal rule (1.7.7)



Implicit trapezoidal rule (1.7.10)

$\diamond$

1.7.2 Stability

For (1.7.1), $\vec{\varphi} = 0$: conservation of energy ➤ no “blow up” of solutions

Is this satisfied for timestepping schemes **?**

1.7.2.1 Spectral decomposition

von Neumann stability analysis: discrete analogue of *diagonalization idea* of Sect. 1.5:

$$\mathbf{A}, \mathbf{M} \quad \text{symmetric positive definite} \quad \Rightarrow \quad \widetilde{\mathbf{A}} = \mathbf{M}^{-1/2}\mathbf{A}\mathbf{M}^{-1/2} \text{ symmetric positive definite} .$$

$$\Rightarrow \quad \exists \text{ orthogonal } \mathbf{T} \in \mathbb{R}^{N,N}: \quad \mathbf{T}^T\mathbf{M}^{-1/2}\mathbf{A}\mathbf{M}^{-1/2}\mathbf{T} = \mathbf{D} := \operatorname{diag}(\lambda_1 \ldots, \lambda_N) ,$$

where the $\lambda_l > 0$ are *generalized eigenvalues* for $\mathbf{A}\vec{\xi} = \lambda\mathbf{M}\vec{\xi}$ ➤ $\lambda_l \geq \gamma$ for all l.

► Transformation ("diagonalization") of (1.7.1): $\vec{\zeta} := \mathbf{T}^T\mathbf{M}^{1/2}\vec{\mu}$

$$\frac{d^2}{dt^2}\vec{\zeta}(t) + \mathbf{D}\vec{\zeta} = \mathbf{T}^T\mathbf{M}^{-1/2}\vec{\varphi}(t) =: \vec{\phi}(t) . \tag{1.7.12}$$

➤ decoupled scalar 2nd-order ODEs (for eigencomponents ζ_i of $\vec{\zeta}$): $\frac{d^2}{dt^2}\zeta_l + \lambda_l\zeta_l = \phi_l(t)$

Same diagonalization applied to two-step method (1.7.5):

$$\vec{\zeta}^{(k+1)} - \vec{\zeta}^{(k-1)} = 2R(\mathbf{D}^{1/2}\Delta t)\zeta^{(k)} , \quad k \in \mathbb{N} \tag{1.7.13}$$

$$\Updownarrow$$

$$\zeta_i^{(k+1)} - \zeta_i^{(k-1)} = 2R(\sqrt{\lambda_i}\Delta t)\zeta_i^{(k)} , \quad i - 1, \ldots, N . \tag{1.7.14}$$

(1.7.14) = linear three-term recurrence

characteristic equation of (1.7.14): $\chi^2 - \alpha\chi + 1 = 0\ ,\quad \alpha := 2R(\sqrt{\lambda_i}\Delta t)\ .$

$|\alpha| \leq 2$: $\chi_\pm = \frac{1}{2}\alpha \pm i\sqrt{4-\alpha^2} \;\Rightarrow\; |\chi_\pm| = 1$

▶ $\zeta_i^{(k)} = A_i\chi_+^k + B_i\chi_-^k \;\Rightarrow\; |\zeta_i^{(k)}| \leq |A_i| + |B_i| \;\forall k \in \mathbb{N}\ .$

$|\alpha| > 2$: $\chi_\pm = \frac{1}{2}\alpha \pm \sqrt{\alpha^2-4} \;\Rightarrow\; |\chi_+| > 1 \vee |\chi_-| > 1$

▶ $|\zeta_i^{(k)}| \to \infty$ for $k \to \infty$.

Stability: explicit trapezoidal rule: $R(x) = 1 - \frac{1}{2}x^2$

$\Big(\, |R(x)| > 1 \Leftrightarrow |x| > 2 \,\Big) \;\Rightarrow$ (1.7.7) unstable, if $\boxed{\sqrt{\lambda_N}\Delta t > 2 \Leftrightarrow \sup_{\vec{\xi}\in\mathbb{R}^N} \frac{\vec{\xi}\cdot\mathbf{A}\vec{\xi}}{\vec{\xi}\cdot\mathbf{M}\vec{\xi}} > \frac{4}{(\Delta t)^2}}$

Remark 15. For Galerkin discretization, Sect. 1.6.2: $\sup_{\vec{\xi}\in\mathbb{R}^N} \frac{\vec{\xi}\cdot\mathbf{A}\vec{\xi}}{\vec{\xi}\cdot\mathbf{M}\vec{\xi}} = \sup_{v_N\in V_N} \frac{\mathsf{a}(v_N,v_N)}{\mathsf{m}(v_N,v_N)}$.

(by definition of $\mathbf{M}$, $\mathbf{A}$)

△

Stability: implicit trapezoidal rule: $R(x) = \dfrac{1 - \frac{1}{4}x^2}{1 + \frac{1}{4}x^2}$

$|R(x)| \leq 1 \quad \forall x \in \mathbb{R} \quad \Rightarrow$ (1.7.10) unconditionally stable

1.7.2.2 Discrete energy estimates

Consider homogeneous transformed system (1.7.2)

➊ Discrete energy estimates for explicit trapezoidal rule: $\dfrac{1}{2\Delta t}(\vec{\nu}^{(k+1)} - \vec{\nu}^{(k-1)}) \cdot$ (1.7.6)

$$\blacktriangleright \quad \frac{1}{2\Delta t}\left(\left| \frac{\vec{\nu}^{(k+1)} - \vec{\nu}^{(k)}}{\Delta t} \right|^2 - \left| \frac{\vec{\nu}^{(k)} - \vec{\nu}^{(k-1)}}{\Delta t} \right|^2 \right) = -\frac{1}{2\Delta t}(\vec{\nu}^{(k+1)} - \vec{\nu}^{(k-1)}) \cdot \widetilde{\mathbf{A}}\vec{\nu}^{(k)} .$$

$$\blacktriangleright \quad E^{(k+1/2)} = E^{(k-1/2)}$$

for discrete pseudo energy $E^{(k+1/2)} := \dfrac{1}{2}\left| \dfrac{\vec{\nu}^{(k+1)} - \vec{\nu}^{(k)}}{\Delta t} \right|^2 + \frac{1}{2}\vec{\nu}^{(k+1)} \cdot \widetilde{\mathbf{A}}\vec{\nu}^{(k)} .$ **(1.7.15)**

Note: $E^{(k+1/2)}$ no "true energy", because $E^{(k+1/2)} < 0$ possible !

However: if $\Delta t \ll 1$ ➤ $\vec{\nu}^{(k)} \approx \vec{\nu}^{(k+1)} \Rightarrow E^{(k+1/2)} > 0$

($E^{(k+1/2)} \hat{=}$ "energy under timestep constraint")

$$E^{(k+1/2)} = \tfrac{1}{2}\left|\frac{\vec{\nu}^{(k+1)} - \vec{\nu}^{(k)}}{\Delta t}\right|^2 + \tfrac{1}{2}\left(\frac{\vec{\nu}^{(k+1)} + \vec{\nu}^{(k)}}{2}\right) \cdot \widetilde{\mathbf{A}}\left(\frac{\vec{\nu}^{(k+1)} + \vec{\nu}^{(k)}}{2}\right) - \frac{(\Delta t)^2}{8}\left(\frac{\vec{\nu}^{(k+1)} - \vec{\nu}^{(k)}}{\Delta t}\right) \cdot \widetilde{\mathbf{A}}\left(\frac{\vec{\nu}^{(k+1)} - \vec{\nu}^{(k)}}{\Delta t}\right) .$$

➤ $$E^{(k+1/2)} \geq \frac{1}{2}\left(1 - \frac{(\Delta t)^2}{4}\left\|\widetilde{\mathbf{A}}\right\|\right)\left|\frac{\vec{\nu}^{(k+1)} - \vec{\nu}^{(k)}}{\Delta t}\right|^2 + \tfrac{1}{2}\vec{\nu}^{(k+1/2)} \cdot \widetilde{\mathbf{A}}\vec{\nu}^{(k+1/2)} ,$$

where $\vec{\nu}^{(k+1/2)} := \frac{\vec{\nu}^{(k+1)}+\vec{\nu}^{(k)}}{2}$, $\|\mathbf{A}\| \hat{=}$ Euklidean matrix norm.

➤ $$\boxed{\frac{(\Delta t)^2}{4}\left\|\widetilde{\mathbf{A}}\right\| \leq 1 \quad\Rightarrow\quad E^{(k+1/2)} \geq 0} .$$

Theorem 1.7.4 (Stability of explicit trapezoidal rule/leap frog)**.**

$$\frac{(\Delta t)^2}{4} \sup_{\vec{\xi}\in\mathbb{R}^N} \frac{\vec{\xi}\cdot\mathbf{A}\vec{\xi}}{\vec{\xi}\cdot\mathbf{M}\vec{\xi}} < 1 \quad \Leftrightarrow \quad \text{(1.7.7) } \textit{stable}$$

❷ Implicit trapezoidal rule: discrete energy estimate:

$$(\vec{\eta}^{(k+1)} + \vec{\eta}^{(k)}) \cdot \text{ (i) of (1.7.11)} + \mathbf{A}(\vec{\mu}^{(k+1)} + \vec{\mu}^{(k)}) \cdot \text{ (ii) of (1.7.11)}$$

$$\blacktriangleright \quad E^{(k+1)} - E^{(k)} = 0\ , \quad \text{with “energy”} \quad E^{(k)} := \vec{\eta}^{(k)} \cdot \mathbf{M}\vec{\eta}^{(k)} + \vec{\mu}^{(k)} \cdot \mathbf{A}\vec{\mu}^{(k)} \geq 0\ .$$

Theorem 1.7.5 (Stability of implicit trapezoidal rule)**.** *The implicit trapezoidal rule (Crank-Nicolson timestepping) is stable for all* $\Delta t > 0$*.*

1.7.3 CFL-conditon

Concrete meaning of stability condition of Thm. 1.7.4 for leap frog timestepping:

Example 16 (CFL-condition for wave equation in 1D).

- 1D wave equation $\dfrac{\partial^2 u}{\partial t^2} - c^2 \dfrac{\partial^2 u}{\partial x^2} = 0$ on $\Omega =]0,1[$, $c > 0$ $\quad\rightarrow$ Ex. 9

- Homogeneous Dirichlet boundary conditions: $\quad u(0) = u(1) = 0$

- FD discretization on equidistant grid $\mathcal{M}$ with meshwidth $h = 1/M$ $\quad\rightarrow$ Sect. 1.6.1

$$\mathbf{M} = \mathbf{I} \quad , \quad \mathbf{A} = \frac{c^2}{h^2} \begin{pmatrix} 2 & -1 & & & & & & \dots & 0 \\ -1 & 2 & -1 & & & & & & \vdots \\ 0 & -1 & 2 & -1 & & & & & \\ \vdots & & \ddots & \ddots & \dots & & & & \\ & & & & & \ddots & \ddots & \ddots & \\ \vdots & & & & & & -1 & 2 & -1 \\ 0 & \dots & & & & & & -1 & 2 \end{pmatrix} \in \mathbb{R}^{M-1,M-1}$$

Eigenvectors/eigenvalues of $\mathbf{A}$: $\quad \vec{\xi}_l = (\sin(\pi l \frac{j}{M}))_{j=1}^{M-1} \quad \sim \quad \lambda_l = 4c^2 M^2 \sin^2(\frac{1}{2}\pi \frac{l}{M})$

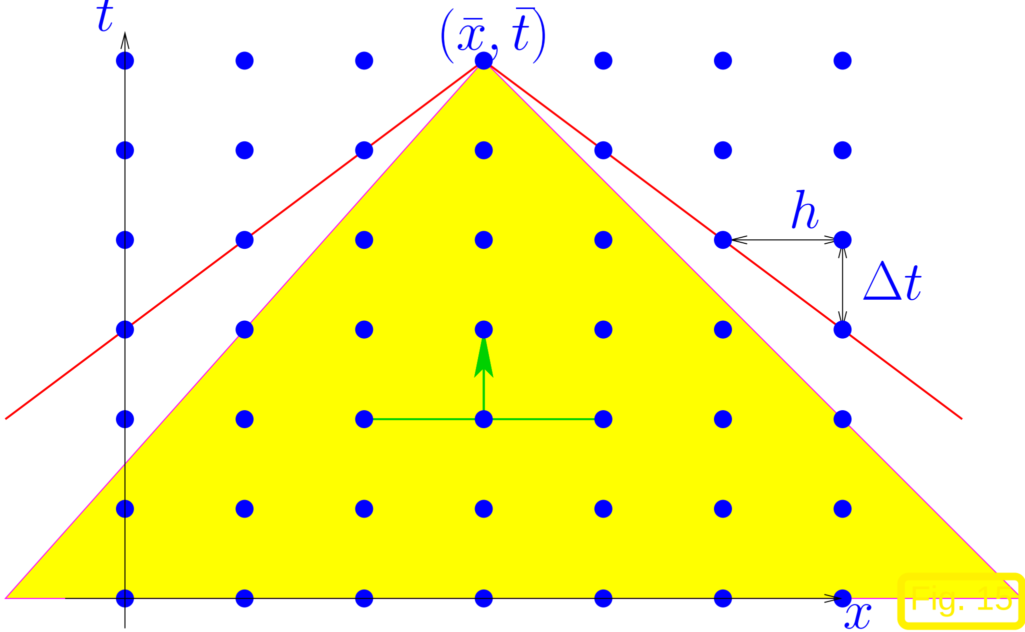
$$c\Delta t \leq h \qquad \Rightarrow \qquad \frac{(\Delta t)^2}{4} \sup_{\vec{\xi} \in \mathbb{R}^N} \frac{\vec{\xi} \cdot \mathbf{A}\vec{\xi}}{\vec{\xi} \cdot \mathbf{M}\vec{\xi}} = (\Delta t)^2 c^2 M^2 \sin^2(\tfrac{1}{2}\pi \frac{M-1}{M}) < 1 \; .$$

Stability limits timestep size in terms of meshwidth of spatial grid !

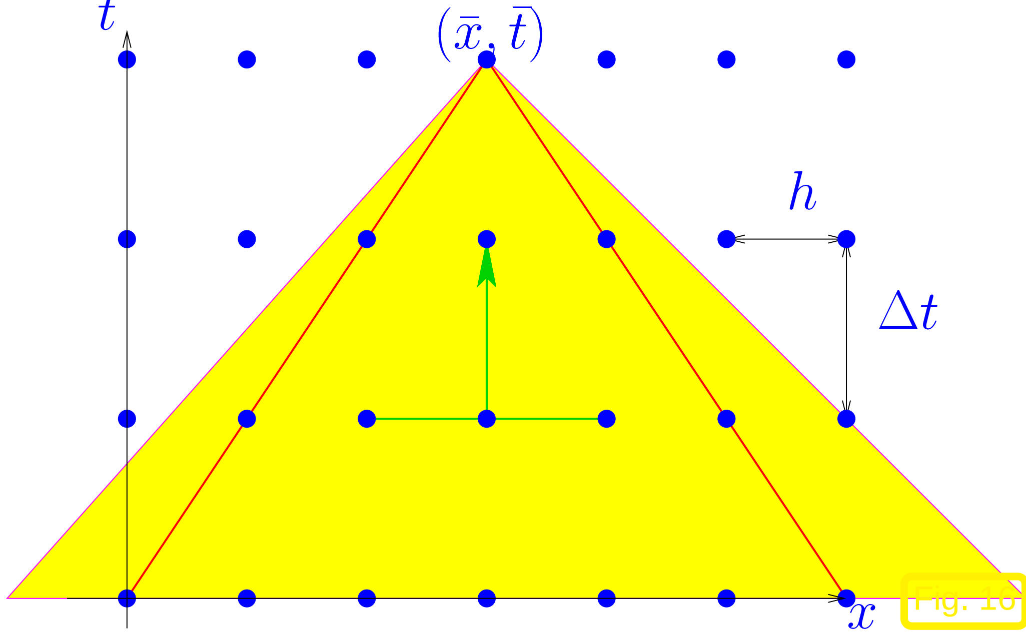
◇

Notion 1.7.6 (CFL-condition I)**.**
Courant-Friedrichs-Levy (CFL-) condition **=** *constraint on timestep size in terms of resolution of spatial discretization to ensure stability for a fully discrete hyperbolic evolution problem.*

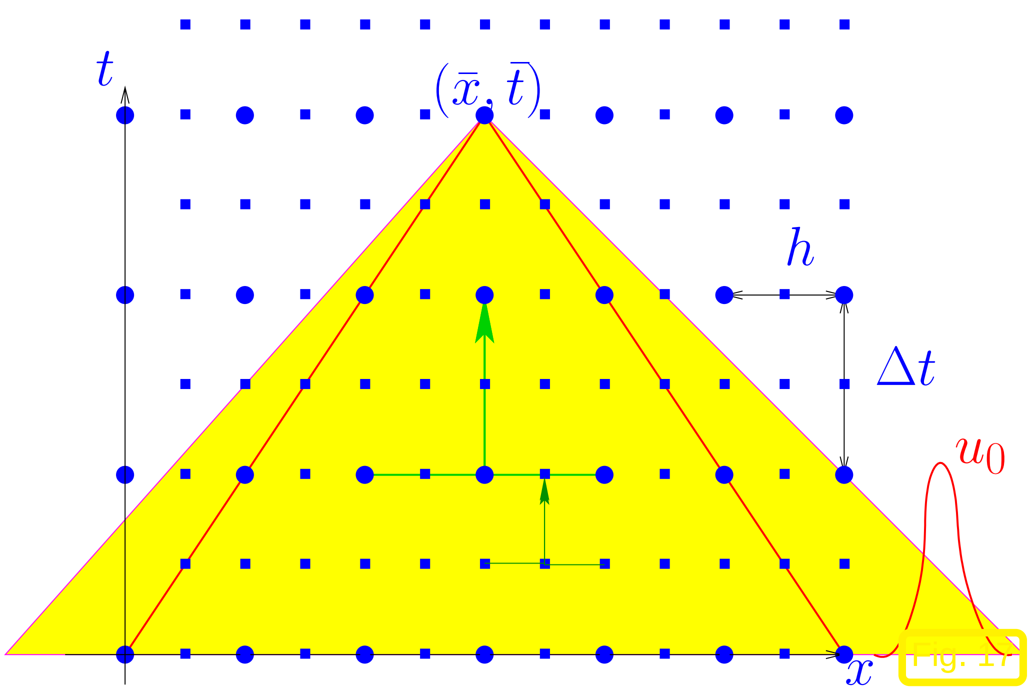
Geometric interpretation in 1D (setting of Ex. 16):



$c\Delta t < h$: numerical domain of dependence (marked —) contained in $D^-(\bar{x}, \bar{t})$
➪ CFL-condition met



$c\Delta t > h$: numerical domain of dependence (marked —) **not** contained in $D^-(\bar{x}, \bar{t})$
➪ CFL-condition violated



(● $\hat{=}$ coarse grid, ■ $\hat{=}$ fine grid, ■ $\hat{=}$ $D^-(\bar{x}, \bar{t})$)

◁ 1D consideration:

sequence of equidistant space-time grids of $\widetilde{\Omega}$ with $\Delta t = \gamma h$ (Δt/h = meshwidth in time/space)

If $\gamma >$ CFL-constraint (here $\gamma > c^{-1}$), then

analytical domain of dependence $\not\subset$ numerical domain of dependence

▲ initial data u_0 outside numerical domain of dependence cannot influence approximation at grid point $(\bar{x}, \bar{t})$ on *any* mesh ▶ no convergence !

CFL-condition $\Leftrightarrow$ analytical domain of dependence $\subset$ numerical domain of dependence

Example 17 (CFL-condition for wave equation in 2D).

$\Omega =]0,1[^2$, wave equation $\dfrac{\partial^2 u}{\partial t^2} - \Delta u = 0$, homogeneous Dirichlet b.c. $u_{|\partial\Omega} = 0$

5-point stencil for discrete Laplacian ▷

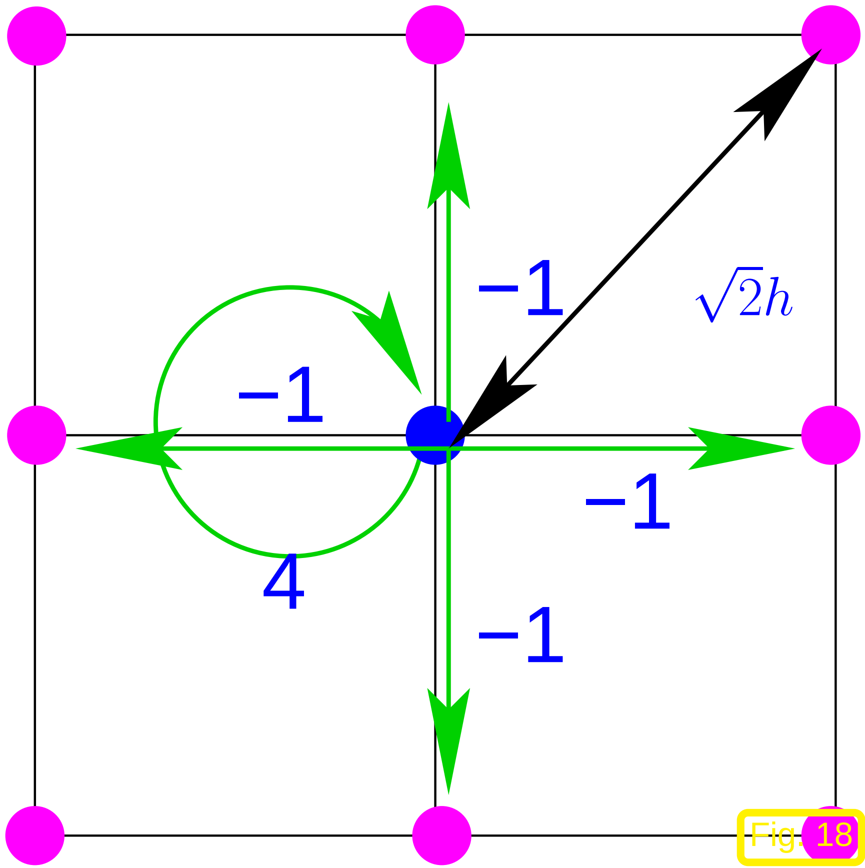


Fig. 18

- Spatial discretization: finite differences → Sect. 1.6.1 on equidistant tensor product grid , meshwidth $h = {}^1/_M$, $M \in \mathbb{N}$
- Temporal discretization: explicit trapezoidal rule (leap frog) (1.7.7)

(1.6.7) ▶ $\mathbf{A} \sim (\mathbf{A}\vec{\mu})_{ij} = \dfrac{1}{h^2}\left(4\mu_{ij} - \mu_{i-1,j} - \mu_{i+1,j} - \mu_{i,j-1} - \mu_{i,j+1}\right)$.

Eigenvectors and eigenvalues of $\mathbf{A}$ [18, Sect. 4.1]:

$$\vec{\xi}_{l_1,l_2} = (\sin(\pi l_1 i/M)\sin(\pi l_2 j/M))_{i,j=1}^{M-1} \quad \longrightarrow \quad \lambda_{l_1,l_2} = \frac{4}{h^2}\sin^2(\tfrac{1}{2}\pi\frac{l_1}{M}) + \sin^2(\tfrac{1}{2}\pi\frac{l_2}{M}) \ .$$

▶ $$\sup_{\vec{\xi}\in\mathbb{R}^N} \frac{\vec{\xi}\cdot\mathbf{A}\vec{\xi}}{\vec{\xi}\cdot\mathbf{M}\vec{\xi}} = \frac{8}{h^2}\sin^2(\tfrac{1}{2}\pi\frac{M-1}{M}) \quad \Rightarrow \quad \boxed{\text{CFL:} \quad \Delta t < \tfrac{1}{\sqrt{2}}h} \ .$$

◇

More general: FE Galerkin discretization of $\dfrac{\partial^2 u}{\partial t^2} - \Delta u = f$, trial/test space $\mathcal{S}^0_{1,0}(\mathcal{M})$
$\longrightarrow$ Sect. 1.6.3

From [27, Sect. 7.3], [27, (7.3.7)]: for $\mathcal{S}^0_{1,0}(\mathcal{M})$-stiffness matrix $\mathbf{A}$ and mass matrix $\mathbf{M}$

$\lambda_{\max}(\mathbf{M}^{-1/2}\mathbf{A}\mathbf{M}^{-1/2}) \approx \min\{h_K : K \in \mathcal{M}\}^{-2}$ (constants depending on shape-regularity)

$\lambda_{\min}(\mathbf{M}^{-1/2}\mathbf{A}\mathbf{M}^{-1/2}) \approx \mathrm{diam}(\Omega)^2$ (constants depending on Ω)

▶ CFL-condition: $\boxed{\Delta t \leq C \min\{h_K : K \in \mathcal{M}\}}$, (1.7.16)

with $C > 0$ depending on Ω **+** shape regularity of FE mesh $\mathcal{M}$.

Note: (1.7.16) ➣ smallest cell size limits timestep **!**
(big obstacle for (adaptive) local mesh refinement)

1.8 Convergence analysis

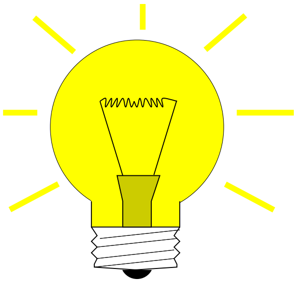
Note: use semi-discrete error estimates, Sect. 1.6.2
➤ only study temporal discretization error for (1.6.15) !

Focus: explicit trapezoidal rule (leap frog) (1.7.7) for (1.7.1)

Natural assumption: CFL-condition ($\longrightarrow$ Thm. 1.7.4) satisfied:

$$1 - \frac{(\Delta t)^2}{4} \left\| \widetilde{\mathbf{A}} \right\| \geq \alpha_0 > 0 \quad \Leftrightarrow \quad (1-\alpha_0)\vec{\xi} \cdot \mathbf{M}\vec{\xi} - \frac{(\Delta t)^2}{4}\vec{\xi} \cdot \mathbf{A}\vec{\xi} \geq 0 \quad \forall \vec{\xi} \in \mathbb{R}^N \ . \tag{1.8.1}$$

Idea: (as in Sect. 1.6.2) Lax equivalence principle



$$\boxed{\text{stability} \;+\; \text{consistency} \;\Rightarrow\; \text{convergence}}$$

$$\mathcal{E}_N(\text{error term}) = \text{residual term} \qquad (1.8.2)$$

operator corresponding to fully discrete timestepping

guaranteed by CFL-condition !

(1.7.1) : $$\mathbf{M}\frac{d^2}{dt^2}\vec{\mu}(t) + \mathbf{A}\vec{\mu}(t) = \vec{\varphi}(t) \;,$$

(1.7.7) : $$\mathbf{M}\,\frac{\vec{\mu}^{(k+1)} - 2\vec{\mu}^{(k)} + \vec{\mu}^{(k-1)}}{(\Delta t)^2} + \mathbf{A}\vec{\mu}^{(k)} = \vec{\varphi}(t_k)$$

discrete leap frog evolution for error: $\vec{\eta}^{(k)} := \vec{\mu}^{(k)} - \vec{\mu}(t_k)$

$$\mathbf{M}\,\frac{\vec{\eta}^{(k+1)} - 2\vec{\eta}^{(k)} + \vec{\eta}^{(k-1)}}{(\Delta t)^2} + \mathbf{A}\vec{\eta}^{(k)} \;=\; \epsilon^{(k)} \;.$$

$$\mathcal{E}_N(\text{error term}) \;=\; \text{residual term}$$

Bound for residual source term

$$\epsilon^{(k)} := -\mathbf{M}\,\frac{\vec{\mu}(t_{k+1}) - 2\vec{\mu}(t_k) + \vec{\mu}(t_{k-1})}{(\Delta t)^2} - \mathbf{A}\vec{\mu}(t_k) + \vec{\varphi}(t_k) \;.$$

by Taylor's formula + (1.7.1)

$$\exists C > 0: \quad |\epsilon^{(k)}| \leq C(\Delta t)^2 \left\| \frac{d^4\vec{\mu}}{dt^4} \right\|_{L^\infty(]0,T[;\mathbb{R}^N)} .$$

CFL-condition ➨ conservation of positive (!) pseudo energy $E^{(k+1/2)} \longrightarrow$ (1.7.15), Sect. 1.7.2

➜ study $\mathfrak{E}^{(k+1/2)} := \frac{1}{2(\Delta t)^2}(\vec{\eta}^{(k+1)} - \vec{\eta}^{(k)}) \cdot \mathbf{M}(\vec{\eta}^{(k+1)} - \vec{\eta}^{(k)}) + \frac{1}{2}\vec{\eta}^{(k+1)} \cdot \mathbf{A}\vec{\eta}^{(k)}$ (1.8.3)

$\hat{=}$ pseudo energy of error .

(1.8.1) ▶

$$\mathfrak{E}^{(k+1/2)} \geq \frac{1}{2(\Delta t)^2}\alpha_0(\vec{\eta}^{(k+1)} - \vec{\eta}^{(k)}) \cdot \mathbf{M}(\vec{\eta}^{(k+1)} - \vec{\eta}^{(k)}) + \frac{1}{2}\vec{\eta}^{(k+1/2)} \cdot \mathbf{A}\vec{\eta}^{(k+1/2)} , \tag{1.8.4}$$

$\vec{\eta}^{(k+1/2)} := \frac{1}{2}(\vec{\eta}^{(k+1)} + \vec{\eta}^{(k)})$, $0 \leq k \leq M-1$.

(1.8.4) ➤ bound for $\mathfrak{E}^{(k+1/2)}$ $\cong$ bound for error $\vec{\eta}^{(k+1/2)}$

Details: for (modified) pseudo energy $\widetilde{\mathfrak{E}}^{(k)} := \mathfrak{E}^{(k+1/2)} +$ "$\mathfrak{E}^{(k)}$" $+ \mathfrak{E}^{(k-1/2)}$ with $C = C(\alpha_0, \vec{\mu}(t))$

$$\frac{1}{\Delta t}(\widetilde{\mathfrak{E}}^{(k+1)} - \widetilde{\mathfrak{E}}^{(k)}) \leq C(\Delta t)^2 \left(\sqrt{\widetilde{\mathfrak{E}}^{(k+1)}} + \sqrt{\widetilde{\mathfrak{E}}^{(k)}}\right) \quad \Rightarrow \quad \sqrt{\widetilde{\mathfrak{E}}^{(k)}} \leq \sqrt{\widetilde{\mathfrak{E}}^{(1)}} + CT \cdot (\Delta t)^2$$

Theorem 1.8.1 (Timestepping error for leap frog)**.** *If the CFL-condition from Thm. 1.7.4 holds* strictly*, the timestepping error* $\vec{\eta}^{(k)} := \vec{\mu}^{(k)} - \vec{\mu}(t_k)$ *for leap frog timestepping* (1.7.7) *for* (1.7.1) *with uniform timestep* Δt *satisfies*

$$\frac{1}{(\Delta t)^2}(\vec{\eta}^{(k)} - \vec{\eta}^{(k-1)}) \cdot \mathbf{M}(\vec{\eta}^{(k)} - \vec{\eta}^{(k-1)}) + \tfrac{1}{2}\vec{\eta}^{(k)} \cdot \mathbf{A}\vec{\eta}^{(k)} \leq C(\Delta t)^4 ,$$

with $C = C(\mathbf{M}, \mathbf{A},$ *"CFL"*$, \vec{\mu}(t))$*.*

2nd-order algebraic convergence of timestepping error for stable leap frog

(total) discretization error $\leq$ spatial discretization error **+** timestepping error

Example 18 (Convergence of fully discrete scheme for 1D wave equation)*.*

- 1D wave equation $\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0$ on $]0,1[\times]0,1[$, homogeneous Dirichlet b.c. $u(0,t) = u(1,t) = 0$, $0 < t < 1$.
- Initial data: compactly supported "pulses": $u_0 = \psi(x)$, $v_0(x) = -\psi'(x)$

(➜ would give rise to solution $u(x,t) = \psi(x-t)$ for Cauchy problem → Sect. 1.3.2)

initial conditions ▷

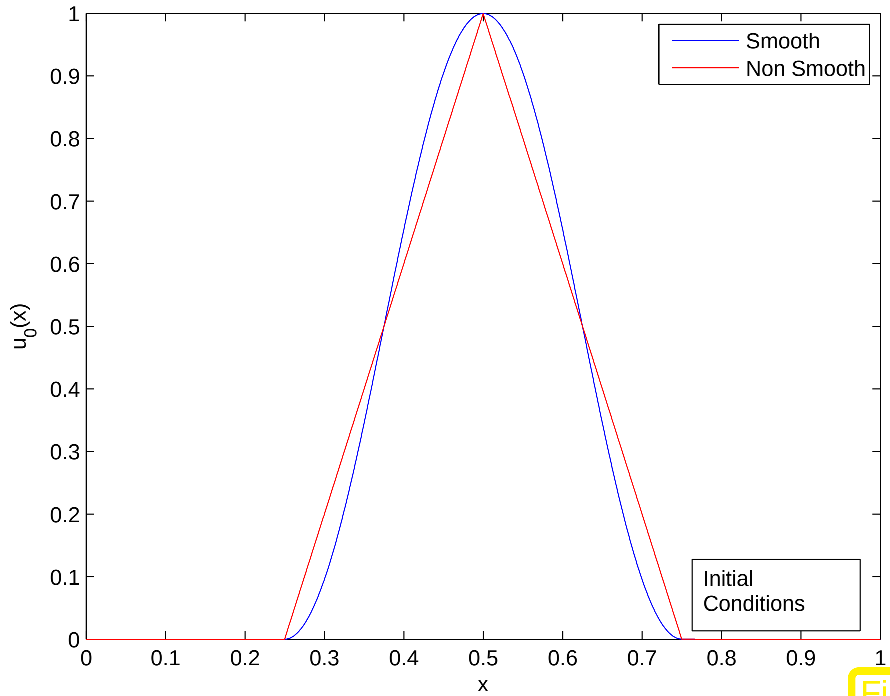
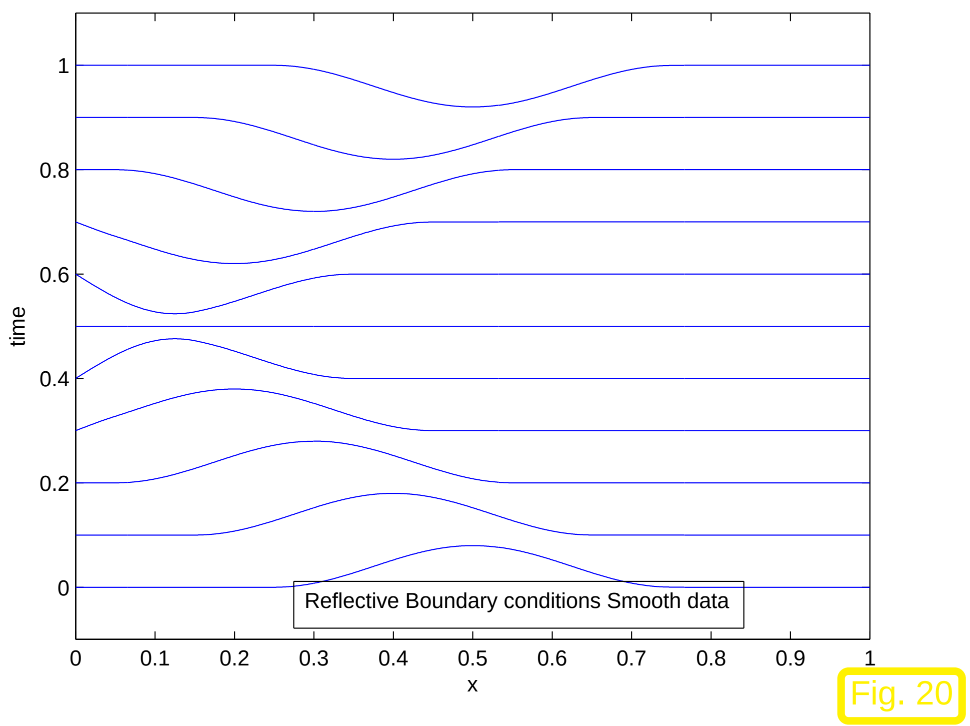
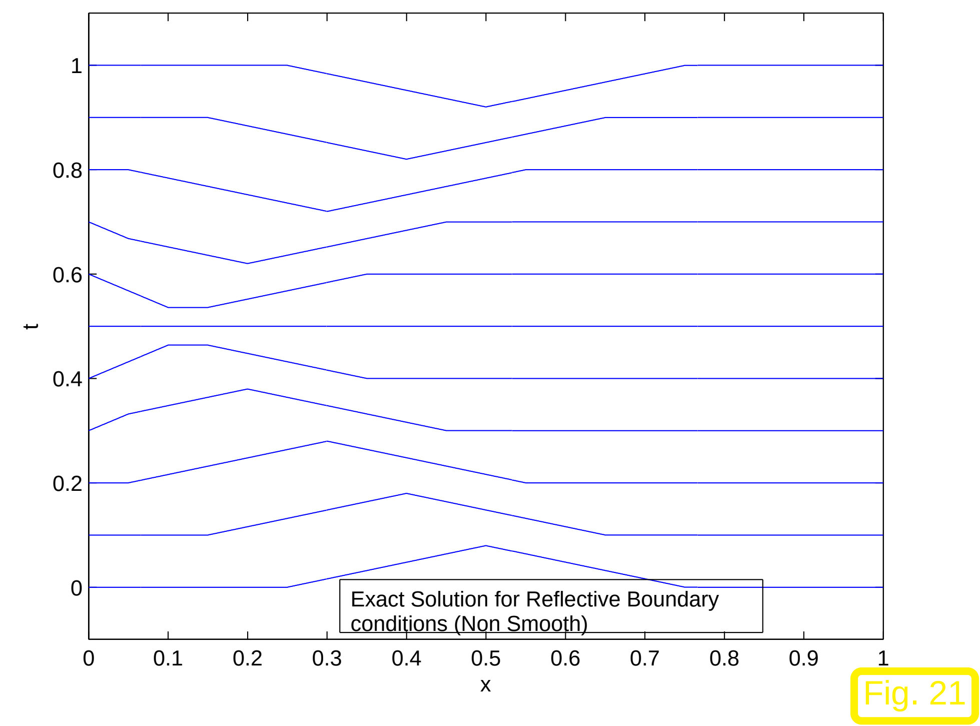


Fig. 19

$$\text{smooth pulse:}\quad \psi(s) = \begin{cases} 1-\cos^2(2\pi(x-0.25)) & \text{, if } x \in [0.25, 0.75] \text{ ,} \\ 0 & \text{,otherwise.} \end{cases} \in C^1(\mathbb{R}) \text{ ,} \tag{1.8.5}$$

$$\text{rough pulse:}\quad \psi(s) = \begin{cases} 4(x-0.25) & \text{, if } x \in [0.25, 0.5] \text{ ,} \\ 0 & \text{, otherwise.} \end{cases} \in C^0(\mathbb{R}) \text{ .} \tag{1.8.6}$$





Exact solution for u_0 = smooth pulse

Exact solution for u_0 = rough pulse

- finite element Galerkin discretization: $\mathcal{S}^0_{1,0}(\mathcal{M})$ on equidistant mesh $\mathcal{M}$ with meshwidth $h = \frac{1}{M}$, $M \in \mathbb{N}$ $\longrightarrow$ Sect. 1.6.3.

- timestepping with (unconditionally stable) implicit trapezoidal rule (1.7.10), uniform timestep Δt

➊ monitored errors:

$$\|u - u_N\|_{L^\infty(]0,T[;L^2(]0,1[))} \approx \max_k \|u(t_k) - u_N(t_k)\|_{L^2(]0,1[)} \; , \tag{1.8.7}$$

$$\|u - u_N\|_{L^\infty(]0,T[;H^1(]0,1[))} \approx \max_k \|u(t_k) - u_N(t_k)\|_{H^1(]0,1[)} \; , \tag{1.8.8}$$

(norms evaluated by means of 2-point Gaussian quadrature on mesh cells)

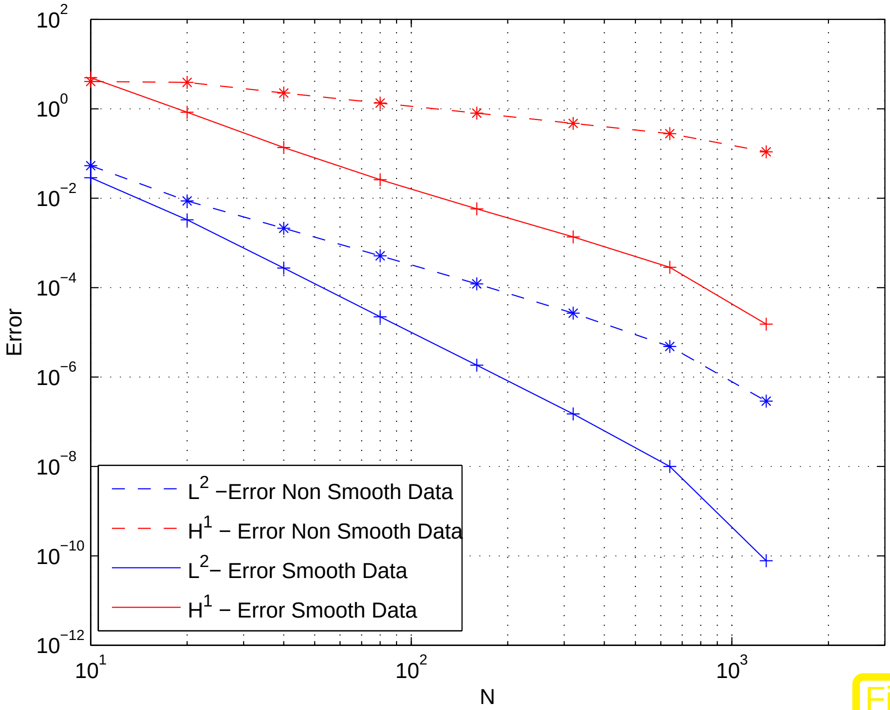


Fig. 22

$\Delta t = 0.0005$ fixed, h is varied

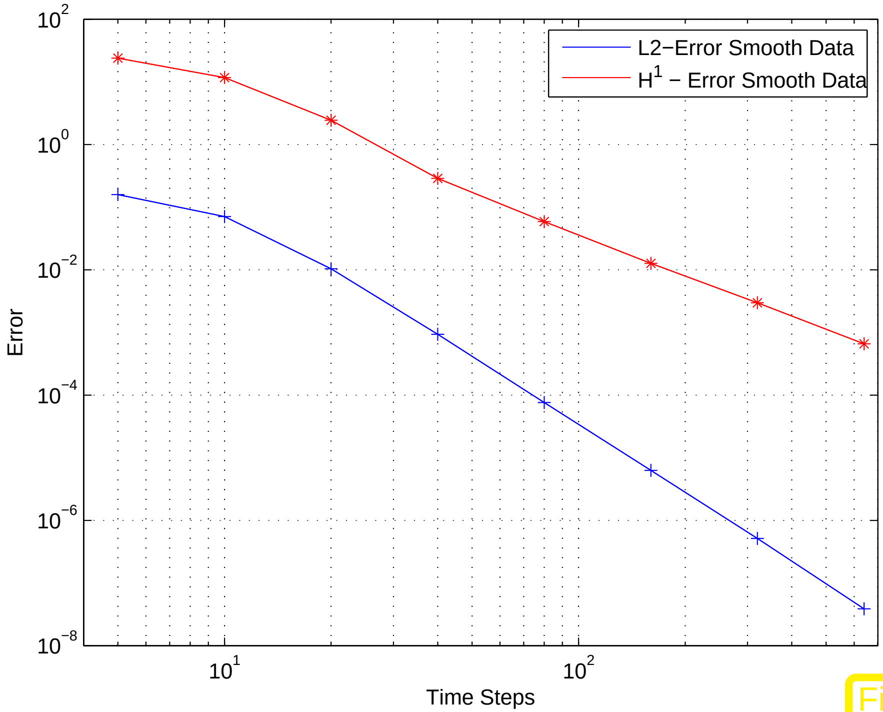
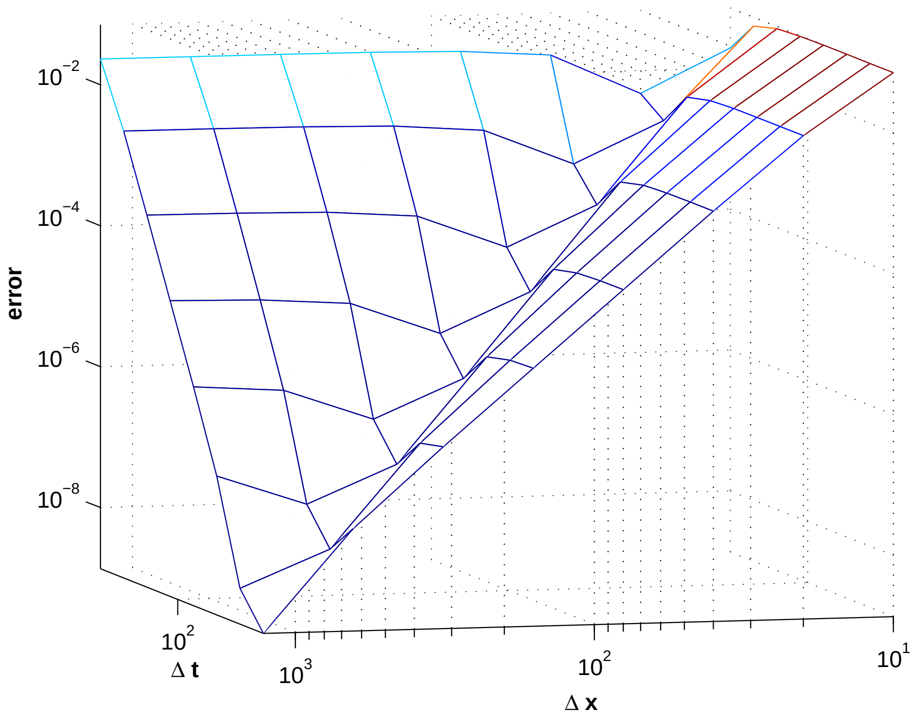


Fig. 23

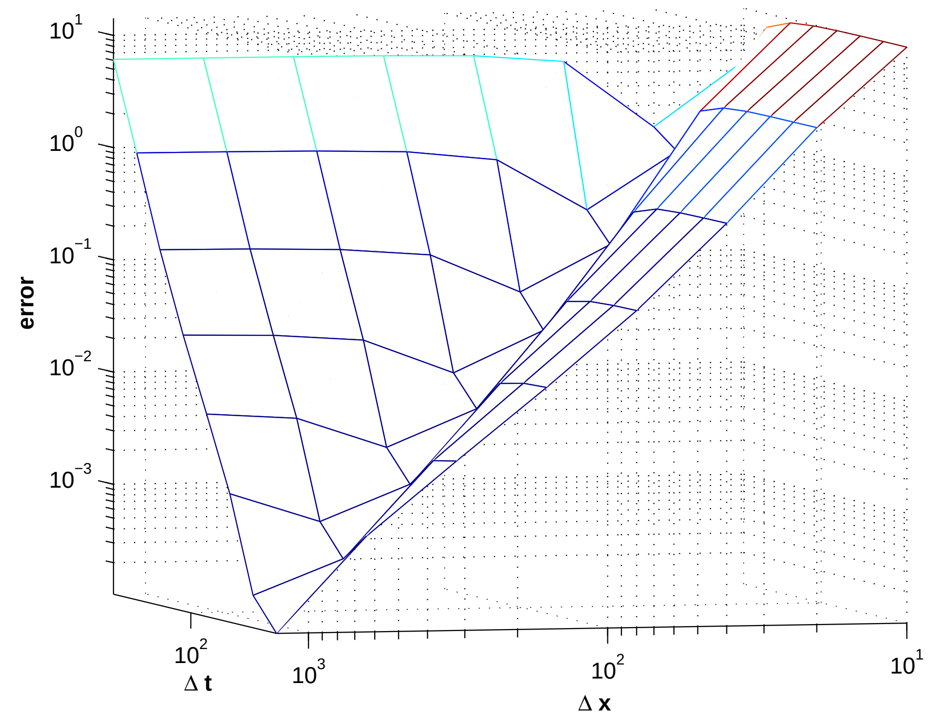
$M = 1280$, Δt varied

☞ Algebraic convergence as $\Delta t, h \to 0$, faster convergence in L^2-norm than in H^1-norm, *cf.* Thm. 1.6.1

❷ monitor errors (1.8.7) for varying Δt and M (smooth pulse initial data):



Approximate $L^{\infty}(]0,T[;L^{2}(]0,1[))$-error



Approximate $L^{\infty}(]0,T[;H^{1}(]0,1[))$-error

$\diamond$

1.9 Numerical Dispersion

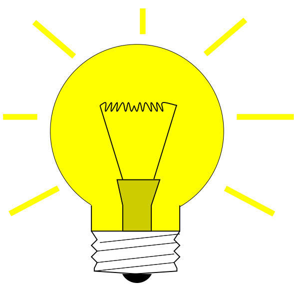
Consider Cauchy problem for 1D wave equation: $\dfrac{\partial^2 u}{\partial t^2} - c^2\dfrac{\partial^2 u}{\partial x^2} = 0$ 1.9

- spatial finite difference discretization on virtual infinite equidistant grid $\mathcal{M}$, gridpoints $x_j = jh$, $j \in \mathbb{Z}$ $\longrightarrow$ Sect. 1.6.1

- leap frog timestepping (1.7.7), timestep Δt, CFL-condition $c\Delta t < h$ $\longrightarrow$ Ex. 16

➤ difference equations for $\mu_j^{(k)} \approx u(x_j, t_k)$:

$$\frac{\mu_j^{(k+1)} - 2\mu_j^{(k)} + \mu_j^{(k-1)}}{(\Delta t)^2} + c^2 \frac{-\mu_{j+1}^{(k)} + 2\mu_j^{(k)} - \mu_{j-1}^{(k)}}{h^2} = 0 \,, \quad k \in \mathbb{N}, j \in \mathbb{Z} \,. \qquad \text{(1.9.1)}$$



Idea: plug (restrictions of) plane waves ($\longrightarrow$ Def. 1.3.2) into (1.9.4)

➤ discrete dispersion relation, see Sect. 1.3.1

plane wave grid function: $(\exp(i(kx - \omega t)))_{x=x_j, t=t_k}$ into (1.9.4)

➤ $$\frac{1}{(\Delta t)^2}\left(e^{i\omega\Delta t} - 2 + e^{-i\omega\Delta t}\right) + \frac{c^2}{h^2}\left(e^{ikh} - 2 + e^{-ikh}\right) = 0$$

$$\Updownarrow$$

Discrete dispersion relation: $$\sin(\tfrac{1}{2}\omega\Delta t) = \pm c\frac{\Delta t}{h}\sin(\tfrac{1}{2}kh) \qquad \text{(1.9.2)}$$

Aliasing: meaningful frequencies/wave vectors

$$-\frac{\pi}{\Delta t} \leq \omega \leq \frac{\pi}{\Delta t}\,,$$
$$-\frac{\pi}{h} \leq k \leq \frac{\pi}{h}$$

(on equidistant grid)

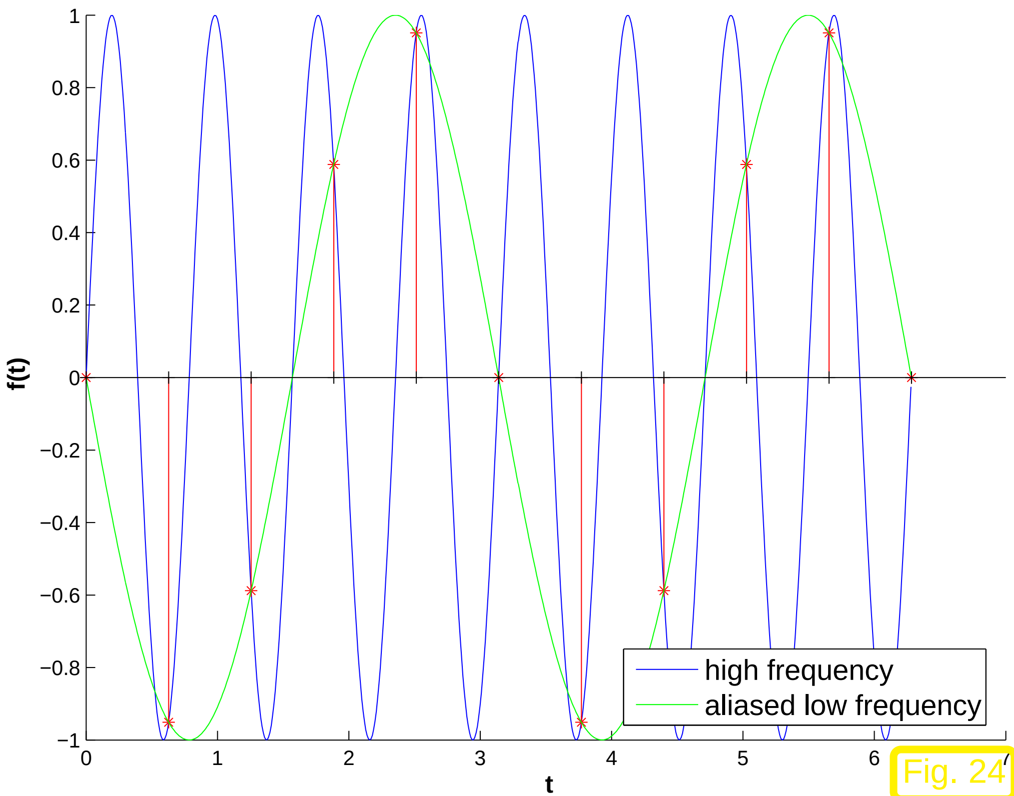


Fig. 24

► (from (1.9.2)): limit frequency for finite differences **+** leap frog on 1D equidistant grid:

$$|\omega| \leq \omega^* := \min\{\frac{2}{\Delta t}\arcsin(c\frac{\Delta t}{h}), \frac{\pi}{\Delta t}\} \tag{1.9.3}$$

$c\Delta t \neq h \quad \Rightarrow$ discrete group velocity $c_g(k) = \dfrac{d\omega}{dk} \neq \text{const}$ numerical dispersion, *cf.* Def. 1.3.3

$c\Delta t \neq h \quad \Rightarrow$ discrete phase speed $c_p(k) = \dfrac{\omega}{k} \neq c$

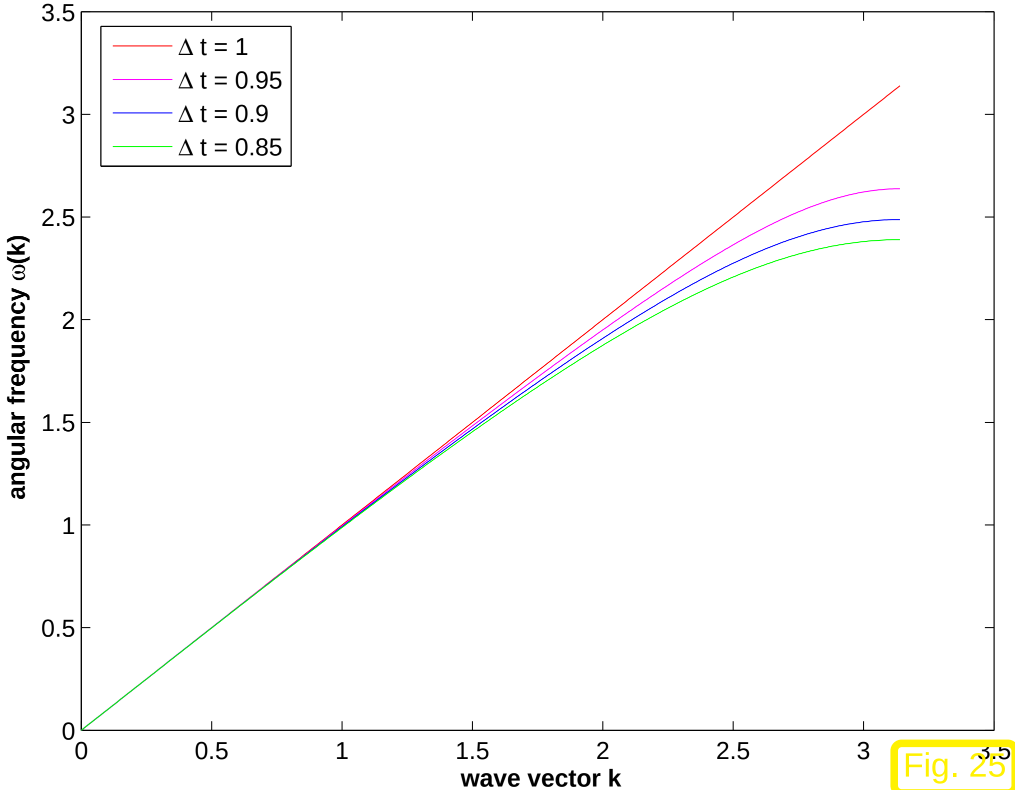


Fig. 25

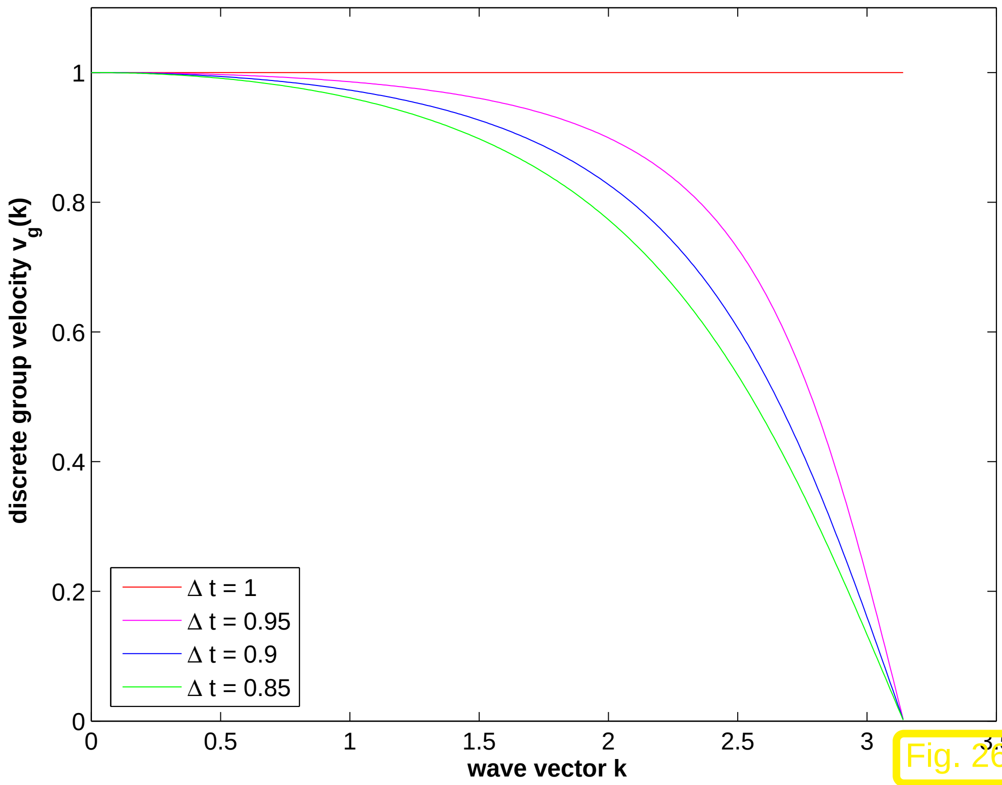


Fig. 26

Discrete dispersion relation (1.9.2), numerical group velocity for $h = c = 1$

▶ "magic timestep" $c\Delta t = h$ ➤ no numerical dispersion

Example 19 (Consequences of numerical dispersion for discrete 1D wave equation).

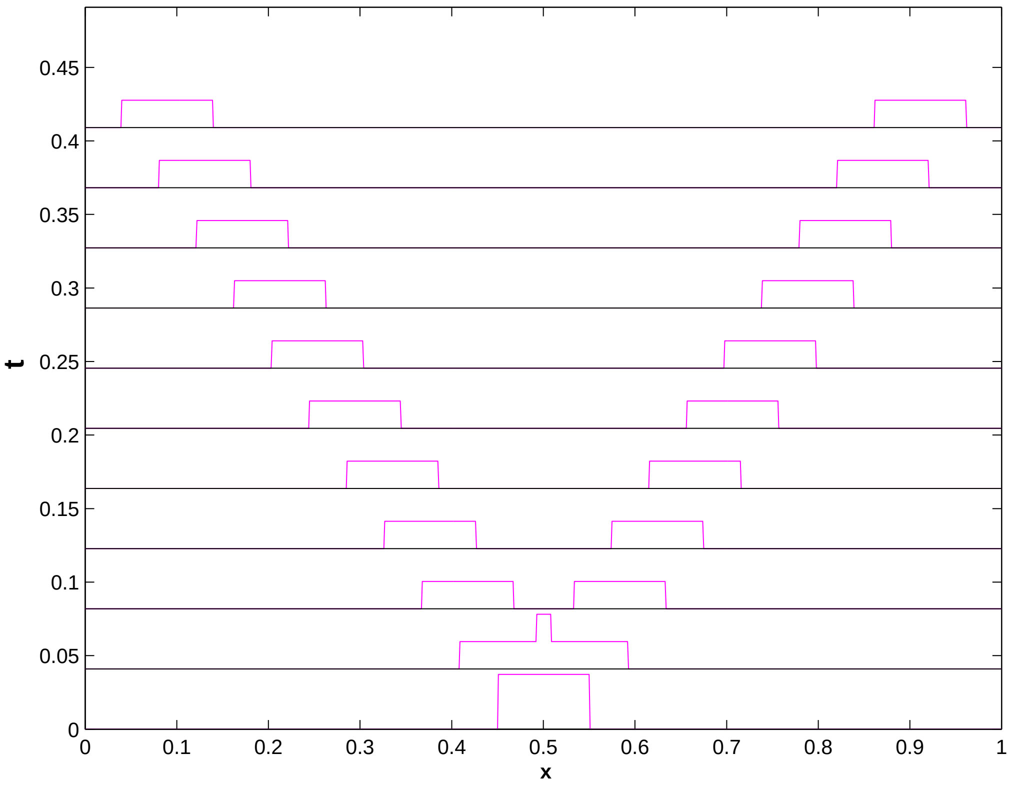
- 1D wave equation $\dfrac{\partial^2 u}{\partial t^2} - \dfrac{\partial^2 u}{\partial x^2} = 0$ on $]0,1[\times]0,\frac{1}{2}[$, homogeneous Dirichlet b.c. $u(0,t) = u(1,t) = 0$, $0 < t < \frac{1}{2}$.

- spatial discretization: finite differences on equidistant grid, meshwidth $h = 10^{-3}$
  temporal discretization: explicit trapezoidal rule (1.7.7), uniform timestep Δt

- initial data: $u_{N,0} \hat{=}$ compactly supported "pulse", $v_{N,0} = 0$

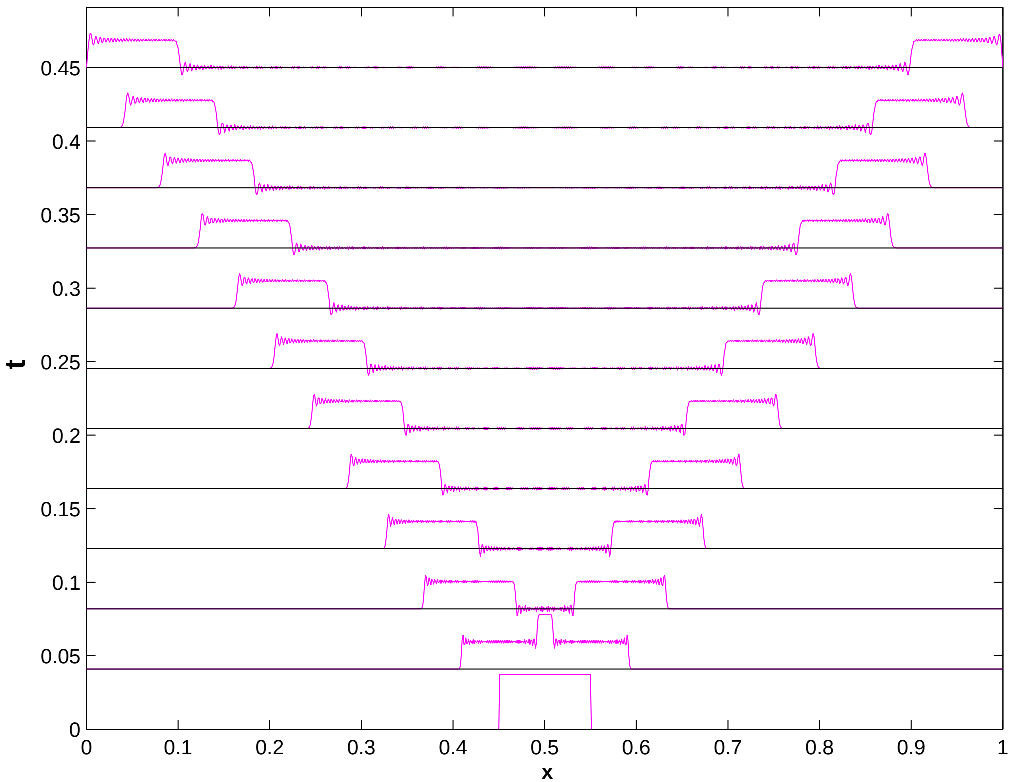
magic timestep $\Delta t = h$ $\triangleright$

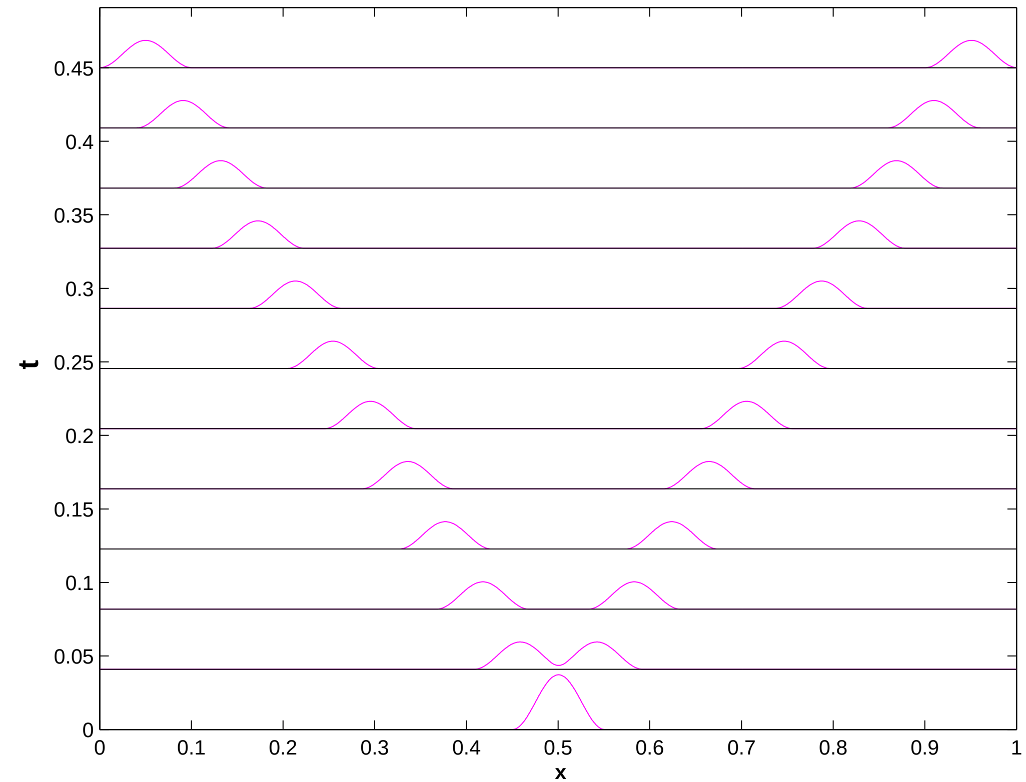
(no dispresion)



Below: $\Delta t = 0.95h$

$\Rightarrow$ numerical dispersion





Rough initial configuration
(broad spatial spectrum)

Smooth initial configuration
(narrow spatial spectrum)

numerical dispersion ➤ different spatial modes travel with different speed
➤ progressive ruffling of wave form

◇

Consider Cauchy problem for 2D wave equation: $$\frac{\partial^2 u}{\partial t^2} - c^2 \Delta u = 0$$

1.9

- spatial finite difference discretization on virtual infinite equidistant grid $\mathcal{M}$, gridpoints $\boldsymbol{x}_{i,j} = (ih, jh)$, $i,j \in \mathbb{Z}$ $\quad\rightarrow$ Sect. 1.6.1

- leap frog timestepping (1.7.7), timestep Δt, CFL-condition $\sqrt{2}c\Delta t < h$ $\quad\rightarrow$ Ex. 17

▶ difference equations for $\mu_{i,j}^{(k)} \approx u(\boldsymbol{x}_{i,j}, t_k)$ $\quad\rightarrow$ Fig. 18

$$\frac{\mu_{i,j}^{(k+1)} - 2\mu_{i,j}^{(k)} + \mu_{i,j}^{(k-1)}}{(\Delta t)^2} + c^2 \frac{4\mu_{i,j}^{(k)} - \mu_{i,j+1}^{(k)} - \mu_{i+1,j}^{(k)} - \mu_{i,j-1}^{(k)} - \mu_{i-1,j}^{(k)}}{h^2} = 0 \;, \quad \begin{array}{l} n \in \mathbb{N}, \\ i,j \in \mathbb{Z}\;. \end{array} \tag{1.9.4}$$

Discrete plane wave in 2D = grid function $\quad \big(\exp(i(\boldsymbol{k}\cdot\boldsymbol{x}_{i,j} - \omega t_k))\big)_{i,j\in\mathbb{Z},k\in\mathbb{N}}$

▶ discrete dispersion relation

$$\sin^2(\tfrac{1}{2}\omega\Delta t) = c^2\frac{(\Delta t)^2}{h^2}\Big(\sin^2(\tfrac{1}{2}k_1 h) + \sin^2(\tfrac{1}{2}k_1 h)\Big)$$

For $c = h = 1$ (scaling !), timestep at CFL limit $\quad \Delta t = 1/\sqrt{2}$:

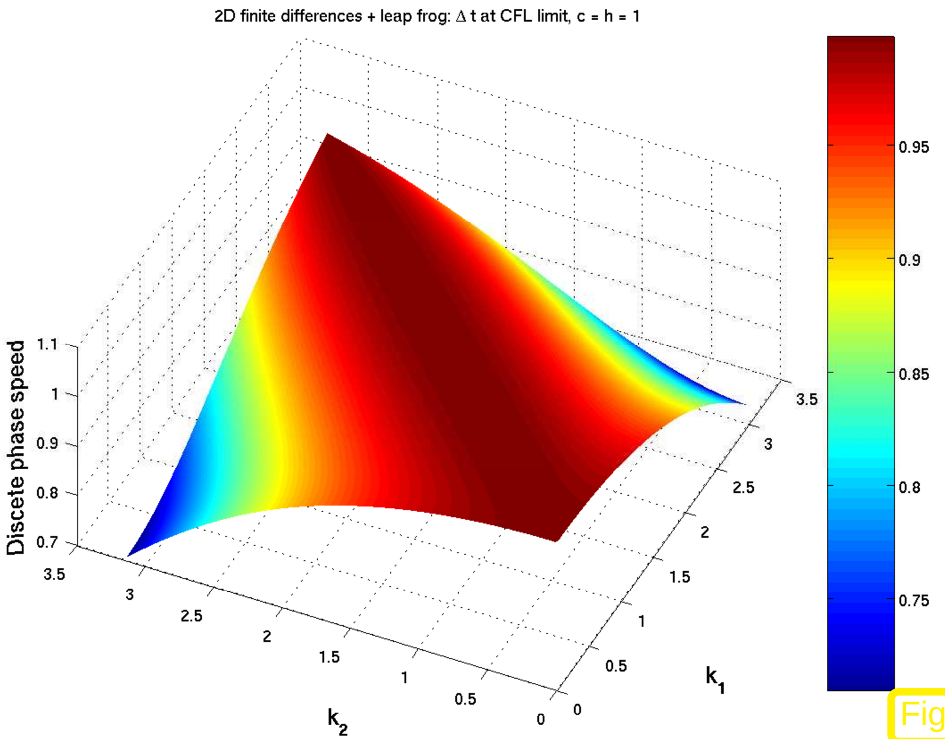


Fig. 27

Discrete phase speed $\frac{|\omega(\boldsymbol{x})|}{|\boldsymbol{k}|}$

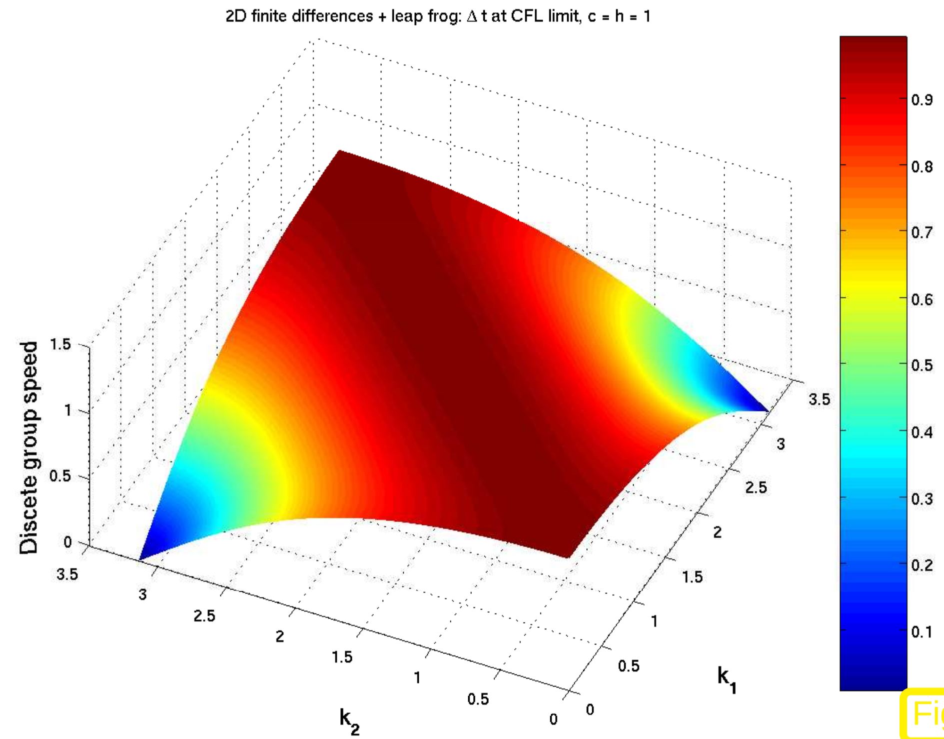


Fig. 28

Discrete group speed $\left|\frac{d\omega(\boldsymbol{x})}{d\boldsymbol{k}}\right|$

In 2D: ☞ phase speed/group speed depend on *direction* of wave vector $\boldsymbol{k}$!

☞ numerical dispersion (in some direction) for all Δt (no magic timestep)

1.10 Reflections

Example 20 (Reflections at "Dirichlet wall").

$d = 1$: consider initial boundary value problem (IBVP) on $\mathbb{R}^+$ with Dirichlet boundary conditions

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0\ , \quad u(x,0) = u_0(x)\ , \quad \frac{\partial u}{\partial t}(x,0) = 0\ , \quad x > 0\ , \quad u(0,t) = 0, t > 0\ . \qquad (1.10.1)$$

Solution via (1.3.3):

$$\widetilde{u}_0(x) = \begin{cases} u_0(x) & \text{, if } x > 0\ , \\ -u_0(-x) & \text{, if } x < 0\ . \end{cases}$$

$$u(x,t) = \tfrac{1}{2}(\widetilde{u}_0(x+t) + \widetilde{u}_0(x-t))\ .$$

"odd" reflection at Dirichlet boundary ▷

(— $\hat{=}\ u(x,t_j)$)

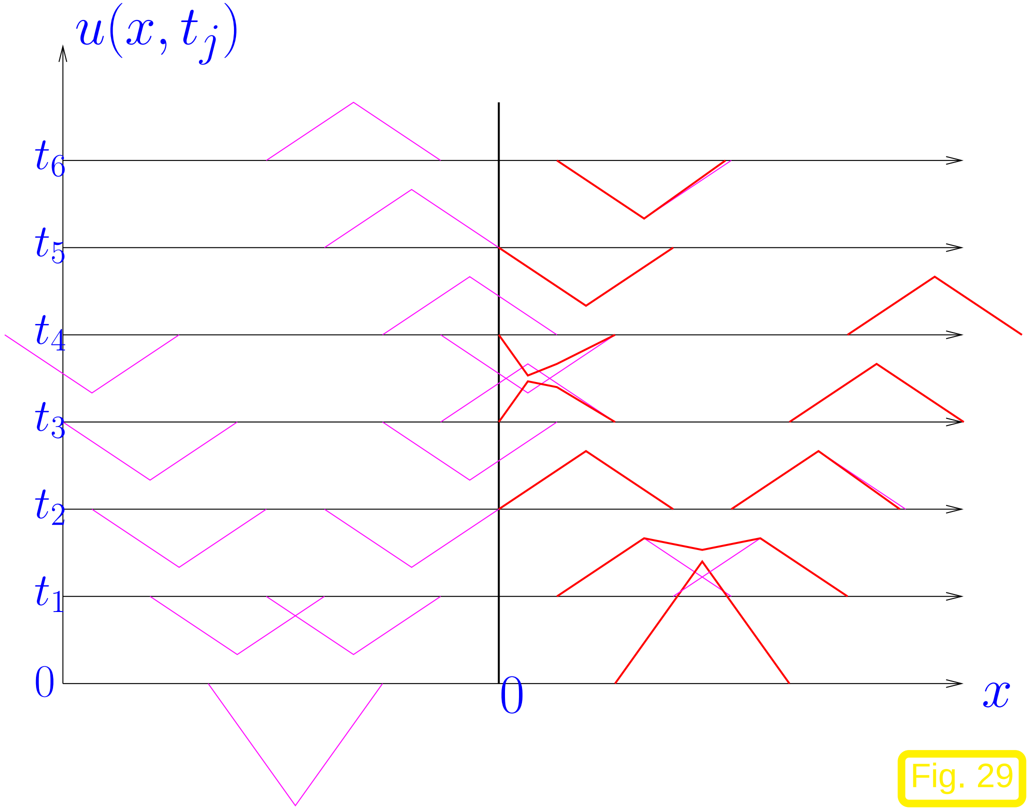


Fig. 29

◇

Example 21 (Reflection at material interface).

Consider plane wave solutions ($\rightarrow$ Sect. 1.3.1) to 1D wave equation on $\mathbb{R} \times \mathbb{R}^+$:

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x}\left(c(x)^2 \frac{\partial u}{\partial x}\right) = 0 \,, \quad c(x) = \begin{cases} c^- > 0 & \text{, if } x < 0 \,, \\ c^+ > 0 & \text{, if } x > 0 \,. \end{cases}$$

Incident wave for $x < 0$: : $\quad u_{\mathrm{inc}}(x,t) := \exp(i(\frac{\omega}{c^-}x - \omega t))$, $x < 0$, $t \geq 0$

in $x < 0$: $\widetilde{u}(x,t) \mathrel{\hat{=}}$ reflected wave
in $x > 0$: $\widetilde{u}(x,t) \mathrel{\hat{=}}$ total wave u

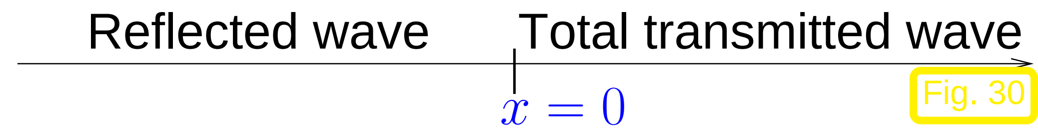


Fig. 30

Transmission jump conditions [27, Sect. 2.9], [27, Lemma 2.9.1], [27, Lemma 2.9.3]

$$\blacktriangleright \qquad [u]_{x=0} = 0 \quad , \quad \left[c^2(x)\frac{\partial u}{\partial x}\right]_{x=0} = 0 \quad \forall t \geq 0 \,.$$

$$\blacktriangleright \qquad [\widetilde{u}]_{x=0} = u_{\mathrm{inc}(0,t)} = e^{-i\omega t} \quad , \quad \left[c^2(x)\frac{\partial \widetilde{u}}{\partial x}\right]_{x=0} = (c^-)^2 \frac{\partial u_{\mathrm{inc}}}{\partial x}(0,t) = (i\omega c^-)e^{-i\omega t} \quad \forall t \geq 0 \,. \tag{1.10.2}$$

Notation: $\quad [\cdot]_{x=0} \mathrel{\hat{=}}$ jump of a function (across $x = 0$)

$$\widetilde{u}(x,t) = \begin{cases} -Re^{i(-\omega/c^- x - \omega t)} & \text{, for } x < 0 \,, \quad \leftarrow \text{left propagating (reflected) wave} \\ Te^{i(\omega/c^+ x - \omega t)} & \text{, for } x > 0 \,, \quad \leftarrow \text{right propagating (transmitted) wave} \,. \end{cases}$$

$\leftarrow$ **+** (1.10.2)

$$\text{reflection coefficient:}\quad R = \frac{c^-/c^+ - 1}{c^-/c^+ + 1}\ , \tag{1.10.3}$$

$$\text{transmission coefficient:}\quad T = \frac{2}{1 + c^-/c^+}\ . \tag{1.10.4}$$

discontinuity in $c(x)$ ➤ reflection of waves

Note: reflection of plane wave does not depend on k, ω !

$\diamond$

Remember $\longrightarrow$ Sect. 1.9: different numerical wave speeds on different spatial grids **!**

$\longrightarrow$ spurious reflections **?**

Consider Cauchy problem for 1D wave equation $$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0$$

Spatial discretization: mass lumped p.w. linear finite elements on non-equidistant infinite mesh

$$\mathcal{M} := \{[x_{j-1}, x_j]: x_j = jH \text{ for } j \in \mathbb{Z}^-,\ x_j = jh \text{ for } j \in \mathbb{Z}_0^+\}\ .$$

Temporal discretization: leap frog timestepping, Sect. 1.7.1.1, fixed timestep $\Delta t \leq \min\{H, h\}$

Difference equations for $\mu_j^{(k)} \approx u(x_j, t_k)$

$$\frac{\mu_j^{(k+1)} - 2\mu_j^{(k)} + \mu_j^{(k-1)}}{(\Delta t)^2} = \begin{cases} \dfrac{\mu_{j+1}^{(k)} - 2\mu_j^{(k)} + \mu_{j-1}^{(k)}}{H^2} & \text{for } j < 0 \,, \\ \dfrac{\frac{1}{h}\mu_1^{(k)} - (\frac{1}{h} + \frac{1}{H})\mu_0^{(k)} + \frac{1}{H}\mu_{-1}^{(k)}}{1/2(H+h)} & \text{for } j = 0 \,, \\ \dfrac{\mu_{j+1}^{(k)} - 2\mu_j^{(k)} + \mu_{j-1}^{(k)}}{h^2} & \text{for } j > 0 \,. \end{cases} \tag{1.10.5}$$

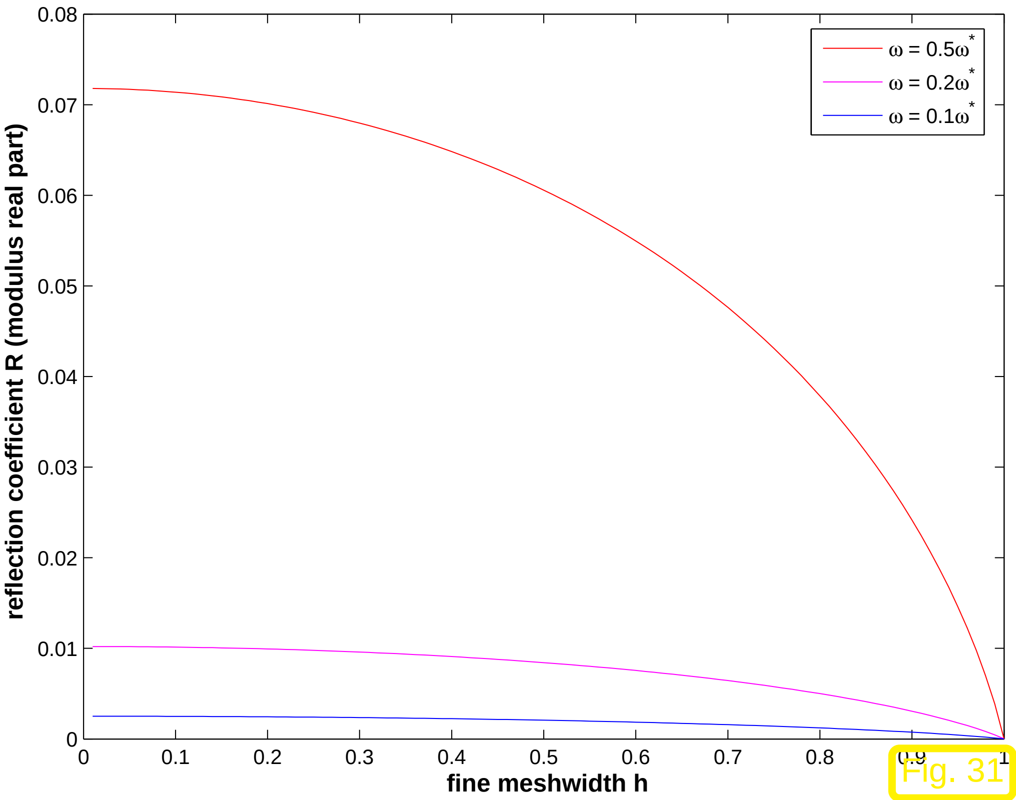
Seek discrete plane wave solution (incident wave)

$$\mu_j^{(k)} = \begin{cases} e^{i(k_H x_j - \omega t_k)} - R e^{i(-k_H x_j - \omega t_k)} & \text{, for } j \leq 0,\ k \in \mathbb{N}_0 \,, \\ T e^{i(k_h x_j - \omega t_k)} & \text{, for } j \geq 0,\ k \in \mathbb{N}_0 \,, \end{cases} \tag{1.10.6}$$

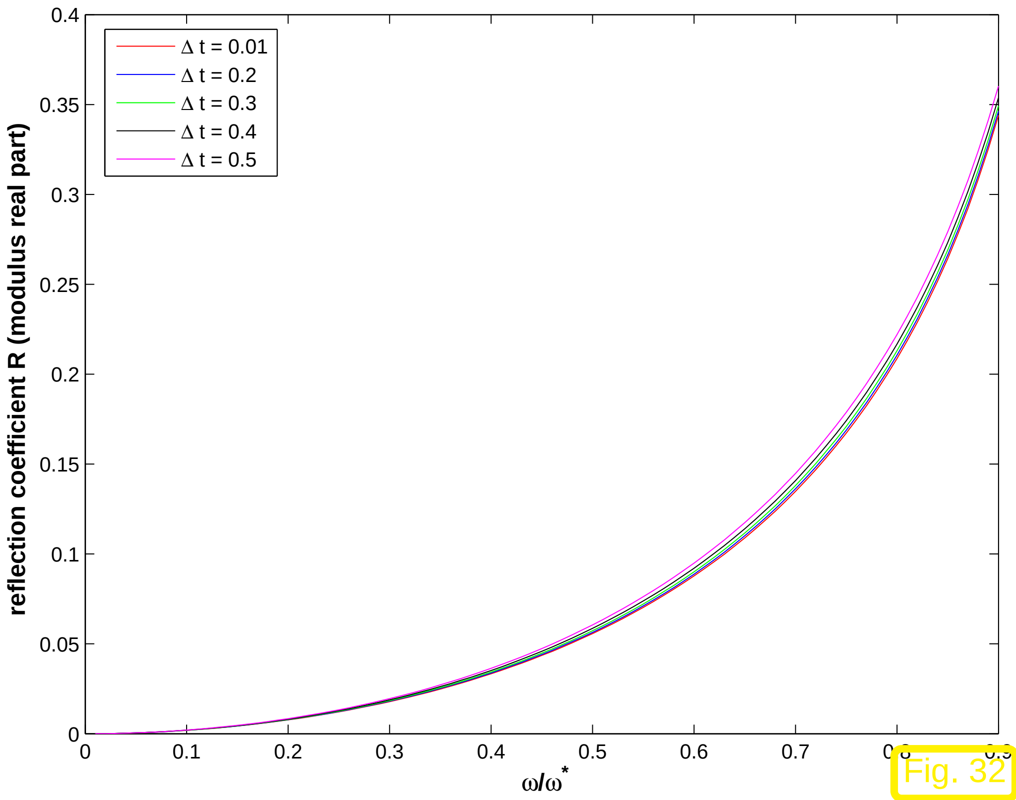
right propagating waves left propagating waves

discrete wave vectors $k_h = k_h(\omega)$ and $k_H = k_H(\omega)$ from discrete dispersion relation (1.9.2).

(1.10.6) well defined **&** (1.10.5) for $j = 0$ ➢ linear equations for R, T



Reflection coefficient $R = R(h)$, $H = 1$, Δt at CFL limit



Reflection coefficient $R = R(\omega/\omega^*)$, $H = 1$, $h = \frac{1}{2}$

Example 22 (Numerical reflections at grid interface).

- 1D wave equation $\dfrac{\partial^2 u}{\partial t^2} - \dfrac{\partial^2 u}{\partial x^2} = 0$ on $]0,1[\times]0,1[$, homogeneous Dirichlet boundary conditions $u(0,t) = u(1,t) = 0$, $0 < t < 1$.

- Initial data: compactly supported "pulses": $u_0 = \psi(x)$, $v_0(x) = -\psi'(x)$
  (➔ would give rise to solution $u(x,t) = \psi(x-t)$ for Cauchy problem $\rightarrow$ Sect. 1.3.2)

  Here: $\psi \hat{=}$ "hat function" pulse supported on two leftmost mesh cells.

- finite element Galerkin discretization ($\rightarrow$ Sect. 1.6.3) in $\mathcal{S}^0_{1,0}(\mathcal{M})$ with mass lumping on *non-equidistant* mesh

$$\mathcal{M} = \mathcal{M}_- \cup \mathcal{M}_+ \quad , \quad \begin{array}{r} \mathcal{M}_-^- := \{]x^-_{j-1}, x^-_j[: x^-_j = \frac{1}{2} j/M_-, j = 1, \ldots, M_-\} \;, \\ \mathcal{M}_+ := \{]x^+_{j-1}, x^+_j[: x^+_j := \frac{1}{2} + \frac{1}{2} j/M_+, j = 1, \ldots, M_+\} \;. \end{array}$$

- leap frog timestepping, Δt at CFL limit (determined by finer mesh !)

Tracking of pulse propagation:

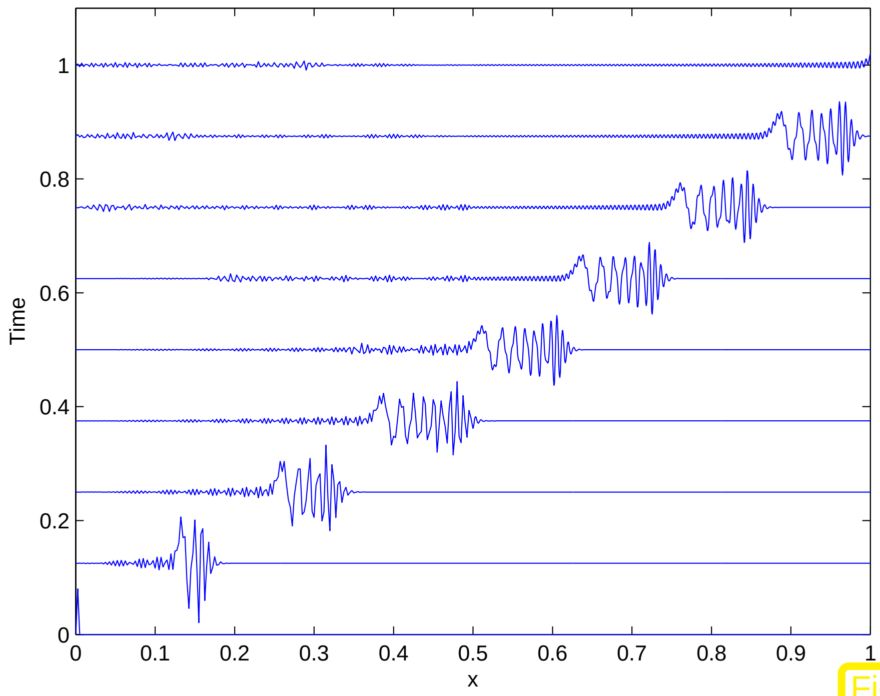


Fig. 33

$M_+ = 4M_- = 800$

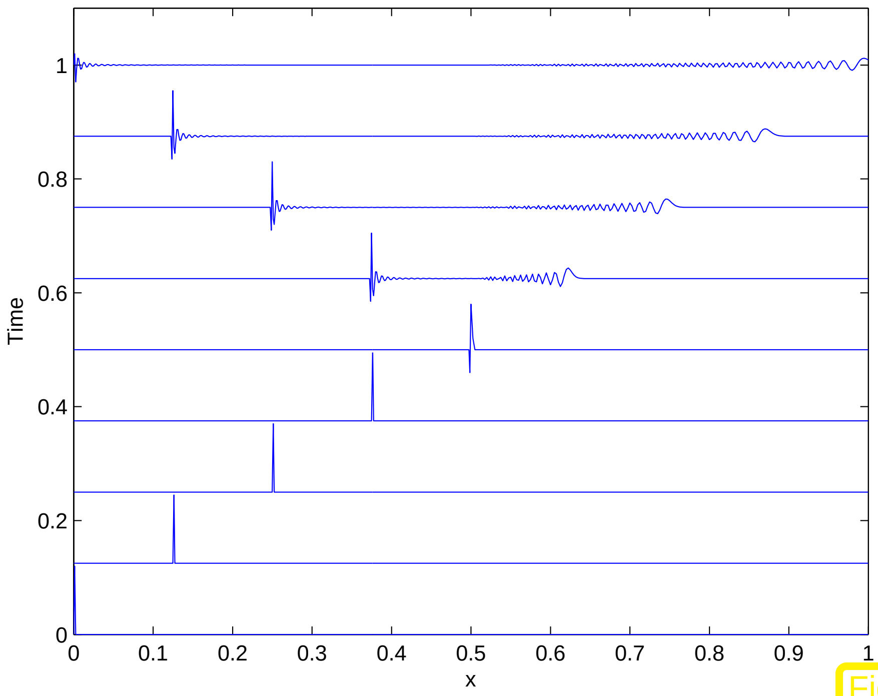


Fig. 34

$M_- = 2M_+ = 800$

◇

Of course: reflection at grid interfaces also in 2D, 3D

simulation of wave propagation on unstructured meshes **?**

1.11 Local timestepping

Resolution of geometry
resolution of materials
$\Downarrow$
locally refined spatial mesh required

BUT: CFL-condition: $\Delta t \sim h_{\min}$, *cf.* (1.7.16)

➤ enforces small *global* timestep

Numerical dispersion (Sect. 1.9): $\Delta t \sim h$

➤ local timesteps adapted to local meshwidth

Locally refined triangular mesh (M. Grote, J. Diaz, Univ. Basel) ▷

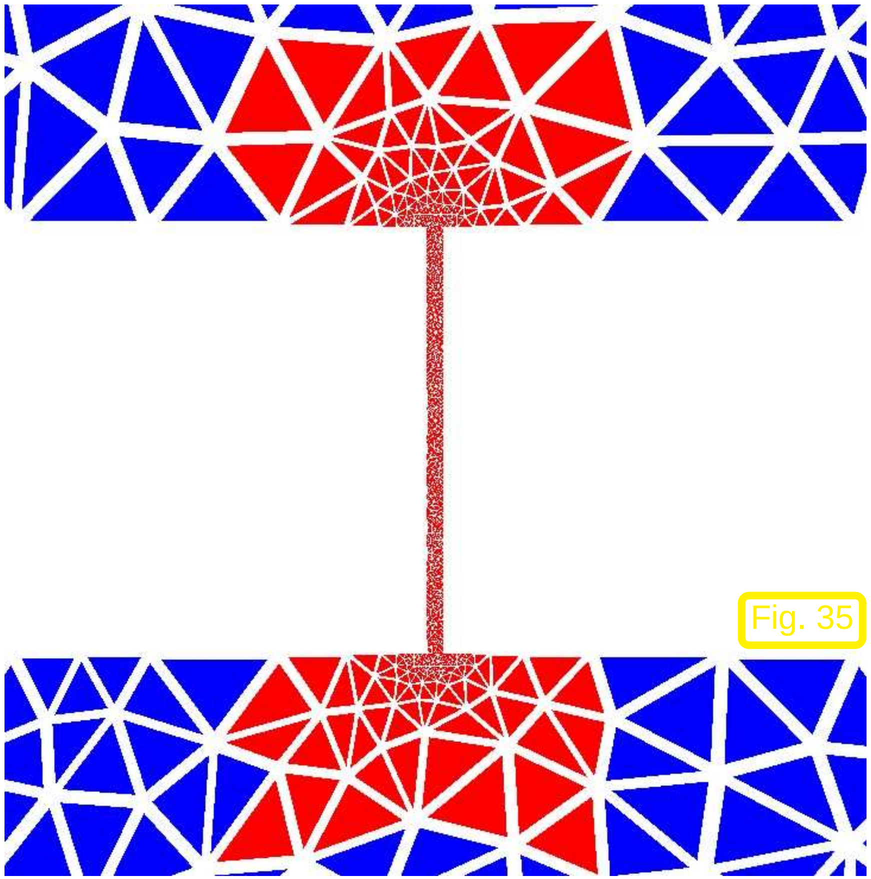


Fig. 35

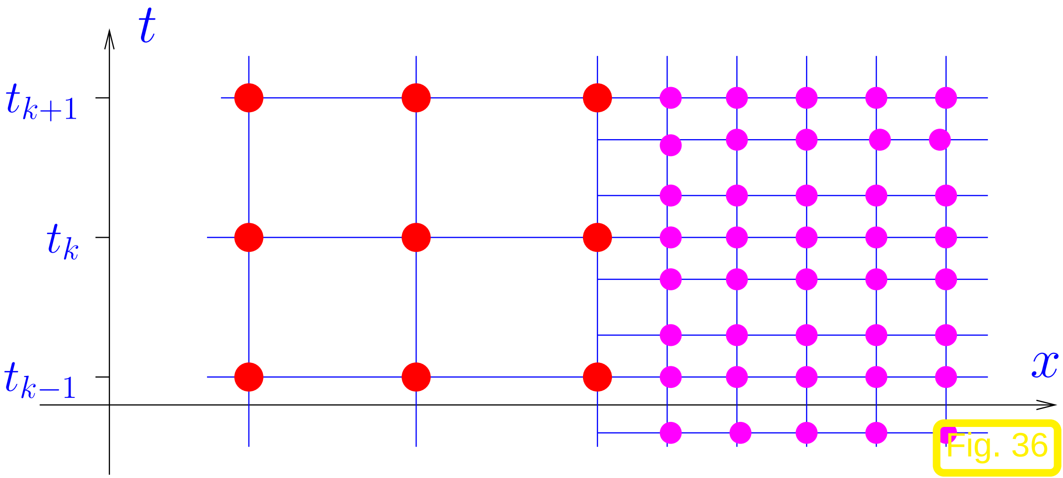


Fig. 36

To control numerical dispersion

► Matched refinement in space **and** time ! (*cf.* magic timestep)

◁ locally refine space-time mesh

Consider (spatially semidiscrete) transformed equation (1.7.2) (for $\vec{\varphi} = 0$):

$$\vec{\nu} := \mathbf{M}^{1/2}\vec{\mu}: \qquad \frac{d^2}{dt^2}\vec{\nu} + \widetilde{\mathbf{A}}\vec{\nu} = 0 \; , \quad \widetilde{\mathbf{A}} := \mathbf{M}^{-\frac{1}{2}}\mathbf{A}\mathbf{M}^{-\frac{1}{2}} \; . \tag{1.11.1}$$

Focus: one timestep of two-step method ($\rightarrow$ Def. 1.7.1):

$\vec{\nu}^{(k-1)} \approx \vec{\nu}(t_{k-1})$, $\vec{\nu}^{(k)} \approx \vec{\nu}(t_k) \rightarrow \vec{\nu}^{(k+1)} \approx \vec{\nu}(t_{k+1})$, fixed timestep Δt

Note: components of $\vec{\nu} \leftrightarrow$ spatial d.o.f. ("nodes")

$$\text{Partitioning:} \quad \vec{\nu}(t) = \begin{pmatrix} \vec{\nu}_c \\ \vec{\nu}_f \end{pmatrix} = \vec{\nu}^c(t) + \vec{\nu}^f(t) = (Id - \mathbf{P})\vec{\nu}(t) + \mathbf{P}\vec{\nu}(t) \; , \tag{1.11.2}$$

$\mathbf{P} \mathrel{\hat{=}}$ diagonal projection matrix, entries $\in \{0, 1\}$.

$\vec{\nu}^c(t) \longleftrightarrow$ nodes located in "coarse zone" ➥ large timestep

$\vec{\nu}^f(t) \longleftrightarrow$ nodes located in "refined zone" ➥ small timestep

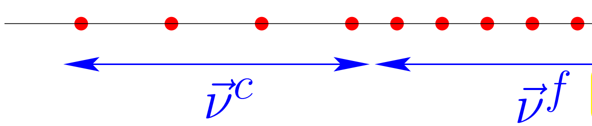
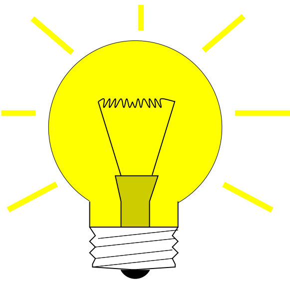


Fig. 37

$$\blacktriangleright \quad (1.11.1) \;\Rightarrow\; \frac{d}{dt^2}\begin{pmatrix} \vec{\nu}_c \\ \vec{\nu}_f \end{pmatrix} + \begin{pmatrix} \mathbf{A}_{cc} & \mathbf{A}_{fc} \\ \mathbf{A}_{cf} & \mathbf{A}_{ff} \end{pmatrix}\begin{pmatrix} \vec{\nu}_c \\ \vec{\nu}_f \end{pmatrix} = 0 \; . \tag{1.11.3}$$

Idea: use solution formula for (1.11.1) ($\Delta t \mathrel{\hat{=}}$ large timestep)



$$\underset{\vec{\nu}^{(k+1)}}{\underset{\updownarrow}{\vec{\nu}(t+\Delta t)}} \; - \; 2\underset{\vec{\nu}^{(k)}}{\underset{\updownarrow}{\vec{\nu}(t)}} \; + \; \underset{\vec{\nu}^{(k-1)}}{\underset{\updownarrow}{\vec{\nu}(t-\Delta t)}} = -(\Delta t)^2 \int_{-1}^{1} (1 - |\xi|)\widetilde{\mathbf{A}}\vec{\nu}(t + \xi\Delta t)\,\mathrm{d}\xi \; . \tag{1.11.4}$$

\+ freezing of $\vec{\nu}^c(t)$: $\boxed{\vec{\nu}^c(t) = (Id - \mathbf{P})\vec{\nu}^{(k)}, \; t_{k-1} \leq t \leq t_{k+1}}$

$$\vec{\nu}^{(k+1)} - 2\vec{\nu}^{(k)} + \vec{\nu}^{(k-1)} = -(\Delta t)^2 \int_{-1}^{1} (1-|\xi|)\Big(\widetilde{\mathbf{A}}(Id-\mathbf{P})\vec{\nu}^{(k)} + \widetilde{\mathbf{A}}\mathbf{P}\vec{\nu}(t_k+\xi\Delta t)\Big)\,\mathrm{d}\xi$$

$$\overset{(1.11.4)}{=} \vec{\rho}(t_k+\Delta t) - 2\rho(\vec{t_k}) + \vec{\rho}(t_k-\Delta t)\ ,$$

where $\vec{\rho}(t)$ solves

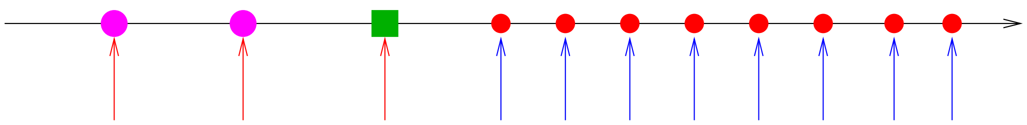
$$\frac{d}{dt^2}\vec{\rho} + \widetilde{\mathbf{A}}\mathbf{P}\vec{\rho} = -\widetilde{\mathbf{A}}(Id-\mathbf{P})\vec{\nu}^{(k)} \quad \text{for} \quad t_k - \Delta t \leq t \leq t_k + \Delta t\ , \tag{1.11.5}$$

$$\Updownarrow$$

$$\frac{d}{dt^2}\begin{pmatrix}\vec{\rho}_c(t)\\ \vec{\rho}_f(t)\end{pmatrix} + \begin{pmatrix}\mathbf{A}_{fc}\\ \mathbf{A}_{ff}\end{pmatrix}\vec{\rho}_f(t) = -\begin{pmatrix}\mathbf{A}_{cc}\\ \mathbf{A}_{cf}\end{pmatrix}\vec{\nu}_c^{(k)}\ . \tag{1.11.6}$$

What do we gain **?**

- ● $\leftrightarrow$ $\vec{\rho}_f$
- ● $\leftrightarrow$ $\vec{\rho}_c$, not connected with $\vec{\rho}_f$
- □ $\leftrightarrow$ "coarse node" linked to $\vec{\rho}_f$



Note: trivial evolution for ● **!**

"Initial" conditions for (1.11.5) **?** $\quad \vec{\rho}(t_k) = \mathbf{P}\vec{\nu}^{(k)},\quad \dfrac{d\vec{\rho}}{dt}(t_k) = 0$

ensures reversibility of timestepping

➢ partitioned leapfrog timestepping

$$\vec{\nu}^{(k+1)} - 2\vec{\nu}^{(k)} + \vec{\nu}^{(k-1)} = \vec{\rho}(t_k + \Delta t) - 2\vec{\rho}(t_k) + \vec{\rho}(t_k - \Delta t)\ . \tag{1.11.7}$$

Approximation of $\vec{\rho}(t)$: leapfrog timestepping for (1.11.5):

- small timestep $\Delta t/M$, $M \in \mathbb{N}$ ($\sim$ magic timestep for fine mesh),
- exploit symmetry $\vec{\rho}(t_k - \Delta t) = \vec{\rho}(t_k + \Delta t)$.

Example 23 (Local timestepping).

- 1D wave equation $\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0$, $0 < x < 1$, homogeneous Dirichlet boundary conditions $u(0,t) = u(1,t) = 0 \hat{=}$ perfectly reflecting b.c.
- initial conditions $\longrightarrow u(x,t)$ = smooth pulse (1.8.5), initially travelling in $+x$-direction, *cf.* Ex. 18.
- "Coarse zone" $]0, \frac{1}{2}[\longrightarrow$ uniform meshwidth H, "refined zone" $]\frac{1}{2}, 1[\longrightarrow$ uniform meshwidth h.

① Simulation: $H = \frac{1}{60}$, at CFL limit $\Delta t : H = 1$!

➨ movie: bouncing bump: accurate solution, little spurious reflections

② largest eigenvalue $\sigma_{\max}$ (in modulus) of discrete evolution operator $\leftrightarrow$ stabiliity, $H = \frac{1}{60}$, for different CFL-numbers $\Delta t : H$

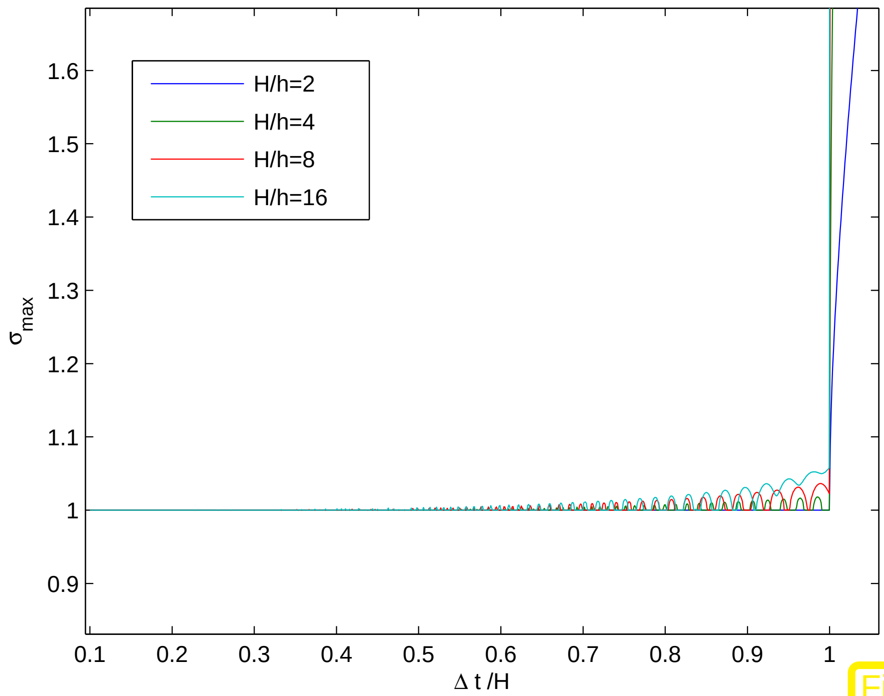


Fig. 38

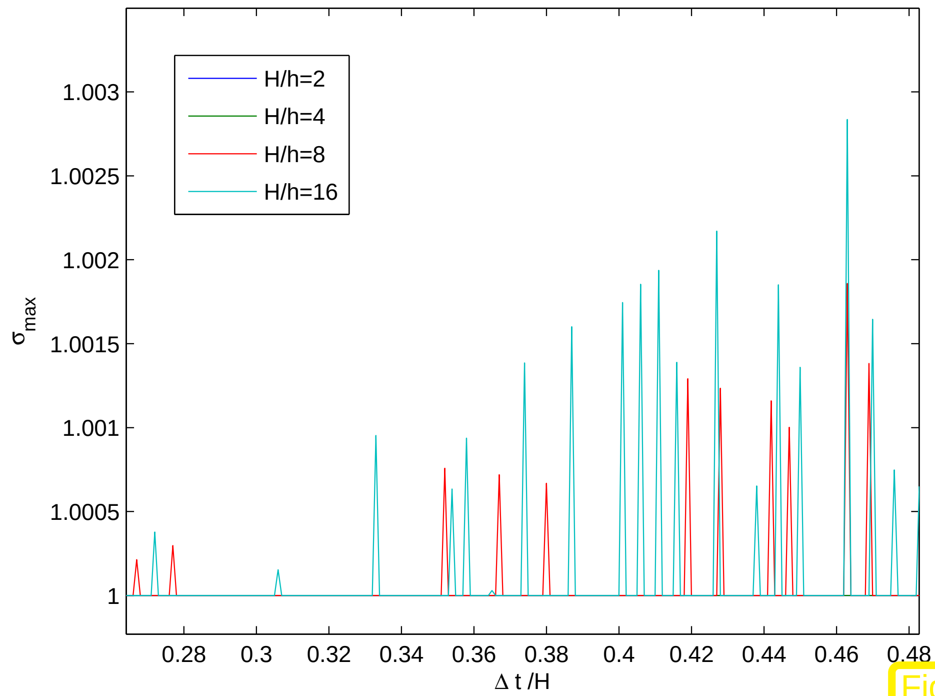


Fig. 39

➡ problematic: stability of local timestepping

◇

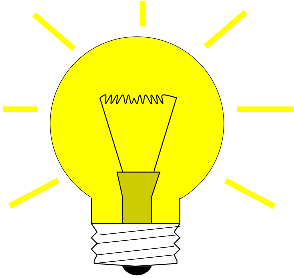
Work in progress: ☞ CFL-conditions for partitioned leapfrog scheme by energy methods
(M. Grote, J. Diaz) ☞ Analysis of numerical dispersion/reflection

1.12 Absorbing boundary conditions

$d = 1$, $\Omega =]0, \infty[$: IBVP for wave equation on **unbounded** spatial domain

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x}\left(c^2(x)\frac{\partial u}{\partial x}\right) = 0 \quad \text{on } \Omega \times]0,T[\ , \quad c(x) = 1 \quad \text{for } x \geq 1\ ,$$
$$\begin{aligned} u(x,0) &= u_0(x)\ , \\ \frac{\partial u}{\partial t}(x,0) &= v_0(x)\ , \end{aligned} \quad x \in \Omega\ , \quad u(0,t) = 0\ , t > 0: \quad \operatorname{supp}(u_0), \operatorname{supp}(v_0) \subset]0,1[\ . \tag{1.12.1}$$

Spatial discretization of (1.12.1) impossible **!**



Idea: ❶ restrict spatial discretization to *convex* "interior region" $D :=]0,1[$ (truncation of Ω)

❷ impose special absorbing boundary conditions (ABCs) at $x = 1$, such that the truncated problem has the same solution as (1.12.1).

$d > 1$: $\Omega = \mathbb{R}^d$ unbounded spatial domain ➞ Cauchy problem

$$\frac{\partial^2 u}{\partial t^2} - \operatorname{div}_{\boldsymbol{x}}(\mathbf{C}(\boldsymbol{x})\,\mathbf{grad}_{\boldsymbol{x}}\,u) = f(\boldsymbol{x},t) \quad \text{in } \mathbb{R}^d \times]0,T[\ , \tag{1.12.2}$$

$$u(\boldsymbol{x},0) = u_0(\boldsymbol{x}) \quad , \quad \frac{\partial u}{\partial t}(\boldsymbol{x},0) = v_0(\boldsymbol{x})\ , \quad \boldsymbol{x} \in \mathbb{R}^d\ , \tag{1.12.3}$$

with $\mathbf{C}(\boldsymbol{x}) = \mathbf{I}$, if $\boldsymbol{x} \notin D$, "interior region" $D \subset \mathbb{R}^d$ *bounded*,
$f(\boldsymbol{x}, t) = 0$, $u_0(\boldsymbol{x}) = 0$, $v_0(\boldsymbol{x}) = 0$ for $\boldsymbol{x} \notin D$

☞ truncation to D: spatial discretization only inside D
\+ absorbing boundary conditions at ∂D

1.12.1 Dirichlet-to-Neumann (DtN) absorbing boundary conditions

Consider $d = 1$, (1.12.1): ABCs have to be *transparent* for outgoing solutions $u(x,t) = \psi(x-t)$:

$$\mathcal{B}\{\psi(x-t)\} = 0 \quad \text{for} \quad \text{spatio-temporal boundary differential operator } \mathcal{B}\,.$$

▶ $\boxed{\mathcal{B} := (\frac{\partial}{\partial x} + \frac{\partial}{\partial t})}$ $\Rightarrow$ ABCs: $\frac{\partial u}{\partial x}(1,t) + \frac{\partial u}{\partial t}(1,t) = 0 \quad \forall t \geq 0\,.$ (1.12.4)

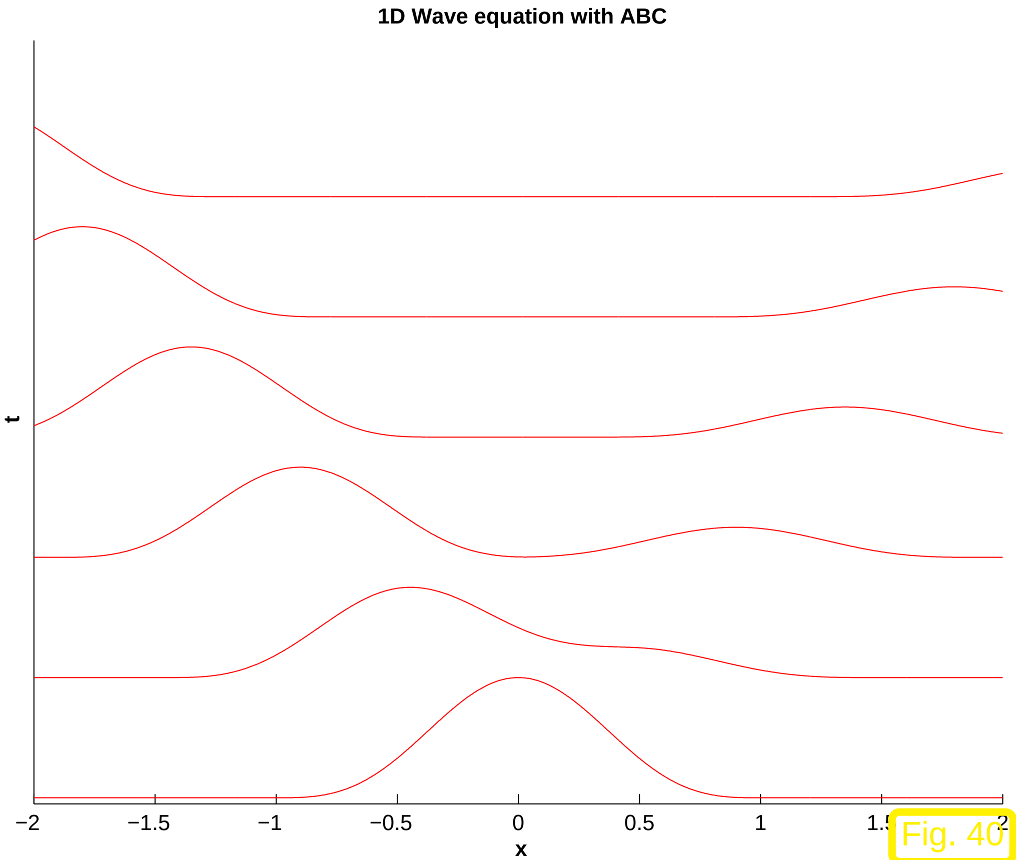
Neumann data at $x = 1$ $\qquad$ $\frac{\partial}{\partial t}$Dirichlet data at $x = 1$

Note: ABCs (1.12.4) are local in space and time
ABCs (1.12.4) = boundary conditions of impedance type ($\leftrightarrow$ DtN)

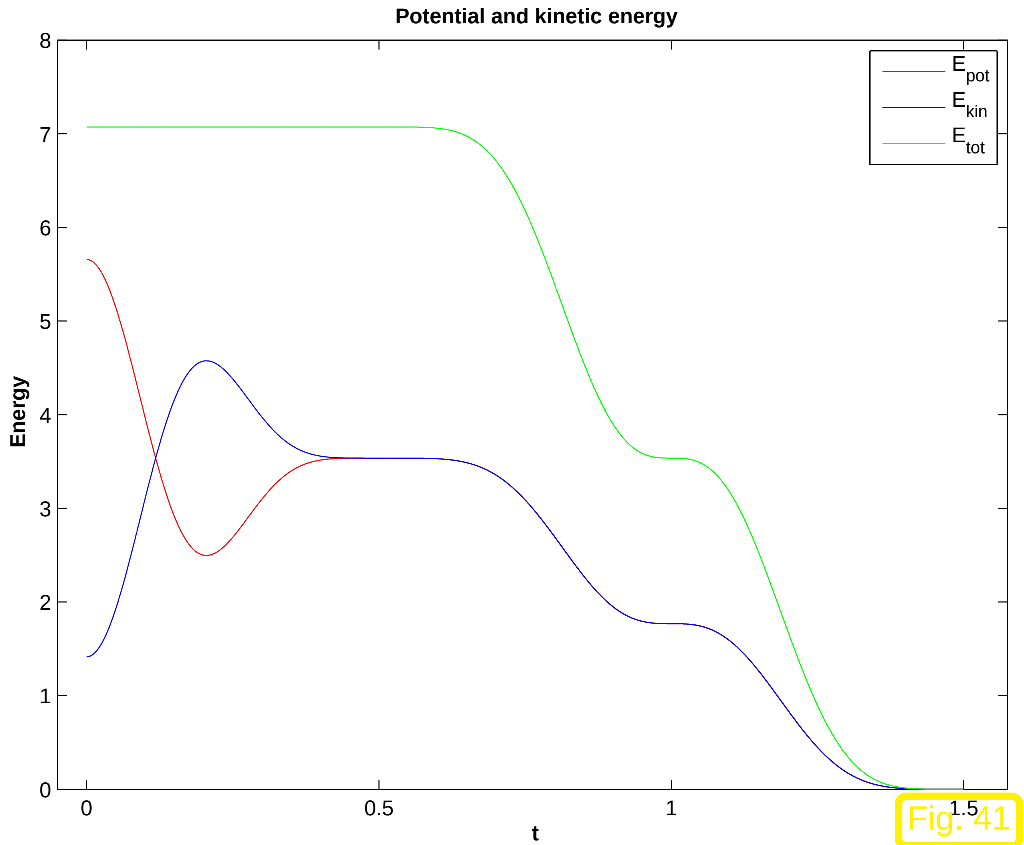
Example 24 (Absorbing boundary conditions for 1D wave equation).

1D wave equation $\frac{\partial^2 u}{\partial t^2} - 4\frac{\partial^2 u}{\partial x^2} = 0$ on $]-2,2[\times]0,1[$, ABC (1.12.4) at $x = -2$, $x = 2$ ($\rightarrow$ Cauchy problem)

- $\varphi(x) = \begin{cases} (1-x^2)^3 \exp(-x^2) & \text{for } -1 < x < 1\,, \\ 0 & \text{for } x \notin]-1,1[\,. \end{cases}$ ➤ $u(x,t) = \frac{3}{4}\varphi(x+2t) + \frac{1}{4}\varphi(x-2t)$
- Finite element Galerkin discretization ($\longrightarrow$ Sect. 1.6.3) on equidistant mesh, $h = \frac{1}{250}$
- timstepping: implicit trapezoidal rule (1.7.10) **+** symmetric finite difference discretization of $\frac{\partial}{\partial t}$.



Snapshots of pulse being absorbed



Behavior of kinetic, potential and total energy (inside D) during absorption.

➡ movie: absorption of propagating bump

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Absorbing boundary conditions in higher dimensions **?**

no local Dirichlet-to-Neumann ABCs for $d > 1$!

wave escaping through $\boldsymbol{x} \in \partial D$ will hit (far away) $\boldsymbol{y} \in \partial D$ *after some time*. ▷

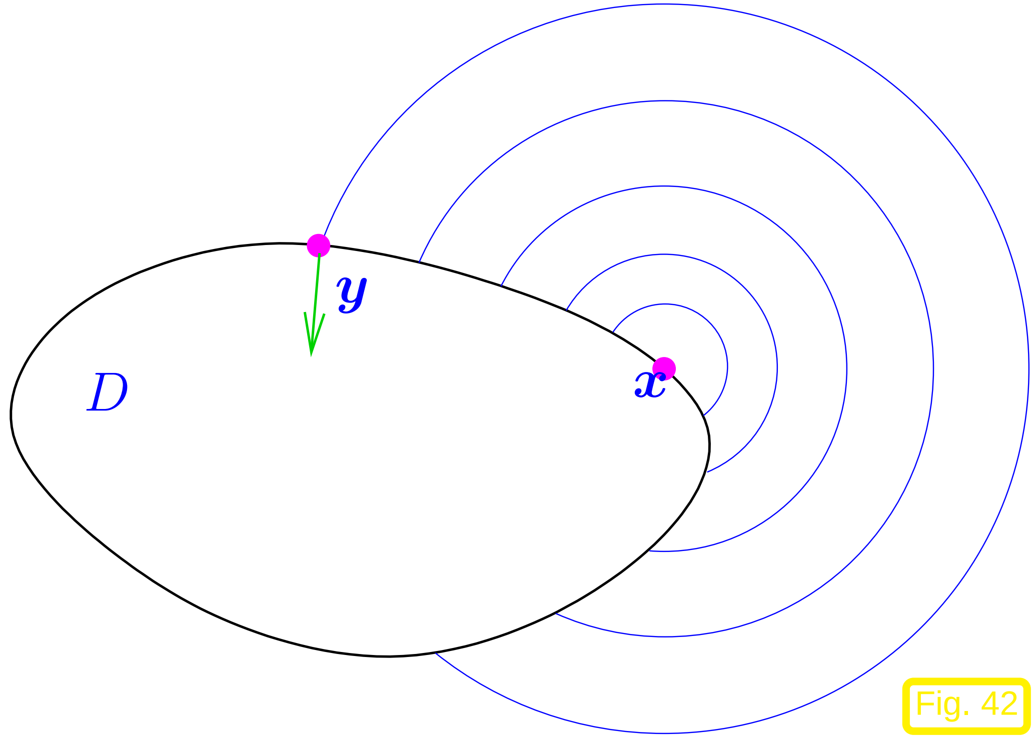


Fig. 42

Approximate Dirichlet-to-Neumann ABCs **?**

❶ Simplest option: use (1.12.4) locally in every $\boldsymbol{x} \in \partial D$

Sommerfeld ABC: $$\mathbf{grad}\, u(\boldsymbol{x},t) \cdot \boldsymbol{n}(\boldsymbol{x}) + \frac{\partial u}{\partial t}(\boldsymbol{x},t) = 0 \quad \text{for } (\boldsymbol{x},t) \in \partial D \times]0,T[\;.$$

$\hat{=}$ first order approximate Dirichlet-to-Neumann absorbing boundary condition (flexible, but inaccurate)

❷ Special option: convolution-based approximate Dirichlet-to-Neumann ABCs

Consider (1.12.2) for $d = 2$ with $D := B_1(0) := \{\boldsymbol{x} \in \mathbb{R}^2 \colon |\boldsymbol{x}| < 1\}$ (unit disk)

➜ solution $u(\boldsymbol{x}, t) = u(r, \varphi, t)$ of (1.12.2) satisfies for $r \geq 1$: (polar coordinates (r, φ))

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial r^2} - \frac{1}{r}\frac{\partial u}{\partial r} - \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2} = 0 \,. \tag{1.12.5}$$

seek causal DtN map $\mathsf{S} : L^2(]0,T[; H^{1/2}(\partial D)) \mapsto L^2(]0,T[; H^{-1/2}(\partial D))$, $\mathsf{S}u := \frac{\partial u}{\partial r}_{|r=1}$

$(\mathsf{S}u)(\cdot, t)$ only depends on "past values" $u(\cdot, \tau)$ for $0 \leq \tau \leq t$

▷ Fourier series expansion w.r.t $\varphi \in [0, 2\pi]$ **+** Laplace transform w.r.t. t ($\alpha \in \mathbb{R}$):
(➵ http://en.wikipedia.org/wiki/Laplace_transform)

$$u(r, \varphi, t) = \tfrac{1}{2\pi i} \sum_{n \in \mathbb{Z}} \int_{\alpha - i\infty}^{\alpha + i\infty} \widehat{u}_n(r, s) e^{in\varphi + st} \,\mathrm{d}s \qquad \widehat{u}_n : \{r \geq 1\} \times \{z \in \mathbb{C} \colon \mathrm{Re}(z) \geq \alpha\} \mapsto \mathbb{C} \,.$$

$\overset{(1.12.5)}{\blacktriangleright}$

$$s^2 \widehat{u}_n(r, s) - \frac{\partial^2 \widehat{u}_n}{\partial r^2}(r, s) - \frac{1}{r}\frac{\partial \widehat{u}_n}{\partial r}(r, s) + \frac{n^2}{r^2}\widehat{u}_n(r, s) = 0 \quad \forall n \in \mathbb{Z},\, r \geq 1,\; \mathrm{Re}(s) \geq \alpha \,. \tag{1.12.6}$$

(1.12.6) $\hat{=}$ modified Bessel differential equation: we seek bounded solutions

$$\widehat{u}_n(r,s) = \frac{K_n(rs)}{K_n(s)}\,\widehat{u}_n(1,s) \quad \Rightarrow \quad \frac{\partial \widehat{u}_n}{\partial r}(1,s) = \underbrace{s\frac{K_n'(s)}{K_n(s)}}_{=:k_n(s)}\ \widehat{u}_n(1,s)\ . \tag{1.12.7}$$

kernel

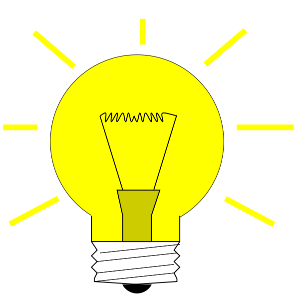
K_n $\hat{=}$ modified Bessel function of order n, $n \in \mathbb{Z}$ [1, Ch. 9]. (MATLAB: `besselk(nu,z)`, ➼ `mathworld.wolfram.com/ModifiedBesselFunctionoftheSecondKind.html`)

$\mathcal{L}^{-1}$ $\hat{=}$ inverse Laplace trf. ($u(r,t) = 0$ for $r \geq 1$ and small t !), $\mathcal{F}$ $\hat{=}$ Fourier series transform.

$$\frac{\partial u}{\partial r}(1,\varphi,t) = \mathcal{F}^{-1}\Big(((\mathcal{L}^{-1}k_n)(t) * (\mathcal{F}u(1,\cdot,t))_n)(t)\Big)_{n\in\mathbb{Z}}\ ,$$

with $*$ $\hat{=}$ temporal convolution: $(f * g)(t) := \int_0^t f(t-\tau)g(\tau)\,\mathrm{d}\tau$.

Temporal convolution **!** What to do **?**



Idea: rational approximation of convolution kernel $k_n(s)$

$$k_n(s) \approx \widetilde{k}_n(s) := \sum_{m=1}^{P} \frac{\beta_{n,m}}{s + z_{n,m}}\ , \quad \beta_{n,m} \in \mathbb{C}, \quad z_{n,m} \in \mathbb{C},\ \mathrm{Re}(z_{n,m}) < 0\ . \tag{1.12.8}$$

residual theorem [39, Ch. 13] $\Rightarrow$ $(\mathcal{L}^{-1}\widetilde{k}_n)(t) = \sum_{m=1}^{P} \beta_{n,m} \int_{-i\infty}^{i\infty} \frac{\exp(st)}{s+z_{n,m}} \,\mathrm{d}s = \sum_{m=1}^{P} \beta_{n,m} e^{-z_{n,m}t}$,

$$((\mathcal{L}^{-1}\widetilde{k}_n) * ((\mathcal{F}u(1,\cdot,t))_n)(t) = \sum_{m=1}^{P} \beta_{n,m} \underbrace{\int_0^t e^{-z_{n,m}(t-\tau)}(\mathcal{F}u(1,\cdot,\tau)_n)\,\mathrm{d}\tau}_{=:I_{n,m}(t)} .$$

$$I_{n,m}(t+\Delta t) \approx e^{-z_{n,m}\Delta t} I_{n,m}(t) + \int_0^{\Delta t} e^{-z_{n,m}(\Delta t-\tau)}((1-\tfrac{\tau}{\Delta t})f(t) + \tfrac{\tau}{\Delta t}f(t+\Delta t))\,\mathrm{d}\tau \;, \tag{1.12.9}$$

$f(t) := ((\mathcal{F}u(1,\cdot,t))_n)(t)$ **=** (implicit) "timestepping formula"

$$(\mathsf{S}u_{|r=1})(\varphi,t) \approx \mathcal{F}^{-1}\left(\sum_{m=1}^{P} \beta_{n,m} I_{n,m}(t)\right)_{n\in\mathbb{Z}} . \tag{1.12.10}$$

Implementation:

(FE Galerkin discretization on triangular mesh of D with uniformly spaced nodes on ∂D)

- $\mathcal{F} \leftrightarrow$ DFT (via FFT) on $\partial D \cap \mathcal{V}(\mathcal{M})$,
  $\sharp\{\partial D \cap \mathcal{V}(\mathcal{M})\}$ Fourier modes
  (another approximation !)
- use (1.12.9) in connection with leapfrog timestepping
- $\beta_{n,m}$ by rational least squares approximation of $k_n(s)$ on imaginary axis [3] by function $p(z)/q(z)$, p, q polynomials, $\deg q = \deg p + 1$.

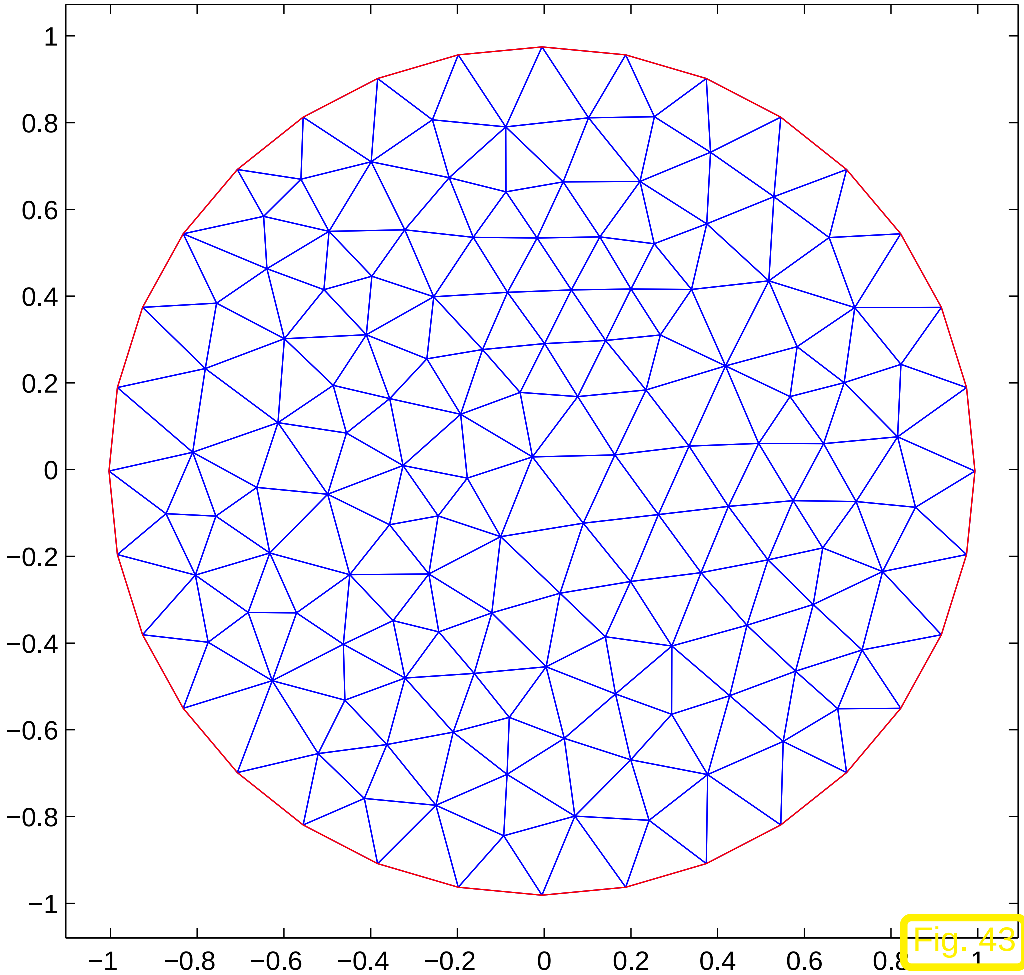


Fig. 43

Rational approximation (1.12.8) possible **?** $k_n(s)$ from (1.12.7): $|k_n(s)| \to \infty$ for $|s| \to \infty$ **!**

Idea: "subtract asymptotics" ➢ modified kernel $\bar{k}_n(s) := k_n(s) + s + \frac{1}{2}$

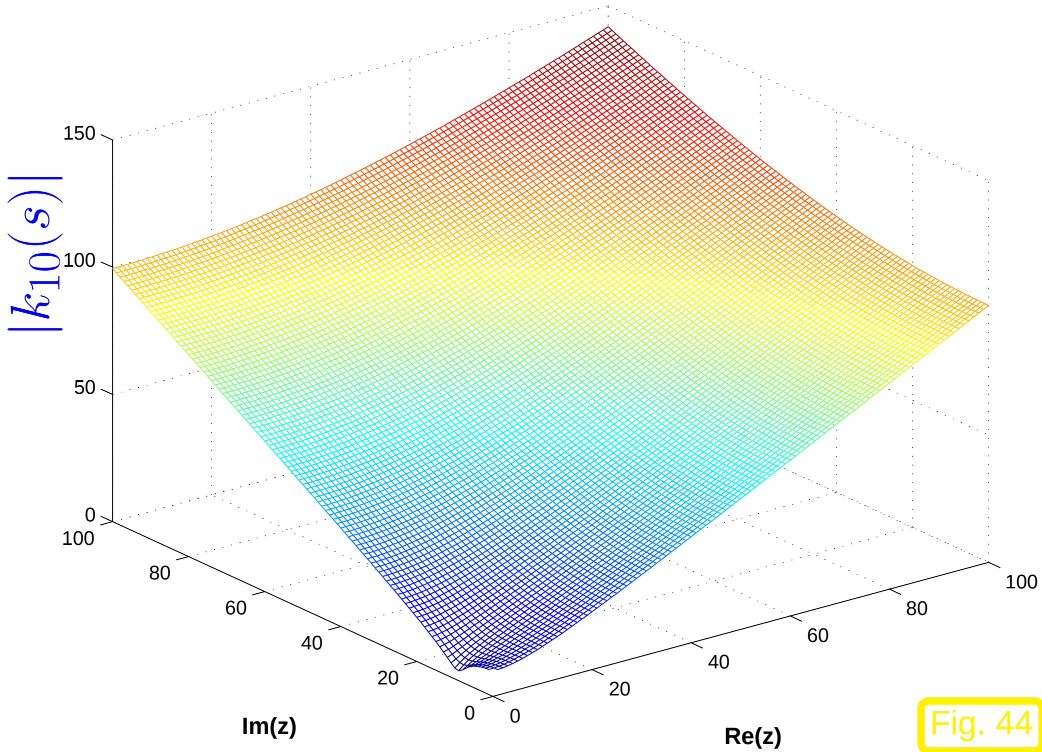


Fig. 44

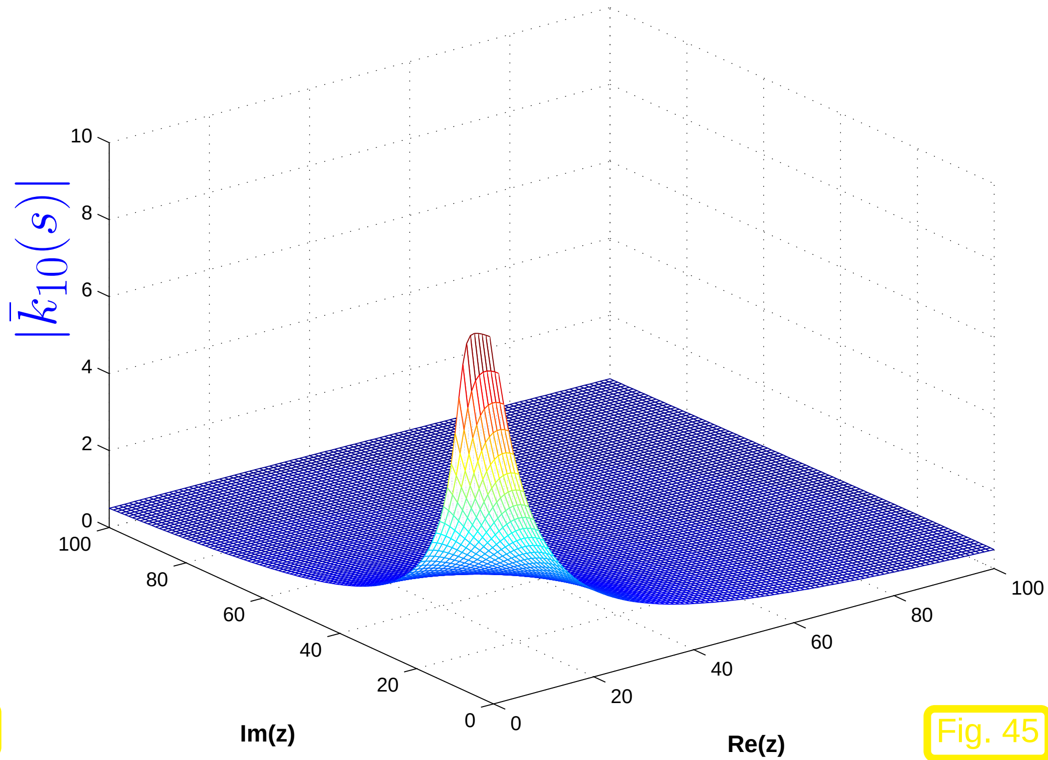


Fig. 45

$$\frac{\partial u_n}{\partial r}(1,s) + \frac{\partial u_n}{\partial t}(1,t) + \tfrac{1}{2}u_n(1,t) = ((\mathcal{L}^{-1}\bar{k}) * u_n(1,\cdot))(t) \;,$$

$$\underbrace{\frac{\partial u}{\partial r}(1,t) + \frac{\partial u}{\partial t}(1,t) + \tfrac{1}{2}u(1,t)}_{\text{local spatio-temporal boundary differential operator}} = \underbrace{\mathcal{F}^{-1}\Big(((\mathcal{L}^{-1}\bar{k}) * \mathcal{F}(u(1,\cdot,t))_n)(t)\Big)_{n\in\mathbb{Z}}}_{\text{non-local boundary operator}} \;.$$

Remark 25 (Required number of poles in rational approximation (1.12.8)).

required number of poles P in (1.12.8) for $T = 10$,

$$\sup_{\xi\in\mathbb{R}} \left|\bar{k}_n(i\xi) - \widetilde{k}_n(i\xi)\right| < \epsilon \;, \qquad (1.12.11)$$

$\epsilon = 10^{-8}$, and rational best approximation $\widetilde{k}_n$ of $\bar{k}_n$.

Lemma 1.12.1 ((n,ϵ)-asymptotic of P [2]).

$$P(\epsilon, n) = \begin{cases} O(\log n \log \frac{1}{\epsilon} + \log^2 \frac{1}{\epsilon}) & \textit{for } n > 0 \;, \\ O(\log T \log \frac{1}{\epsilon}) & \textit{for } n = 0 \;. \end{cases}$$

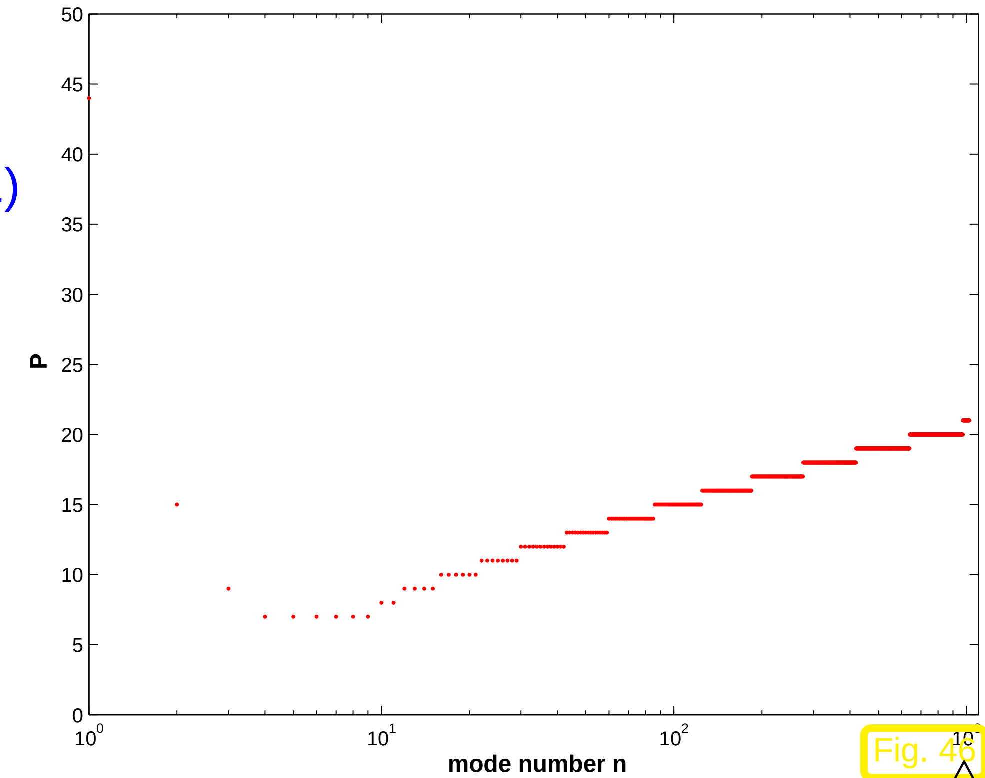
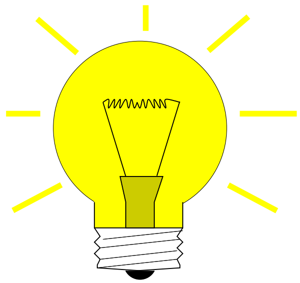


Fig. 46

1.12.2 Perfectly matched layers (PML)



Idea: "absorbing material" in exterior region:

- **!** no reflections at interface
- **!** fast decay (attenuation) of outgoing waves (away from D)

(material $\hat{=}$ coefficients $\rho(\boldsymbol{x})$, $\mathbf{C}(\boldsymbol{x})$ in (1.1.3)/(1.1.1))

Design of absorbing material in 1D:

$d = 1$: Cauchy problem for wave equation with variable coefficients:

$$\rho(x)\frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x}\left(\gamma(x)\frac{\partial u}{\partial x}\right) = 0 \,,$$

$$\rho(x) = \begin{cases} 1 & \text{, for } x < 0 \,, \\ \rho^* & \text{, for } x > 0 \,, \end{cases} \quad , \quad \gamma(x) = \begin{cases} 1 & \text{, for } x < 0 \,, \\ \gamma^* & \text{, for } x > 0 \,. \end{cases}$$

▶ reflection coefficient, *cf.* (1.10.3): $$R = \frac{1 - \sqrt{\rho^*\gamma^*}}{1 + \sqrt{\rho^*\gamma^*}} \,. \qquad (1.12.12)$$

(Terminology: $\sqrt{\rho^*\gamma^*}$ = wave impedance of material (ρ^*, μ^*))

$$\rho^*\gamma^* = 1 \quad \Leftrightarrow \quad \text{no reflections at } x = 0\,!$$

Dispersion relation for $x > 0$ (plane wave $u(x,t) = \exp(i(k(\omega)x - \omega t))$)

$$\rho^*\omega^2 - \gamma^* k(\omega)^2 = 0 \quad \overset{\rho^*\gamma^*=1}{\Longrightarrow} \quad k(\omega) = \pm\rho^*\omega \,. \qquad (1.12.13)$$

➤ $\boxed{\mathrm{Im}(k(\omega)) > 0}$ $\Leftrightarrow$ exponential decay of outgoing waves for $x \to \infty$

Desirable: attenuation independent of frequency ω ➤ $\rho^* = 1 + i\dfrac{\sigma_0}{\omega}\,,\quad \sigma_0 > 0\,.$ (1.12.14)

How to make sense of **complex** ρ^*, γ^* **?**

time domain frequency domain

! perspective: frequency domain $\leftrightarrow$ temporal Fourier transform $u(x,t)$ $\circ\!\!-\!\!\bullet$ $\widehat{u}(x,\omega)$

$$\frac{\partial}{\partial t} \quad \circ\!\!-\!\!\bullet \quad \cdot\,(-i\omega)$$

➤ in frequency domain: $-\,\omega^2(1 + i\sigma_0/\omega)\widehat{u}(x,\omega) - \dfrac{\partial}{\partial x}\left(\dfrac{1}{1 + i\sigma_0/\omega}\dfrac{\partial \widehat{u}}{\partial x}\right) = 0$ (1.12.15)

$\Updownarrow$ new variable $\widehat{v} := \dfrac{1}{-i\omega + \sigma_0}\dfrac{\partial \widehat{u}}{\partial x}$

$$\begin{aligned}(-i\omega + \sigma_0)\widehat{u}(x,\omega) - \frac{\partial \widehat{v}}{\partial x}(x,\omega) &= 0\,,\\ (-i\omega + \sigma_0)\widehat{v}(x,\omega) - \frac{\partial \widehat{u}}{\partial x}(x,\omega) &= 0\,.\end{aligned} \qquad (1.12.16)$$

in time domain: (1.12.16) $\bullet\!\!-\!\!\!-\!\!\circ$

$$\left(\frac{\partial}{\partial t}+\sigma_0\right)u(x,t)-\frac{\partial v}{\partial x}(x,t) = 0\;,\qquad \left(\frac{\partial}{\partial t}+\sigma_0\right)v(x,t)-\frac{\partial u}{\partial x}(x,t) = 0\;. \tag{1.12.17}$$

(1.12.17)**=** wave equation for perfectly matched layer (PML) in 1D

Coupling: PML **+** wave equation (1.12.1): a single 1st-order system ! ($\longrightarrow$ 1.5.2)

$$\boxed{\begin{aligned}\frac{\partial u}{\partial t}(x,t)+\sigma(x)u(x,t)-\frac{\partial v}{\partial x}(x,t) &= v_0\;,\\ \frac{\partial v}{\partial t}(x,t)+\sigma(x)v(x,t)-c^2(x)\frac{\partial u}{\partial x}(x,t) &= 0\;,\end{aligned}} \tag{1.12.18}$$

$$c(x)=\begin{cases}\text{uniformly positive} & \text{, if } 0<x<1\;,\\ 1 & \text{, if } x>1\;,\end{cases}\qquad \sigma(x)=\begin{cases}0 & \text{, if } 0<x<1\;,\\ >0 & \text{, if } x>1\;.\end{cases} \tag{1.12.19}$$

$\Uparrow$ generalization: variable absorption coefficient: $\sigma_0 \quad\longrightarrow\quad \sigma(x)$

Again, spatial discretization of (1.12.18) requires truncation of spatial domain

Idea: Truncation (mostly) harmless !

✌ outgoing waves decay *exponentially* away from D
(setting $u \leftarrow 0$ has “exponentially small impact”)

➤ “practical” PML system $\leftrightarrow$ (1.12.1)

$$\begin{aligned}\frac{\partial u}{\partial t}(x,t)+\sigma(x)u(x,t)-\frac{\partial v}{\partial x}(x,t) &= v_0\ ,\\ \frac{\partial v}{\partial t}(x,t)+\sigma(x)v(x,t)-c^2(x)\frac{\partial u}{\partial x}(x,t) &= 0\ ,\end{aligned}\quad \text{in }]0,L[\times]0,T[\ ,\quad L>1\ , \tag{1.12.20}$$

$$u(L,t)=0 \quad \text{for } 0<t<T\ ,\quad u(x,0)=u_0(x)\ ,\quad v(x,0)=0\ ,\quad 0<x<L\ .$$

➤ PML reflection coefficient
(no reflections at $x=1$!)

$$\boxed{R_{\mathrm{PML}}=\exp(-2\int_1^L \sigma(x)\,\mathrm{d}x)} \tag{1.12.21}$$

Note: no equivalent 2nd-order wave equation for $\sigma=\sigma(x)$: spatial discretization **?**

➥ Hybrid variational formulation (in space) of system (1.12.20):

seek $u :]0,T[\mapsto H_0^1(]0,L[)$, $v :]0,T[\mapsto L^2(]0,L[)$

$$\begin{aligned}
\int_0^L \frac{\partial u}{\partial t}\, w \,\mathrm{d}x + \int_0^L \sigma(x) u\, w \,\mathrm{d}x + \int_0^L v \frac{\partial w}{\partial x} \,\mathrm{d}x &= \int_0^L v_0\, w \,\mathrm{d}x \quad \forall w \in H_0^1(]0,L[)\,, \\
\int_0^L \frac{\partial v}{\partial t}\, q \,\mathrm{d}x + \int_0^L \sigma(x) v\, q \,\mathrm{d}x - \int_0^L c^2(x) \frac{\partial u}{\partial x}\, q \,\mathrm{d}x &= 0 \quad \forall q \in L^2(]0,L[)\,.
\end{aligned} \tag{1.12.22}$$

(Simplest) spatial Galerkin FE semi-discretization on mesh $\mathcal{M}$ of $]0,L[$:

- $u(t) \rightarrow u_N(t) \in \mathcal{S}_{1,0}^0(\mathcal{M}) \subset H_0^1(]0,L[)$ ($\rightarrow$ Sect. 1.6.3)
- $v(t) \rightarrow v_N(t) \in \mathcal{S}_0^{-1}(\mathcal{M}) \subset L^2(]0,L[)$ $\hat{=}$ p.w. constants on $\mathcal{M}$

Timestepping: semi-explicit trapezoidal rule, *cf.* (1.7.7) ("dissipative" leap frog):

$$\begin{aligned}
\int_0^L \frac{u_N^{(k+1)} - u_N^{(k)}}{\Delta t}\, w_N \,\mathrm{d}x + \int_0^L \sigma \frac{u_N^{(k+1)} + u_N^{(k)}}{2}\, w_N \,\mathrm{d}x + \int_0^L v_N^{(k)} \frac{\partial w_N}{\partial x} \,\mathrm{d}x &= \int_0^L v_0\, w_N \,\mathrm{d}x\,, \\
\int_0^L \frac{v_N^{(k+1)} - v_N^{(k)}}{\Delta t}\, q_N \,\mathrm{d}x + \int_0^L \sigma \frac{v_N^{(k+1)} + v_N^{(k)}}{2}\, q_N \,\mathrm{d}x - \int_0^L c^2 \frac{\partial u_N^{(k+1)}}{\partial x}\, q_N \,\mathrm{d}x &= 0\,.
\end{aligned} \tag{1.12.24}$$

for all $w_N \in \mathcal{S}_{1,0}^0(\mathcal{M})$, $q_N \in \mathcal{S}_0^{-1}(\mathcal{M})$.

Example 26 (Perfectly matched layer in 1D).

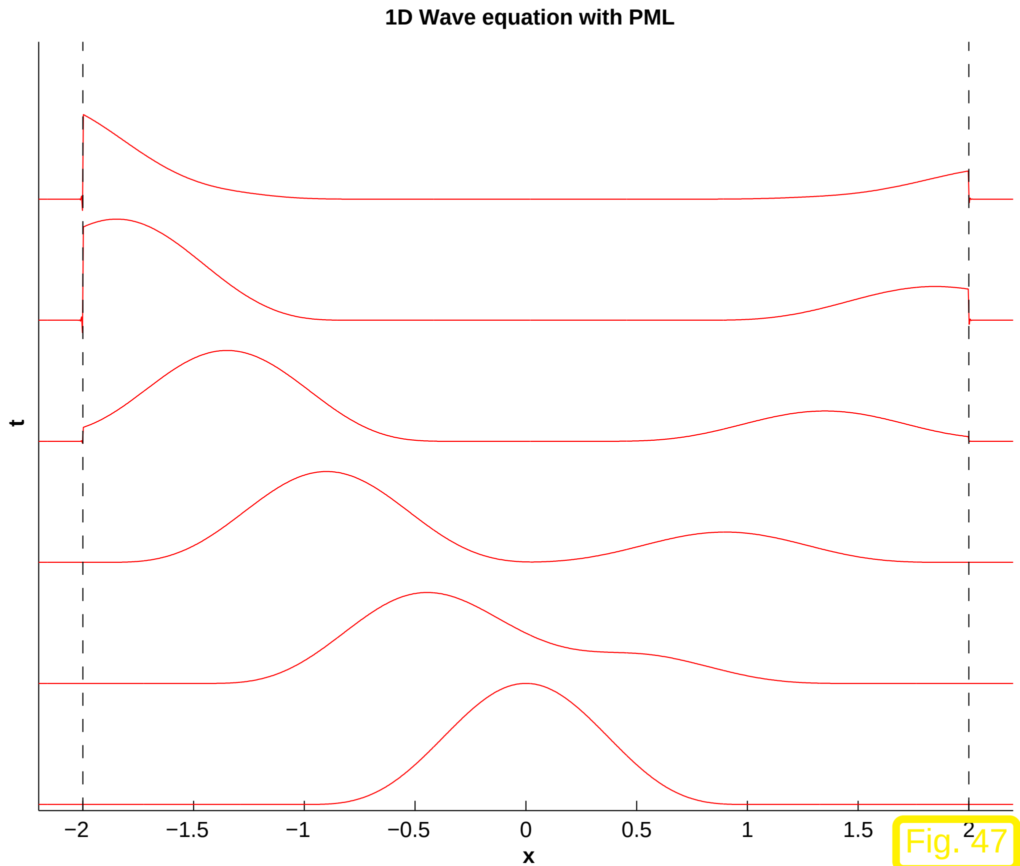
Cauchy problem for 1D wave equation, $c \equiv 2$, interior region $D =]-2, 2[$, $u(x,t)$ as in Ex. 24.

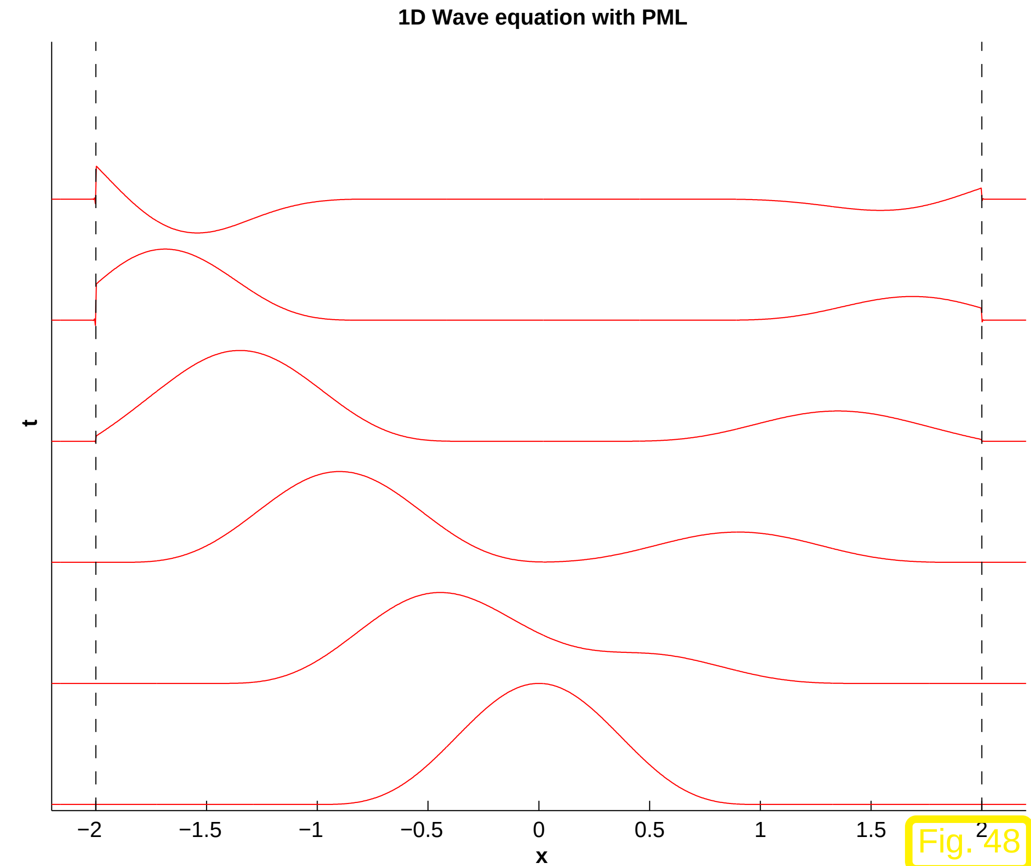
- PML layer: $L = 2.2$ ➢ computational domain $]-2.2, 2.2[$, $\sigma(x) = \sigma_0$ for $2 < |x| < 2.2$, $\sigma(x) = 0$ elsewhere.

- Galerkin (lowest order hybrid mixed) finite element discretization (see above) on equidistant mesh, meshwidth $h = 0.0044$

- uniform dissipative leap frog timestepping (1.12.24), uniform timestep $\Delta t = 1.5 \cdot 10^{-4}$.

Monitored: fully discrete evolution of $u(x,t)$, $-2.2 < x < 2.2$, for different absorption coefficients σ_0





$\sigma_0 = 1000$ $\qquad$ $\sigma_0 = 6000$ ▷ reflections !

➡ movie: $\sigma_0 = 100$, movie: $\sigma_0 = 1400$

Observation: large jump in $\sigma(x)$ ➪ spurious reflections at PML boundary
(artifact of discretization $\longrightarrow$ Sect. 1.10, Ex. 22) ◇

Remark 27 (Practical choice of PML absorption coefficient).

Unless spurious (numerical) reflections interfered:

$\sigma \rightarrow \infty \Rightarrow$ Any thin PML layer already perfectly transparent, *cf.* (1.12.21)

Practice:

- $\sigma'(x)$ small where waves still strong
- $\sigma'(x)$ large where waves already damped

Choice: parabolic profile ▷

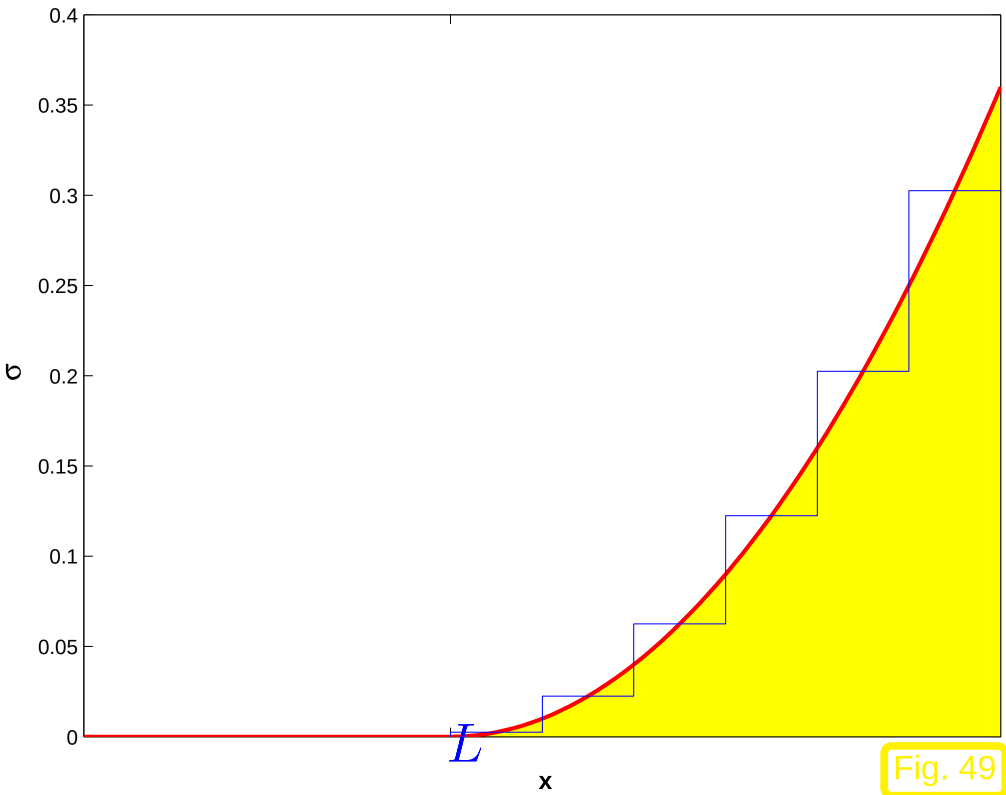


Fig. 49

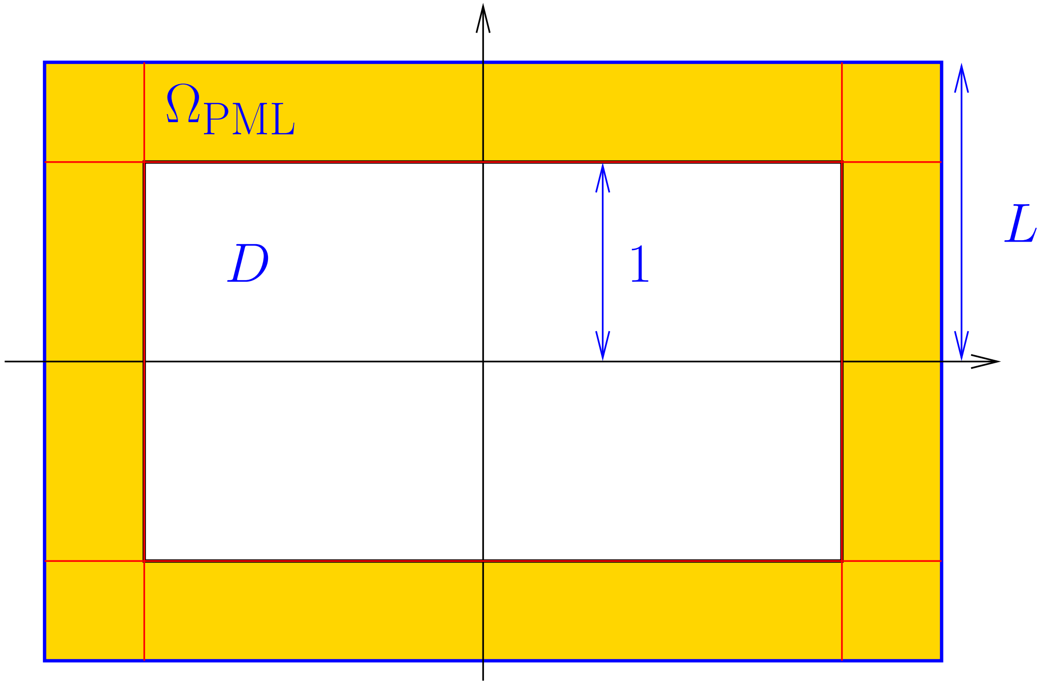
△

(Heuristic) PML in 2D for simple D:

rectangular interior region $\quad D :=]-1,1[^2 \quad \triangleright$

$$\Omega_{\mathrm{PML}} :=]-L,L[^2 \setminus D \;,$$

$L > 1$.



Approach: use 1D PML (1.12.17) (in x_1-direction, x_2-direction, or both) inside Ω_{PML}

Technique: split $\; u = u_1 + u_2 \quad \rightarrow$ split PML

$$\begin{aligned}
\frac{\partial u_1}{\partial t}(\boldsymbol{x},t) + \sigma_1(\boldsymbol{x}) u_1(\boldsymbol{x},t) - \frac{\partial v_1}{\partial x_1}(\boldsymbol{x},t) &= \tfrac{1}{2} v_0 \;, \\
\frac{\partial u_2}{\partial t}(\boldsymbol{x},t) + \sigma_2(\boldsymbol{x}) u_2(\boldsymbol{x},t) - \frac{\partial v_2}{\partial x_2}(\boldsymbol{x},t) &= \tfrac{1}{2} v_0 \;, \quad \text{in } \Omega_{\mathrm{PML}} \times]0,T[\;, \\
\frac{\partial \mathbf{v}}{\partial t}(\boldsymbol{x},t) + \begin{pmatrix} \sigma_1(\boldsymbol{x}) & 0 \\ 0 & \sigma_2(\boldsymbol{x}) \end{pmatrix} \mathbf{v}(\boldsymbol{x},t) - \mathbf{C}(\boldsymbol{x}) \, \mathbf{grad}(u_1+u_2)(\boldsymbol{x},t) &= 0 \;.
\end{aligned}$$

$$u_1(\boldsymbol{x},t) = 0 \quad , \quad u_2(\boldsymbol{x},t) = 0 \quad \text{on } \partial\Omega_{\mathrm{PML}} \times]0,T[\; ,$$
$$u_1(\boldsymbol{x},0) = u_2(\boldsymbol{x},0) = \tfrac{1}{2} u_0(\boldsymbol{x}) \; , \quad \mathbf{v}(\boldsymbol{x},0) = 0 \quad \text{for } \boldsymbol{x} \in \Omega_{\mathrm{PML}} \; .$$

Choice of absorption tensor $\begin{pmatrix} \sigma_1(\boldsymbol{x}) & 0 \\ 0 & \sigma_2(\boldsymbol{x}) \end{pmatrix}$:

$$\sigma_1 > 1 \quad \text{for } 1 < |x_1| < L \; ,$$
$$\sigma_2 > 1 \quad \text{for } 1 < |x_2| < L \; ,$$
$$\sigma_1 \equiv 0 \; , \quad \sigma_2 \equiv 0 \quad \text{elsewhere.}$$

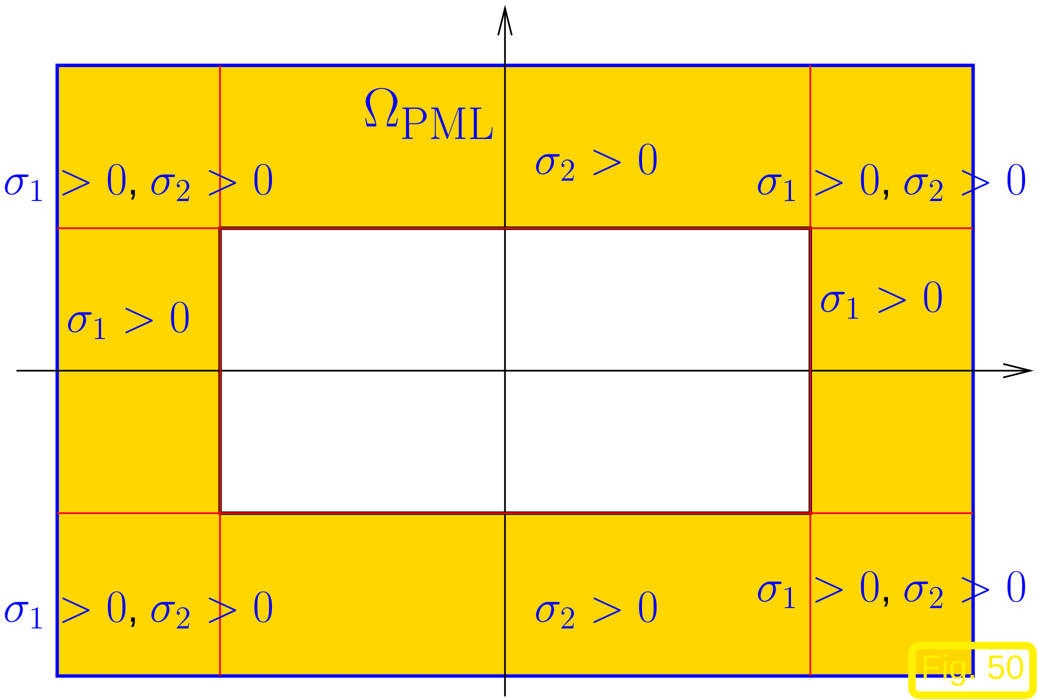


Fig. 50

➤ Discretization: hybrid variational formulation, *cf.* (1.12.22) **+** dissipative leap frog, *cf.* (1.12.24)

Example 28 (Rectangular PML in 2D)*.*

Cauchy problem for $\dfrac{\partial^2 u}{\partial t^2} u - 4\Delta u = 0$ in $\mathbb{R}^2 \times]0,T[$

$$u_0(\boldsymbol{x}) = \begin{cases} (1 - r/r_0)^3 \cdot \exp(-0.0001 r^2) & \text{, if } r < r_0 \; , \\ 0 & \text{, if } r > r_0 \; , \end{cases} \quad r := \left|\boldsymbol{x} - \begin{pmatrix} 0.2 \\ 0.2 \end{pmatrix}\right| \; , \quad \frac{\partial u}{\partial t}(\boldsymbol{x},0) = 0 \; .$$

- interior region $D =]-0.75, 0.75[^2$, computational domain $\Omega_{\mathrm{PML}} =]-1,1[^2$

- Galerkin (lowest order hybrid mixed) FE discretization on "structured" triangular mesh $\longrightarrow$ Fig. 11 ($u_N(t) \in \mathcal{S}^0_{1,0}(\mathcal{M})$ $\longrightarrow$ Sect. 1.6.3, $\mathbf{v}_N$ p.w. constant)

- uniform dissipative leap frog timestepping, timestep $\Delta t = 1.5 \cdot 10^{-3}$

➨ movie: constant $\sigma = 100$, movie: parabolic profile for absorption coefficient

$\diamond$

2 One-dimensional scalar conservation laws

2.1 Conservation laws

- $\Omega \subset \mathbb{R}^d \hat{=}$ fixed (bounded/unbounded) spatial domain ($\Omega = \mathbb{R}^d$ = Cauchy problem)
- computational domain: space-time cylinder $\widetilde{\Omega} := \Omega \times]0,T[$, $T > 0$ final time
- $U \subset \mathbb{R}^m$ ($m \in \mathbb{N}$) $\hat{=}$ phase space (state space) for extensive quantitities u_i (usually $U = \mathbb{R}^m$)

Conservation law for transient state distribution $\mathbf{u} : \widetilde{\Omega} \mapsto U$: $\mathbf{u} = \mathbf{u}(\boldsymbol{x}, t)$
for (almost) all $t \in]0,T[$

$$\frac{d}{dt}\int_V \mathbf{u}\,\mathrm{d}\boldsymbol{x} + \int_{\partial V} \mathbf{F}(\mathbf{u},\boldsymbol{x}) \cdot \mathbf{n}\,\mathrm{d}S(\boldsymbol{x}) = \int_V \mathbf{s}(\mathbf{u},\boldsymbol{x},t)\,\mathrm{d}\boldsymbol{x} \quad \forall \text{ "control volumes" } V \subset \Omega\,. \quad (2.1.1)$$

change of amount inflow/outflow production term

▷ Flux function $\mathbf{F} : U \times \Omega \mapsto \mathbb{R}^{m,d}$:

Assumption: $\boxed{\mathbf{F} \text{ only depends on local state } \mathbf{u}\text{, not on derivatives of } \mathbf{u}!}$

▷ source function $\mathbf{s} : U \times \Omega \times]0,T[\mapsto \mathbb{R}^m$ ($\mathbf{s} = 0 \leftrightarrow$ homogeneous conservation law, will mainly be considered)

► Integral form of (2.1.1):

$$\int_V \mathbf{u}(\boldsymbol{x}, t_1)\,\mathrm{d}\boldsymbol{x} - \int_V \mathbf{u}(\boldsymbol{x}, t_0)\,\mathrm{d}\boldsymbol{x} + \int_{t_0}^{t_1}\int_{\partial V} \mathbf{F}(\mathbf{u}, \boldsymbol{x}) \cdot \boldsymbol{n}\,\mathrm{d}S(\boldsymbol{x})\mathrm{d}t = \int_{t_0}^{t_1}\int_V \mathbf{s}(\mathbf{u}, \boldsymbol{x}, t)\,\mathrm{d}\boldsymbol{x}\mathrm{d}t \qquad (2.1.2)$$

for all $V \subset \Omega$, $0 < t_0 < t_1 < T$, $\boldsymbol{n} \mathrel{\hat{=}}$ exterior unit normal at ∂V

► [Gauss theorem] (local) differential form of (2.1.1):

$$\boxed{\frac{\partial}{\partial t}\mathbf{u} + \mathrm{div}_{\boldsymbol{x}}\mathbf{F}(\mathbf{u}, \boldsymbol{x}) = \mathbf{s}(\mathbf{u}, \boldsymbol{x}, t) \quad \text{in } \widetilde{\Omega}}\ . \qquad (2.1.3)$$

div acting on the rows of matrix $\mathbf{F}$

+ initial condition $\mathbf{u}(\boldsymbol{x}, 0) = \mathbf{u}_0(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega$

Suitable boundary values on $\partial\Omega \times]0,T[$ **?** $\rightarrow$ usually tricky question ($\mathbf{F}$-dependent)

Example 29 (Advection of a density).

Given: (stationary) velocity field $\mathbf{v} : \Omega \mapsto \mathbb{R}^d$, $\mathbf{v} = \mathbf{v}(\boldsymbol{x})$,

density (concentration) $u : \widetilde{\Omega} \mapsto \mathbb{R}$ ($u = u(\boldsymbol{x}, t)$): $\int_V u(\boldsymbol{x}, t)\,\mathrm{d}\boldsymbol{x}$ = mass in $V \subset \Omega$ at time t.

Conservation of mass ➤ (linear) advection equation

$$\int_V u(\boldsymbol{x}, t_1) - u(\boldsymbol{x}, t_0)\,\mathrm{d}\boldsymbol{x} + \int_{t_0}^{t_1} \int_{\partial V} u(\boldsymbol{x}, t)\mathbf{v}(\boldsymbol{x}) \cdot \mathbf{n}\,\mathrm{d}S(\boldsymbol{x})\mathrm{d}t = 0 \quad \forall V \subset \Omega,\ 0 < t_0 < t_1 < T$$

$$\Updownarrow$$

$$\frac{\partial u}{\partial t} + \mathrm{div}_{\boldsymbol{x}}(u\,\mathbf{v}) = 0 \quad \text{in } \widetilde{\Omega}\,. \tag{2.1.4}$$

(2.1.4) = scalar ($m = 1$), linear conservation law with flux function $\mathbf{F}(u, \boldsymbol{x}) = u\,\mathbf{v}(\boldsymbol{x})$
(describes distribution of matter carried along by velocity field $\mathbf{v}$)

Boundary conditions: prescribe $u(\cdot, t)$ at inflow boundary $\Gamma_{\mathrm{in}} := \{\boldsymbol{x} \in \partial\Omega \colon \mathbf{v}(\boldsymbol{x}) \cdot \boldsymbol{n}(\boldsymbol{x}) < 0\}$ ◇

Remark 30 (“Elliptic” flux functions).

If $m = 1$, $\mathbf{F}(u, \boldsymbol{x}) = -\,\mathbf{grad}\, u$ ➨ (2.1.3) becomes parabolic heat equation, *cf.* [27, Sect. 7.1].

☞ if $\mathbf{F}(\mathbf{u}, \boldsymbol{x}) = \mathbf{F}(D\mathbf{u}, \boldsymbol{x})$ ➤ $\mathrm{div}_{\boldsymbol{x}}\, \mathbf{F}(\mathbf{u}, \boldsymbol{x})$ (non-linear) (potentially) elliptic differential operator
$\longrightarrow$ “elliptic flux”/“diffusive flux”

➨ theory and numerical treatment of (non-linear) parabolic evolution problems $\longrightarrow$ [27, Ch. 7]

$\triangle$

$d = 1$, $m = 1$ $\leftrightarrow$ (2.1.3) = one-dimensional scalar conservation law for “density” $u : \widetilde{\Omega} \mapsto \mathbb{R}$

$$\frac{\partial u}{\partial t}(x,t) + \frac{\partial}{\partial x}(f(u(x,t),x)) = s(u(x,t),x,t) \quad \text{in }]\alpha,\beta[\times]0,T[,\ \alpha,\beta \in \mathbb{R}\cup\{\pm\infty\}\ . \tag{2.1.5}$$

Simplest case, *cf.* Ex. 29: constant linear advection:

$$\frac{\partial u}{\partial t}(x,t) + v\frac{\partial u}{\partial x}(x,t) = 0 \quad \text{in }]\alpha,\beta[\times]0,T[\ . \tag{2.1.6}$$

Example 31 (Burgers equation). ($m = 1$, $d = 1$)

$u = u(x,t)$ = velocity of fluid with constant density (confined to "1D container" $\Omega :=]\alpha,\beta[\subset \mathbb{R}$)

➢ flux of linear momentum $f(u) = \frac{1}{2}u \cdot u$ ("momentum u advected by velocity u")

Conservation of linear momentum ($\sim u$): for all $V :=]x_0, x_1[\subset \Omega]$

$$\underbrace{\int_{x_0}^{x_1} u(x,t_1) - u(x,t_0)\,\mathrm{d}x}_{\text{change of momentum in } V} + \underbrace{\int_{t_0}^{t_1} \tfrac{1}{2}u^2(x_1,t) - \tfrac{1}{2}u^2(x_0,t)}_{\text{outflow of momentum}} = 0 \quad \forall 0 < t_0 < t_1 < T$$

$$\Updownarrow$$

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\tfrac{1}{2}u^2\right) = 0 \quad \text{in } \Omega \times]0,T[\,. \tag{2.1.7}$$

(2.1.7) = Burgers equation: homogeneous one-dimensional scalar conservation law, $f(u) = \frac{1}{2}u^2$

boundary conditions: depend on direction of velocity: "$u(\alpha,t) = u_0(t)$, if $u(\alpha,t) > 0$"

◇

Remark 32 (Particle model for Burgers equation).

- particles with velocities $v_i \in \mathbb{R}$ and trajectories $x_i : [0,T] \mapsto \mathbb{R}$, $i \in I \subset \mathbb{N}$.

  no collision ➤ $x_i(t+\Delta t) = x_i(t) + v_i \Delta t, \quad \Delta t > 0$

- size of particle i: $h_i(t) = \mathrm{diam}\{x \in \mathbb{R}\colon |x - x_i(t)| < |x - x_j(t)| \; \forall j \neq i\}$

- perfectly inelastic collisions of particles i and j: i,$j \mapsto k$: $v_k = \dfrac{h_i v_i + h_j v_j}{h_i + h_j}$

reconstruction: $u(x_i(t), t) = v_i$

△

2.2 Characteristics

Focus: Cauchy problem ($\Omega = \mathbb{R}$) for one-dimensional scalar conservation law (2.1.5):

$$
\begin{aligned}
\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u) &= 0 && \text{in } \mathbb{R}\times]0,T[\;, \\
u(x,0) &= u_0(x) && \text{in } \mathbb{R}\;.
\end{aligned}
\qquad (2.2.1)
$$

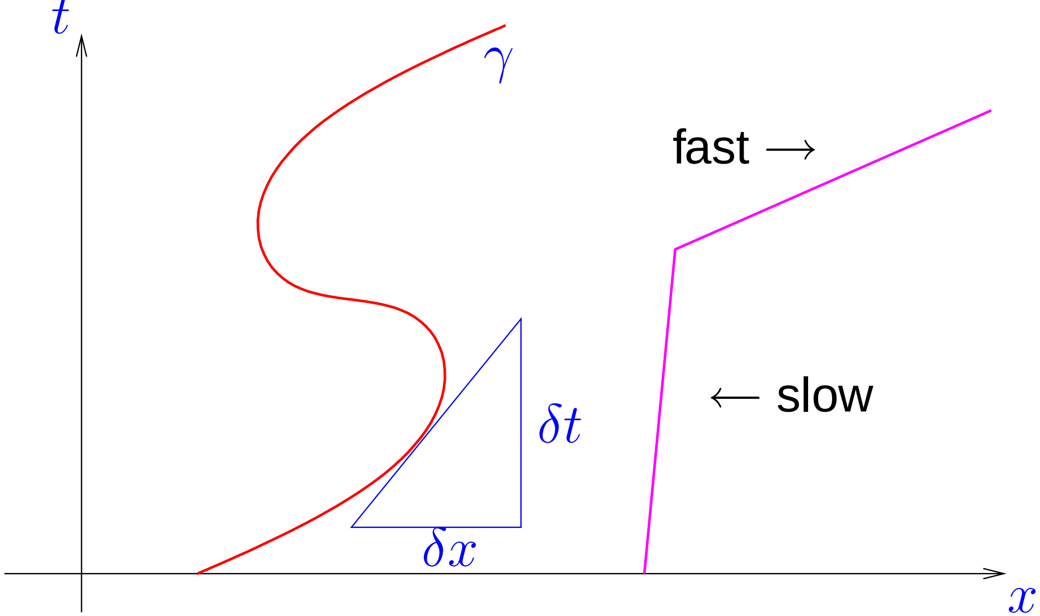
Assumption: flux function $f : \mathbb{R} \mapsto \mathbb{R}$ smooth ($f \in C^2$) and convex

Definition 2.2.1 (Classical solution of Cauchy problem)**.** $u \in C^1(\mathbb{R} \times [0,T])$ *is a classical solution of* (2.2.1)*, if* (2.2.1) *is satisfied pointwise.*

Definition 2.2.2 (Characteristic curve for one-dimensional scalar conservation law)**.**
A curve $\Gamma := (\gamma(\tau), \tau) : [0,T] \mapsto \mathbb{R}\times]0,T[$ *in the* (x,t)*-plane is a characteristic curve of* (2.2.1)*, if*

$$\frac{d}{d\tau}\gamma(\tau) = f'(u(\gamma(\tau),\tau)) \;, \quad 0 \leq \tau \leq T \;, \tag{2.2.2}$$

where u *is a classical solution (*$\longrightarrow$ *Def. 2.2.1) of* (2.2.1)



◁ $x - t$-diagram

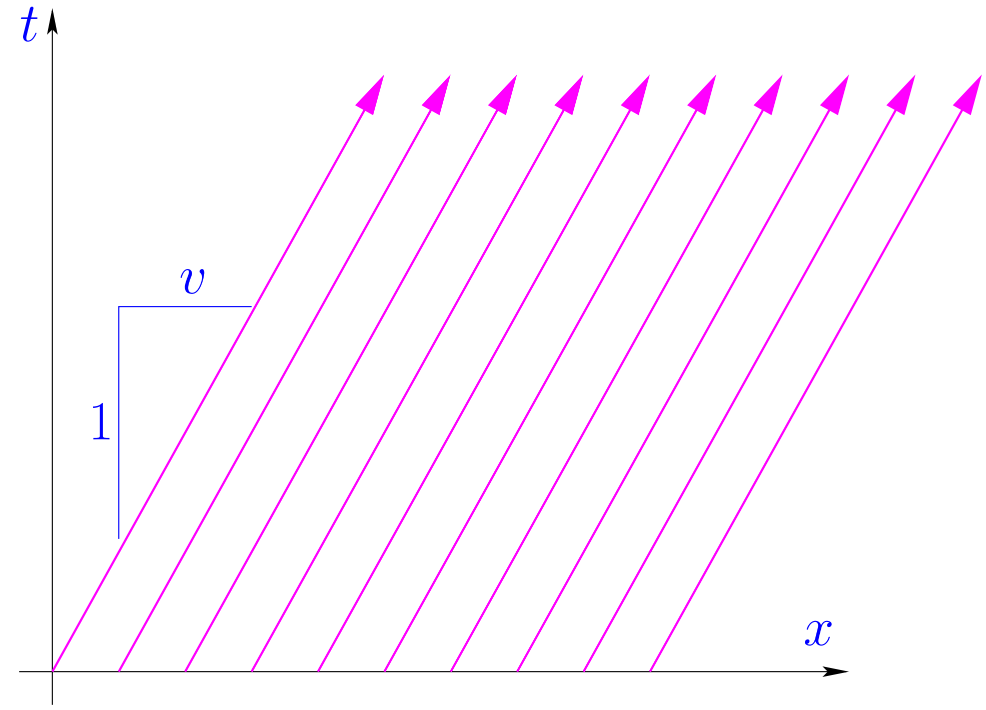
$$\frac{d}{d\tau}\gamma(\tau) = s(x,t)\ .$$

speed of interface γ

Example 33 (Characteristics for advection). Constant linear advection (2.1.6):

➽ characteristics $\gamma(\tau) = v\tau + c$, $c \in \mathbb{R}$.

▷



solution $\quad u(x,t) = u_0(x - vt)$

meaningful for *any* u_0 **!**
(*cf.* Sect. 1.3.2)

◇

Lemma 2.2.3 (Classical solutions and characteristic curves)**.** *Classical solutions of* (2.2.1) *are constant along characteristic curves.*

➤ Characteristic curve through $(x_0, 0)$ **=** straight line $(x_0 + f'(u_0(x_0))\tau, \tau)$, $0 \leq \tau \leq T$ **!**

!? implicit solution formula for (2.2.1) (f' monotone !):

$$u(x,t) = u_0(x - f'(u(x,t))t) \ . \tag{2.2.3}$$

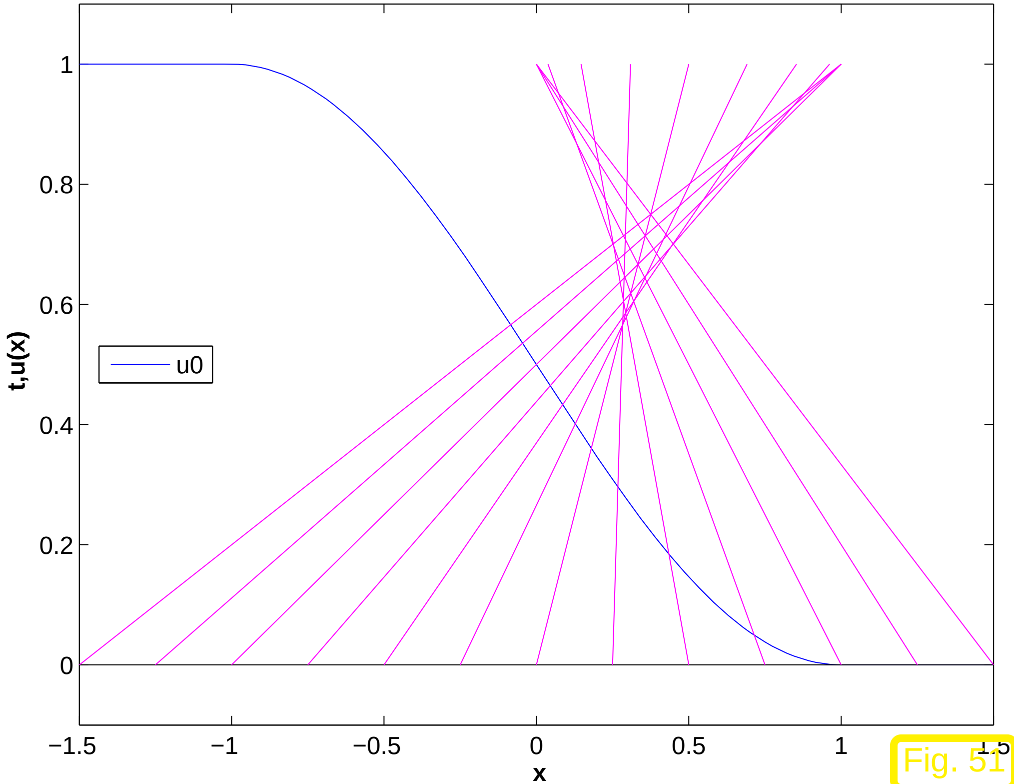


Fig. 51

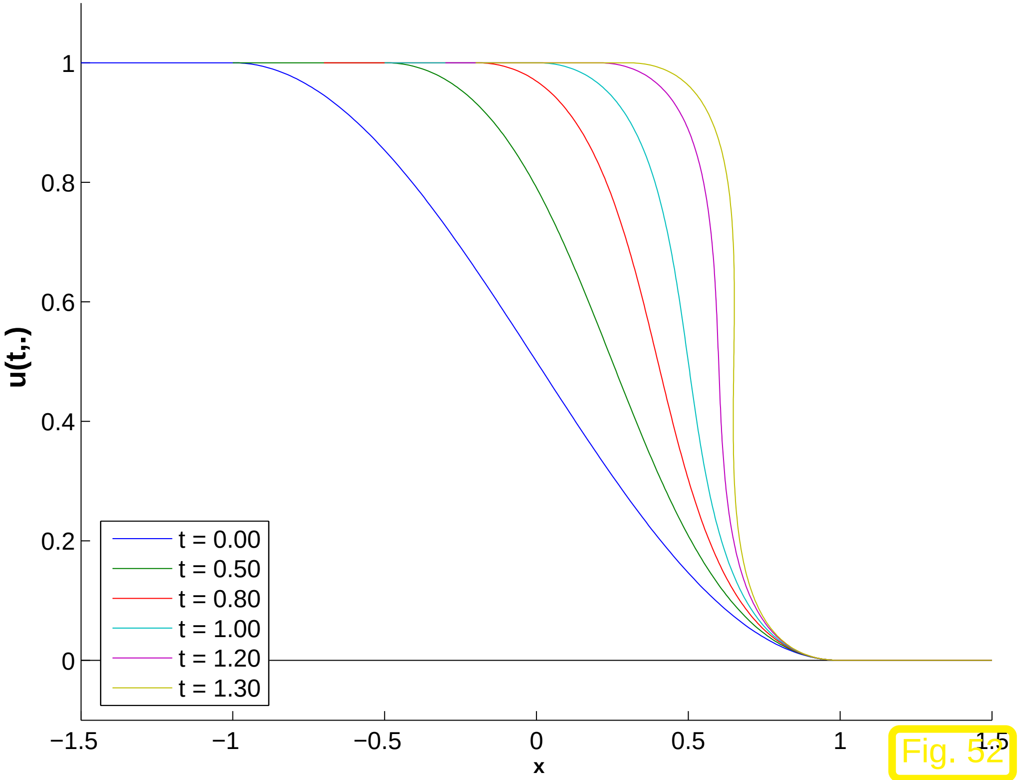
for Burger's equation (2.1.7):
($f(u) = \frac{1}{2}u^2$ smooth and strictly convex)

▷ $f'(u) = u$ (increasing)

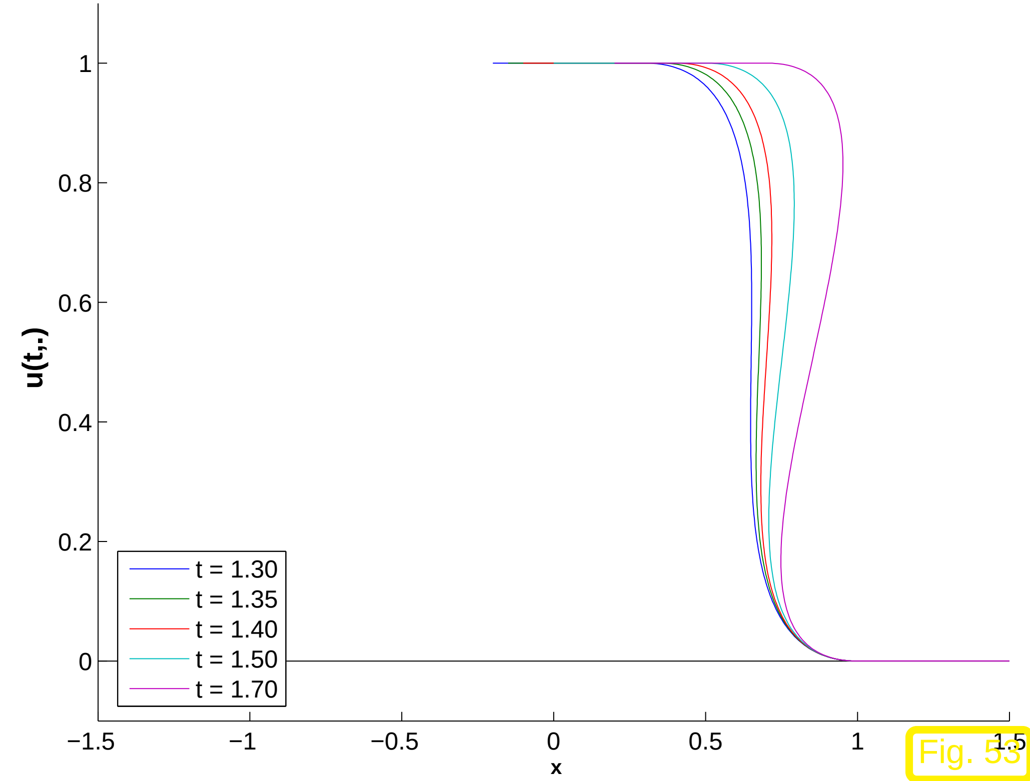
◁ if u_0 smooth and decreasing

➤ characteristic curves intersect **!**

➤ solution formula (2.2.3) becomes invalid



$t < 1.3$: solution by (2.2.3)



the wave breaks: “multivalued solution”

Theorem 2.2.4 (Local in time existence of classical solutions). $\rightarrow$ *[29, Lemma 2.1.2]*

$u_0 \in C^1(\mathbb{R})$, $f \in C^2(\mathbb{R})$ *convex: a classical solution of* (2.2.1) *exists for*

$$0 \leq t < T_\infty := \begin{cases} \infty & \text{, if } \kappa \geq 0 \,, \\ -\kappa^{-1} & \text{, if } \kappa < 0 \end{cases} \quad , \quad \kappa := \inf_{x \in \mathbb{R}} \{ f''(u_0(x)) u_0'(x) \} \,.$$

If $\kappa < 0$, *‘blow-up”* $\left\| \frac{\partial u}{\partial x}(\cdot, t) \right\|_{L^\infty(\mathbb{R})} \rightarrow \infty$ *for* $t \rightarrow T_\infty$.

Proof. T_∞ = earliest time at which characteristic curves intersect, [8, Thm. 6.1.1]. □

breakdown of classical solutions & Ex. 33 ➥ new concept of solution of (2.2.1)

Remark 34. Breakdown of classical solutions even for smooth u_0 = *non-linear* effect (does not occur with (2.1.6)).

△

Example 35 (Solution of particle model for Burgers equation). → Rem. 32

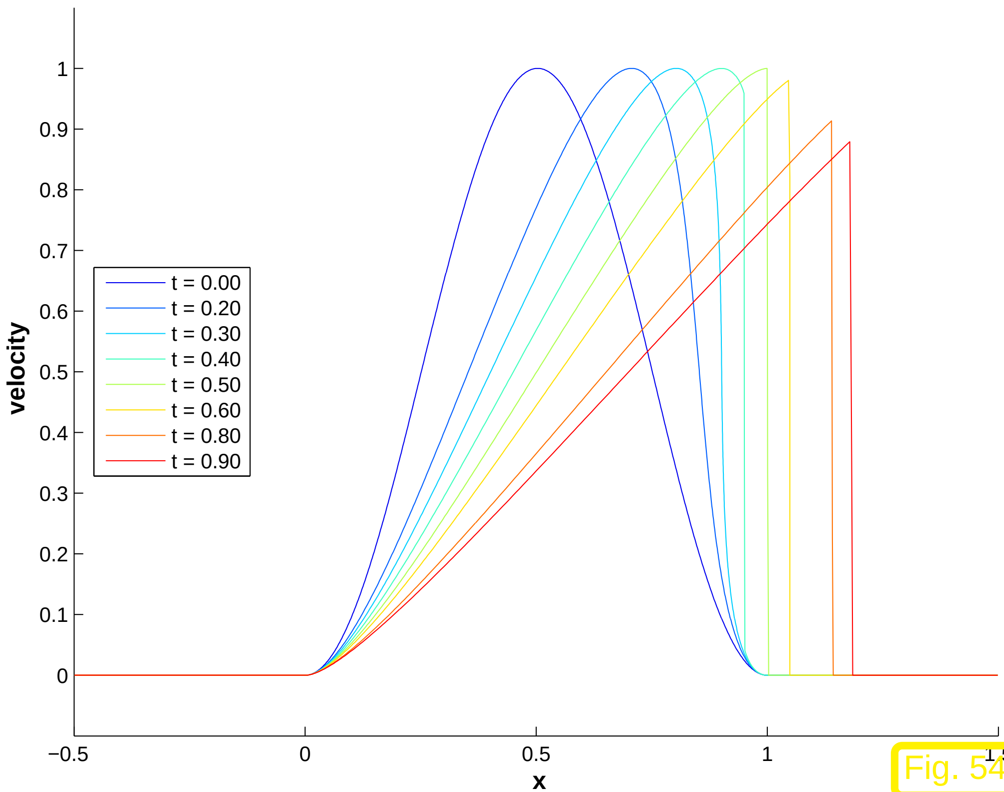


Fig. 54

Cauchy problem for Burgers equation (2.1.7):

◁ $$u_0(x) = \begin{cases} \cos^2 x & \text{for } 0 \leq x \leq 1\,, \\ 0 & \text{elsewhere} \end{cases}$$

Simulation for $T = 1$ based on particle model, 1000 particles, $x_i(0) = -\frac{1}{2} + 2i/1000$, $i = 0, \dots, 999$, $v_i(0) = u_0(x_i(0))$

◁ linear interpolation of $(x_i(t), v_i(t))$, t fixed

movie: evolution of particle solution

2.3 Weak solutions

Idea: weak (distributional) interpretation of partial derivatives in (2.1.3)
$\longrightarrow$ [27, Sect. 2.6], [27, Def. 2.6.1]

Definition 2.3.1 (Weak solution of Cauchy problem for scalar conservation law)**.**
Let $u_0 \in L^\infty(\mathbb{R})$*.* $u : \mathbb{R}\times]0,T[\mapsto \mathbb{R}$ *is a weak solution (solution in the sense of distributions) of the Cauchy problem* (2.2.1)*, if*

$$u \in L^\infty(\mathbb{R}\times]0,T[) \quad \wedge \quad \int_{-\infty}^{\infty}\int_{0}^{T}\left\{u\frac{\partial\Phi}{\partial t} + f(u)\frac{\partial\Phi}{\partial x}\right\}\mathrm{d}t\mathrm{d}x + \int_{-\infty}^{\infty} u_0(x)\Phi(x,0)\,\mathrm{d}x = 0\;,$$

for all $\Phi \in C_0^\infty(\mathbb{R}\times[0,T[)$*.*

u weak solution of (2.2.1) **&** $u \in C^1$ $\Longleftrightarrow$ u classical solution of (2.2.1)

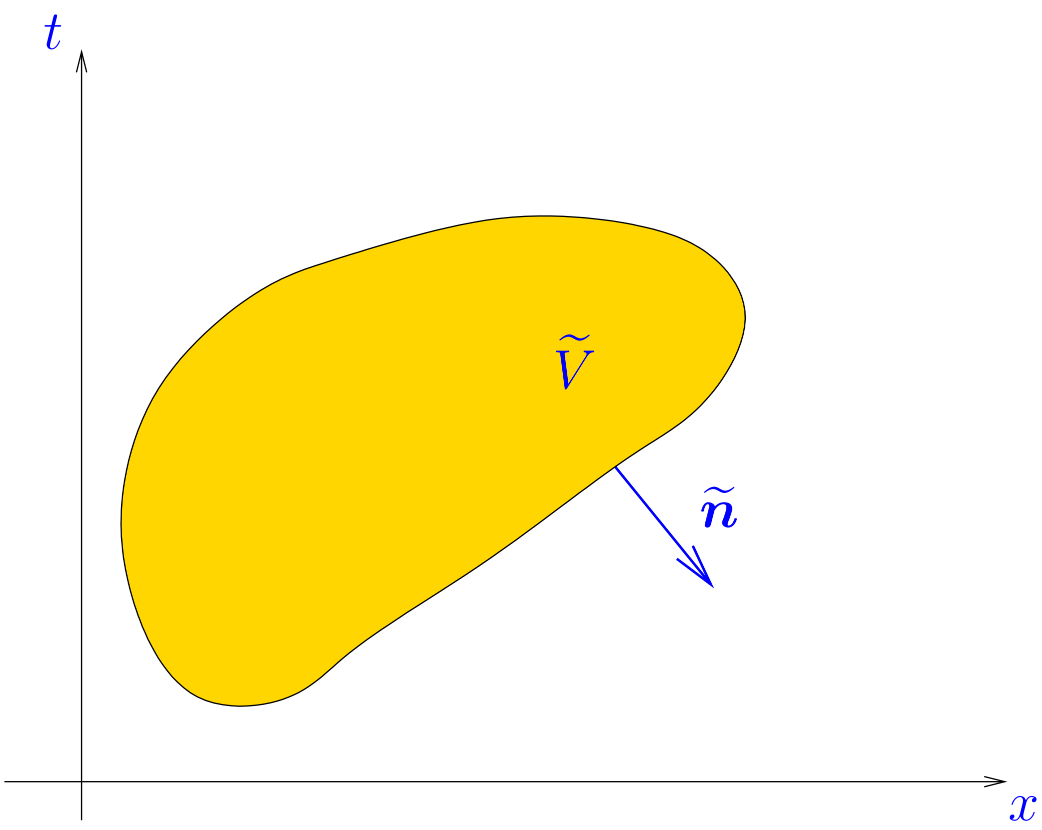
Remark 36*.* $\forall u_0 \in L^\infty(\mathbb{R})$: $u(x,t) = u_0(x - vt)$ = weak solution of Cauchy problem for constant advection $\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial x} = 0 \longrightarrow$ Ex. 33 △

"Space-time Gaussian theorem"

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u) = 0 \qquad (2.3.1)$$

$$\Updownarrow$$

$$\mathrm{div}_{(x,t)} \begin{pmatrix} f(u) \\ u \end{pmatrix} = 0 \quad \text{in } \widetilde{\Omega} \,. \qquad (2.3.2)$$



➤ for any "control volume" $\widetilde{V} \subset \widetilde{\Omega}$:

$$\int_{\partial\widetilde{V}} \begin{pmatrix} f(u(\widetilde{\boldsymbol{x}})) \\ u(\widetilde{\boldsymbol{x}}) \end{pmatrix} \cdot \begin{pmatrix} n_x(\widetilde{\boldsymbol{x}}) \\ n_t(\widetilde{\boldsymbol{x}}) \end{pmatrix} \mathrm{d}S(\widetilde{\boldsymbol{x}}) = 0 \,,$$

$\widetilde{\boldsymbol{n}} := (n_x, n_t)^T \mathrel{\hat{=}}$ space-time unit normal

➤ weak solution of (2.3.1) satisfies (2.3.2) in weak sense $\longrightarrow$ [27, Def. 2.6.1]

(2.3.2) for space-time rectangle $\widetilde{V} =]x_0, x_1[\times]t_0, t_1[$ ➤ integral form of (2.3.1), *cf.* (2.1.2):

$$\int_{x_0}^{x_1} u(x, t_1)\,\mathrm{d}x - \int_{x_0}^{x_1} u(x, t_0)\,\mathrm{d}x = \int_{t_0}^{t_1} f(u(x_0, t))\,\mathrm{d}t - \int_{t_0}^{t_1} f(u(x_1, t))\,\mathrm{d}t \,. \qquad (2.3.3)$$

$u \in L^\infty_{\mathrm{loc}}(\mathbb{R}\times]0, T[)$ weak solution of (2.2.1) $\Rightarrow$ u satisfies integral form (2.3.3) for almost all $x_0 < x_1$, $0 < t_0 < t_1 < T$.

Theorem 2.3.2 (Rankine-Hugoniot jump conditions)**.**
C^1*-curve* $\Gamma := (\gamma(\tau), \tau)$, $0 \leq \tau \leq T$,

$$\widetilde{\Omega}_l := \{(x,t) \in \mathbb{R}\times]0,T[: x < \gamma(t)\} \quad , \quad \widetilde{\Omega}_r := \{(x,t) \in \mathbb{R}\times]0,T[: x > \gamma(t)\} \; .$$

$u \in L^1_{\text{loc}}(\mathbb{R}\times]0,T[)$ *and* $u_{|\widetilde{\Omega}_l}$/$u_{|\widetilde{\Omega}_r}$ *can be extended to* $u_l \in C^1(\overline{\widetilde{\Omega}_l})$, $u_r \in C^1(\overline{\widetilde{\Omega}_r})$ *satisfy* $\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u) = 0$ *in a classical sense* (→ *Def. 2.2.1) in* $\overline{\widetilde{\Omega}_l}$/$\overline{\widetilde{\Omega}_r}$. *Then* u *is a weak solution* (→ *Def. 2.3.1) of* (2.2.1), ***if and only if***

$$\frac{d\gamma}{d\tau}(\tau)\,(u_l(\gamma(\tau),\tau) - u_r(\gamma(\tau),\tau)) = f(u_l(\gamma(\tau),\tau)) - f(u_r(\gamma(\tau),\tau)) \quad \forall 0 < \tau < T \; .$$

Terminology: (2.3.4) **=** Rankine-Hugoniot (jump) condition, shorthand notation:

$$\boxed{\dot{s}(u_l - u_r) = f_l - f_r} \quad , \quad \dot{s} := \frac{d\gamma}{d\tau} \quad \text{“propagation speed of discontinuity”} \tag{2.3.4}$$

Proof. $\longrightarrow$ [29, Lemma 2.1.4]

Existence of weak (space-time) divergence

$\Updownarrow$

"normal continuity" of piecewise smooth vectorfield $(f(u), u)^T$!

➜ *cf.* compatibility condition for Sobolev space $\boldsymbol{H}(\mathrm{div}, \Omega)$, [27, Lemma 2.9.3]

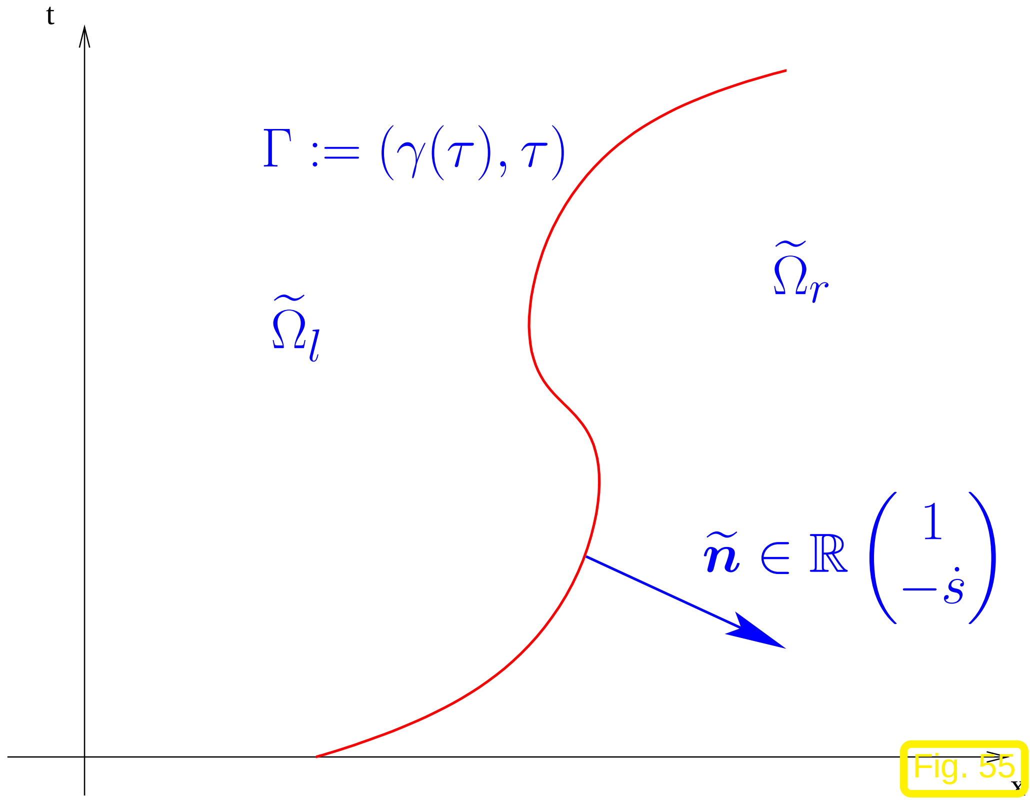


Fig. 55

Remark 37. Thm. 2.3.2 generalizes to partitioning of $\widetilde{\Omega}$ into several 'sub-domains" △

Caution when "manipulating" conservation laws:

Burgers equation $\longrightarrow$ Ex. 31: $$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(\tfrac{1}{2}u^2) = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$$

$\overset{\cdot 2u}{\blacktriangleright}$ $$2u\frac{\partial u}{\partial t} + 2u^2\frac{\partial u}{\partial x} = \frac{\partial}{\partial t}u^2 + \frac{\partial}{\partial x}(\tfrac{2}{3}u^3) = 0 \,.$$

$w := u^2$: Burgers equation (2.1.7) equivalent to

$$\frac{\partial w}{\partial t} + \frac{\partial}{\partial x}\hat{f}(w) = 0 \quad , \quad \hat{f}(w) = \tfrac{2}{3}w^{3/2} \; ? \tag{2.3.5}$$

Discontinuity separating two states $u_l = 1$, $u_r = 0$ Thm. 2.3.2 ➣ speed $\dot{s} = \begin{cases} 1/2 & \text{for (2.1.7)} \; , \\ 2/3 & \text{for (2.3.5)} \; . \end{cases}$

Manipulations involving differentiation (chain rule) may only be valid for classical solutions **!**

2.4 The Riemann problem

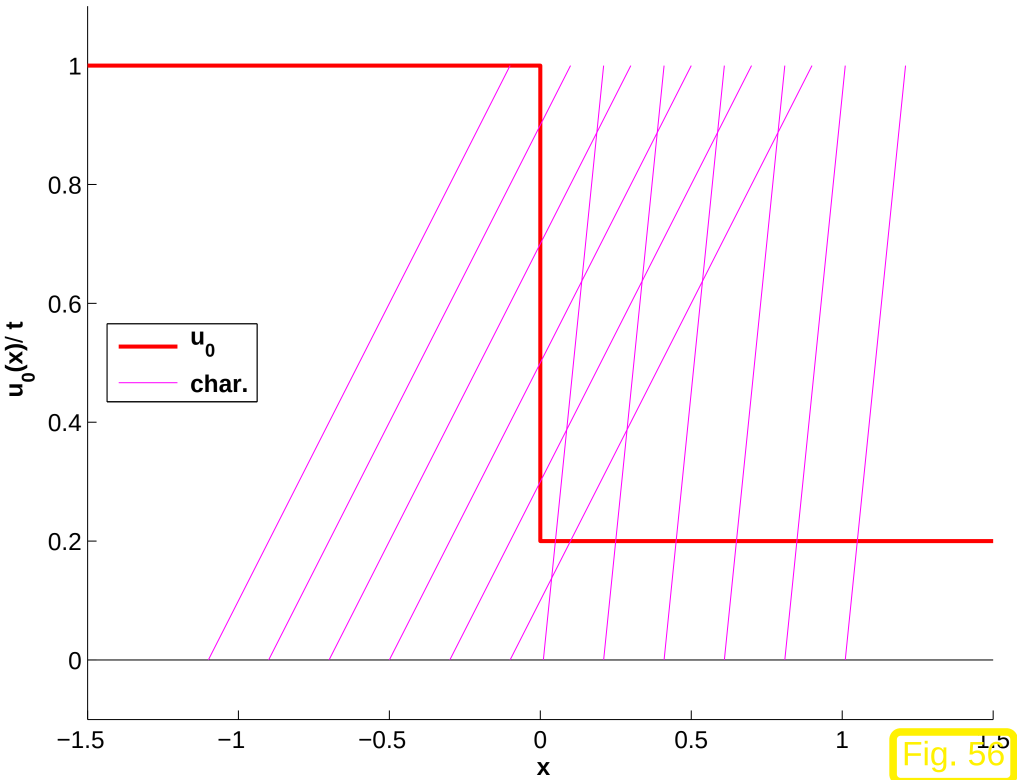
Consider: Cauchy-problem (2.2.1) for piecewise constant initial data u_0

Definition 2.4.1 (Riemann problem)**.**

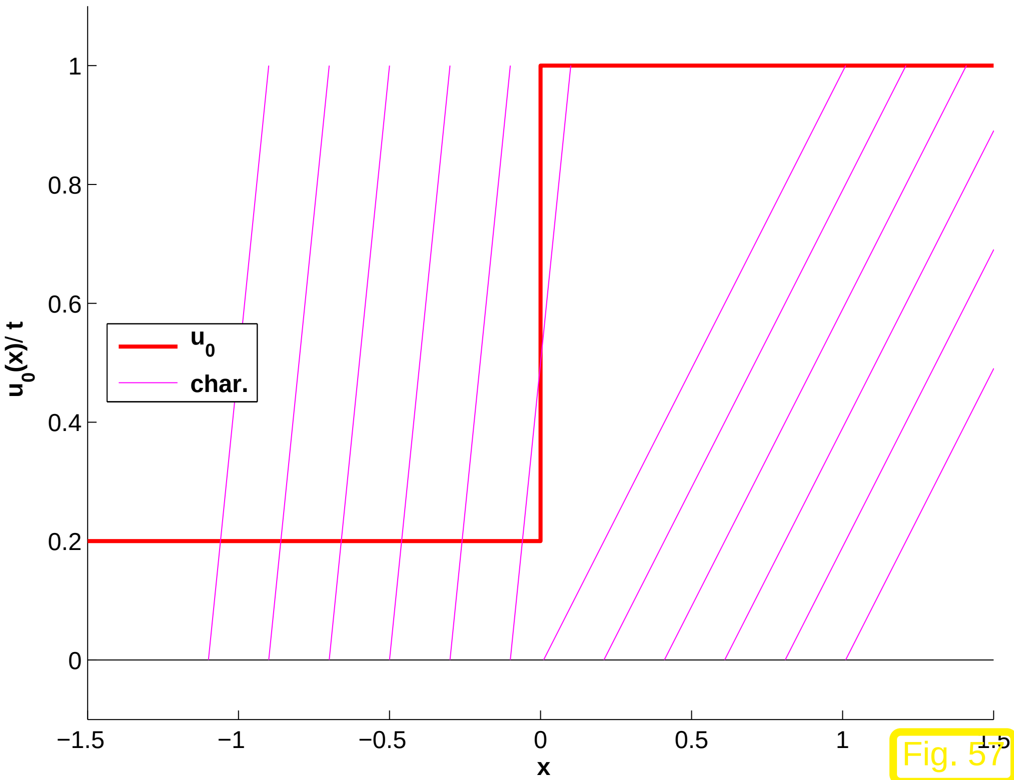
$$u_0(x) = \begin{cases} u_l \in \mathbb{R} & , \text{ if } x < 0 \; , \\ u_r \in \mathbb{R} & , \text{ if } x > 0 \; . \end{cases} \quad \hat{=} \quad \textit{Riemann problem for } (2.2.1)$$

Assumption, *cf.* Sect. 2.2: flux function $f : \mathbb{R} \mapsto \mathbb{R}$ smooth **&** convex

▶ f' non-decreasing ➤ pattern of characteristic curves for Riemann problem:



intersecting characteristics



diverging characteristics

2.4.1 Shocks

Definition 2.4.2 (Shock)**.** *If Γ is a smooth curve in the (x,t)-plane and u a weak solution of* (2.2.1)*, a discontinuity of u across Γ is called a shock.*

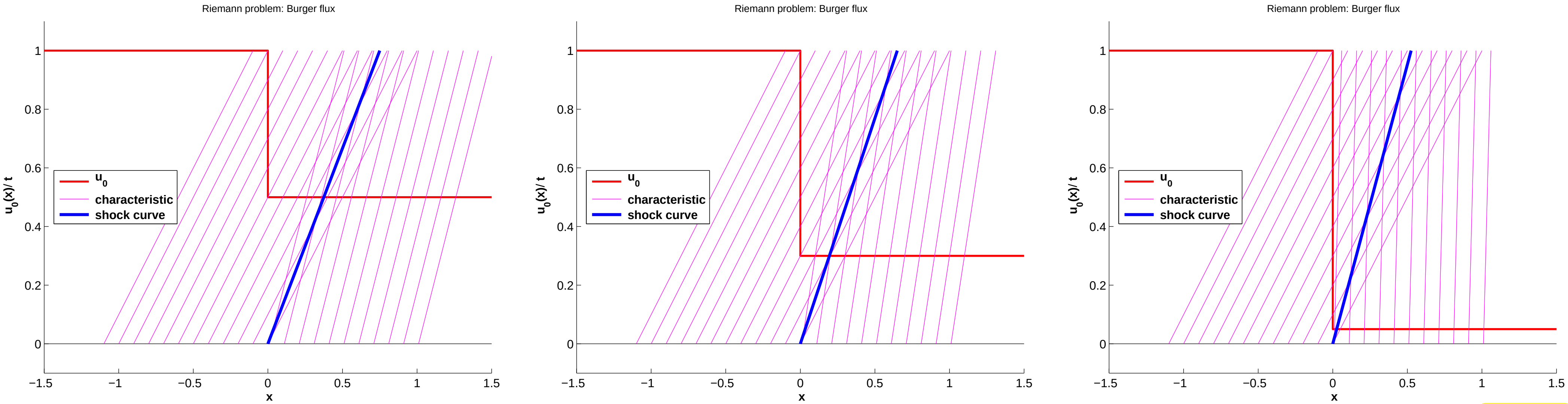
Thm. 2.3.2 ➤ shock speed s $\leftrightarrow$ Rankine-Hugoniot jump conditions:

$$(x_0,t_0)\in\Gamma: \quad \dot{s} = \frac{f(u_l)-f(u_r)}{u_l-u_r} \quad , \qquad \begin{aligned} u_l &:= \lim_{\epsilon\to 0} u(x_0-\epsilon,t_0) \; , \\ u_r &:= \lim_{\epsilon\to 0} u(x_0+\epsilon,t_0) \; . \end{aligned} \tag{2.4.1}$$

Lemma 2.4.3 (Shock solution of Riemann problem)**.**

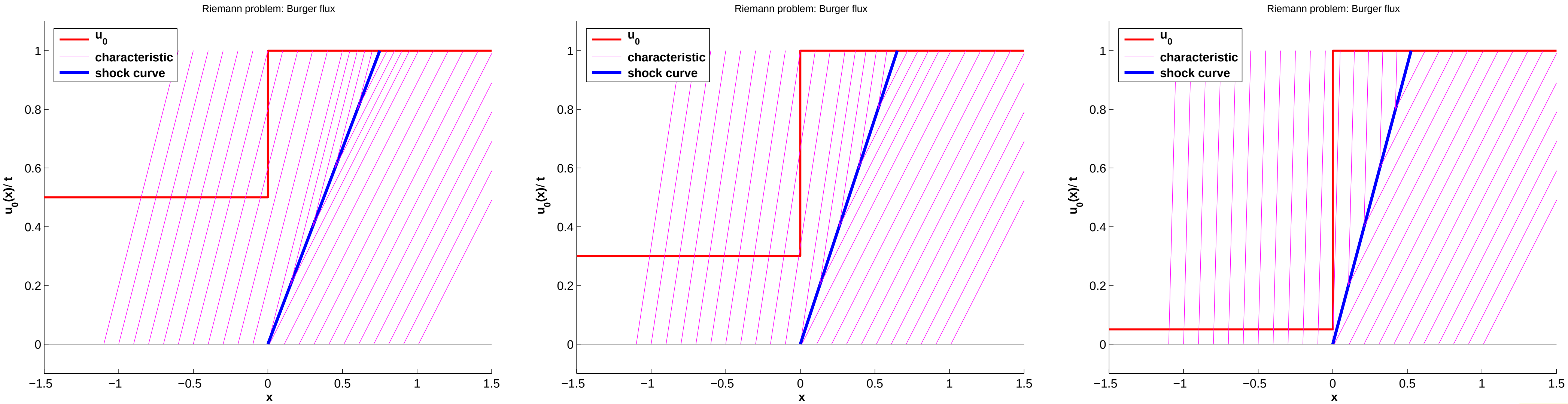
$$u(x,t) = \begin{cases} u_l & \text{for } x < \dot{s}t \; , \\ u_r & \text{for } x > \dot{s}t \; , \end{cases} \quad \dot{s} := \frac{f(u_l)-f(u_r)}{u_l-u_r} \; , \quad x\in\mathbb{R},\ 0<t<T \; ,$$

is weak solution of Riemann problem ($\longrightarrow$ Def. 2.4.1) for (2.2.1)*.*



Burgers flux $f(u) = \frac{1}{2}u^2$, $u_l > u_r$: characteristic curves impinge on shock

Fig. 58



Burgers flux $f(u) = \frac{1}{2}u^2$, $u_l < u_r$: characteristic curves emanate from shock (expansion shock)

Fig. 59

2.4.2 Rarefaction waves

Conservation law (2.3.1) homogeneous in spatial/temporal derivatives:

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u) = 0 \quad \text{in } \mathbb{R} \times \mathbb{R}^+ \quad \Rightarrow \quad \frac{\partial u_\lambda}{\partial t} + \frac{\partial}{\partial x} f(u_\lambda) = 0 \quad \text{in } \mathbb{R} \times \mathbb{R}^+ ,$$

$u_\lambda(x,t) := u(\lambda x, \lambda t)$, $\lambda > 0$.

- try similarity solution: $u(x,t) = \psi(x/t)$

$\leftarrow$ insert in $\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u) = 0$

$$f'(\psi(x/t))\psi'(x/t) = (x/t)\psi'(x/t) \quad \forall x \in \mathbb{R},\ 0 < t < T \ .$$

- $\psi' \equiv 0 \quad \vee \quad f'(\psi(w)) = w \Leftrightarrow \psi(w) = (f')^{-1}(w)$.

f' strictly monotone !

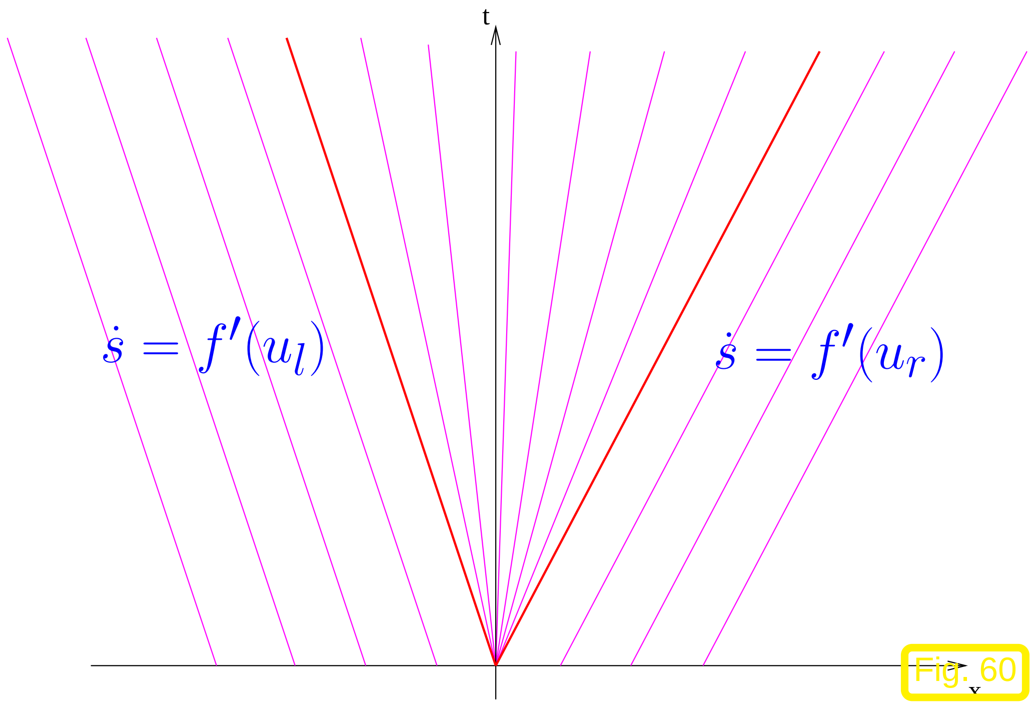


Fig. 60

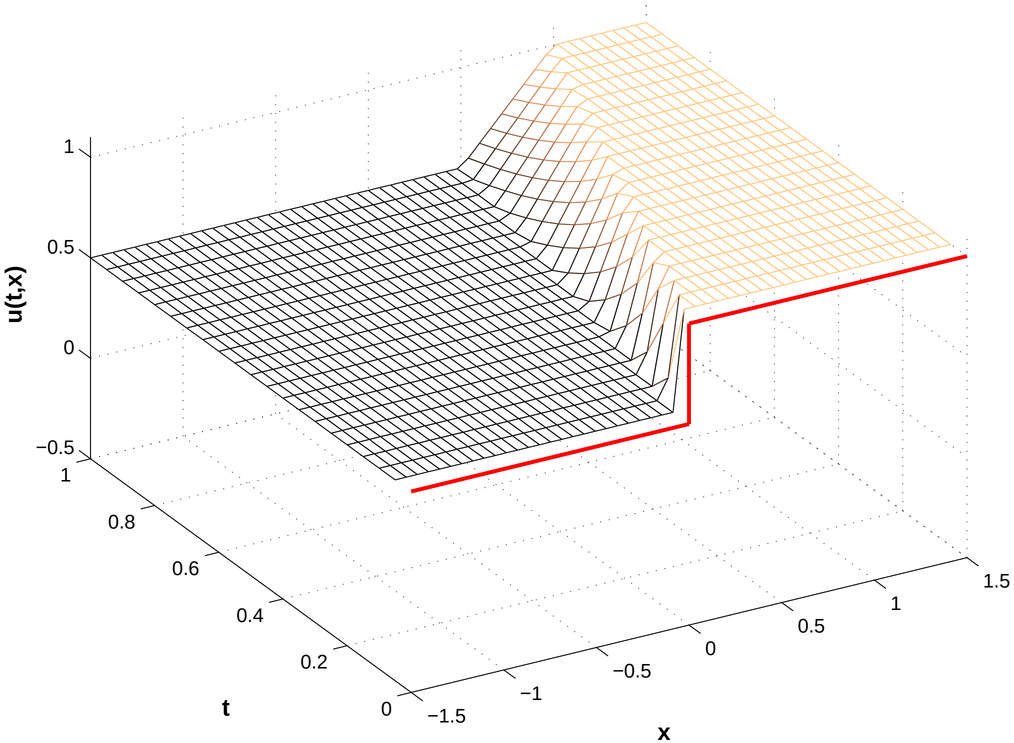
Lemma 2.4.4. ***(Rarefaction solution of Riemann problem)***

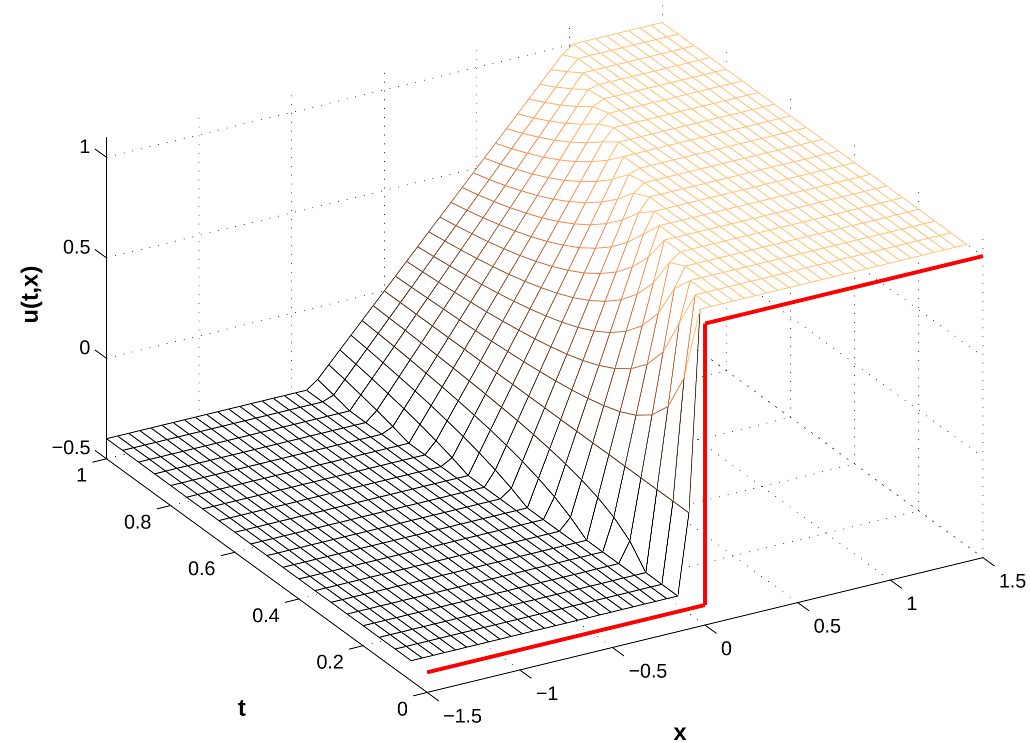
If $f \in C^2(\mathbb{R})$ strictly convex, $u_l < u_r$, then

$$u(x,t) := \begin{cases} u_l & \text{for } x < f'(u_l)t \;, \\ g(\frac{x}{t}) & \text{for } f'(u_l) < \frac{x}{t} < f'(u_r) \;, \\ u_r & \text{for } x > f'(u_r)t \;, \end{cases}$$

$g := (f')^{-1}$, is a weak solution of the Riemann problem ($\rightarrow$ Def. 2.4.1).

Terminology: solution of Lemma 2.4.4 **=** rarefaction wave: *continuous solution* !





Burger flux function $f(u) = \frac{1}{2}u^2$, $u_l < u_r$: rarefaction wave solutions

Remark 38. All weak solutions u of the Riemann problem ($\longrightarrow$ Lemmas 2.4.3, 2.4.4) are similarity solutio
$u(x,t) = \psi(x/t)$ a.e. in $\mathbb{R}\times]0,T[$.

$\triangle$

2.5 Entropy conditions

Sect 2.4 ➤ *Non-uniqueness* of weak solutions:

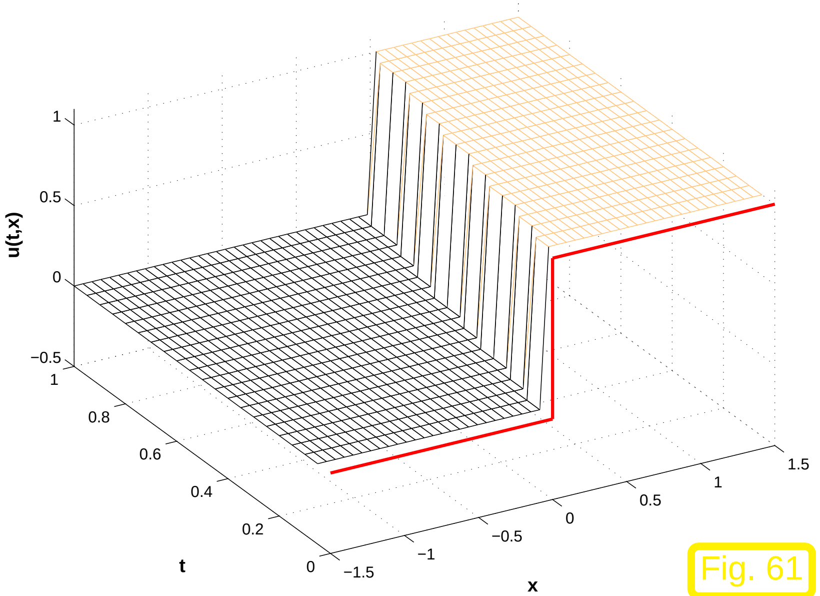


Fig. 61

Riemann solution (Burgers equation):
shock

?

$\longleftrightarrow$

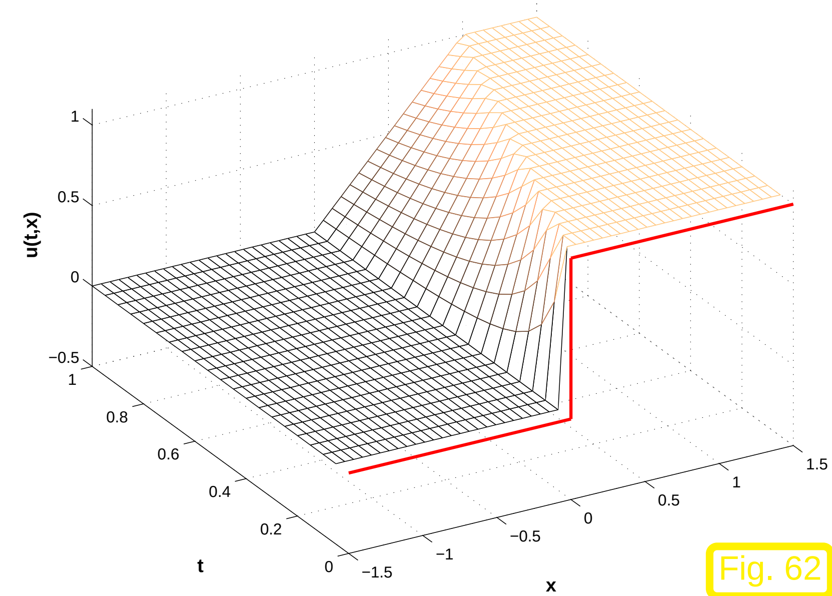


Fig. 62

Riemann solution (Burgers equation):
rarefaction wave

How to select "physically meaningful" **=** admissible solution **?**

Example 39 (Riemann solution by means of particle method). $\rightarrow$ Rem. 32, Ex. 35

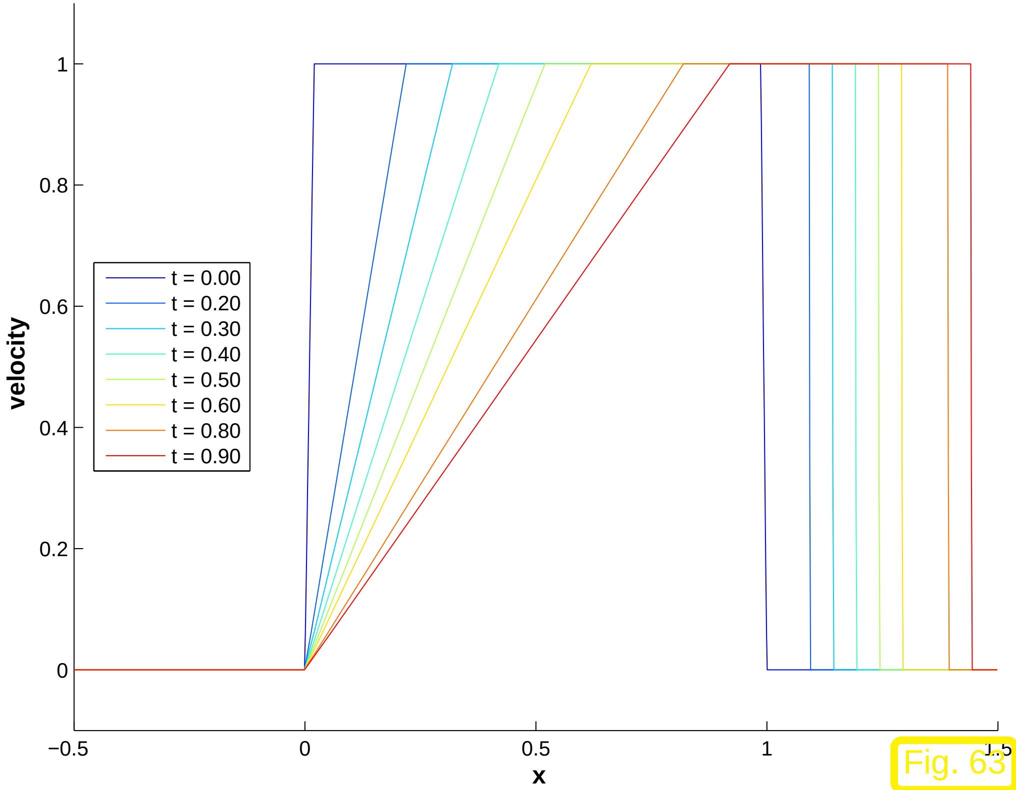


Fig. 63

Cauchy problem for Burgers equation (2.1.7):

◁ $u_0(x) = \max(0, \min(1, 30 - 60 * |x - \frac{1}{2}|))$.

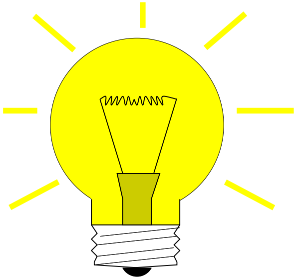
Simulation for $T = 1$ based on particle model, 1000 particles, $x_i(0) = -\frac{1}{2} + 2i/1000$, $i = 0, \ldots, 999$, $v_i(0) = u_0(x_i(0))$

◁ linear interpolation of $(x_i(t), v_i(t))$, t fixed

➨ movie: Riemann solution by particle method

◇

2.5.1 Vanishing viscosity



Idea: conservation law $\leftrightarrow$ limit of extended model for dissipation/friction/viscosity $\to 0$

modelled by elliptic spatial differential operator

Example 40 (Vanishing viscosity for Burgers equation).

Viscous Burgers equation: $$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\tfrac{1}{2}u^2\right) = \epsilon \frac{\partial^2 u}{\partial x^2} \,. \qquad (2.5.1)$$

dissipative term

Travelling wave solution of Riemann problem for (2.5.1) via Cole-Hopf transform $\longrightarrow$ [14, Sect. 4.4.1]

$$u_\epsilon(x,t) = w(x - \dot{s}t) \quad , \quad w(\xi) = u_r + \tfrac{1}{2}(u_l - u_r)\left(1 - \tanh\left(\frac{\xi(u_l - u_r)}{4\epsilon}\right)\right) \;, \quad \dot{s} = \tfrac{1}{2}(u_l + u_r) \;.$$

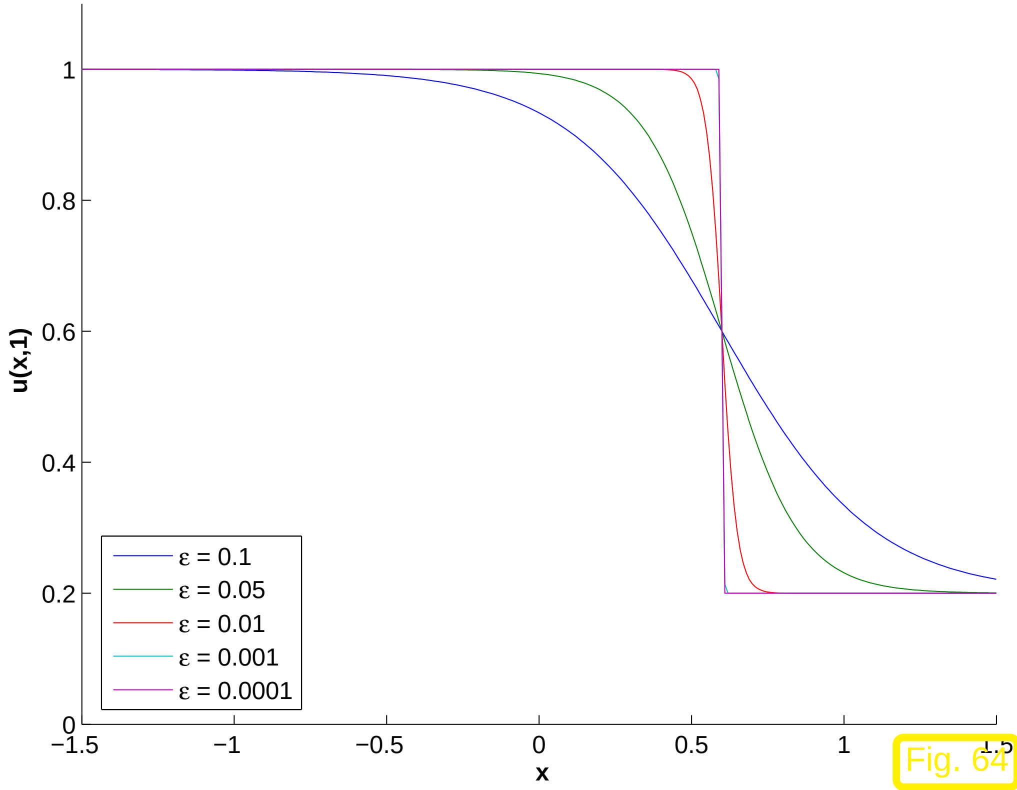


Fig. 64

$u_\epsilon(x,t)$ = classical solution of (2.5.1) for all $t > 0$, $x \in \mathbb{R}$ (only for $u_l > u_r$!).

$\lhd$ $u_l > u_r$, $t = 0.5$

emerging shock for $\epsilon \rightarrow 0$

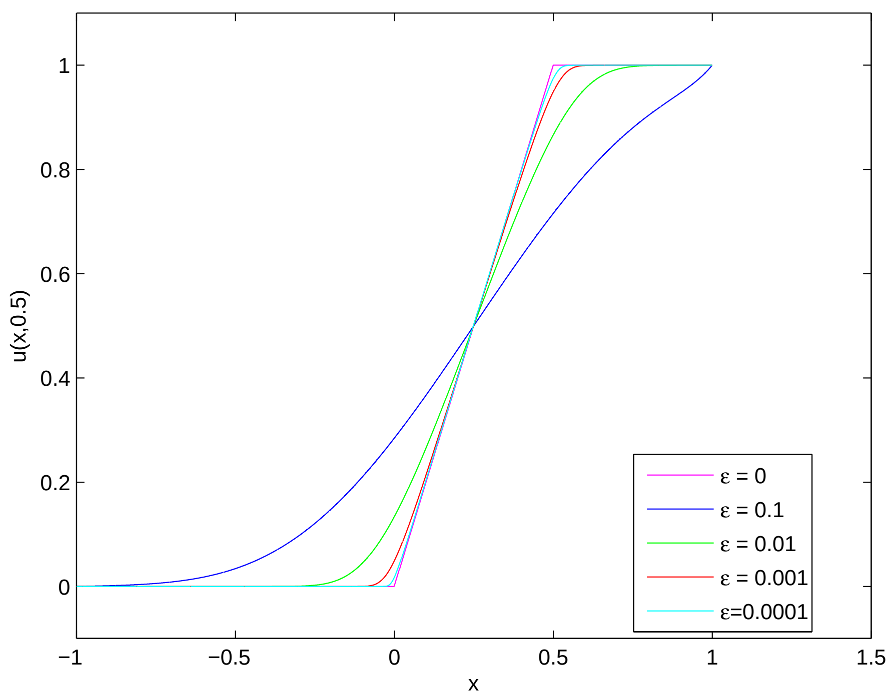
$u_\epsilon \rightarrow u$ from Lemma 2.4.3 in $L^\infty(\mathbb{R})$.

Highly accurate numerical solution $u_\epsilon(x, 0.5)$ of Riemann problem for (2.5.1) ▷

$u_l < u_r$ ▷

emerging rarefaction wave as $\epsilon \longrightarrow 0$

$u_\epsilon \longrightarrow u$ from Lemma 2.4.4 a.e.



◇

Generalization: one-dimensional scalar conservation law with dissipative term:

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u) = \epsilon \frac{\partial^2 u}{\partial x^2} , \quad \epsilon > 0 \; . \tag{2.5.2}$$

(2.5.2) **=** quasi-linear parabolic evolution problem (linear principal part)

► Existence & uniqueness of classical solutions of Cauchy problem $\forall t > 0$ **!**

Theorem 2.5.1 (Vanishing viscosity solution). $\longrightarrow$ *[29, Thm. 2.1.7]*

If $u_0 \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$, $f \in C^2(\mathbb{R})$, f'' bounded, then

- *for any $\epsilon > 0$ $\exists$ classical solution $u_\epsilon \in C^2(\mathbb{R} \times \mathbb{R}^+)$ of the Cauchy problem for* (2.5.2),
- *$u_\epsilon \to u$ a.e. in $\mathbb{R} \times \mathbb{R}^+$, where the viscosity solution u is a weak solution of the Cauchy problem* (2.2.1),
- $\exists C > 0$: $\quad \left\| \frac{\partial}{\partial x} u_\epsilon \right\|_{L^\infty(\mathbb{R})} (\cdot, t) \leq C\epsilon^{-1/2} \quad \forall t > 0$

▶ existence of weak solutions of (2.2.1) !

2.5.2 Entropies

Definition 2.5.2 (Pair of entropy functions).

$\eta, \psi \in C^2(\mathbb{R})$ **=** *pair of entropy functions ($\eta \mathrel{\hat{=}}$ entropy, $\psi \mathrel{\hat{=}}$ entropy flux) for conservation law* $\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u) = 0$, *if*

$$\eta \text{ is strictly convex} \quad \text{and} \quad \psi'(w) = \eta'(w) f'(w) \quad \text{for all } w \in \mathbb{R}.$$

Motivation: for pair (η, ψ) of entropy functions & solutions u_ϵ from Thm. 2.5.1

$$\int_{t_0}^{t_1}\int_{x_0}^{x_1} \frac{\partial}{\partial t}\eta(u_\epsilon(x,t)) + \frac{\partial}{\partial x}\psi(u_\epsilon(x,t))\,\mathrm{d}t =$$

$$\underbrace{\epsilon\int_{t_0}^{t_1} \eta'(u_\epsilon(x_1,t))\frac{\partial u_\epsilon}{\partial x}(x_1,t) - \eta'(u_\epsilon(x_0,t))\frac{\partial u_\epsilon}{\partial x}(x_0,t)\,\mathrm{d}t}_{\to 0 \text{ for } \epsilon\to 0} - \underbrace{\epsilon\int_{t_0}^{t_1}\int_{x_0}^{x_1} \overset{\geq 0}{\eta''(u_\epsilon)}\left(\frac{\partial u_\epsilon}{\partial x}\right)^2 \mathrm{d}x\mathrm{d}t}_{\text{bounded for } \epsilon\to 0}\ .$$

viscosity solution $u := \lim_{\epsilon\to 0} u_\epsilon$ of Cauchy problem ($\to$ Thm. 2.5.1) satisfies

$$\int_{t_0}^{t_1}\int_{x_0}^{x_1} \frac{\partial}{\partial t}\eta(u(x,t)) + \frac{\partial}{\partial x}\psi(u(x,t))\,\mathrm{d}x\mathrm{d}t \begin{cases} = 0 & \text{, if } u \text{ smooth in space ,} \\ \leq 0 & \text{, if } u \text{ non-smooth.} \end{cases} \tag{2.5.3}$$

- if $\psi(0) = 0$, $u(\cdot,t)$ compactly supported $\forall t > 0$ $\Rightarrow$ total entropy $\int_{-\infty}^{\infty}\eta(u)\,\mathrm{d}x$ non-increasing in time ($\to$ name !)

How to find pairs of entropy functions **?** Easy, there are infinitely many **!**

$$\eta \in C^2(\mathbb{R})\ , \quad \eta'' > 0 \quad , \quad \psi(w) = \int_0^w \eta'(\xi)f'(\xi)\,\mathrm{d}\xi \quad \Rightarrow \quad (\eta,\psi)$$ = pair of entropy functions.

Definition 2.5.3 (Weak entropy inequality). *For* $\eta, \psi \in C^2(\mathbb{R})$, $u \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$ *satisfies the entropy inequality*

$$\frac{\partial}{\partial t}\eta(u(x,t)) + \frac{\partial}{\partial x}\psi(u(x,t)) \leq 0 \quad \textit{in } \mathbb{R}\times]0,T[$$

weakly, if

$$\int_{-\infty}^{\infty}\int_{0}^{T} \eta(u(x,t))\frac{\partial\Phi}{\partial t} + \psi(u(x,t))\frac{\partial\Phi}{\partial x}\,\mathrm{d}t\mathrm{d}x \geq 0 \quad \forall\Phi \in C_0^\infty(\mathbb{R}\times]0,T[),\ \Phi \geq 0\ .$$

☞ u weak solution of the Cauchy problem (2.2.1) and
u satisfies weak entropy inequality ($\longrightarrow$ Def. 2.5.3)
for **any** pair of entropy functions (η, ψ) ($\longrightarrow$ Def. 2.5.2)
➤ u = entropy solution

Theorem 2.5.4 (Uniqueness of entropy solutions).

Entropy solutions of (2.2.1) *are unique.*

(For Lipschitz-continous flux function $f : \mathbb{R} \mapsto \mathbb{R}$ *and each* $u_0 \in L^\infty(\mathbb{R})$ *there exists a unique entropy solution* $u \in C^0(]0,T[; L^1_{\mathrm{loc}}(\mathbb{R}))$ *of* (2.2.1) $\longrightarrow$ *[8, Thm. 6.2.1])*

In special cases existence of a *single* entropy pair ($\longrightarrow$ Def. 2.5.2) already characterizes the entropy solution:

Theorem 2.5.5 (Single pair entropy condition)**.** *[15, Thm. 3.4], [11]*
If f *is* strictly convex/concave*, then a piecewise smooth solution of* (2.2.1) *satisfies a weak entropy inequality (*$\longrightarrow$ *Def. 2.5.3) for all pairs of entropy functions (*$\longrightarrow$ *Def. 2.5.2), if it is satisfied for a particular pair.*

2.5.3 Lax entropy condition

Consider setting of Thm. 2.3.2: u p.w. smooth weak solution with discontinuity along curve $\Gamma := (\gamma(\tau), \tau)$ in (x,t)-plane

$$u \text{ entropy solution} \quad \Leftrightarrow \quad \dot{s}(\eta(u_r) - \eta(u_l)) \geq \psi(u_r) - \psi(u_l), \quad \dot{s} := \frac{d\gamma}{d\tau}. \tag{2.5.4}$$

Example 41 (Entropy violating shock for Burgers equation).

Pair of entropy functions: $\eta(w) = w^2 \quad , \quad \psi(w) = \frac{2}{3}w^3$

$$(2.5.4) \quad \Leftrightarrow \quad \tfrac{1}{2}(u_l + u_r)(u_r^2 - u_l^2) \geq \tfrac{2}{3}(u_r^3 - u_l^3) \quad \Leftrightarrow \quad (u_l - u_r)^3 \geq 0 \; .$$

- $u_l > u_r$ ➤ (compression) shock complies with entropy inequality $\to$ Fig. 58.
  $u_l < u_r$ ➤ (expansion) shock violates entropy inequality $\to$ Fig. 59

$\diamond$

Lemma 2.5.6 (Jump conditions for entropy solutions). $\to$ *[29, Thm. 2.1.12]*
For C^1-curve. $\Gamma := (\gamma(\tau), \tau)$, $0 \leq \tau \leq T$, let u be a weak solution of (2.2.1) *(with convex flux function $f \in C^2(\mathbb{R})$) that is piecewise smooth and bounded outside Γ.*
For a pair of entropy functions (η, ψ) ($\to$ Def. 2.5.2) we assume $\frac{\partial}{\partial t}\eta(u) + \frac{\partial}{\partial x}\psi(u) \leq 0$ weakly ($\to$ Def. 2.5.3). Then across Γ (notations $\to$ (2.4.1)*)*

$$f'(u_l) > \dot{s} > f'(u_r) \quad , \quad \dot{s} := \frac{d\gamma}{d\tau} \; .$$

Proof. ➔ proof of Rankine-Hugoniot jump conditions, Thm. 2.3.2 $\square$

Definition 2.5.7 (Lax entropy condition)**.**
$u \hat{=}$ *weak solution of* (2.2.1), *piecewise classical solution in a neigborhood of* C^2*-curve* $\Gamma := (\gamma(\tau), \tau)$, $0 \leq \tau \leq T$, *discontinuous across* Γ.

u *satisfies the Lax entropy condition in* $(x_0, t_0) \in \Gamma$ $:\Leftrightarrow$ $f'(u_l) > \dot{s} := \dfrac{f(u_l) - f(u_r)}{u_l - u_r} > f'(u_r)$.

$\Updownarrow$

Characteristic curves must not emanate from shock $\leftrightarrow$ no "generation of information"

Parlance: shock satisfying Lax entropy condition **=** physical shock

Note: f' increasing ➤ Lemma 2.5.6: necessary for physical shock $\boxed{u_l > u_r}$

Remark 42. For *concave* f: reduction to the case of convex f by $x \leftrightarrow -x$ (swapping of u_l/u_r) △

Theorem 2.5.8 (Equivalence of entropy conditions)**.**
For piecewise classical solution u *of the Cauchy problem* (2.2.1) *on* $\mathbb{R}\times]0, T[$.

u *entropy solution* $\Leftrightarrow$ *Lax entropy condition* ($\longrightarrow$ *Def. 2.5.7*) *holds a.e. on discontinuities.*

Remark 43 (General entropy solution for 1D scalar Riemann problem). $\rightarrow$ [36]

Entropy solution of Riemann problem ($\rightarrow$ Def. 2.4.1) for (2.2.1) with arbitrary $f \in C^1(\mathbb{R})$:

$$u(x,t) = \psi(x/t) \quad , \quad \psi(\xi) := \begin{cases} \underset{u_l \leq u \leq u_r}{\operatorname{argmin}} \, (f(u) - \xi u) & \text{, if } u_l < u_r \; , \\ \underset{u_r \leq u \leq u_l}{\operatorname{argmax}} \, (f(u) - \xi u) & \text{, if } u_l \geq u_r \; . \end{cases} \tag{2.5.5}$$

△

Remark 44 (Oleinik's entropy condition).

For general flux function f (neither convex nor concave):

➤ role of Lax entropy condition ($\rightarrow$ Def. 2.5.7 is played by the Oleinik entropy condition:

$$\frac{f(u) - f(u_l)}{u - u_l} \leq \dot{s} \leq \frac{f(u) - f(u_r)}{u - u_r} \quad \forall \min\{u_l, u_r\} < u < \max\{u_l, u_r\} \; , \tag{2.5.6}$$

locally at discontinuity connecting states u_l, u_r.

△

2.6 Properties of entropy solutions

Setting: $u \in L^\infty(\mathbb{R}\times]0,T[)$ (weak $\longrightarrow$ Def. 2.3.1) entropy solutions $\longrightarrow$ Def. 2.5.3 of Cauchy problem

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u) = 0 \quad \text{in } \mathbb{R}\times]0,T[\quad , \quad u(\cdot,0) = u_0 \in L^\infty(\mathbb{R}) \ , \tag{2.2.1}$$

with flux function $f \in C^1(\mathbb{R})$ (not necessarily convex/concave).

$\bar{u} \in L^\infty(\mathbb{R}\times]0,T[) \hat{=}$ entropy solution w.r.t. initial data $\bar{u}_0 \in L^\infty(\mathbb{R})$.

2.6.1 Stability

Notation: $\xi^+ := \max\{\xi, 0\}$ for $\xi \in \mathbb{R}$.

Lemma 2.6.1. $\longrightarrow$ *[8, Thm. 6.2.2]. There is* $\dot{s} > 0$ *such that for all* $t \in]0,T[$, $R > 0$

$$\int_{|x|<R} (u(x,t) - \bar{u}(x,t))^+ \, \mathrm{d}x \leq \int_{|x|<R+\dot{s}t} (u_0(x) - \bar{u}_0(x))^+ \, \mathrm{d}x \ .$$

Corollary 2.6.2 (Maximum principle for scalar conservation laws)**.**

If $u_0 \leq \bar{u}_0$ *a.e. on* $\mathbb{R}$ $\quad\Rightarrow\quad$ $u \leq \bar{u}$ *a.e. on* $\mathbb{R}\times]0,T[$

▶ $u_0(x) \in [\alpha, \beta]$ a.e. on $\mathbb{R}$ $\quad\Rightarrow\quad$ $u_0(x,t) \in [\alpha, \beta]$ a.e. on $\mathbb{R}\times]0,T[$

▶ L^∞-stability:

$$\forall 0 \leq t \leq T\colon \quad \|u(\cdot,t)\|_{L^\infty(\mathbb{R})} \leq \|u_0\|_{L^\infty(\mathbb{R})} \ . \tag{2.6.1}$$

Corollary 2.6.3 (L^1-contractivity of evolution for scalar conservation law)**.**

$$\forall t \in]0,T[,\ R > 0\colon \quad \int\limits_{|x|<R} |u(x,t) - \bar{u}(x,t)| \,\mathrm{d}x \leq \int\limits_{|x|<R+\dot{s}t} |u_0(x) - \bar{u}_0(x)| \,\mathrm{d}x \ ,$$

with *maximal speed of propagation*

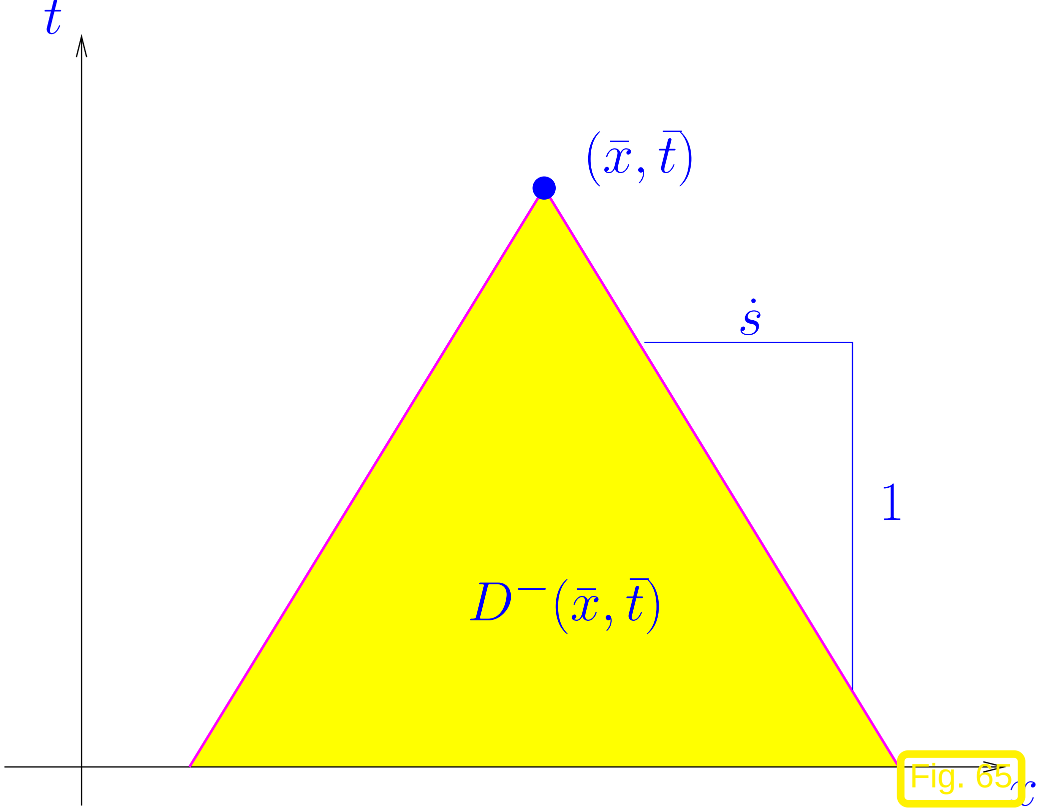
$$\dot{s} := \max\{|f'(\xi)|\colon \operatorname*{essinf}_{x\in\mathbb{R}} u_0(x) \leq \xi \leq \operatorname*{esssup}_{x\in\mathbb{R}} u_0(x)\} \ . \tag{2.6.2}$$

$$\forall t \in]0,T[,\ R>0\colon \quad \int\limits_{|x|<R} |u(x,t)|\,\mathrm{d}x \leq \int\limits_{|x|<R+\dot{s}t} |u_0(x)|\,\mathrm{d}x\ . \qquad (2.6.3)$$

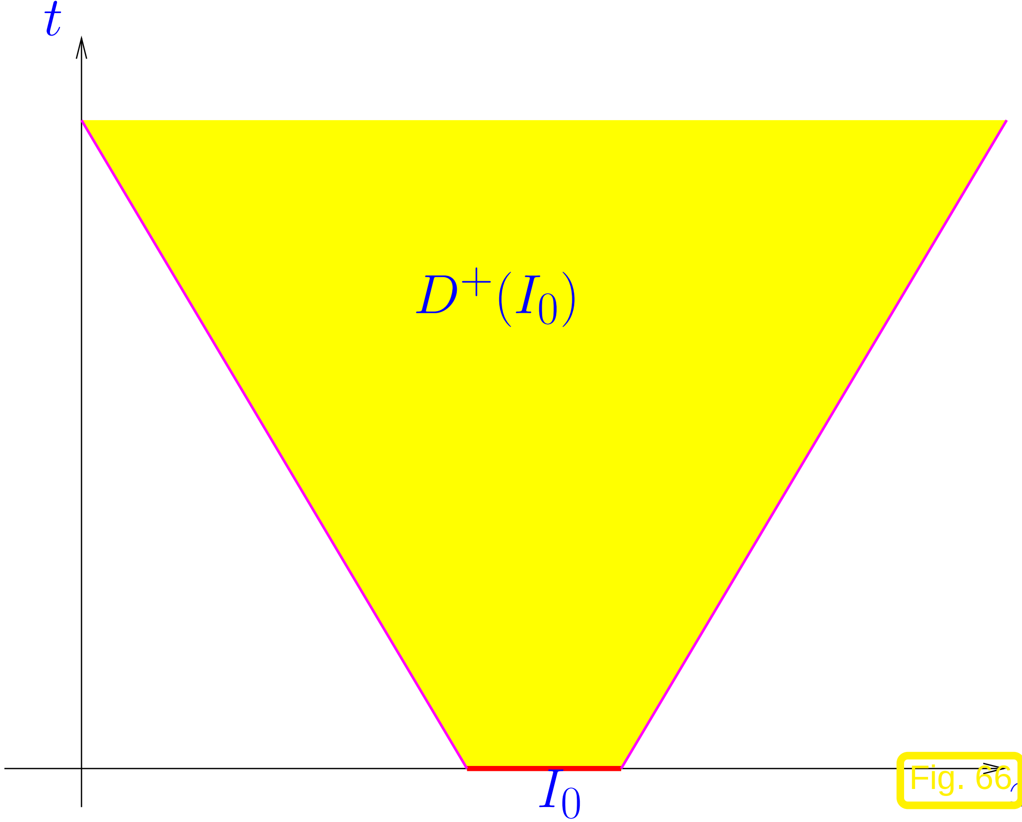
2.6.2 Domains of dependence and influence

Cor. 2.6.3 ➤ finite speed of propagation in conservation law, bounded by $\dot{s}$ from (2.6.2):

▶ As in the case of the wave equation $\longrightarrow$ Sect. 1.4:



domain of dependence of $(\bar{x},\bar{t}) \in \widetilde{\Omega}$



domain of influence of $I_0 \subset \mathbb{R}$

Analoguous to Thm. 1.4.1:

Corollary 2.6.4 (Domain of dependence for scalar conservation law). $\longrightarrow$ *[8, Cor. 6.2.2]*
The value of the entropy solution at $(\bar{x},\bar{t}) \in \widetilde{\Omega}$ *depends only on the restriction of the initial data to* $\{x \in \mathbb{R}\colon |x-\bar{x}| < \dot{s}\bar{t}\}$.

2.6.3 Monotonicity preservation

For solutions of Riemann problem ($\rightarrow$ Def. 2.4.1), Lemmas 2.4.3, 2.4.4:

$$u_0 \text{ monotone} \quad \Rightarrow \quad u(\cdot,t) \text{ monoton for all } 0 \leq t \leq T$$

Definition 2.6.5 (Total variation). $\rightarrow$ *http://mathworld.wolfram.com/BoundedVariation.html*
The total variation $TV_{]a,b[}(u)$ *of a function* $u :]a,b[\subset \mathbb{R} \mapsto \mathbb{R}$ *is*

$$TV_{]a,b[}(u) := \sup\{\sum_{i=1}^{K} |u(x_i) - u(x_{i-1})| : a \leq x_0 \leq x_1 \leq x_2 \leq \cdots \leq x_K \leq b, K \in \mathbb{N}\}$$

► $TV_{]a,b[}$ is a seminorm on the space of functions $]a,b[\subset \mathbb{R} \mapsto \mathbb{R}$

Definition 2.6.6 (Functions of bounded variation). *For open set* $\Omega \subset \mathbb{R}$

$$BV_{\mathrm{loc}}(\Omega) := \{u \in L^\infty(\Omega) : TV_I(u) < \infty \quad \forall \text{ compact } I \subset \Omega\} .$$

Lemma 2.6.7. $\rightarrow$ *[8, Thm. 1.7.1] If $u \in BV_{\mathrm{loc}}(\Omega)$, then*

$$TV_K(u) = \limsup_{h\to 0} \frac{1}{h} \int_K |u(x+h) - u(x)| \,\mathrm{d}x \quad \forall \text{ compact } K \subset \Omega\ .$$

Theorem 2.6.8 (Total variation stability of evolution for scalar conservation law)**.** $\rightarrow$ *[8, Thm. 6.2.3]*
If $u_0 \in BV_{\mathrm{loc}}(\mathbb{R}) \cap L^\infty(\mathbb{R})$, then $u \in BV_{\mathrm{loc}}(\mathbb{R}\times]0,T[)$ and

$$TV_{\{|x|<R\}}(u(\cdot,t)) \le TV_{\{|x|<R+\dot{s}t\}}(u_0) \quad \forall R > 0,\ 0 < t < T\ ,$$

with $\dot{s}$ from (2.6.2)*.*

Note: $u \in C^0([a,b])$: $\quad TV_{[a,b]}(u) = |u(b) - u(a)| \quad \Leftrightarrow \quad u$ monotone !

► If u_0 monotone & constant outside compact set $\Rightarrow$ $u(\cdot,t)$ monotone $\forall t$!

Note: $TV_{[a,b]}(u)$ large for oscillatory functions

► u_0 non-oscillatory ➤ $u(\cdot,t)$ non-oscillatory $\forall t$

Remark 45 (Local monotonicity preservation).

Above statement can be made sharper:

u solves (2.2.1) ➤ No. of local extrema (in space) of $u(\cdot, t)$ decreasing

△

Remark 46 (Total oscillation diminishing property). $\longrightarrow$ [38]

Under the assumptions of Thm. 2.6.8 holds for *any* Lipschitz-continuous *monotone* function $\Phi : \mathbb{R} \mapsto \mathbb{R}$

$$TV_{\{|x|<R\}}(\Phi(u(\cdot, t))) \leq TV_{\{|x|<R+\dot{s}t\}}(\Phi(u_0)) \quad \forall R > 0, 0 < t < T \; ,$$

with $\dot{s}$ from (2.6.2).

➤ allows to zoom in on local oscillations !

△

2.7 Supplement: Multidimensional scalar conservation laws

Cauchy problem for multidimensional scalar conservation law, flux function $\mathbf{f} : \mathbb{R} \mapsto \mathbb{R}^d$,

$$\frac{\partial u}{\partial t} + \operatorname{div}_{\boldsymbol{x}} \mathbf{f}(u) = 0 \quad \text{in } \mathbb{R}^d \times]0,T[\quad , \quad u(x,0) = u_0(x) \ , \quad x \in \mathbb{R}^d \ . \tag{2.7.1}$$

Which results for $d = 1 \leftrightarrow$ (2.2.1) carry over to (2.7.1) for $d > 1$ **?**

➊ Characteristic curves $\Gamma = (\boldsymbol{\gamma}(\tau), \tau)$, $0 \leq \tau \leq T$, $\frac{d}{d\tau}\boldsymbol{\gamma}(\tau) = \mathbf{f}'(u(\boldsymbol{\gamma}(\tau), \tau))$, $u \mathrel{\hat{=}}$ classical solution of (2.7.1) ($\longrightarrow$ Def. 2.2.2):

☞ Classical solution constant on characteristic curves, *cf.* Lemma 2.2.3

☞ Characteristic curves are straight lines in space-time.

➊ Notion of weak solution **=** $u \in L^\infty(\mathbb{R} \times]0,T[)$ satisfying

$$\int_{-\infty}^{\infty} \int_0^T \left\{ u \frac{\partial \Phi}{\partial t} + \mathbf{f}(u) \cdot \mathbf{grad}_{\boldsymbol{x}} \Phi \right\} \mathrm{d}t \mathrm{d}\boldsymbol{x} + \int_{-\infty}^{\infty} u_0(x) \Phi(x,0) \, \mathrm{d}\boldsymbol{x} = 0 \quad \forall \Phi \in C_0^\infty(\mathbb{R}^d \times [0,T[) \ .$$

❷ Generalization of Rankine-Hugoniot jump condition, Thm. 2.3.2: $\longrightarrow$ [15, Sect. I.2]

$\Sigma \subset \mathbb{R}^d \times]0,T[\;\hat{=}$ *surface* of discontinuity: $\Sigma = \{(\boldsymbol{x},\tau) \colon \Phi(\boldsymbol{x},\tau) = 0\}$, $0 \leq \tau \leq T$

$$\dot{s}(u_l - u_r) = (\mathbf{f}(u_l) - \mathbf{f}(u_r)) \cdot \boldsymbol{n}\,, \quad \boldsymbol{n} := \frac{\mathbf{grad}_{\boldsymbol{x}}\,\Phi}{|\,\mathbf{grad}_{\boldsymbol{x}}\,\Phi|} \quad = \text{spatial unit normal}\,, \tag{2.7.2}$$

$\dot{s} \;\hat{=}$ normal speed of surface: $\dot{s} = -\dfrac{\frac{\partial}{\partial\tau}\Phi}{|\,\mathbf{grad}_{\boldsymbol{x}}\,\Phi|}$

❸ Same definitions: pairs of entropy functions $\longrightarrow$ Def. 2.5.2

➤ weak entropy inequality $\longrightarrow$ Def. 2.5.3

Existence & uniqueness of entropy solutions of (2.7.1), *cf.* Thm. 2.5.4

❹ Entropy solution of (2.7.1) satisfies

- maximum principle, see Cor. 2.6.2
- L^1-contractivity, see Cor. 2.6.3
- TV-contractivity, see Thm. 2.6.8

$$\Omega \subset \mathbb{R}^d: \quad TV_\Omega(u) := \sup\left\{\int_\Omega u \operatorname{div}\boldsymbol{\Phi}\,\mathrm{d}\boldsymbol{x} \colon \boldsymbol{\Phi} \in (C_0^\infty(\Omega))^d,\ |\boldsymbol{\Phi}| \leq 1 \text{ a.e. in } \Omega\right\}\,.$$

maximal speed of propagation: $\dot{s} \leq \sup\{|\mathbf{f}'(\xi)| \colon \operatorname{essinf} u_0 \leq \xi \leq \operatorname{esssup} u_0\}$

➤ domains of dependence/influence, *cf.* Sect. 2.6.2

3 Finite volume methods for scalar conservation laws

Consider: Cauchy problem for 1D scalar conservation law:

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u) = 0 \quad \text{in } \mathbb{R}\times]0,T[\quad , \quad u(\cdot,0) = u_0 \quad \text{in } \mathbb{R} \,, \tag{2.2.1}$$

flux function $f : \mathbb{R} \mapsto \mathbb{R}$, $f \in C^1(\mathbb{R})$.

Model problems: ① linear advection with constant velocity: $f(u) = vu$, $v \in \mathbb{R}$ $\longrightarrow$ (2.1.6)

② Burgers equation: $f(u) = \frac{1}{2}u^2$ $\longrightarrow$ (2.1.7)

3.1 Space-time finite differences in 1D

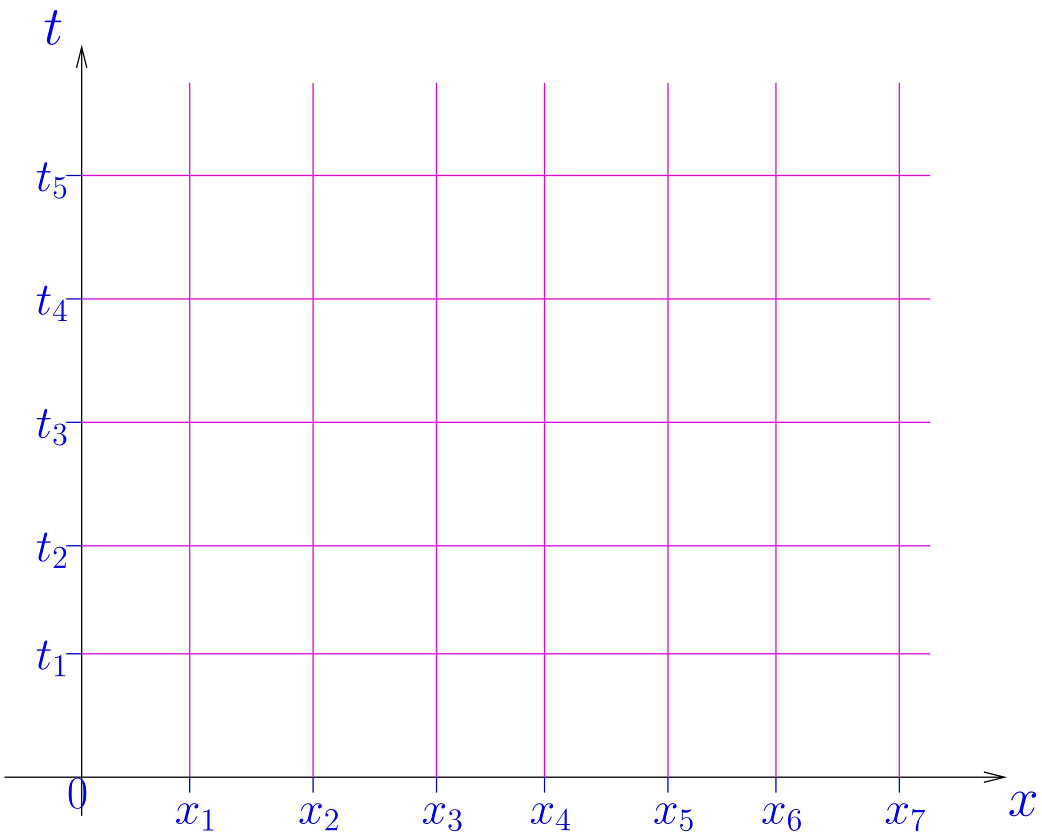
$\hat{=}$ fully discrete schemes for Cauchy problem (2.2.1)

Tool: infinite space-time tensor product grid:

$$\mathcal{M} := \{]x_{j-1}, x_j[\times]t_{k-1}, t_k[,\ j \in \mathbb{Z}, k \in \mathbb{N}\}\ , \tag{3.1.1}$$

spatial gridpoints: $\mathcal{G}_{\Delta x} := \{x_j \in \mathbb{R}\colon x_{j-1} < x_j, j \in \mathbb{Z}\}$,

temporal gridpoints: $\mathcal{G}_{\Delta t} := \{0 = t_0 < t_1 < \cdots < t_M = T\}$, $M \in \mathbb{N}$.



meshwidths: $\Delta x_j := x_j - x_{j-1} > 0$, $j \in \mathbb{Z}$

timesteps: $\Delta t_k := t_k - t_{k-1} > 0$, $k \in \{1, \ldots,$

Focus: equidistant grids:

$$\Delta x_j = \Delta x > 0\ , \quad \forall j \in \mathbb{Z}\ ,$$
$$\Delta t_k = \Delta t := T/M\ , \quad \forall k \in \mathbb{N}\ .$$

Vector space of (spatial) grid functions

$$C^0(\mathcal{G}_{\Delta x}) := \{\mathcal{G}_{\Delta x} \mapsto \mathbb{R}\}\ .$$

➡ notation: $\vec{\mu}^{(k)}$, $\vec{\zeta}^{(k)}$, etc.

➡ notation: $\vec{\mu}^{(\cdot)} \hat{=}$ grid function $\mathcal{M} \to \mathbb{R}$

Single step, *time-invariant* discrete evolution based on discrete evolution operator $\mathcal{H} : C^0(\mathcal{G}_{\Delta x}) \mapsto C^0(\mathcal{G}_{\Delta x})$

$$\vec{\mu}^{(k)} := \mathcal{H}\vec{\mu}^{(k-1)} \ , \quad k = 1, \ldots, M \ , \tag{3.1.2}$$

with initial value $\vec{\mu}^{(0)} \in C^0(\mathcal{G}_{\Delta x})$.

Relationship: $\vec{\mu}^{(k)} = (\mu_j^{(k)})_{j\in\mathbb{Z}} \longleftrightarrow$ function $u(x,t)$ **=** solution of (2.2.1))

Different interpretations:

$$\mu_j^{(k)} \approx u(x_j, t_k) \qquad \text{or} \qquad \mu_j^{(k)} \approx \frac{2}{\Delta x_j + \Delta x_{j+1}} \int_{x_{j-1/2}}^{x_{j+1/2}} u(x, t_k)\,\mathrm{d}x \ ,$$

with $x_{j-1/2} := 1/2(x_{j-1} + x_j)$,
$x_{j+1/2} = 1/2(x_j + x_{j+1})$.

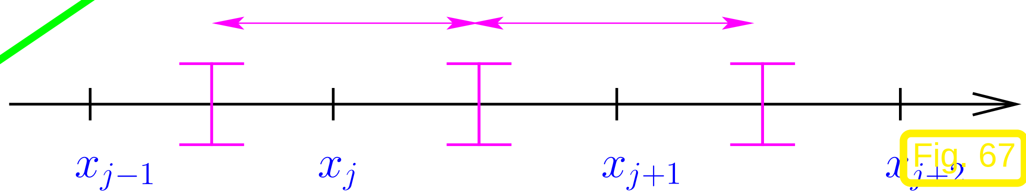
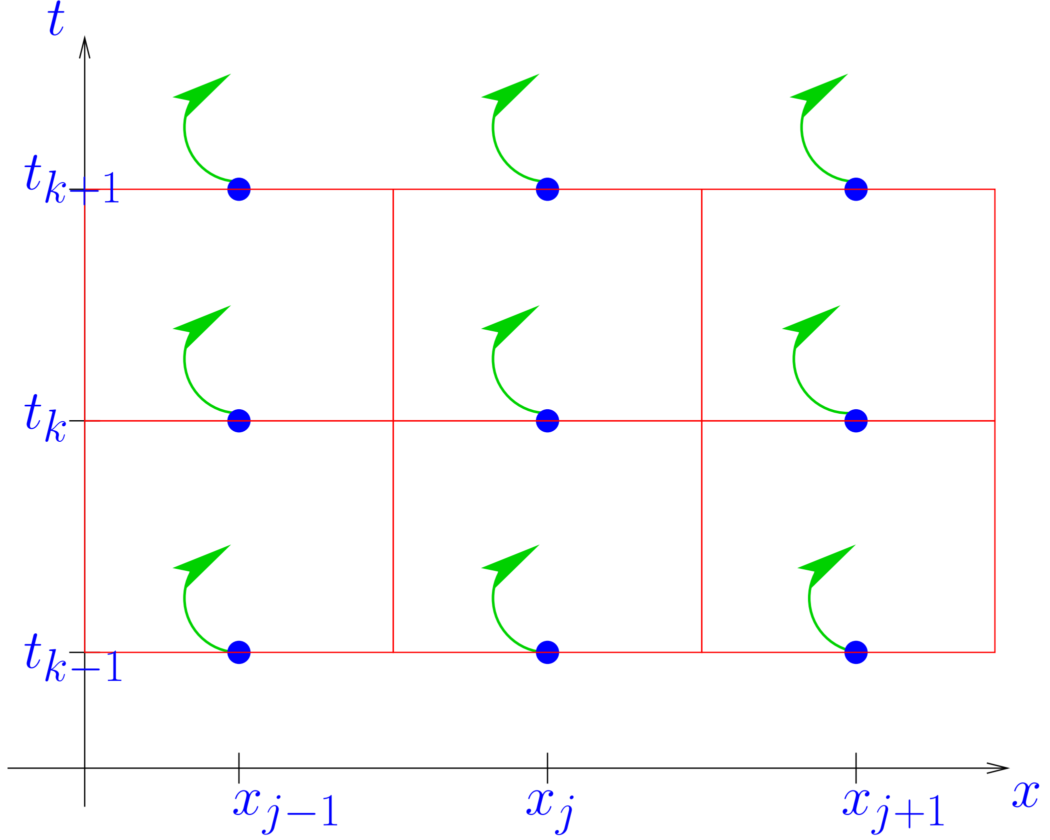


Fig. 67

$[x_{j-1/2}, x_{j+1/2}] \hat{=}\ j$-th cell ➤ cell average



$\triangleleft$ (piecewise constant) reconstruction:
$\vec{\mu}^{(\cdot)} \in \mathbb{R}^{\mathbb{Z}\times\{0,\dots,M\}} \longrightarrow$ function on $\mathbb{R}\times]0,T[$:

$$\mathcal{C}\vec{\mu}^{(\cdot)} \in L^\infty(\mathbb{R}\times]0,T[) \;,$$

$$\mathcal{C}\vec{\mu}^{(\cdot)}(x,t) = \mu_j^{(k)} \quad \text{for} \quad \begin{array}{r} x_{j-1/2} < x < x_{j+1/2} \;, \\ t_k < t < t_{k+1} \;. \end{array}$$

$\vec{\mu} \in C^0(\mathcal{G}_{\Delta x}) \longrightarrow$ function on $\mathbb{R}$:

$$\mathcal{C}\vec{\mu}(x) = \mu_j \quad \text{for } x_{j-1/2} < x < x_{j+1/2} \;.$$

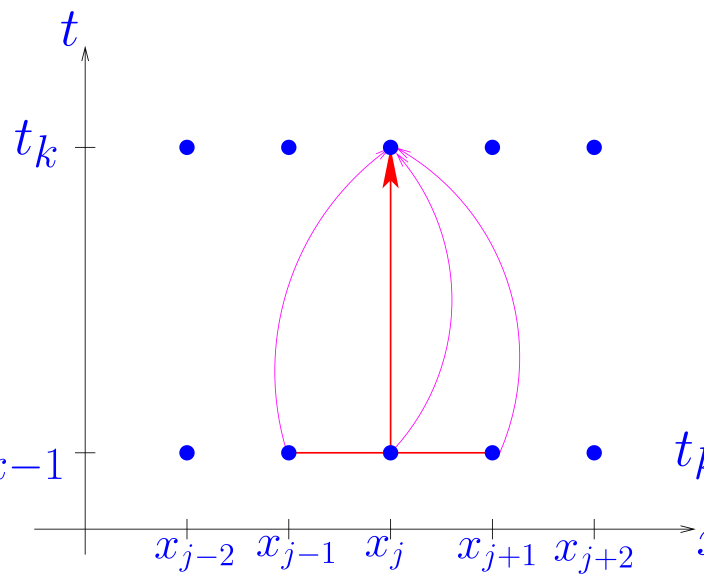
Definition 3.1.1 (Explicit finite difference timestepping)**.**
A single step time-invariant discrete evolution (3.1.2) *is an explicit finite difference method (FDM), if* $\mathcal{H}$ *is* local *in the sense that*

$$\exists m_l, m_r \in \mathbb{N}_0\colon \quad (\mathcal{H}\vec{\mu})_j = \mathsf{H}_j(\mu_{j-m_l}, \mu_{j-m_l+1}, \dots, \mu_{j+m_r-1}, \mu_{j+m_r}) \quad \forall j \in \mathbb{Z} \;,$$

with functions $\mathsf{H}_j : \mathbb{R}^{m_l+m_r+1} \mapsto \mathbb{R}$, $j \in \mathbb{Z}$

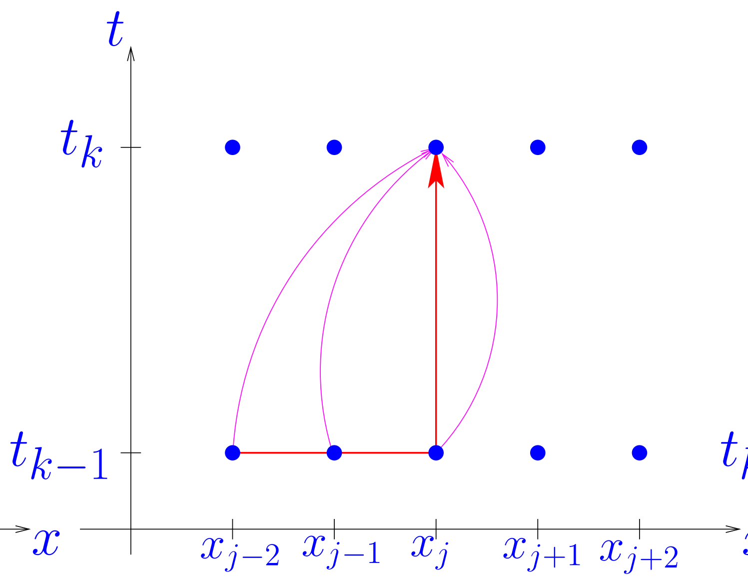
☞ "explicit", *cf.* discussion of explicit/implicit trapezoidal rule, Sect. 1.7.1

☞ Stencil notation, *cf.* Figs. 13, 14:



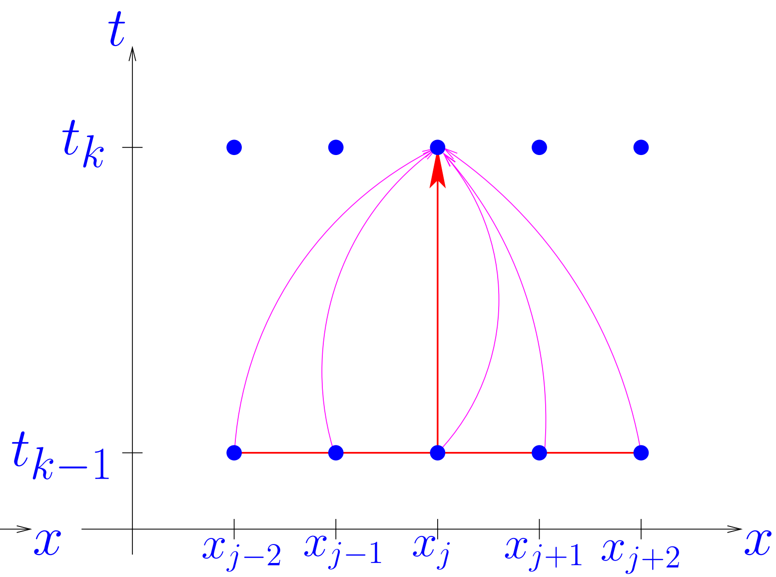
3-point explicit FDM

($m_l = 1$, $m_r = 1$)



one-sided 3-point stencil

($m_l = 2$, $m_r = 0$)



5-point explicit FDM

($m_l = 2$, $m_r = 2$)

Definition 3.1.2 (Linear finite difference methods)**.**

A discrete evolution (3.1.2) *is linear, if $\mathcal{H}$ is a linear operator.*

Definition 3.1.3 (Translation invariant FDM)**.**

An explicit finite difference method (→ Def. 3.1.1) is translation invariant, if $\mathsf{H}_j = \mathsf{H}$ for all $j \in \mathbb{Z}$.

Note: natural requirement for FDM for (2.2.1), because $\frac{\partial}{\partial t} + \frac{\partial}{\partial x} f(\cdot)$ independent of x

---

Consider: explicit finite difference method ($\rightarrow$ Def. 3.1.1) on equidistant tensor product grid

▶ Discrete domain of dependence for gridpoint (x_j, t_k), $j \in \mathbb{Z}$, $k = 0, \dots, M$:

$$D^-_{\mathcal{M}}(x_j, t_k) = \{(x_i, t_l) \colon -m_l \cdot (k-l) \leq i - j \leq m_r \cdot (k-l), 0 \leq l \leq k\} . \tag{3.1.3}$$

Notation ($\rightarrow$ Fig. 65): $D^-(\bar{x}, \bar{t}) \hat{=}$ domain of dependence of $(\bar{x}, \bar{t})$ w.r.t. (2.2.1), see Sect. 2.6.2

Definition 3.1.4 (CFL-condition II)**.**

An explicit FDM ($\rightarrow$ Def. 3.1.1) satisfies the CFL-condition, if *domain of dependence*

$$D^-(x_j, t_k) \subset \mathrm{convex}\{D^-_{\mathcal{M}}(x_j, t_k)\}$$

for all j, k ($\mathrm{convex} \leftrightarrow$ convex hull).

➔ Sect. 1.7.3 for more explanations.

If $\dot{s}$ **=** maximal speed of propagation for (2.2.1) $\rightarrow$ Cor. 2.6.3

(symmetric) 3-point explicit FDM $\dot{s}\Delta t \leq \Delta x$
(symmetric) 5-point explicit FDM $\dot{s}\Delta t \leq 2\Delta x$ $\Rightarrow$ CFL-condition ($\longrightarrow$ Def. 3.1.4) satisfied.

3.1.1 Abstract convergence theory

Asymptotic perspective: family $\{\mathcal{M}_{\Delta x, \Delta t}\}$ of equidistant tensor product grids, see (3.1.1), with meshwidths Δx, timesteps Δt

▶ Family of time-invariant single step discrete evolutions

$$\vec{\mu}^{(k)} = \mathcal{H}\vec{\mu}^{(k-1)} \;, \quad k = 1, \dots, M := T/\Delta t \quad , \quad \mathcal{H} = \mathcal{H}(\Delta x, \Delta t) \;. \tag{3.1.4}$$

Tool: restriction operators, *cf.* interpretation of $\mu^{(k)}$, Sect. 3.1:

$$\mathsf{R} : \begin{cases} C^0(\mathbb{R}) & \mapsto & C^0(\mathcal{G}_{\Delta x}) \\ u & \mapsto & (u(x_j))_{j\in\mathbb{Z}} \end{cases} \quad \text{or} \quad \mathsf{R} : \begin{cases} L^1(\mathbb{R}) & \mapsto & C^0(\mathcal{G}_{\Delta x}) \\ u & \mapsto & \left(\frac{2}{\Delta x_j + \Delta x_{j+1}} \int_{x_{j-1/2}}^{x_{j+1/2}} u(x)\,\mathrm{d}x \right)_{j\in\mathbb{Z}} \end{cases} .$$

(depend on spatial grid $\leftrightarrow$ Δx !)

u (sufficiently smooth) solution of (2.2.1) ➤ error $\vec{\eta}^{(k)} := \vec{\mu}^{(k)} - \mathsf{R}(u(\cdot, t_k)) \in C^0(\mathcal{G}_{\Delta x})$

Below: $\|\cdot\|_{\Delta x} \hat{=}$ (grid-dependent) norm on $C^0(\mathcal{G}_{\Delta x})$

Example 47 (Grid dependent norms).

Maximum norm $$\left\|\vec{\xi}\right\|_{l^\infty(\mathbb{Z})} = \sup_{j\in\mathbb{Z}} |\xi_j|$$

l^p-norm $$\left\|\vec{\xi}\right\|_{l^p(\mathbb{Z})} = \left(\Delta x \sum_{j\in\mathbb{Z}} |\xi_j|^p\right)^{1/p}, \quad 1 \le p < \infty \, .$$

Note: related to norms of p.w. constants functions on spatial grid cells.

◇

Definition 3.1.5 (Convergence of discrete evolution)**.**
A discrete evolution converges *to the solution* u *of* (2.2.1) *in norm* $\|\cdot\|_{\Delta x}$

$$:\Leftrightarrow \quad \left\|\vec{\eta}^{(k)}\right\|_{\Delta x} \to 0 \quad \textit{for } \max\{\Delta x, \Delta t\} \to 0 \quad \text{uniformly } \textit{in} \quad k \in \{0, \ldots, T/\Delta t\}.$$

Convergence is of order $(p, q) \in \mathbb{N}^2$ *(order* p *in time, order* q *in space)* $\quad :\Leftrightarrow \quad$ *for all* $\Delta x, \Delta t$ *sufficiently small*

$$\exists C_t, C_x > 0: \quad \left\|\vec{\eta}^{(k)}\right\|_{\Delta x} \leq C_t(\Delta t)^p + C_x(\Delta x)^q \quad \forall k \in \{0, \ldots, T/\Delta t\}\ .$$

Recursion for error, *cf.* numerical analysis of ODE

$$\begin{aligned}
\vec{\eta}^{(k)} &= \vec{\mu}^{(k)} - \mathsf{R}(u(\cdot, t_k)) \\
&= \mathcal{H}(\mathsf{R}(u(\cdot, t_{k-1})) + \vec{\eta}^{(k-1)}) - \mathsf{R}(u(\cdot, t_k)) \\
&= \underbrace{\mathcal{H}(\mathsf{R}(u(\cdot, t_{k-1})) + \vec{\eta}^{(k-1)}) - \mathcal{H}(\mathsf{R}(u(\cdot, t_{k-1})))}_{\text{propagated error}} \\
&\quad + \underbrace{\Delta t \frac{\mathcal{H}(\mathsf{R}(u(\cdot, t_{k-1}))) - \mathsf{R}(u(\cdot, t_k))}{\Delta t}}_{\text{one step error}}
\end{aligned}$$

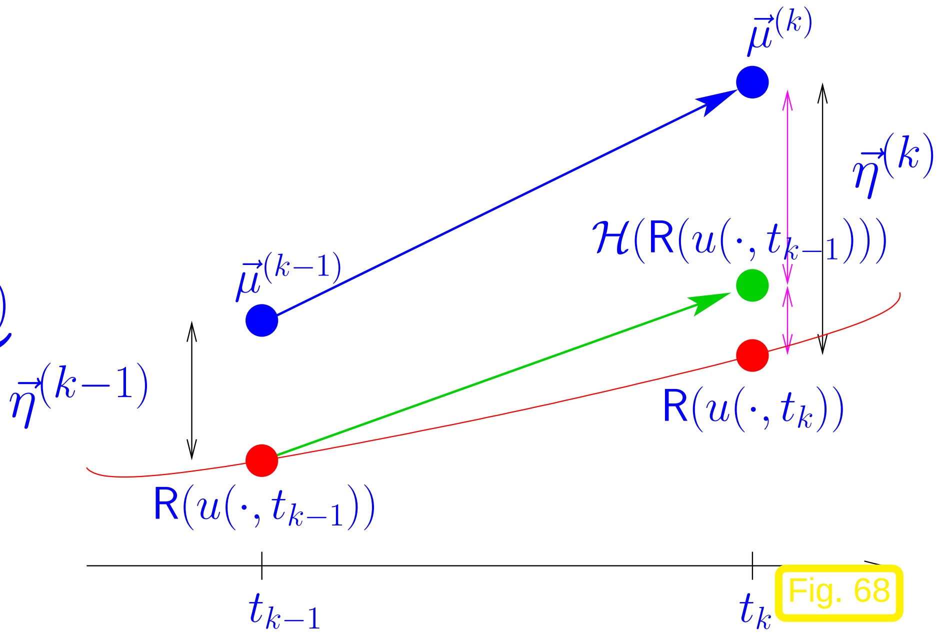


Fig. 68

Definition 3.1.6 ((Local) truncation error)**.**
For the (sufficiently smooth) solution u *of* (2.2.1)*, the (time-local) truncation error of the time-invariant single step discrete evolution* (3.1.4) *is*

$$\vec{\tau}^{(k)} := \frac{1}{\Delta t}\left(\mathcal{H}(\mathsf{R}(u(\cdot,t_{k-1}))) - \mathsf{R}(u(\cdot,t_k))\right) \;, \quad k = 1,\dots,M \;.$$

Definition 3.1.7 (Consistency)**.** *(→ Def. 3.1.5)*
A discrete evolution (3.1.4) *is consistent with* (2.2.1)

$$:\Leftrightarrow \quad \left\|\vec{\tau}^{(k)}\right\|_{\Delta x} \to 0 \quad \text{for } \max\{\Delta x,\Delta t\} \to 0 \quad \text{uniformly in } k \in \{0,\dots,T/\Delta t\}.$$

It is consistent of order $(p,q) \in \mathbb{N}^2$ $:\Leftrightarrow$

$$\exists C_t, C_x > 0: \quad \left\|\vec{\tau}^{(k)}\right\|_{\Delta x} \leq C_t(\Delta t)^p + C_x(\Delta x)^q \quad \forall k \in \{1,\dots,T/\Delta t\} \;,$$

for all $\Delta x, \Delta t$ *sufficiently small.*

Definition 3.1.8 (Non-linear stability). *A time-invariant single step discrete evolution* (3.1.4) *is (non-linearly) stable*

$$:\Leftrightarrow \quad \exists c > 0: \quad \left\| \mathcal{H}(\Delta x, \Delta t)\vec{\xi} - \mathcal{H}(\Delta x, \Delta t)\vec{\zeta} \right\|_{\Delta x} \leq (1 + c\Delta t) \left\| \vec{\xi} - \vec{\zeta} \right\|_{\Delta x} \quad \forall \vec{\xi}, \vec{\zeta} \in C^0(\mathcal{G}_{\Delta x}) ,$$

for all sufficiently small Δx, Δt.

Theorem 3.1.9 ($\boxed{\text{Consistency \& non-linear stability} \Rightarrow \text{convergence}}$).

$$\left. \begin{array}{l} \left\| \vec{\mu}^{(0)} - \mathsf{R}u_0 \right\|_{\Delta x} \to 0 \text{ for } \Delta x \to 0 \\ \text{(3.1.4) consistent with (2.2.1) } (\to \text{Def. 3.1.7}) \\ \text{(3.1.4) non-linearly stable } (\to \text{Def. 3.1.8}) \end{array} \right\} \Longrightarrow \begin{array}{c} \text{discrete evolution convergent} \\ (\to \text{Def. 3.1.5}) \end{array}$$

If $\left\| \vec{\mu}^{(0)} - \mathsf{R}u_0 \right\|_{\Delta x} \leq C_0(\Delta x)^q$, (3.1.2) *consistent with* (2.2.1) *of order* (p, q), *and non-linearly stable, then* (3.1.2) *is convergent of order* (p, q).

Stronger result: (➜ convergence analysis for wave equation in Sect. 1.8!)

Theorem 3.1.10 (Lax equivalence theorem)**.**

For a consistent *(→ Def. 3.1.7)* linear *(→ Def. 3.1.2) time-invariant single step discrete evolution* (3.1.4)

$$\begin{array}{l} \exists C>0\colon \quad \left\|\mathcal{H}^k\right\|_{\Delta x} \leq C \quad \forall k \textit{ (uniformly in } \Delta x, \Delta t\textit{)} \\ \textit{and} \quad \left\|\vec{\eta}^{(0)}\right\|_{\Delta x} \to 0 \quad \textit{for} \quad \Delta x \to 0 \end{array} \quad \Longrightarrow \quad \textit{convergence}$$

3.1.2 Consistency

Setting:
- Cauchy problem (2.2.1), flux function $f \in C^1(\mathbb{R})$
- families of equidistant infinite tensor product grids (meshwidths Δx, timesteps Δt)
- $\boxed{\text{fixed ratio } \gamma := \Delta t : \Delta x = \text{const}}$ motivated by CFL-condition (→ Def. 3.1.4)
- operators $\mathcal{H} = \mathcal{H}(\Delta x, \Delta t)$ from explicit translation-invariant finite difference method:

$$\mu_j^{(k)} = \mathsf{H}(\mu_{j-m_l}^{(k-1)}, \dots, \mu_{j+m_r}^{(k-1)}; \Delta x, \Delta t)\ , \quad \mathsf{H}(\cdot; \Delta x, \Delta t) : \mathbb{R}^{m_r+m_l+1} \mapsto \mathbb{R} \text{ smooth}\ . \tag{3.1.6}$$

Focus: interpretation $\mu_j^{(k)} \approx u(x_j, t_k)$ (→ Sect. 3.1), maximum norm $\|\cdot\|_{l^\infty(\mathbb{Z})}$

Goal: bound local truncation error (⟶ Def. 3.1.6)

$$\tau_j^{(k)} = \frac{1}{\Delta t}\left(\mathsf{H}(u(x_j - m_l\Delta x, t_{k-1}), \dots, u(x_j + m_r\Delta x, t_{k-1}); \Delta x, \Delta t) - u(x_j, t_k)\right) \tag{3.1.7}$$

in terms of Δx, Δt.

Technique: Taylor expansion (in x and t) of **smooth** solution of (2.2.1)

First special case: linear (⟶ Def. 3.1.2) explicit 3-point FDM for linear advection (2.1.6)

$$\mu_j^{(k)} = \alpha_{-1}\mu_{j-1}^{(k-1)} + \alpha_0\mu_j^{(k-1)} + \alpha_1\mu_{j+1}^{(k-1)}\,, \quad \alpha_{-1}, \alpha_0, \alpha_1 \in \mathbb{R}\,. \tag{3.1.8}$$

Taylor expansion: 1st/2nd-order consistency $\leftrightarrow$ linear conditions on α_{-1}, α_0, α_1

(3.1.8) 2nd-order $\Leftrightarrow$ $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha_{-1} \\ \alpha_0 \\ \alpha_1 \end{pmatrix} = \begin{pmatrix} 1 \\ \gamma v \\ (\gamma v)^2 \end{pmatrix}$

□ $\hat{=}$ 1st-order conditions

- first-order centered finite differences

$$\mu_j^{(k)} = \mu_j^{(k-1)} - \tfrac{1}{2}\gamma v(\mu_{j+1}^{(k-1)} - \mu_{j-1}^{(k-1)})\,. \tag{3.1.9}$$

- first-order forward differencing ("magic timestep": exact for $\gamma v = -1$):

$$\mu_j^{(k)} = (1 + \gamma v)\mu_j^{(k-1)} - \gamma v \mu_{j+1}^{(k-1)} \ . \tag{3.1.10}$$

- first-order backward differencing (("magic timestep": exact for $\gamma v = 1$):

$$\mu_j^{(k)} = (1 - \gamma v)\mu_j^{(k-1)} + \gamma v \mu_{j+1}^{(k-1)} \ . \tag{3.1.11}$$

- 2nd-order Lax-Wendroff-scheme ("magic timestep": exact for $\gamma v = \pm 1$):

$$\mu_j^{(k)} = (1 - (\gamma v)^2)\mu_j^{(k-1)} + \tfrac{1}{2}\gamma v(\gamma v + 1)\mu_{j-1}^{(k-1)} + \tfrac{1}{2}\gamma v(\gamma v - 1)\mu_{j+1}^{(k-1)} \tag{3.1.12}$$

(**only** 2nd-order linear 3-point FDM for constant advection !)

In all cases (3.1.9)-(3.1.12): CFL-condition ($\rightarrow$ Def. 3.1.4) $\Leftrightarrow$ $|\gamma v| \leq 1$

Example 48 (Accuracy of 2-point and 3-point schemes for constant linear advection).

- (2.1.6) with advection velocity $v = 1$, $T = 1$ ➤ $u(x,t) = u_0(x - t)$

- smooth, non-smooth and discontinuous initial data, supported in $[0, 1]$:

$$u_0(x) = 1 - \cos^2(\pi x)\ , \quad 0 \leq x \leq 1\ , \quad 0 \text{ elsewhere}\ , \tag{3.1.13}$$

$$u_0(x) = 1 - 2 * |x - \tfrac{1}{2}|\ , \quad 0 \leq x \leq 1\ , \quad 0 \text{ elsewhere}\ , \tag{3.1.14}$$

$$u_0(x) = 1\ , \quad 0 \leq x \leq 1\ , \quad 0 \text{ elsewhere}\ . \tag{3.1.15}$$

Monitored: convergence of (3.1.11) and Lax-Wendroff-scheme w.r.t. to norms $\max_k \left\|\vec{\mu}^{(k)} - \mathsf{R}u(\cdot,t_k)\right\|_{l^2(\mathbb{Z})}$, $\max_k \left\|\vec{\mu}^{(k)} - \mathsf{R}u(\cdot,t_k)\right\|_{l^1(\mathbb{Z})}$, ($\max_k \left\|\vec{\mu}^{(k)} - \mathsf{R}u(\cdot,t_k)\right\|_{l^\infty(\mathbb{Z})}$) for $\gamma = 0.8$ and different initial data u_0.

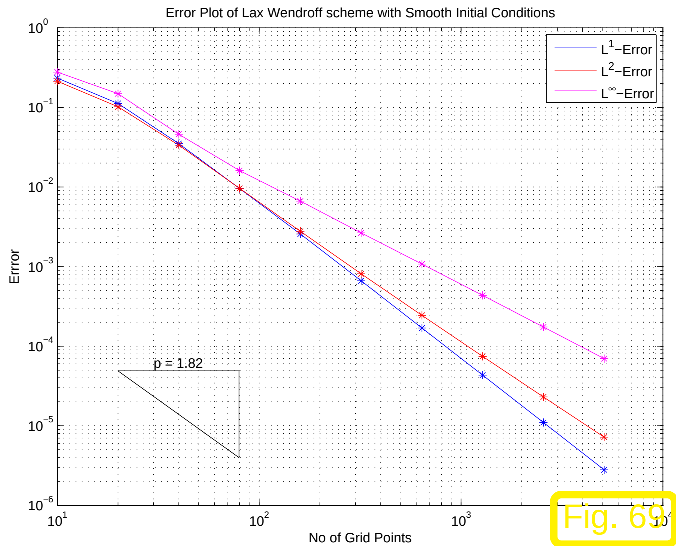


Fig. 69

$u_0 \leftrightarrow$ (4.2.3)

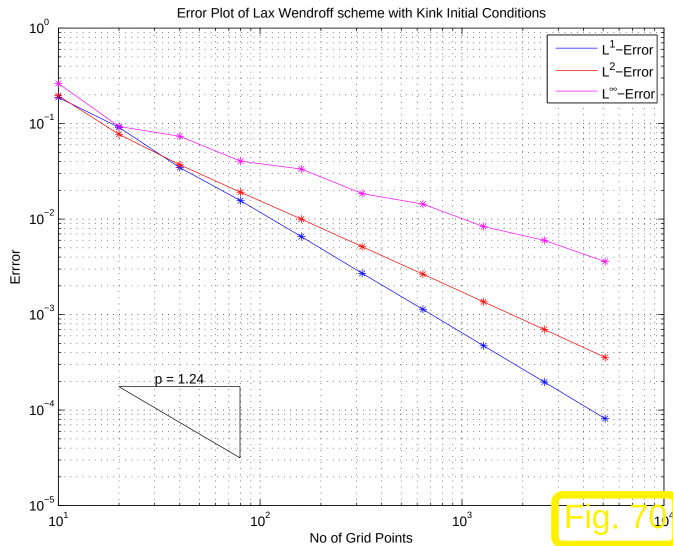


Fig. 70

$u_0 \leftrightarrow$ (4.2.4)

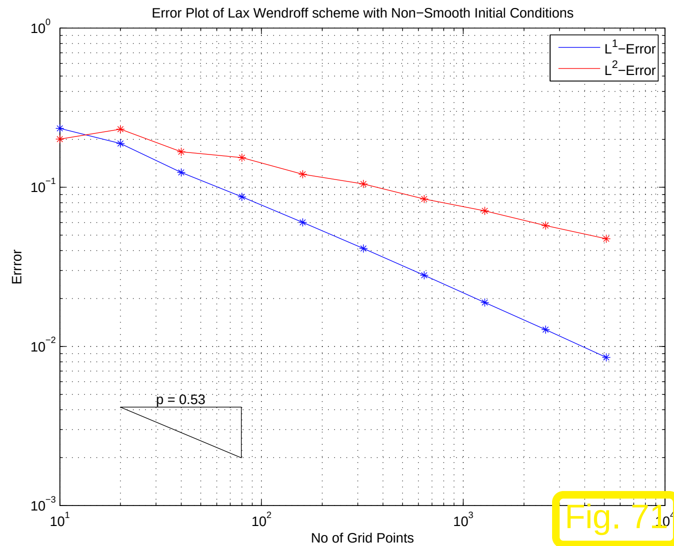


Fig. 71

$u_0 \leftrightarrow$ (4.2.5)

Observation:
- 2nd-order algebraic convergence (for smooth u) w.r.t. $\Delta t = \gamma \Delta x$
- order of consistency $=$ order of convergence for smooth solutions
- lower order of convergence for non-smooth solutions

$\diamond$

Special case: general explicit 3-point FDM: $\mu_j^{(k)} = \mathsf{H}(\mu_{j-1}^{(k-1)}, \mu_j^{(k-1)}, \mu_{j+1}^{(k-1)}; \Delta x, \Delta t)$ (3.1.16)

Assume: H differentiable in $\mu_{j-1}^{(k-1)}, \mu_j^{(k-1)}, \mu_{j+1}^{(k-1)}$

Lemma 3.1.11 (Consistency of 3-point FDM)**.**
If $u \in C^2(\mathbb{R}\times]0,T[)$*, then a 3-point FDM* (3.1.16) *is first order consistent with* (2.2.1)*, if*

$$(i)\ \mathsf{H}(u,u,u) = u \quad \forall u \in \mathbb{R}\ , \quad \forall \Delta x > 0\ ,$$
$$(ii)\ \partial_{-1}\mathsf{H}(u,u,u) - \partial_1 \mathsf{H}(u,u,u) = \gamma f'(u) \quad \forall u \in \mathbb{R}\ , \quad \forall \Delta x > 0\ ,$$

➡ notation: $\partial_l \mathsf{H} =$ partial derivative of H w.r.t. to $l+2$-th argument, $l = -1, 0, 1$

- first-order centered finite differences for (2.2.1):

$$\mu_j^{(k)} = \mu_j^{(k-1)} - \tfrac{1}{2}\gamma \left(f(\mu_{j+1}^{(k-1)}) - f(\mu_{j-1}^{(k-1)}) \right)\ , \qquad (3.1.17)$$
$$\mu_j^{(k)} = \mu_j^{(k-1)} - \tfrac{1}{2}\gamma f'(\mu_j^{(k-1)})(\mu_{j+1}^{(k-1)} - \mu_{j-1}^{(k-1)})\ . \qquad (3.1.18)$$

* first-order forward finite differences for (2.2.1):

$$\mu_j^{(k)} = \mu_j^{(k-1)} - \gamma \left(f(\mu_{j+1}^{(k-1)}) - f(\mu_j^{(k-1)}) \right) . \tag{3.1.19}$$

* first-order backward finite differences for (2.2.1):

$$\mu_j^{(k)} = \mu_j^{(k-1)} - \gamma \left(f(\mu_j^{(k-1)}) - f(\mu_{j-1}^{(k-1)}) \right) . \tag{3.1.20}$$

Remark 49 (Viscous modification).

Given 3-point FDM (3.1.16), 1st-order consistent with (2.2.1), $q \in C^1(\mathbb{R}^3, \mathbb{R})$ with $q(u,v,w) = q(w,v,u)$ for all $u,v,w \in \mathbb{R}$, then

$$\widetilde{\mathsf{H}}(\mu_{-1}, \mu_0, \mu_1) := \mathsf{H}(\mu_{-1}, \mu_0, \mu_1) + q(\mu_{-1}, \mu_0, \mu_1) \frac{\mu_1 - 2\mu_0 + \mu_{-1}}{\Delta x^2} \tag{3.1.21}$$

defines another 3-point FDM $\xrightarrow{\text{Lemma 3.1.11}}$ first order consistent with (2.2.1).

Sect. 1.6.1 ➢ $\dfrac{\mu_{j+1}^{(k)} - 2\mu_j^{(k)} + \mu_{j-1}^{(k)}}{\Delta x^2} \approx \dfrac{\partial^2 u}{\partial x^2}(x_j, t) \longleftarrow$ viscous term, *cf.* Sect. 2.5.1

△

Example 50 (Convergence of 3-point FDM for Burgers equation).

- Cauchy problem for Burgers equation (2.1.7)
- initial data u_0 as in Ex. 48 ➢ $0 \leq u(x,t) \leq 1$ a.e. in $\mathbb{R}\times]0,T[$
- backward 3-point FDM (3.1.20) with $\gamma = 1$ ➢ CFL-condition satisfied

Monitored: (algebraic) convergence w.r.t. norms $\max_k \left\| \vec{\mu}^{(k)} - \mathsf{R}u(\cdot,t_k) \right\|_{l^2(\mathbb{Z})}$,
$\max_k \left\| \vec{\mu}^{(k)} - \mathsf{R}u(\cdot,t_k) \right\|_{l^1(\mathbb{Z})}$ for different u_0 from (4.2.3)-(4.2.5).

Backward 3-point FDM (3.1.20):

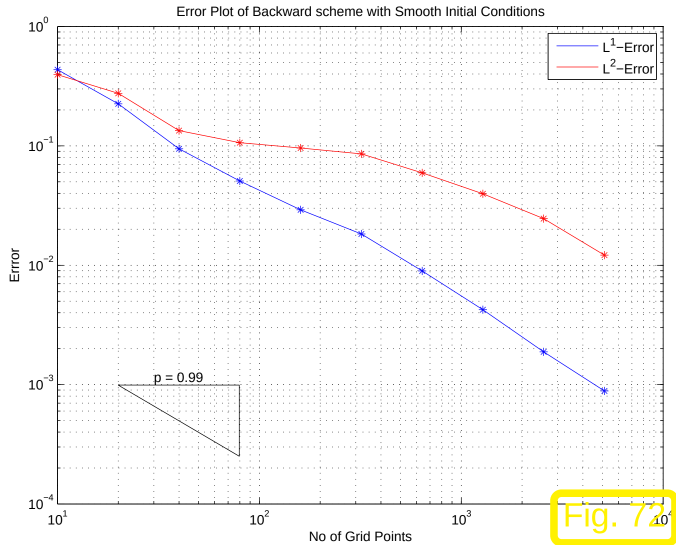


Fig. 72

smooth initial data (4.2.3)

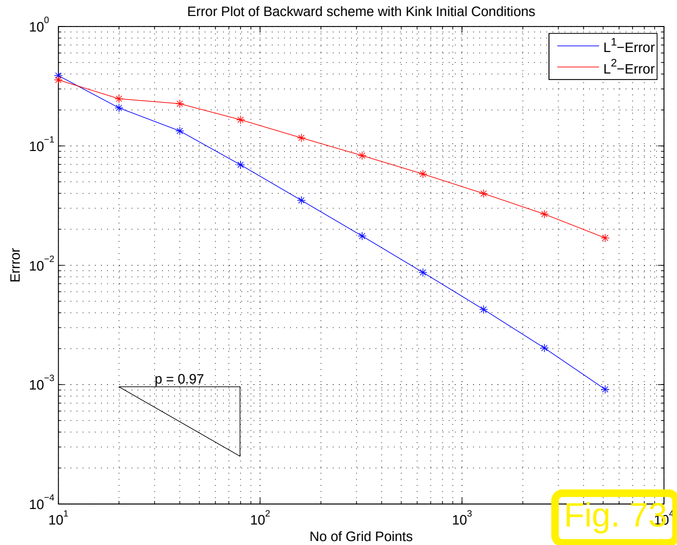


Fig. 73

non-smooth initial data (4.2.4)

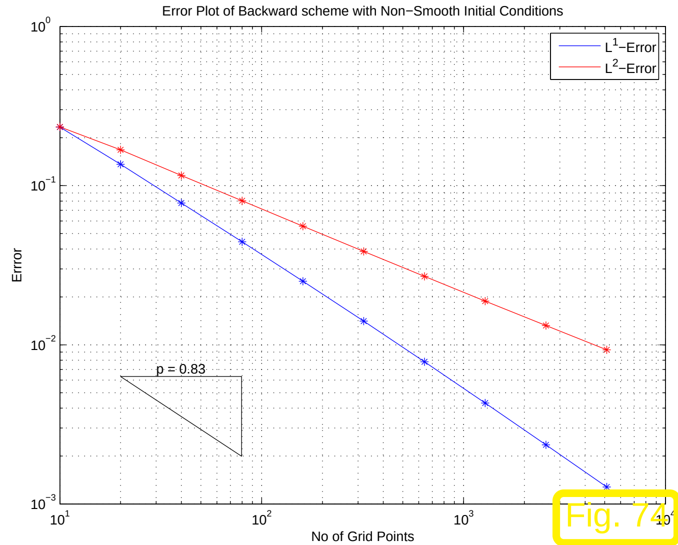


Fig. 74

discontinuous initial data (4.2.5)

Observation:
- first order convergence in $l^1(\mathbb{Z})$-norm in any case
- slightly slower convergence in $l^2(\mathbb{Z})$-norm

◇

(Order of) consistency ↔ power of FDM to approximate **smooth** solutions of conservation law

Remark 51. Strongly linked to consistency of scheme (3.1.6): (⟶ Lemma 3.1.11)

$$\text{local preservation of constants} \quad :\leftrightarrow \quad \mathsf{H}(u,\dots,u;\Delta x,\Delta t) = u \quad \forall u \in \mathbb{R}\,, \quad \forall \Delta x, \Delta t\,.$$

△

3.1.3 Stability

Goal: verification of non-linear stability (⟶ Def. 3.1.8),
stronger: contraction properties of $\mathcal{H}$

rule of thumb: CFL-condition (⟶ Def. 3.1.4) necessary for stability of explicit discrete evolution

Note: for non-linear discrete evolutions: stability also depends on solution $\vec{\mu}^{(k)}$!

Setting for FDM: equidistant meshes, spatial meshwidth Δx, timestep Δt, $\gamma := \Delta t / \Delta x$

3.1.3.1 Linear stability

- targets **linear** discrete evolutions
- focus on $l^2(\mathbb{Z})$-norm **=** von Neumann stability analysis, *cf.* Sect. 1.5.1
- tool: diagonalization of $\mathcal{H}$ by Fourier transform on $\mathbb{Z}$

$$\vec{\mu} \in l^1(\mathbb{Z}) \longleftrightarrow \hat{\mu} \in C^0(]-\pi,\pi[)$$

$$\mu_j = (\mathcal{F}^{-1}\hat{\mu})_j := \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{\mu}(\xi) e^{i\xi j}\,\mathrm{d}\xi \longleftrightarrow \hat{\mu}(\xi) = (\mathcal{F}\vec{\mu})(\xi) := \sum_{j\in\mathbb{Z}} \mu_j e^{-ij\xi} \,. \qquad (3.1.22)$$

Lemma 3.1.12 (Fourier series transform is isometry)**.**

$$\|\widehat{\mu}\|_{L^2(]-\pi,\pi[)} = \|\vec{\mu}\|_{l^2(\mathbb{Z})} \quad \forall \vec{\mu} \in l^2(\mathbb{Z}) \ .$$

Representation of linear (→ Def. 3.1.2), translation-invariant (→ Def. 3.1.3) finite difference method

$$(\mathcal{H}\vec{\mu})_j = \sum_{l=-m_l}^{m_r} \alpha_l \mu_{j+l} \ , \quad \vec{\mu} \in C^0(\mathcal{G}_{\Delta x}) \ , \quad \alpha_l \in \mathbb{R} \ . \tag{3.1.23}$$

▶ $$\mathcal{H}\vec{\mu} = \mathcal{F}^{-1}\left(\rho(\cdot)\cdot(\mathcal{F}\vec{\mu})\right) \ , \quad \forall \vec{\mu} \in l^1(\mathbb{Z}) \ , \quad \rho(\xi) := \sum_{l=-m_l}^{m_r} \alpha_l e^{il\xi} \tag{3.1.24}$$

Symbol of $\mathcal{H}$

⯆ ← Lemma 3.1.12

Corollary 3.1.13 (l^2-norm of linear, translation-invariant FDM evolution operator)**.** *For the linear, translation-invariant finite difference method* (3.1.23)

$$\|\mathcal{H}\|_{l^2(\mathbb{Z})} = \|\rho\|_{L^\infty(]-\pi,\pi[)} \ , \quad \rho = \textit{symbol of } \mathcal{H} \ .$$

$$\boxed{|\rho(\xi)| \leq 1 \quad \forall \xi \in]-\pi, \pi[\quad \Longrightarrow \quad \text{linear FDM stable, } \textit{cf.} \text{ Thm. 3.1.10}}$$

Example 52 (Symbols for linear translation-invariant FDM)*.*

- Constant linear advection (2.1.6), velocity $v > 0$

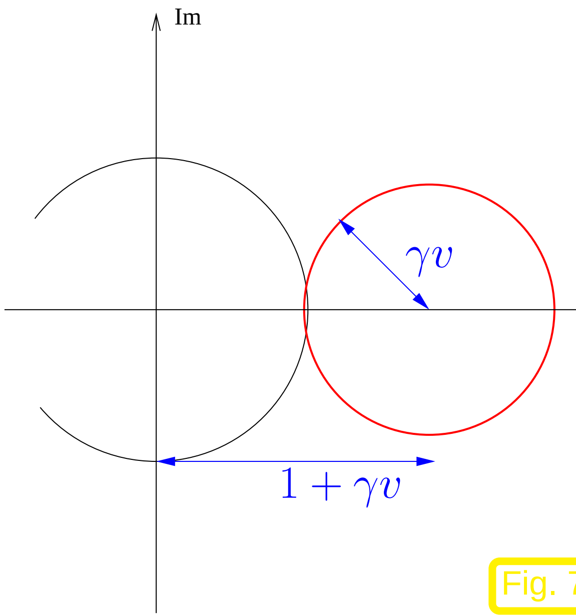


Fig. 75

$\rho(\xi)$ for (3.1.10)

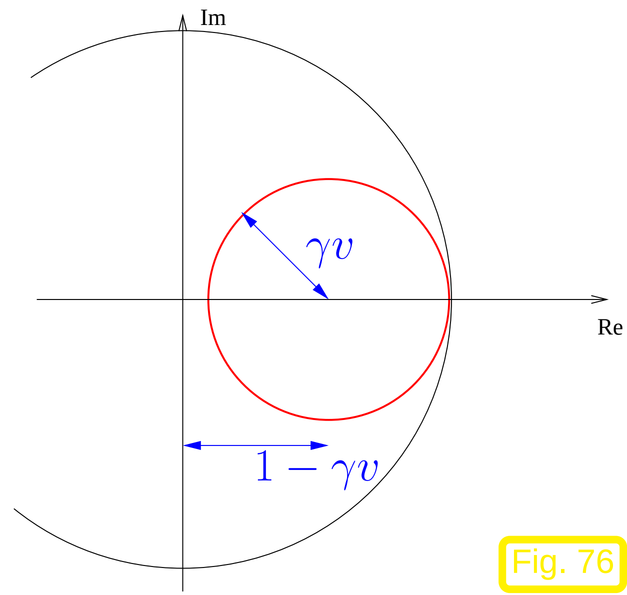


Fig. 76

$\rho(\xi)$ for (3.1.11)

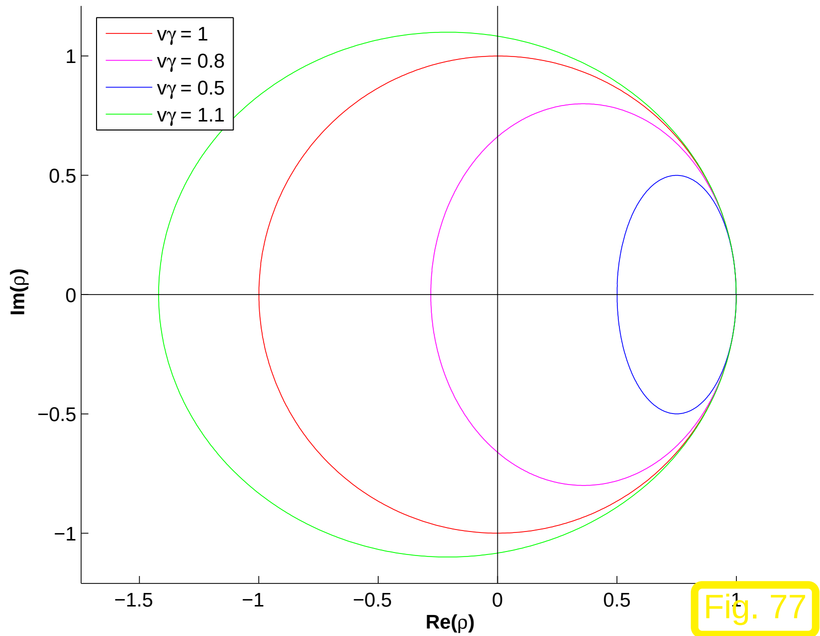


Fig. 77

$\rho(\xi)$ for Lax-Wendroff (3.1.12)

▶ in all cases: CFL-condition $\Longleftrightarrow$ stability

BUT: symbol for centered finite differences (3.1.9):

$$\rho(\xi) = 1 - iv\gamma \sin(\xi) \quad \Rightarrow \quad \max_{-\pi \leq \xi \leq \pi} |\rho(\xi)| > 1$$ ▶ (3.1.9) unconditionally unstable !

◇

3.1.3.2 Nonlinear stability

Policy: target norms pivotal in stability theory for scalar conservation laws $\longrightarrow$ Sect. 2.6:

Theoretical result	(semi-)norm	norm on $C^0(\mathcal{G}_{\Delta x})$
Maximum principle, Cor. 2.6.2	$\|\cdot\|_{L^\infty(\mathbb{R})}$	$\|\cdot\|_{l^\infty(\mathbb{Z})}$
L^1-contractivity, Cor. 2.6.3	$\|\cdot\|_{L^1(\mathbb{R})}$	$\|\cdot\|_{l^1(\mathbb{Z})}$
Total variation stability, Thm. 2.6.8	$TV_{\mathbb{R}}(\cdot)$	$TV_{\Delta x}(\cdot)$

try to find criteria for discrete counterparts of Cor. 2.6.2, Cor. 2.6.2, Thm. 2.6.8 for FDM

Note: function space norms $\leftrightarrow$ grid dependent norms: via interpretation of $\vec{\mu} \in C^0(\mathcal{G}_{\Delta x})$ as cell-p.w. constant function

➢ $TV_{\Delta x}(\vec{\mu})$ = total variation of function $u(x) = \mu_j$, $x_{j-1/2} \leq x < x_{j+1/2}$:

$$TV_{\Delta x}(\vec{\mu}) = \sum_{j\in\mathbb{Z}} |\mu_j - \mu_{j-1}| \,. \tag{3.1.25}$$

Definition 3.1.14 (Monotone discrete evolution)**.**
Discrete evolution (3.1.2) *is monotone, if*

$$\zeta_j \geq \mu_j \quad \forall j \in \mathbb{Z} \quad \Rightarrow \quad (\mathcal{H}\vec{\zeta})_j \geq (\mathcal{H}\vec{\mu})_j \quad \forall j \in \mathbb{Z}\ .$$

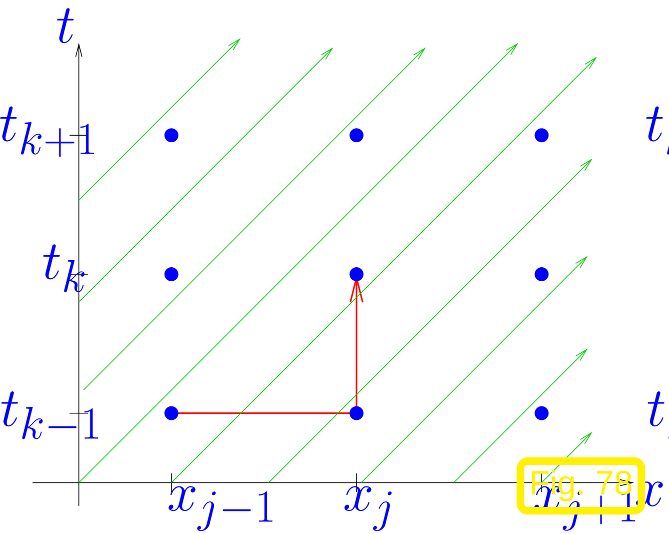
▶ discrete evolution monoton $\iff$ $\mathcal{H}$ non-decreasing in all its arguments

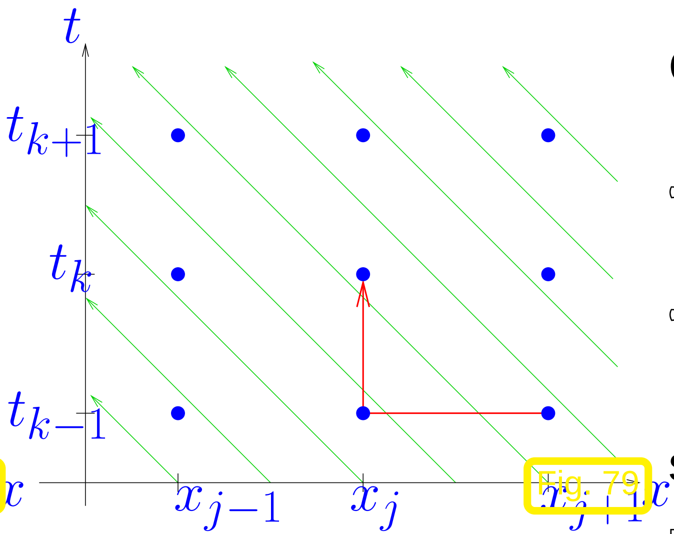
Lemma 3.1.15 (Monotone FDM are (linearly) L^∞-stable)**.**
$\mathcal{H}$ **=** *single step, time-invariant, translation-invariant, explicit finite difference method (→ Def. 3.1.1) with* $\mathsf{H}(u,\dots,u) = u$ *for all* $u \in \mathbb{R}$

$$\mathcal{H} \text{ monotone } (\rightarrow \text{Def. 3.1.14}) \quad \Longrightarrow \quad \begin{aligned} &\min_l \mu_l^{(0)} \leq \mu_j^{(k)} \leq \max_l \mu_l^{(0)} \quad \forall j,k\ , \\ &\left\|\vec{\mu}^{(k)}\right\|_{l^\infty(\mathbb{Z})} \leq \left\|\vec{\mu}^{(0)}\right\|_{l^\infty(\mathbb{Z})} \quad \forall k\ . \end{aligned}$$

Example 53 (Upwinding for linear advection).

Consider: constant linear advection (2.1.6) $\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial x} = 0$





— = characteristic curves $x - vt = \text{const}$.

CFL-condition (⟶ Def. 3.1.4) requires

☞ $v > 0$: backward differences (3.1.11)

☞ $v < 0$: forward differences (3.1.10)

stencil towards "upstream" direction

⇨ upwinding

$$\mu_j^{(k)} = \mathsf{H}_{\mathrm{uw}}(\mu_{j-1}^{(k-1)}, \mu_j^{(k-1)}, \mu_{j+1}^{(k-1)}) := \begin{cases} \mu_j - \gamma v\left(\mu_j^{(k)} - \mu_{j-1}^{(k-1)}\right) & \text{, if } v > 0 \text{ ,} \\ \mu_j - \gamma v\left(\mu_{j+1}^{(k+1)} - \mu_j^{(k)}\right) & \text{, if } v < 0 \text{ .} \end{cases} \tag{3.1.26}$$

$$\blacktriangleright \quad \zeta_j^{(k)} - \mu_j^{(k)} = \begin{cases} (1-\gamma v)(\zeta_j^{(k-1)} - \mu_j^{(k-1)}) + \gamma v(\zeta_{j-1}^{(k-1)} - \mu_{j-1}^{(k-1)}) & \text{, if } v > 0 \text{ ,} \\ (1+\gamma v)(\zeta_j^{(k-1)} - \mu_j^{(k-1)}) - \gamma v(\zeta_{j+1}^{(k-1)} - \mu_{j+1}^{(k-1)}) & \text{, if } v < 0 \text{ .} \end{cases}$$

▷ if $|v\gamma| \leq 1$ (⟷ CFL-condition) ⇒ 3-point upwind FDM (3.1.26) monotone (⟶ Def. 3.1.14) ◇

Example 54 (Monotonicity of non-linear upwind FDM).

Consider Cauchy problem (2.2.1) for $\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u) = 0$, $f \in C^1(\mathbb{R})$

Assumptions:
- $u_0 \in [0, u_{\max}]$ ➢ $0 \le u(x,t) \le u_{\max}$ a.e. in $\mathbb{R}\times]0,T[$
- $f'(u) \ge 0 \quad \Leftrightarrow \quad u \ge 0$ ➢ propagation only in $+x$-direction

▶ CFL-condition (→ Def. 3.1.4): use backward finite differences (3.1.20)
timestep constraint: $\gamma \max\limits_{0 \le u \le u_{\max}} f'(u) \le 1$

Monotonicity: if $(\zeta_j^{(k-1)} - \mu_j^{(k-1)}) \ge 0$ for all $j \in \mathbb{Z}$

$$
\begin{aligned}
\zeta_j^{(k)} - \mu_j^{(k)} &= \zeta_j^{(k-1)} - \mu_j^{(k-1)} - \gamma\left(f(\zeta_j^{(k-1)}) - f(\mu_j^{(k-1)})\right) + \gamma\left(f(\zeta_{j-1}^{(k-1)}) - f(\mu_{j-1}^{(k-1)})\right) \\
&\ge \underbrace{(1 - \gamma \max_{\mu_j^{(k-1)} \le u \le \zeta_j^{(k-1)}} f'(u))}_{\ge 0 \text{ by CFL-condition}}(\zeta_j^{(k-1)} - \mu_j^{(k-1)}) + \gamma \min_{\mu_{j-1}^{(k-1)} \le u \le \zeta_j^{(k-1)}} f'(u)(\zeta_{j-1}^{(k-1)} - \mu_{j-1}^{(k-1)}) \\
&\ge 0 \,.
\end{aligned}
$$

What to do, in case f' changes sign **?** → Sect. 3.2.2

◇

Simple criterion for translation-invariant explicit FDM:

$$\mathcal{H} \text{ monotone} \quad \Longrightarrow \quad \mathsf{H} \text{ non-decreasing in each } \mu\text{-argument}$$
$$\overset{\mathsf{H}\ C^1\text{-smooth}}{\Longrightarrow} \quad \partial_l \mathsf{H} \geq 0 \text{ everywhere, } l = -m_l + 1, \ldots, m_r \;. \tag{3.1.27}$$

Known: monotonicity holds for (discrete) parabolic evolutions

➤ can we use viscous modification (→ Rem. 49) to enforce monotonicity **?**

Approach (→ Sect. 3.2.3): start from first-order centered FDM (3.1.17) **+** viscous modification

$$\mu_j^{(k)} = \mu_j^{(k-1)} - \tfrac{1}{2}\gamma \left(f(\mu_{j+1}^{(k-1)}) - f(\mu_{j-1}^{(k-1)}) \right) + q \frac{\mu_{j+1}^{(k-1)} - 2\mu_j^{(k-1)} + \mu_{j-1}^{(k-1)}}{\Delta x^2} \tag{3.1.28}$$
$$=: \mathsf{H}(\mu_{j-1}^{(k-1)}, \mu_j^{(k-1)}, \mu_{j+1}^{(k-1)}) \;,$$

and choose $q = q(\mu_{j-1}^{(k-1)}, \mu_j^{(k-1)}, \mu_{j+1}^{(k-1)})$ such that $\rightarrow$ (3.1.27)

$$\partial_{-1}\mathsf{H}(\mu_{j-1}^{(k-1)}, \mu_j^{(k-1)}, \mu_{j+1}^{(k-1)}) = \tfrac{1}{2}\gamma f'(\mu_{j-1}^{(k-1)}) + \partial_{-1}q \frac{\mu_{j+1}^{(k-1)} - 2\mu_j^{(k-1)} + \mu_{j-1}^{(k-1)}}{\Delta x} + \frac{q}{\Delta x} \geq 0 \,,$$

$$\partial_0\mathsf{H}(\mu_{j-1}^{(k-1)}, \mu_j^{(k-1)}, \mu_{j+1}^{(k-1)}) = 1 + \partial_0 q \frac{\mu_{j+1}^{(k-1)} - 2\mu_j^{(k-1)} + \mu_{j-1}^{(k-1)}}{\Delta x} - \frac{2q}{\Delta x} \geq 0 \,,$$

$$\partial_1\mathsf{H}(\mu_{j-1}^{(k-1)}, \mu_j^{(k-1)}, \mu_{j+1}^{(k-1)}) = -\tfrac{1}{2}\gamma f'(\mu_{j-1}^{(k-1)}) + \partial_1 q \frac{\mu_{j+1}^{(k-1)} - 2\mu_j^{(k-1)} + \mu_{j-1}^{(k-1)}}{\Delta x} + \frac{q}{\Delta x} \geq 0 \,.$$

Simplest choice $q = \frac{1}{2}$: conditions met, because $|\gamma f'(u)| \leq 1$ for all possible u (CFL-condition !)

► (under CFL-condition *monotone*) Lax-Friedrichs 3-point FDM:

$$\mu_j^{(k)} = \tfrac{1}{2}(\mu_{j+1}^{(k-1)} + \mu_{j-1}^{(k-1)}) - \tfrac{1}{2}\gamma\left(f(\mu_{j+1}^{(k-1)}) - f(\mu_{j-1}^{(k-1)})\right) \,. \qquad (3.1.29)$$

Definition 3.1.16 (FDM in viscous form)**.**

Explicit, time-invariant, translation-invariant (→ Def. 3.1.3, Def. 3.1.1) FDM in viscous form reads

$$\mu_j^{(k)} = \underbrace{\mu_j^{(k-1)} - \tfrac{1}{2}\gamma\left(f(\mu_{j+1}^{(k-1)}) - f(\mu_{j-1}^{(k-1)})\right)}_{\textit{centered scheme } (3.1.17)} + \tfrac{1}{2}q_{j+1/2}(u_{j+1}^{(k-1)} - \mu_j^{(k-1)}) -$$
$$\tfrac{1}{2}q_{j-1/2}(u_j^{(k-1)} - \mu_{j-1}^{(k-1)}) \;, \quad j \in \mathbb{Z} \;,$$

where $q_{j+1/2} = q_{j+1/2}(\mu_{j-m_l+1}, \dots, \mu_{j+m_r})$.

Theorem 3.1.17 (l^∞-stability of FDM in viscous form)**.**

An explicit, time-invariant, translation-invariant finite difference method in viscous form (→ Def. 3.1.16) satisfies $\left\|\vec{\mu}^{(k)}\right\|_{l^\infty(\mathbb{Z})} \leq \left\|\vec{\mu}^{(0)}\right\|_{l^\infty(\mathbb{Z})}$ *for all* k*, if*

$$\gamma\left|\frac{f(\mu_{j+1}) - f(\mu_j)}{\mu_{j+1} - \mu_j}\right| \leq q_{j+1/2}(\mu_{j-m_l+1}, \dots, \mu_{j+m_r}) \leq \frac{1}{2} \quad \forall \mu \in l^\infty(\mathbb{Z}) \;.$$

Proof. $\mu_j^{(k)}$ = convex combination of $\mu_{j-1}^{(k-1)}$, $\mu_j^{(k-1)}$, $\mu_{j+1}^{(k-1)}$:

$$\mu_j^{(k)} = (1 - 1/2q_{j+1/2} + 1/2\gamma b_{j+1/2} - 1/2q_{j-1/2} + 1/2\gamma b_{j-1/2})\mu_j^{(k-1)} + \\ (1/2q_{j+1/2} - 1/2\gamma b_{j+1/2})\mu_{j+1}^{(k-1)} + (1/2q_{j-1/2} - 1/2\gamma b_{j-1/2})\mu_{j-1}^{(k-1)} ,$$

$b_{j+1/2} := \frac{f(\mu_{j+1})-f(\mu_j)}{\mu_{j+1}-\mu_j}$ ➤ $|\gamma b_{j+1/2}| \leq 1$ (CFL-conditioon). □

l^1-stability

If u_0 constant outside bounded interval ➤ conservation property of solution u of (2.2.1):

$$\frac{d}{dt}\int_{-\infty}^{\infty} u(x,t)\,\mathrm{d}x = -\int_{-\infty}^{\infty} \frac{\partial}{\partial x} f(u)\,\mathrm{d}x = f(u_{-\infty}) - f(u_{\infty}) \,.$$

Definition 3.1.18 (Conservative discrete evolution)**.**
Discrete evolution (3.1.2) *for* (2.2.1) *(on equidistant grid) is conservative*

$$\sum_{j\in\mathbb{Z}} (\mathcal{H}\vec{\mu})_j = \sum_{j\in\mathbb{Z}} \vec{\mu}_j + \gamma(f(\mu_{-\infty}) - f(\mu_{\infty})) \quad \forall \vec{\mu} \in C^0(\mathcal{G}_{\Delta x}) \text{ constant for } |j| > R \,.$$

Theorem 3.1.19 (conservative & monontone FDM are l^1-contracting). $\longrightarrow$ *[7]*

If a discrete evolution (3.1.2) *for* (2.2.1) *is monotone (*$\rightarrow$ *Def. 3.1.14) and conservative (*$\rightarrow$ *Def. 3.1.18), then*

$$\left\|\mathcal{H}\vec{\mu}-\mathcal{H}\vec{\zeta}\right\|_{l^1(\mathbb{Z})} \leq \left\|\vec{\mu}-\vec{\zeta}\right\|_{l^1(\mathbb{Z})} \quad \forall \vec{\mu},\vec{\zeta}\in l^1(\mathbb{Z}) \quad \begin{array}{l}\vec{\mu},\vec{\zeta}\equiv \mathrm{const} \text{ for } |j|>R\ ,\\ \mu_{\pm\infty}=\zeta_{\pm\infty}\ .\end{array}$$

Notations: $\alpha^+ := \max\{\alpha,0\}$, $\alpha\in\mathbb{R}$, $\max\{\vec{\mu},\vec{\zeta}\}\in C^0(\mathcal{G}_{\Delta x})$, $(\max\{\vec{\mu},\vec{\zeta}\})_j := \max\{\mu_j,\zeta_j\}$, $j\in\mathbb{Z}$

Total variation stability

Discrete counterpart of total variation stability of evolution for (2.2.1), Thm. 2.6.8:

Definition 3.1.20 (TVD-property).

A discrete evolution (3.1.2) *is called TVD (total variation decreasing), if*

$$TV_{\Delta x}(\mathcal{H}\vec{\mu}) \leq TV_{\Delta x}(\vec{\mu}) \quad \forall \vec{\mu}\in l^1(\mathbb{Z})\ .$$

Lemma 3.1.21 (l^1-contracting FDM are TVD). $\longrightarrow$ *[30, Thm. 15.4]*
If a discrete evolution (3.1.2) *is translation-invariant and* $l^1(\mathbb{Z})$*-contracting (*$\longrightarrow$ *Thm. 3.1.19), then it is TVD*

CFL-condition ▶ Lax-Friedrichs FDM (3.1.29) **&** upwind FDM (setting of Ex. 54) are TVD

Other criteria for TVD:

Incremental form of explicit, time-invariant, translation-invariant ($\longrightarrow$ Def. 3.1.3, Def. 3.1.1) FDM:

$$\begin{aligned}\mu_j^{(k)} = \mu_j^{(k-1)} &- c_{j-1/2}(\mu_{j-m_l+1},\dots,\mu_{j+m_r+1})(\mu_j^{(k-1)} - \mu_{j-1}^{(k-1)}) \\ &+ d_{j+1/2}(\mu_{j-m_l},\dots,\mu_{j+m_r})(\mu_{j+1}^{(k-1)} - \mu_j^{(k-1)})\ ,\end{aligned} \tag{3.1.30}$$

with functions $c_{j+1/2}, d_{j+1/2} : \mathbb{R}^{m_l+m_r+1} \mapsto \mathbb{R}$, $j \in \mathbb{Z}$.

Example 55 (3-point FDM in incremental form).

- backward finite differences (3.1.20)

$$\mu_j^{(k)} = \mu_j^{(k-1)} - \left(\gamma \frac{f(\mu_j^{(k-1)}) - f(\mu_{j-1}^{(k-1)})}{\mu_j^{(k-1)} - \mu_{j-1}^{(k-1)}}\right)(\mu_j^{(k-1)} - \mu_{j-1}^{(k-1)})$$

$$\Rightarrow \quad c_{j-1/2} = \gamma \frac{f(\mu_j^{(k-1)}) - f(\mu_{j-1}^{(k-1)})}{\mu_j^{(k-1)} - \mu_{j-1}^{(k-1)}} \quad , \quad d_{j+1/2} = 0 \; .$$

- Lax-Friedrichs 3-point FDM (3.1.29) for (2.2.1):

$$\begin{aligned} \mu_j^{(k)} = \mu_j^{(k-1)} &- \underbrace{\tfrac{1}{2}\left(1 + \gamma \frac{f(\mu_j^{(k-1)}) - f(\mu_{j-1}^{(k-1)})}{\mu_j^{(k-1)} - \mu_{j-1}^{(k-1)}}\right)}_{=c_{j-1/2}}(\mu_j^{(k-1)} - \mu_{j-1}^{(k-1)}) \\ &+ \underbrace{\tfrac{1}{2}\left(1 - \gamma \frac{f(\mu_{j+1}^{(k-1)}) - f(\mu_j^{(k-1)})}{\mu_{j+1}^{(k-1)} - \mu_j^{(k-1)}}\right)}_{=d_{j+1/2}}(\mu_{j+1}^{(k-1)} - \mu_j^{(k-1)}) \; . \end{aligned}$$

$\diamond$

Theorem 3.1.22 (Harten's theorem). $\longrightarrow$ *[22]*

An explicit, time-invariant, translation-invariant FDM in incremental form (3.1.30) *is TVD, if*

$$c_{j+1/2} \geq 0 \quad , \quad d_{j+1/2} \geq 0 \quad , \quad c_{j+1/2} + d_{j+1/2} \leq 1 \quad \forall j \in \mathbb{Z} \, .$$

Theorem 3.1.23 (TVD-FDM in viscous form).

An explicit, time-invariant, translation-invariant finite difference method in viscous form ($\longrightarrow$ Def. 3.1.16) satisfies is TVD, if

$$\gamma \left| \frac{f(\mu_{j+1}) - f(\mu_j)}{\mu_{j+1} - \mu_j} \right| \leq q_{j+1/2}(\mu_{j-m_l+1}, \dots, \mu_{j+m_r}) \leq 1 \quad \forall \mu \in l^\infty(\mathbb{Z}) \, .$$

Proof. Viscous form ➞ incremental form (3.1.30):

$$d_{j+1/2} := \tfrac{1}{2} \left(q_{j+1/2} - \gamma \frac{f(\mu_{j+1}^{(k-1)}) - f(\mu_j^{(k-1)})}{\mu_{j+1}^{(k-1)} - \mu_j^{(k-1)}} \right) \, ,$$

$$c_{j+1/2} := \tfrac{1}{2} \left(q_{j+1/2} + \gamma \frac{f(\mu_{j+1}^{(k-1)}) - f(\mu_j^{(k-1)})}{\mu_{j+1}^{(k-1)} - \mu_j^{(k-1)}} \right) \, .$$

Then apply Thm. 3.1.22. $\square$

Definition 3.1.24 (Monotonicity preservation)**.**

A discrete evolution is monotonicity preserving, if

$$\vec{\mu} \in C^0(\mathcal{G}_{\Delta x})\colon \quad \mu_{j-1} \underset{(\geq)}{\overset{\leq}{}} \mu_j \quad \forall j \in \mathbb{Z} \quad \Rightarrow \quad (\mathcal{H}\vec{\mu})_{j-1} \underset{(\geq)}{\overset{\leq}{}} (\mathcal{H}\vec{\mu})_j \quad \forall j \in \mathbb{Z} \; .$$

FDM is TVD & preserves constants $\Longrightarrow$ FDM is monotonicity preserving

Theorem 3.1.25 (Godunov's theorem)**.**

A linear *monotonicity preserving (→ Def. 3.1.24) discrete evolution is monotone (→ Def. 3.1.14)*

Proof. $\vec{\mu}$, $\vec{\xi}$ with $\mu_j \leq \xi_j$ allow representation

$$\xi_j = \mu_j + (\zeta_j - \zeta_{j-1}) \; , \quad \zeta_j = \zeta_{j-1} + \underbrace{\xi_j - \mu_j}_{\geq 0} \quad \Rightarrow \quad \vec{\zeta} \text{ non-decreasing} \; .$$

$\square$

3.2 Finite volume discretization 1D

➧ special class of translation invariant FDM ($\longrightarrow$ Def. 3.1.3) for (2.2.1)

Assume: equidistant tensor product grid, fixed ratio $\gamma := \Delta t / \Delta x > 0$

Adopt interpretation ($\longrightarrow$ Sect. 3.1): $$\mu_j^{(k)} \approx \frac{1}{\Delta x} \int_{x_{j-1/2}}^{x_{j+1/2}} u(x, t_k) \, \mathrm{d}x$$ (cell average)

(2.3.3) for $\widetilde{V} =]x_{j-1/2}, x_{j+1/2}[\times]t_{k-1}, t_k[$: ▷

$$
\begin{aligned}
\mu_j^{(k)} = \;& \mu_j^{(k-1)} \\
& - \frac{1}{\Delta x} \int_{t_{k-1}}^{t_k} f(u(x_{j+1/2}, t))\,\mathrm{d}t \\
& + \frac{1}{\Delta x} \int_{t_{k-1}}^{t_k} f(u(x_{j-1/2}, t))\,\mathrm{d}t \;.
\end{aligned}
\tag{3.2.1}
$$

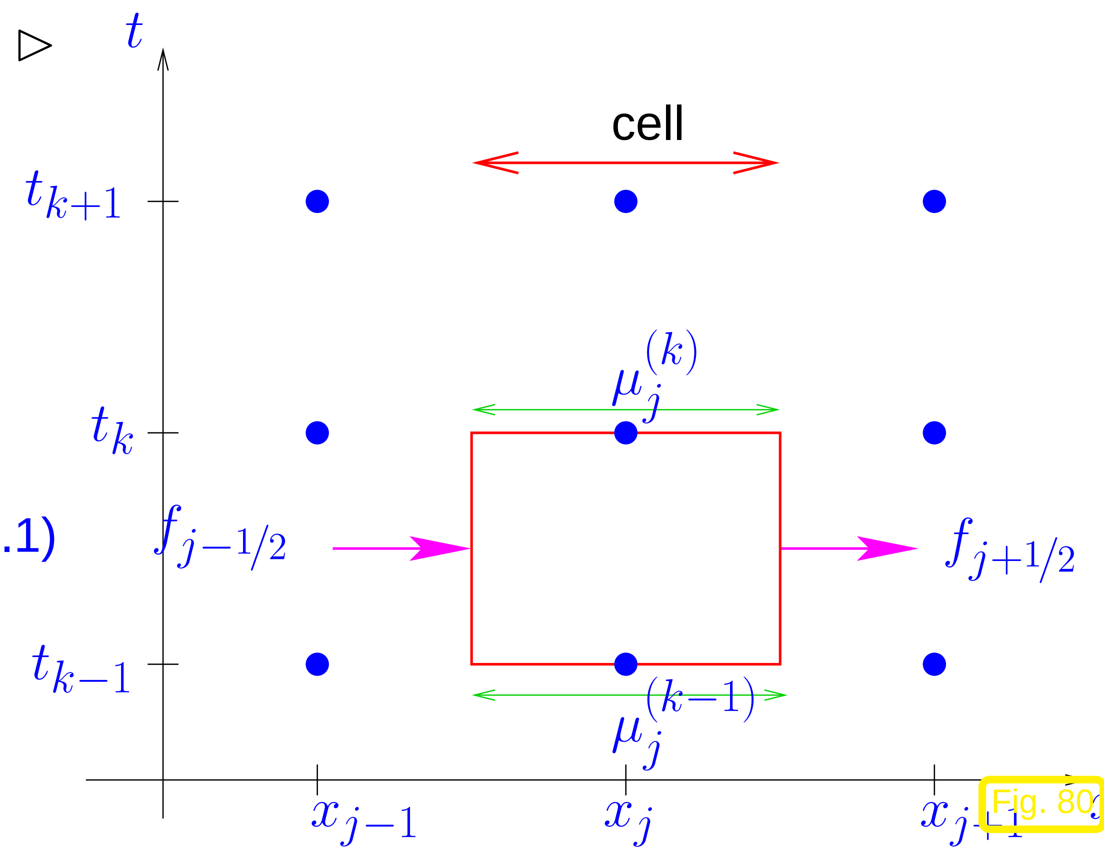


Fig. 80

Definition 3.2.1 (FDM in conservation form)**.**
Explicit, time-invariant, translation-invariant finite difference scheme (⟶ Def. 3.1.3) in conservation form

$$\mu_j^{(k)} = \mu_j^{(k-1)} - \gamma \left(f_{j+1/2}^{(k-1)} - f_{j-1/2}^{(k-1)} \right)$$

with numerical fluxes $f_{j+1/2}^{(k-1)} = F(\mu_{j-m_l+1}^{(k-1)}, \dots, \mu_{j+m_r}^{(k-1)})$*, and numerical flux function* $F : \mathbb{R}^{m_l+m_r} \mapsto \mathbb{R}$*.*

Terminology: FDM in conservation form = finite volume method (FVM)

☛ Def. 80 ▶ 3-point finite volume method:

$$\mu_j^{(k)} = \mu_j^{(k-1)} - \gamma(F(\mu_j^{(k-1)}, \mu_{j+1}^{(k-1)}) - F(\mu_{j-1}^{(k-1)}, \mu_j^{(k-1)})) \;, \tag{3.2.2}$$

☛ for theory: initial values for discrete evolution for FVM always obtained through

$$\mu_j^{(0)} := \frac{1}{\Delta x} \int_{x_{j-1/2}}^{x_{j+1/2}} u_0(x)\,\mathrm{d}x \;, \quad j \in \mathbb{Z} \;. \tag{3.2.3}$$

3.2.1 Consistent numerical flux functions

Consider: FDM in conservation form ($\longrightarrow$ Def. 3.2.1), consistent with $\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u) = 0$

- desirable approximation: $f_{j+1/2} \approx \frac{1}{\Delta t} \int_{t_{k-1}}^{t_k} f(u(x_{j+1/2}, t))\, \mathrm{d}t$

- F will always be assumed to be Lipschitz-continuous

- Focus on 3-point FDM: $\boxed{F = F(v, w),\ v, w \in \mathbb{R}}$

$$\mu_j^{(k)} = \underbrace{\mu_j^{(k-1)} - \gamma \left(F(\mu_j^{(k-1)}, \mu_{j+1}^{(k-1)}) - F(\mu_{j-1}^{(k-1)}, \mu_j^{(k-1)}) \right)}_{= \mathsf{H}(\mu_{j-1}^{(k-1)}, \mu_j^{(k-1)}, \mu_{j+1}^{(k-1)})} . \tag{3.2.4}$$

❶ Numerical flux function $F = F(v, w)$ smooth:
Lemma 3.1.11 $\Rightarrow$ necessary $\partial_l F(u, u) + \partial_r F(u, u) = f'(u), \quad u \in \mathbb{R}$

❷ Assume $u(x, t_{k-1}) = u^*$, $x_{j-1/2} < x < x_{j+3/2}$ & CFL-condition $\max_u f'(u) \cdot \Delta t \leq \Delta x$

➤ $f_{j+1/2} = F(u^*, u^*) = \frac{1}{\Delta t} \int_{t_{k-1}}^{t_k} f(u^*)\, \mathrm{d}t = f(u^*)$

Definition 3.2.2 (Consistent numerical flux functions)**.**
A numerical flux function $F : \mathbb{R}^{m_l+m_r} \mapsto \mathbb{R}$ *is consistent with the flux function* $f : \mathbb{R} \mapsto \mathbb{R}$, *if*

$$\exists C > 0,\ \delta > 0:\quad |F(u_{-m_l+1}, \dots, u_{m_r}) - f(u)| \leq C \sum_{k=-m_l+1}^{m_r} |u_k - u|$$

for all $u, u_{-m_l}, \dots, u_{m_r} \in \mathbb{R}$, $\sum_{k=-m_l}^{m_r} |u_k - u| \leq \delta$. *In particular,*

$$F(u, \dots, u) = f(u) \quad \forall u \in \mathbb{R} .$$

▶ FDM in conservation form with consistent numerical flux function are consistent ($\longrightarrow$ Def. 3.1.7)

Example 56 (Upwind flux)*.* $\quad\longrightarrow$ Ex. 54

Setting of Ex. 54: backward difference formula (3.1.20) in conservation form:

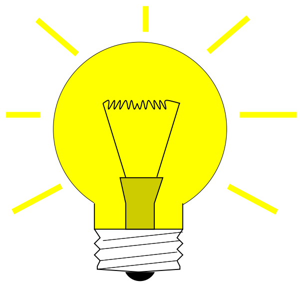
$$\mu_j^{(k)} = \mu_j^{(k-1)} - \gamma \left(f(\mu_j^{(k-1)}) - f(\mu_{j-1}^{(k-1)} \right)$$

$$\Updownarrow$$

$$\mu_j^{(k)} = \mu_j^{(k-1)} - \gamma \left(F(\mu_j^{(k-1)}, \mu_{j+1}^{(k-1)}) - F(\mu_{j-1}^{(k-1)}, \mu_j^{(k-1)}) \right) , \quad \boxed{F(v,w) = f(v)} .$$

Upwind flux for propagation in $+x$-direction

For propagation in $-x$-direction: use $F(v,w) = f(w)$ $\diamond$



Idea: Numerical flux $f_{j+1/2}$ depends on two states $\mu_j^{(k-1)}$, $\mu_{j+1}^{(k-1)}$:

- if $\mu_j^{(k-1)} \approx \mu_{j+1}^{(k-1)}$ ➤ same $f_{j+1/2}$ for any consistent numerical flux function
- if $v := \mu_j^{(k-1)}$, $w := \mu_{j+1}^{(k-1)}$ differ much (➪ discontinuity !)

  ► shock speed $\dot{s} = \frac{f(w)-f(v)}{w-v}$ $\approx$ local speed of propagation (**?**)

General upwind flux (Roe flux) for 1D scalar conservation lawn

$$F_{\mathrm{uw}}(v,w) := \begin{cases} f(v) & \text{, if } \dot{s} > 0 \text{ ,} \\ f(w) & \text{, if } \dot{s} < 0 \text{ ,} \end{cases} \qquad \dot{s} := \frac{f(w)-f(v)}{w-v} \text{ .} \tag{3.2.5}$$

$$\blacktriangleright \qquad \mu_j^{(k)} = \begin{cases} \mu_j^{(k-1)} - \gamma(f(\mu_j^{(k-1)}) - f(\mu_{j-1}^{(k-1)})) & \text{, if } \dot{s} > 0 \text{ ,} \\ \mu_j^{(k-1)} - \gamma(f(\mu_{j+1}^{(k-1)}) - f(\mu_j^{(k-1)})) & \text{, if } \dot{s} < 0 \text{ .} \end{cases} \tag{3.2.6}$$

Alternative upwind-type numerical flux function: Enquist-Osher flux

$$F_{\mathrm{EO}}(v,w) = \tfrac{1}{2}(f(v)+f(w)) - \tfrac{1}{2}\int_v^w |f'(\xi)|\,\mathrm{d}\xi \text{ .} \tag{3.2.7}$$

$$F_{\mathrm{EO}}(v,w) = \begin{cases} f(v) & \text{, if } \min_{u\in I} f'(u) > 0 \;, \\ f(w) & \text{, if } \max_{u\in I} f'(u) < 0 \;, \end{cases} \qquad I := [\min\{v,w\}, \max\{v,w\}] \;.$$

unambiguous direction

Example 57 (Centered flux).

(3.1.17): $$\mu_j^{(k)} = \mu_j^{(k-1)} - \tfrac{1}{2}\gamma \left(f(\mu_{j+1}^{(k-1)}) - f(\mu_{j-1}^{(k-1)}) \right)$$

$$\Updownarrow$$

$$\mu_j^{(k)} = \mu_j^{(k-1)} - \gamma \left(F_{\mathrm{c}}(\mu_j^{(k-1)}, \mu_{j+1}^{(k-1)}) - F_{\mathrm{c}}(\mu_{j-1}^{(k-1)}, \mu_j^{(k-1)}) \right) \;, \quad \boxed{F_{\mathrm{c}}(v,w) = \tfrac{1}{2}(f(v)+f(w))} \;.$$

centered flux

Ex. 52: moot point: stability of FDM in conservation form **not** guaranteed !

◇

Example 58 (Diffusive flux). $\longrightarrow$ Rem. 30

Simple *explicit* FDM on equidistant grid for parabolic Cauchy problem

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0 \quad \text{in } \mathbb{R}\times]0,T[\;, \quad u(\cdot,0) = u_0 \quad \text{in } \mathbb{R} \;.$$

$$\triangleright \quad \frac{\mu_j^{(k)} - \mu_j^{(k-1)}}{\Delta t} = \frac{\mu_{j+1}^{(k-1)} - 2\mu_j^{(k-1)} + \mu_{j-1}^{(k-1)}}{(\Delta x)^2} ,$$

$$\triangleright \quad \mu_j^{(k)} = \mu_k^{(k-1)} + \gamma \left(\frac{\mu_{j+1}^{(k-1)} - \mu_j^{(k-1)}}{\Delta x} - \frac{\mu_j^{(k-1)} - \mu_{j-1}^{(k-1)}}{\Delta x} \right)$$

➤ diffusive/viscous flux function: $$\boxed{F_{\mathrm{diff}}(v, w) = -\frac{1}{\Delta x}(w - v)} \tag{3.2.8}$$

◇

Example 59 (Lax-Friedrichs numerical flux function).

Lax-Friedrichs FDM (3.1.29) on equidistant grid:

$$\mu_j^{(k)} = \tfrac{1}{2}(\mu_{j+1}^{(k-1)} + \mu_{j-1}^{(k-1)}) - \tfrac{1}{2}\gamma \left(f(\mu_{j+1}^{(k-1)}) - f(\mu_{j-1}^{(k-1)}) \right)$$

$$\Updownarrow$$

$$\mu_j^{(k)} = \mu_j^{(k-1)} - \gamma \left(F_{\mathrm{LF}}(\mu_j^{(k-1)}, \mu_{j+1}^{(k-1)}) - F_{\mathrm{LF}}(\mu_{j-1}^{(k-1)}, \mu_j^{(k-1)}) \right) ,$$

$$\boxed{F_{\mathrm{LF}}(v, w) = \tfrac{1}{2}(f(v) + f(w)) - \frac{1}{2\gamma}(w - v)} . \tag{3.2.9}$$

Lax-Friedrichs flux = centered flux **+** diffusive flux

$\longleftrightarrow$ *cf.* construction of Lax-Friedrichs FDM by viscous modification ($\rightarrow$ Rem. 49)

Alternative: in light of CFL-condition $\max_u \gamma |f'(u)| < 1$ ($\rightarrow$ Def. 3.1.4)

$$F_{\mathrm{LF}}(v,w) = \tfrac{1}{2}(f(v)+f(w)) - \tfrac{1}{2}C(w-v) \;, \quad C := \max_{\inf u_0 < u < \sup u_0} |f'(u)| \;. \tag{3.2.10}$$

$\hat{=}$ local Lax-Friedrichs flux

$\Diamond$

Example 60 (Flux profiles).

Different numerical flux functions for Burgers equation (2.1.7)

transonic rarefaction region:

$$f'(v) < 0 < f'(w)$$

◇

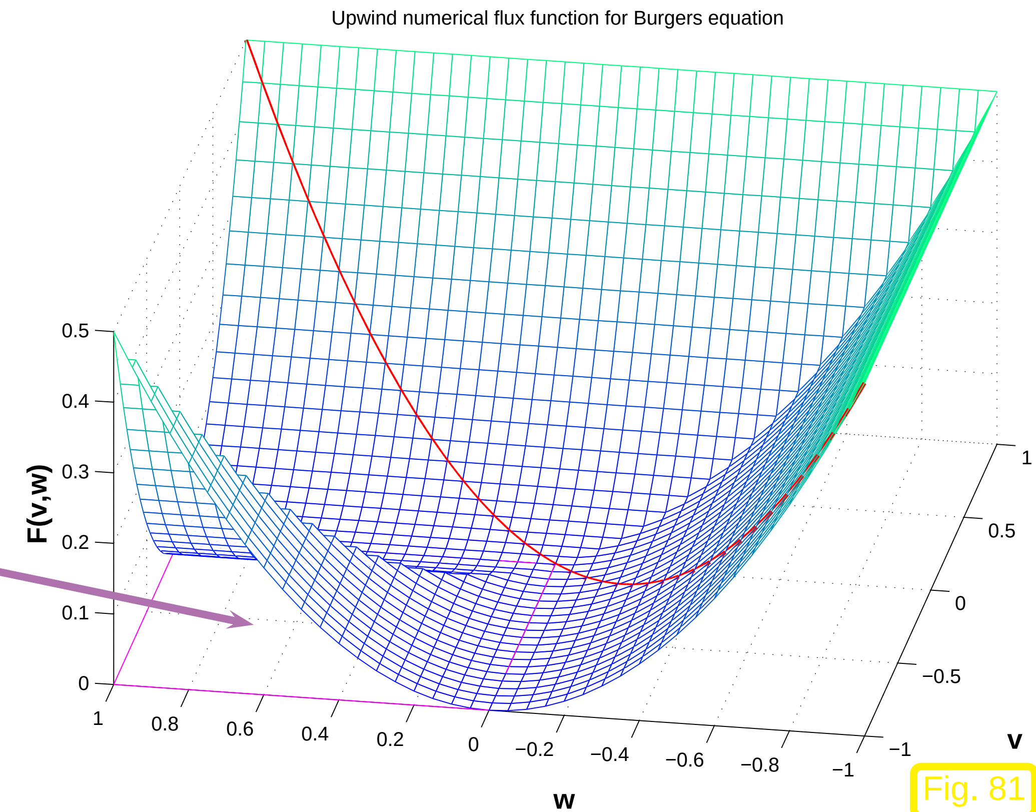


Fig. 81

Roe flux F_{uw}

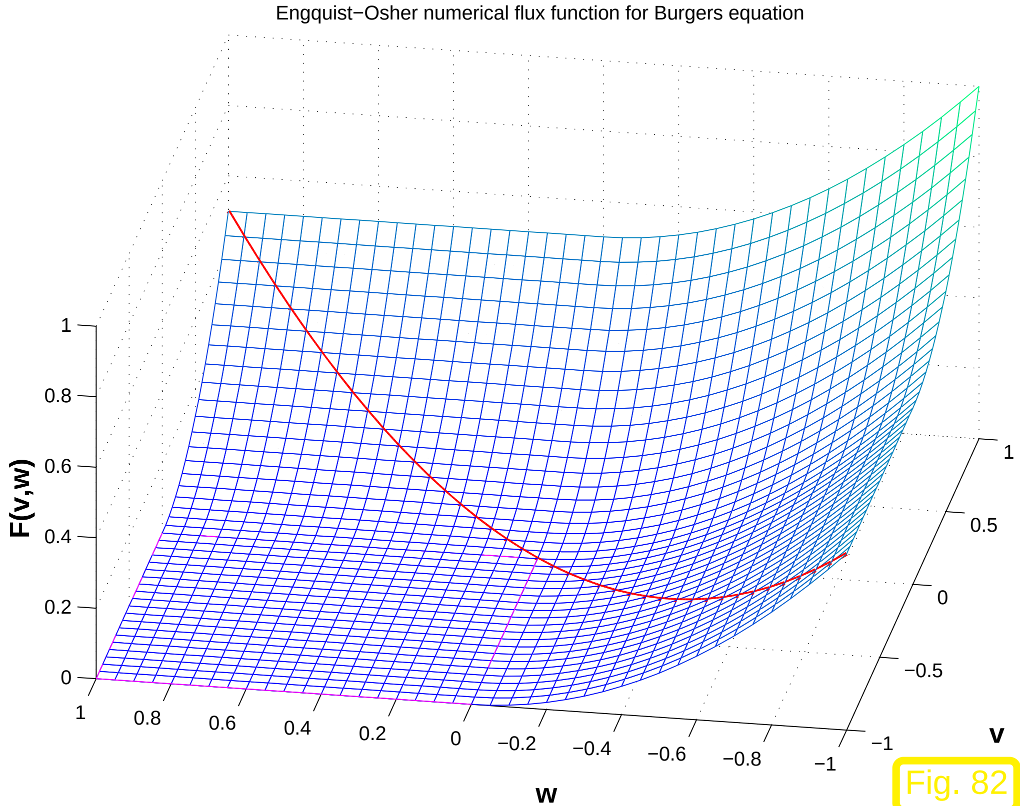


Fig. 82

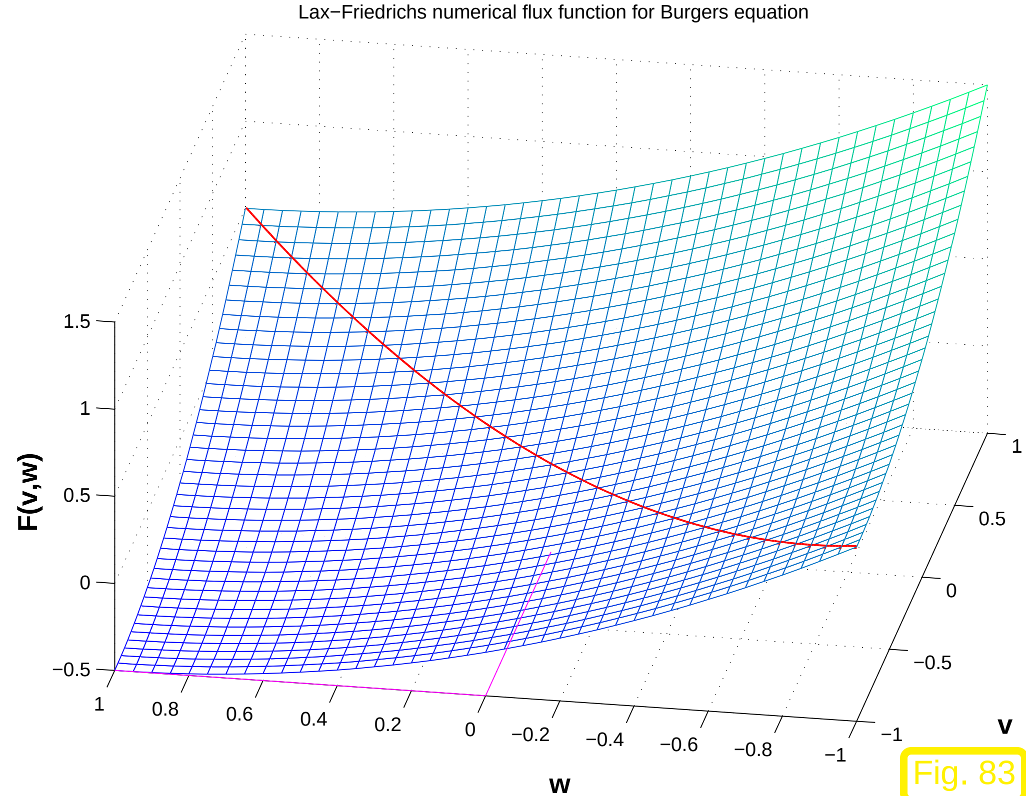


Fig. 83

E.-O. flux F_{EO}, $\gamma = 1$ L.-F. flux F_{LF}, $\gamma = 1$

Remark 61 (Viscous modification in conservation form). $\longrightarrow$ Rem. 49

$F \hat{=}$ numerical flux function for FDM in conservation form ($\rightarrow$ Def. 3.2.1)

☛ augmentation by diffusive flux: $\widetilde{F}(v,w) = F(v,w) - Q(v,w)(w-v)$, $Q : \mathbb{R}^2 \mapsto \mathbb{R}$

$$\mu_j^{(k)} = \mathsf{H}(\mu_{j-1}^{(k-1)}, \mu_j^{(k-1)}, \mu_{j+1}^{(k-1)}) + \gamma\Big(Q(\mu_j^{(k-1)}, \mu_{j+1}^{(k-1)})(\mu_{j+1}^{(k-1)} - \mu_{j-1}^{(k-1)}) - Q(\mu_{j-1}^{(k-1)}, \mu_j^{(k-1)})(\mu_j^{(k-1)} - \mu_{j-1}^{(k-1)})\Big) \ . \quad (3.2.11)$$

original method $\longleftrightarrow F$

$Q \geq 0$ $\leftrightarrow$ extra diffusion, $Q < 0$ $\leftrightarrow$ anti-diffusion

$\triangle$

3.2.2 Godunov's method

Still pending ($\rightarrow$ Sect. 54, *cf.* (3.2.5)): correct non-linear upwinding **?**

Consider Cauchy problem (2.2.1) for 1D scalar conservation law, flux function $f \in C^1(\mathbb{R})$

(Setting for discretization: : equidistant tensor product mesh $\mathcal{M} = \mathcal{G}_{\Delta x} \times \mathcal{G}_{\Delta t}$, $\gamma := \Delta x/\Delta t$)

Godunov's method: **=** pieceweise constant REA-algorithm for discrete evolution

➡ given $\vec{\mu}^{(k-1)}$ obtain $\vec{\mu}^{(k)}$ in 3 steps:

① **R**econstruct: here (interpretation → Sect. 3.1): $w_0 := \mathcal{C}\vec{\mu}^{(k-1)}$ p.w. constant on $\mathcal{G}_{\Delta x}$

② **E**volve: solve the Cauchy problem

$$\frac{\partial w}{\partial t} + \frac{\partial}{\partial x} f(w) = 0 \quad \text{in } \mathbb{R}\times]0,\Delta t[\;, \quad w(x,0) = w_0(x) \;,\; x \in \mathbb{R} \;. \tag{3.2.12}$$

③ **A**verage: get $\vec{\mu}^{(k)}$ from cell avarages: $\mu_j^{(k)} := \dfrac{1}{\Delta x}\displaystyle\int_{x_{j-1/2}}^{x_{j+1/2}} w(x,\Delta t)\,\mathrm{d}x$ (3.2.13)

Theorem 3.2.3 (Properties of Godunov's method)**.**
Godunov's method yields a time-invariant, translation-invariant, monotone (→ Def. 3.1.14) discrete evolution.

Observation: Godunov's method is in conservation form **!** (→ Def. 3.2.1)

$$\overset{(3.2.1)}{\Longrightarrow} \quad \mu_j^{(k)} = \mu_j^{(k-1)} - \frac{1}{\Delta x}\int_0^{\Delta t} f(w(x_{j+1/2},t))\,\mathrm{d}t + \frac{1}{\Delta x}\int_0^{\Delta t} f(w(x_{j-1/2},t))\,\mathrm{d}t \;. \tag{3.2.14}$$

Godunov numerical flux function

$$F_{\mathrm{GD}}(,\dots,\mu_{-1},\mu_0,\mu_1,\dots) := \frac{1}{\Delta t}\int_0^{\Delta t} f(w(0,t))\,\mathrm{d}t\ , \quad (3.2.15)$$

where $w = w(x,t)$ solves Cauchy problem (3.2.12) with p.w. constant initial data $w_0(x) = \mu_j$ for $(j-1)\Delta x < x < j\Delta x$, $j = -m_l+1,\dots,m_r$, $w_0 \equiv 0$ elsewhere.

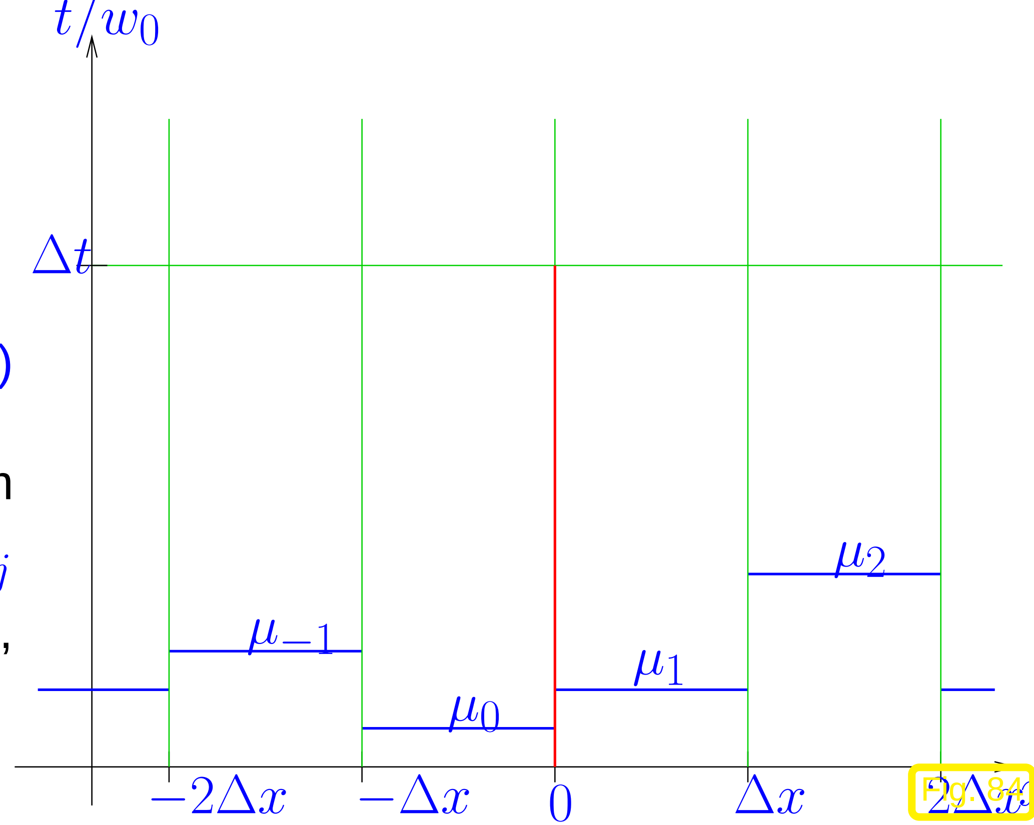
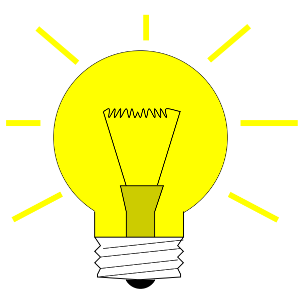


Fig. 84

Nice try ! BUT, how do you want to realize "**E**volve" **?**

Idea: finite speed of propagation **!**



If $\Delta t \le \dot{s}\Delta x$ ➤ F_{GD} only depends on two (adjacent) states μ_0, μ_1 !

➤ evolution (3.2.12) $\longleftrightarrow$ *local* ("non-interacting") Riemann problems ($\longrightarrow$ Def. 2.4.1)

($\dot{s} \hat{=}$ maximal speed of propagation,

$\dot{s} := \max\{|f'(\xi)|\colon \operatorname{essinf}_{x\in\mathbb{R}} u_0(x) \le \xi \le \operatorname{esssup}_{x\in\mathbb{R}} u_0(x)\}$, Cor. 2.6.3)

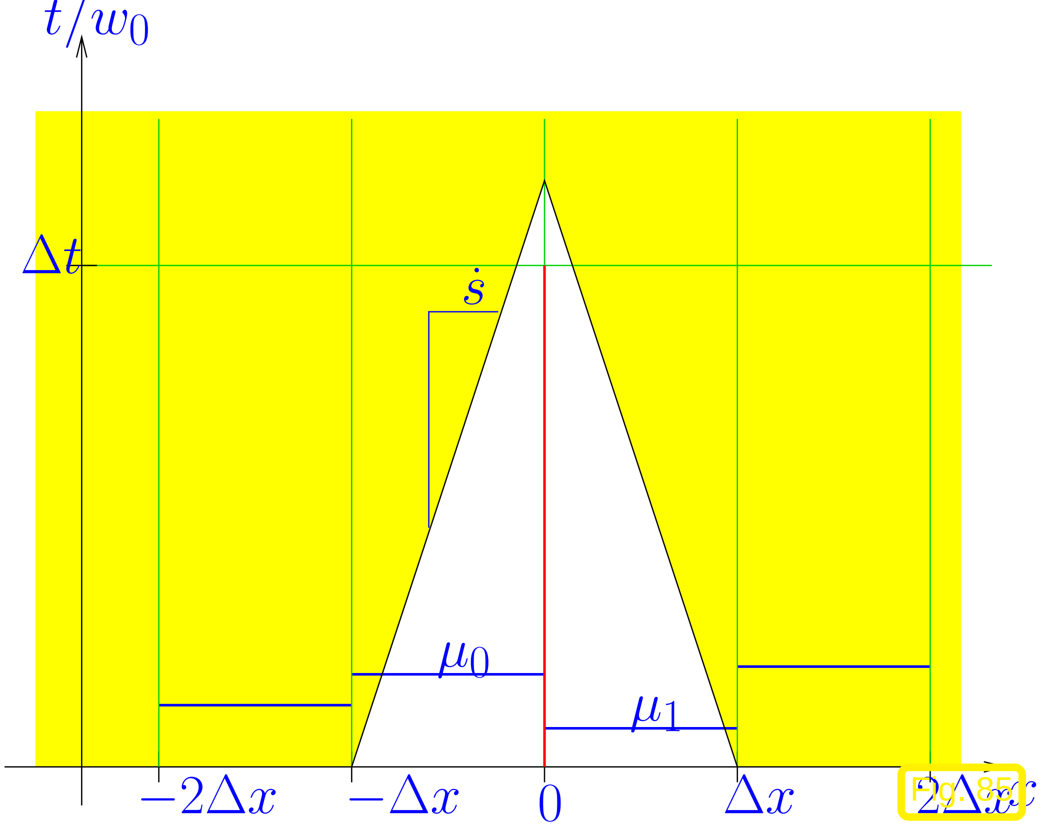


Fig. 85

Assume: $\dot{s}\Delta t \leq \Delta x$

$\triangleleft$ domain of influence of non-adjacent grid cells

For Godunov flux from (3.2.15):

$$F_{\mathsf{GD}} = F_{\mathsf{GD}}(\mu_0, \mu_1) \; .$$

► Godunov's method
= 3-point FDM in conservation form

CFL-condition ($\to$ Def. 3.1.4) $\Rightarrow$ solution w of (3.2.12) agrees with solution of Riemann problem at $x = x_{j-1/2}$ ($\to$ Def. 2.4.1) with $u_l = \mu_{j-1}^{(k)}$, $u_r = \mu_j^{(k)}$ on $(x_{j-1/2}, t)$, $0 \leq t \leq \Delta t$!

Entropy solutions of Riemann problems are similarity solutions:
(*cf.* Lemma 2.4.3, Lemma 2.4.4, Rem. 43)

$$u \text{ solves Riemann problem} \quad \Longrightarrow \quad u(x,t) = \psi(x/t) \quad \text{a.e. in } \mathbb{R}\times]0,T[\; .$$

➤ $\boxed{F_{\mathrm{GD}}(v,w) = f(u(0,t)) = f(\psi(0))}$, $u \hat{=}$ Riemann solution for $u_l = v$, $u_r = w$.

✎ Notation: $u^{\downarrow}(v,w) := u(0,t) = \psi(0)$ for entropy solution u of Riemann problem with $u_l = v$, $u_r = w$

Special case: $f : \mathbb{R} \mapsto \mathbb{R}$ strictly convex & smooth (e.g. Burgers equations (2.1.7))

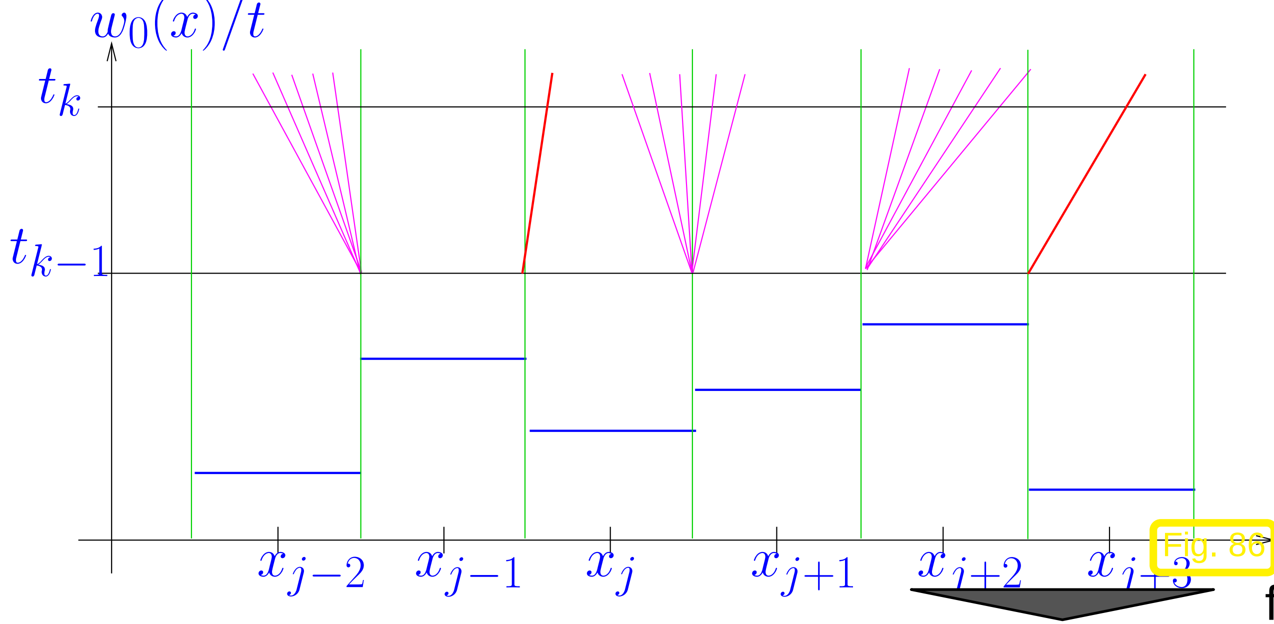
➤ Riemann problem ($\longrightarrow$ Def. 2.4.1) for (2.2.1) has the solution:

❶ If $u_l > u_r$ ➤ discontinuous solution, shock ($\longrightarrow$ Lemma 2.4.3)

$$u(t,x) = \begin{cases} u_l & \text{if } x < \dot{s}t \,, \\ u_r & \text{if } x > \dot{s}t \,, \end{cases} \qquad \dot{s} = \frac{f(u_l) - f(u_r)}{u_l - u_r} \,.$$

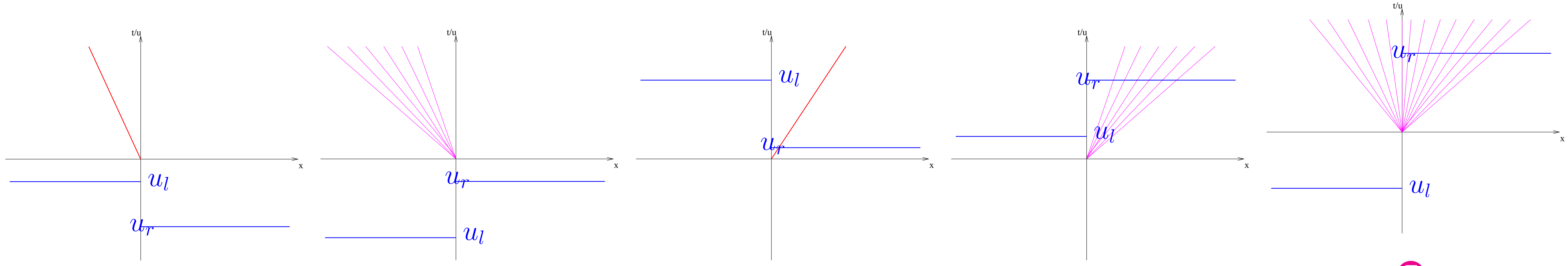
❷ If $u_l \le u_r$ ➤ continuous solution, rarefaction wave ($\longrightarrow$ Lemma 2.4.4)

$$u(t,x) = \begin{cases} u_l & \text{if } x < f'(u_l)t \,, \\ g(x/t) & \text{if } f'(u_l) \le x/t \le f'(u_r) \,, \\ u_r & \text{if } x > f'(u_r)t \,, \end{cases} \qquad g := (f')^{-1} \,.$$



for *convex* flux function f

$$u^{\downarrow}(u_l, u_r) = \begin{cases} u_r & \text{, if } \begin{array}{l} u_l > u_r \wedge \dot{s} < 0 \text{ (shock ❶) ,} \\ u_l < u_r \wedge f'(u_r) < 0 \text{ (rarefaction ❷) ,} \end{array} \\ u_l & \text{, if } \begin{array}{l} u_l > u_r \wedge \dot{s} > 0 \text{ (shock ❸),} \\ u_l < u_r \wedge f'(u_l) > 0 \text{ (rarefaction ❹) ,} \end{array} \\ (f')^{-1}(0) & \text{, if } u_l < u_r \wedge f'(u_l) \leq 0 \leq f'(u_r) \text{ (rarefaction ❺).} \end{cases} \qquad (3.2.16)$$



❶: subsonic shock ❷: subsonic rarefaction ❸: supersonic shock ❹: supersonic rarefaction ❺: transonic rarefaction

Using general Riemann solution (2.5.5): for any flux function

▶ Godunov numerical flux function

$$F_{\mathsf{GD}}(v,w) = \begin{cases} \min\limits_{v \leq u \leq w} f(u) & \text{, if} \quad v < w \;, \\ \max\limits_{w \leq u \leq v} f(u) & \text{, if} \quad w \leq v \;. \end{cases} \tag{3.2.17}$$

for Burgers equation (2.1.7) ▷
(*c.f.* Ex. 60)

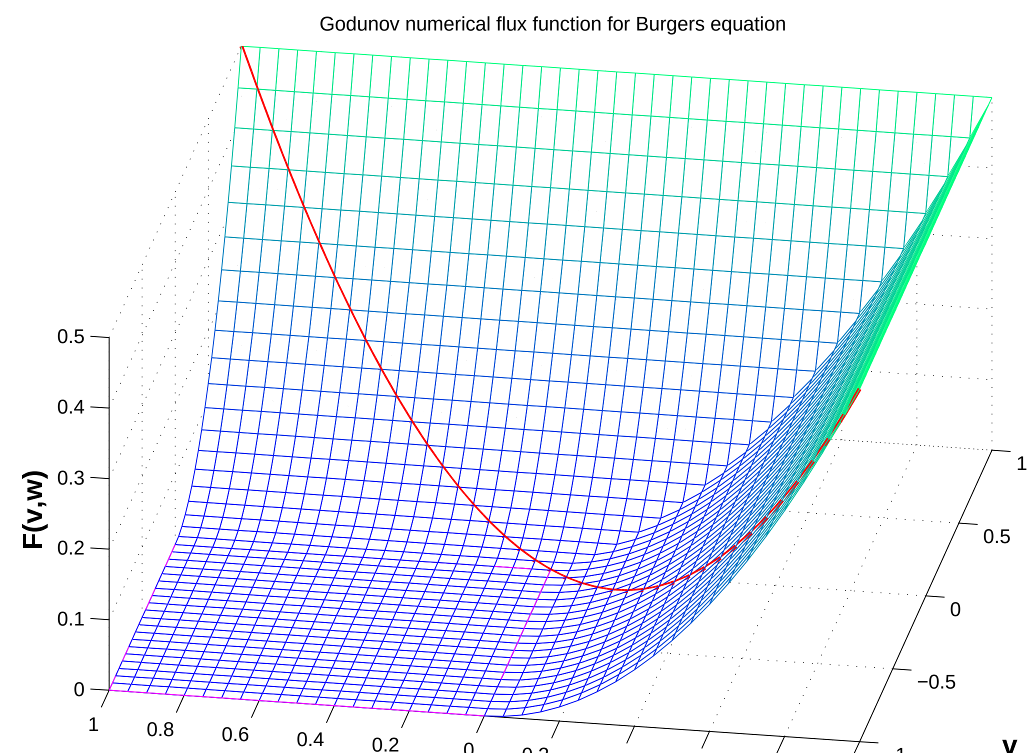


Fig. 87

Remark 62 (Simple upwinding as REA-method).

General 2-point upwind scheme (3.2.6) **=**

- REA-algorithm under CFL-condition ($|\gamma f'(u)| \leq 1$ for all possible u) with
- only (even entropy violating !) shock solutions of local Riemann problems (3.2.12) ($\rightarrow$ Lemma. 2.4.3) taken into account.

➤ (Roe) upwinding (3.2.6) is *monotone* ($\rightarrow$ Def. 3.1.14) (Thm. 3.1.23 $\rightarrow$ alterative proof)

$\triangle$

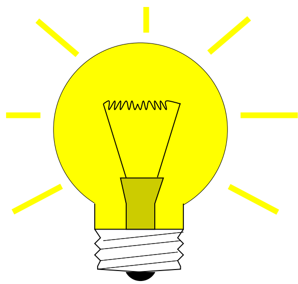
3.2.3 Modified equations

Setting of Sect. 3.1.2 (➔ equidistant tensor product grids, $\gamma := \Delta t/\Delta x > 0$ fixed !):

➔ explicit translation-invariant finite volume discretization ($\rightarrow$ Def. 3.2.1) of (2.2.1)

Assume ($\rightarrow$ Sect. 3.1.2): solution $u = u(x,t)$ of (2.2.1) “sufficiently” smooth (in space & time)

Definition 3.2.4 (Modified equation)**.**
Let a finite difference method (FDM) (3.1.6) *be consistent with* (2.2.1) *of order* p, $p \in \mathbb{N}$, *in space and time (*$\longrightarrow$ *Def. 3.1.7). Any PDE, to which it is consistent of order* $p+1$ *in space and time (*$\longrightarrow$ *Def. 3.1.7), is called a modified equation (ME) for the FDM.*



Idea:
- FDM yields "better" solutions of modified equation than of (2.2.1)
  (➢ discrete solution will display features of solution of ME)
- study solutions of modified equation (qualitatively)

➢ qualitative insights into discretization error for (3.1.6)

Lemma 3.2.5 (Modified equation for first-order 3-point FVM)**.** $\longrightarrow$ *[24, Sect. 2]*
Explicit 3-point FDM (3.1.16) *in conservation form (*$\longrightarrow$ *Def. 3.2.1,* (3.2.4)*) with* C^2 *numerical flux function* F *and first-order consistent with* (2.2.1), *is second order consistent with*

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u) = \Delta t \frac{\partial}{\partial x}\left(b(u,\gamma) \frac{\partial u}{\partial x} \right) , \tag{3.2.18}$$

with

$$\begin{aligned} b(u,\gamma) &:= \frac{1}{2\gamma^2}\left(\partial_{-1}\mathsf{H}(u,u,u) + \partial_1 \mathsf{H}(u,u,u)\right) - \tfrac{1}{2}(f'(u))^2 \\ &= \frac{1}{2\gamma}(\partial_l F(u,u) - \partial_r F(u,u) - \gamma f'(u)^2) . \end{aligned} \tag{3.2.19}$$

➡ Notation: $\partial_l F$, $\partial_r F \hat{=}$ partial derivative of numerical flux function for 3-point FVM

Proof. Lemma 3.1.11 $\Rightarrow$
$$\begin{aligned}\mathsf{H}(u,u,u) &= u,\\ \partial_{-1}\mathsf{H}(u,u,u)-\partial_1\mathsf{H}(u,u,u) &= \gamma f'(u),\end{aligned}$$
$\forall u\in\mathbb{R}$, with $\mathsf{H}(u,v,w):=v-\gamma(F(v,w)-F(u,v))$.

$$D^2\mathsf{H}(u,u,u)=\begin{pmatrix}\gamma\partial_l^2F(u,u) & \gamma\partial_l\partial_rF(u,u) & 0\\ \gamma\partial_l\partial_rF(u,u) & \gamma(-\partial_l^2F(u,u)+\partial_r^2F(u,u)) & -\gamma\partial_l\partial_rF(u,u)\\ 0 & -\gamma\partial_l\partial_rF(u,u) & \gamma\partial_r^2F(u,u)\end{pmatrix}\ .$$

Tool: Taylor expansion of local truncation error $\tau_j^{(k)}$ (3.1.7) $\rightarrow$ Sect. 3.1.2, up to terms $O((\Delta x)^3)$

$$\mathsf{H}(u(x-\Delta x,t),u(x,t),u(x+\Delta x,t))\doteq$$
$$\begin{aligned}&\doteq \mathsf{H}(u,u,u)+\partial_{-1}\mathsf{H}(u,u,u)(u(x-\Delta x,t)-u(x,t))+\partial_1\mathsf{H}(u,u,u)(u(x+\Delta x,t)-u(x,t))+\\ &\quad \tfrac{1}{2}\partial_{-1}^2\mathsf{H}(u,u,u)(u(x-\Delta x,t)-u(x,t))^2+\tfrac{1}{2}\partial_1^2\mathsf{H}(u,u,u)(u(x+\Delta x,t)-u(x,t))^2+O((\Delta x)^3)\\ &\doteq u+\Delta x u_x(\partial_1\mathsf{H}-\partial_{-1}\mathsf{H})(u,u,u)+\\ &\quad \tfrac{1}{2}(\Delta x)^2\Big(u_{xx}(\partial_{-1}\mathsf{H}+\partial_1\mathsf{H})(u,u,u)+(u_x)^2(\partial_{-1}^2\mathsf{H}+\partial_1^2\mathsf{H})(u,u,u)\Big)\\ &\doteq\ u-\gamma\Delta x u_x f'(u)+\tfrac{1}{2}(\Delta x)^2\big(\tfrac{\partial}{\partial x}((\partial_{-1}\mathsf{H}+\partial_1\mathsf{H})(u,u,u)\cdot u_x)-\\ &\quad \underbrace{(\partial_0\partial_{-11}\mathsf{H}+\partial_0\partial_1\mathsf{H})(u,u,u)}_{=0}(u_x)^2\big),\end{aligned}$$

where $u:=u(x,t)$, $u_x:=\frac{\partial u}{\partial x}(x,t)$, $u_{xx}:=\frac{\partial^2u}{\partial x^2}(x,t)$, $u_t:=\frac{\partial u}{\partial t}(x,t)$, $u_{tt}:=\frac{\partial^2 u}{\partial t^2}(x,t)$.

$$u(x,t+\Delta t)\doteq u+\Delta t u_t+\tfrac{1}{2}u_{tt}(\Delta t)^2=u-\Delta t f'(u)u_x+\tfrac{1}{2}(\Delta t)^2\frac{\partial}{\partial x}((f'(u))^2u_x)\ .$$

$$\blacktriangleright \quad \tau_j^{(k)} = \Delta t \frac{\partial}{\partial x}\left(\left(\frac{1}{2\gamma^2}(\partial_{-1}\mathsf{H} + \partial_1\mathsf{H})(u,u,u) - \tfrac{1}{2}(f'(u))^2\right)\frac{\partial u}{\partial x}\right) + O((\Delta t)^2) . \qquad \Box$$

Example 63 (Modified equations for simple 3-point FDM).

- first-order backward finite differences (3.1.20) for $\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}f(u) = 0$

$$\begin{aligned}\partial_{-1}\mathsf{H}(u,u,u) &= \gamma f'(u) \\ \partial_1\mathsf{H}(u,u,u) &= 0\end{aligned} \quad \Rightarrow \quad b(u,\gamma) = \frac{1}{2\gamma}f'(u)(1-\gamma f'(u)) \tag{3.2.20}$$

$$\blacktriangleright \quad \text{Modified equation:} \quad \frac{\partial u}{\partial t} + \frac{\partial}{\partial x}F(u) = \tfrac{1}{2}\Delta x f'(u)(1-\gamma f'(u)) . \tag{3.2.21}$$

$$f'(u) > 0 \wedge |\gamma f'(u)| \leq 1 \quad \blacktriangleright \quad b(u,\gamma) \geq 0$$

- first-order centered finite differences (3.1.17) for $\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}f(u) = 0$

$$\begin{aligned}\partial_{-1}\mathsf{H}(u,u,u) &= \tfrac{1}{2}\gamma f'(u) \\ \partial_1\mathsf{H}(u,u,u) &= -\tfrac{1}{2}\gamma f'(u)\end{aligned} \quad \Rightarrow \quad b(u,\gamma) = -\tfrac{1}{2}(f'(u))^2 \leq 0 . \tag{3.2.22}$$

- first-order Lax-Friedrichs scheme (3.1.29) for $\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}f(u) = 0$

$$\begin{aligned}\partial_l F_{\mathrm{LF}}(v,w) &= \tfrac{1}{2}f'(v) + 1/2\gamma , \\ \partial_r F_{\mathrm{LF}}(v,w) &= \tfrac{1}{2}f'(v) - 1/2\gamma ,\end{aligned} \quad \Rightarrow \quad b(u,\gamma) = \frac{1}{2\gamma^2}(1-(\gamma f'(u))^2) \geq 0 \quad \text{(CFL !)} . \tag{3.2.23}$$

$\diamond$

What does the modified equation (3.2.18) tell us **?**

$b(u,\gamma) > 0$ ➩ (3.2.18) **=** quasi-linear *parabolic evolution problem* ("heat equation"), *cf.* (2.5.2), Sect. 2.5.1:

- stable evolution: existence & uniqueness of smooth solutions $\forall t > 0$
- evolution diffusive/dissipative: has smoothing effect $\longrightarrow$ Ex. 40 ➢ shock smearing

$b(u,\gamma) < 0$ ➩ (3.2.18) **=** ill-posed IBVP for *"backward heat equation"*

- unconditionally unstable: exponential blow-up of solutions
- $b(u,\gamma) < 0 \leftrightarrow$ instability of discrete evolution (3.1.16) ($\longrightarrow$ Sect. 3.1.3)

Example 64 (Diffusive 3-point schemes)*.*

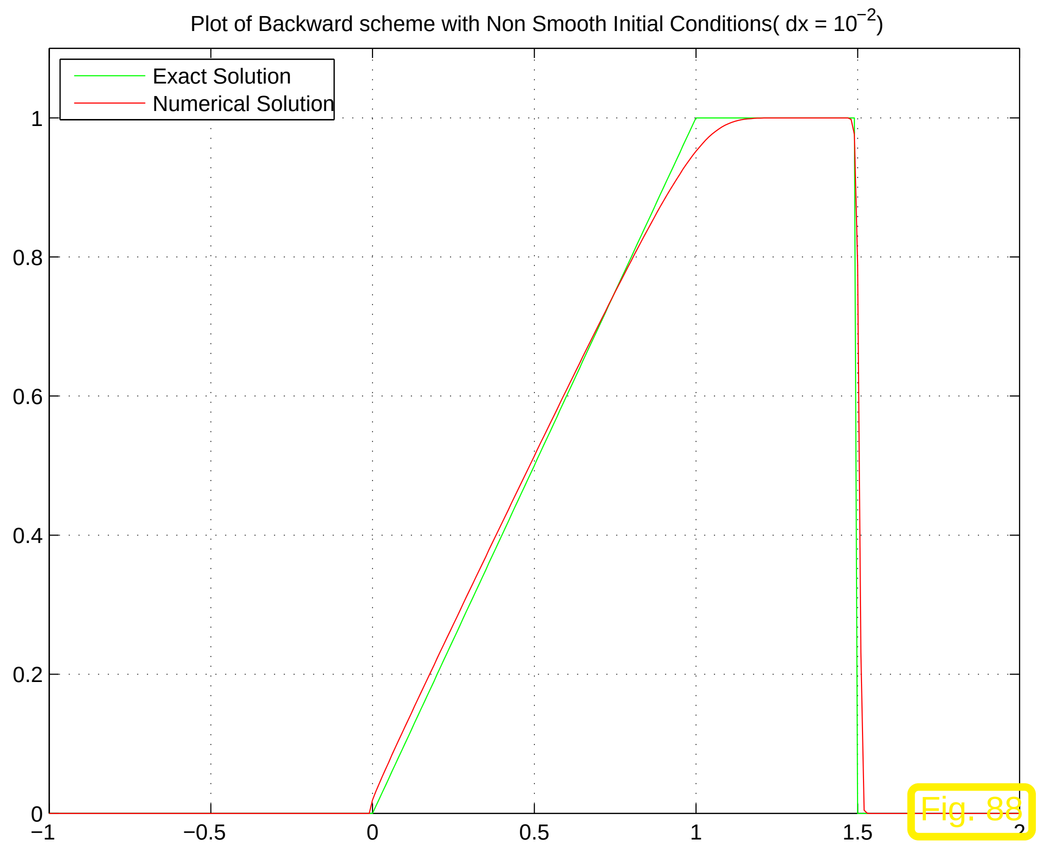
- Cauchy problem for Burgers equation (2.1.7)
- initial data: C^1-"bump" (4.2.3), "box function" $u_0 = \chi_{]0,1[}$ (4.2.5)

- equidistant grid $\mathcal{M} = \mathcal{G}_{\Delta x} \times \mathcal{G}_{\Delta t}$, $\gamma := {}^{\Delta t}/_{\Delta x} = 0.5$

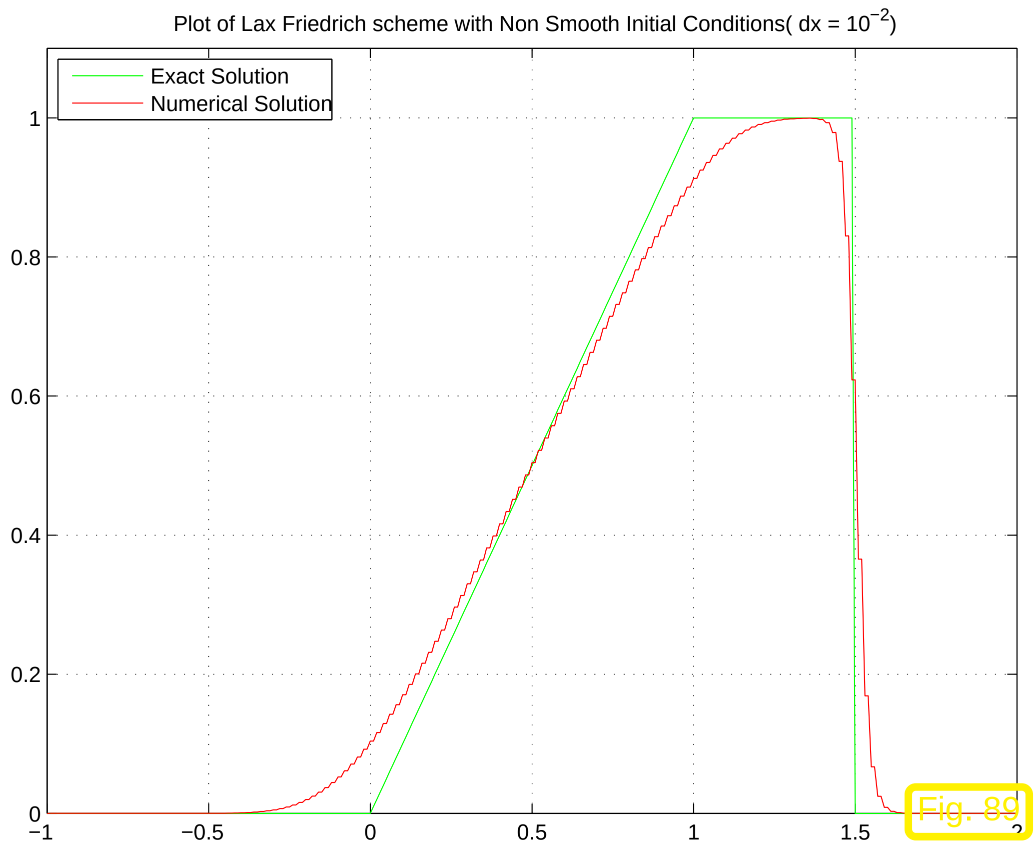
- FDM: backward finite differences (3.1.20), Lax-Friedrich scheme (3.1.29)

Monitored: Approximate solutions for $T = 1$ and animated discrete evolutions for $\Delta x = 10^{-2}$,

➤ movie: burger_godunov_box.avi, movie: burger_lf_box.avi



$u(x,1)$ for backward FD

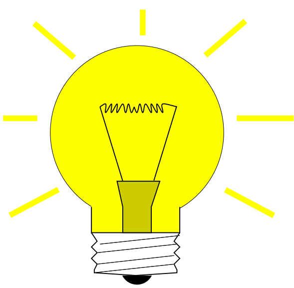


$u(x,1)$ for Lax-Friedrichs

Observation: smoothing of shock discontinuity due to diffusive character (shock smearing)

different amounts of *diffusivity* in the schemes $\longrightarrow$ Sect. 3.2.9 $\diamond$

Second order schemes for non-linear conservation laws **?**



Idea: Lemma 3.2.5: $b(u,\gamma) = 0$ ➤ 2nd-order 3-pt FDM for $\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u) = 0$

▶ use (3.2.19) to construct 2nd-order 3-point FDM (for non-linear case)

(Lax-Wendroff-scheme for non-linear conservation law)

Recall: 2nd-order Lax-Wendroff scheme for constant advection: (3.1.12) rewritten

$$\boxed{\mu_j^{(k)} = \mu_j^{(k-1)} - \tfrac{1}{2}\gamma v(\mu_{j+1}^{(k-1)} - \mu_{j-1}^{(k-1)})} + \boxed{\tfrac{1}{2}\gamma^2 v^2(\mu_{j+1}^{(k-1)} - 2\mu_j^{(k-1)} + \mu_{j-1}^{(k-1)})} \qquad (3.2.24)$$

centered finite differences (3.1.9) discrete diffusive term $\longrightarrow$ Rem. 49

▶ obtain $b(u,\gamma) = 0$ through *viscous modification* of first-order centered FDM (3.1.17) !

Preserve conservation form !

➤ viscous augmentation of centered flux (→ Rem. 61)

► $F(v,w) = \frac{1}{2}(v+w) - q(v,w)(w-v)$,

with $q : \mathbb{R}^2 \mapsto \mathbb{R}$ C^1-smooth.

$$\gamma^{-1} q(u,u) - \tfrac{1}{2}(f'(u))^2 = 0$$

$$\overset{\text{Lemma 3.2.5}}{\Longrightarrow} \quad b(u,\gamma) = 0 \quad \Longrightarrow \quad \text{2nd-order .}$$

F_{LW} for Burgers equation, $\gamma = 1$ ▷

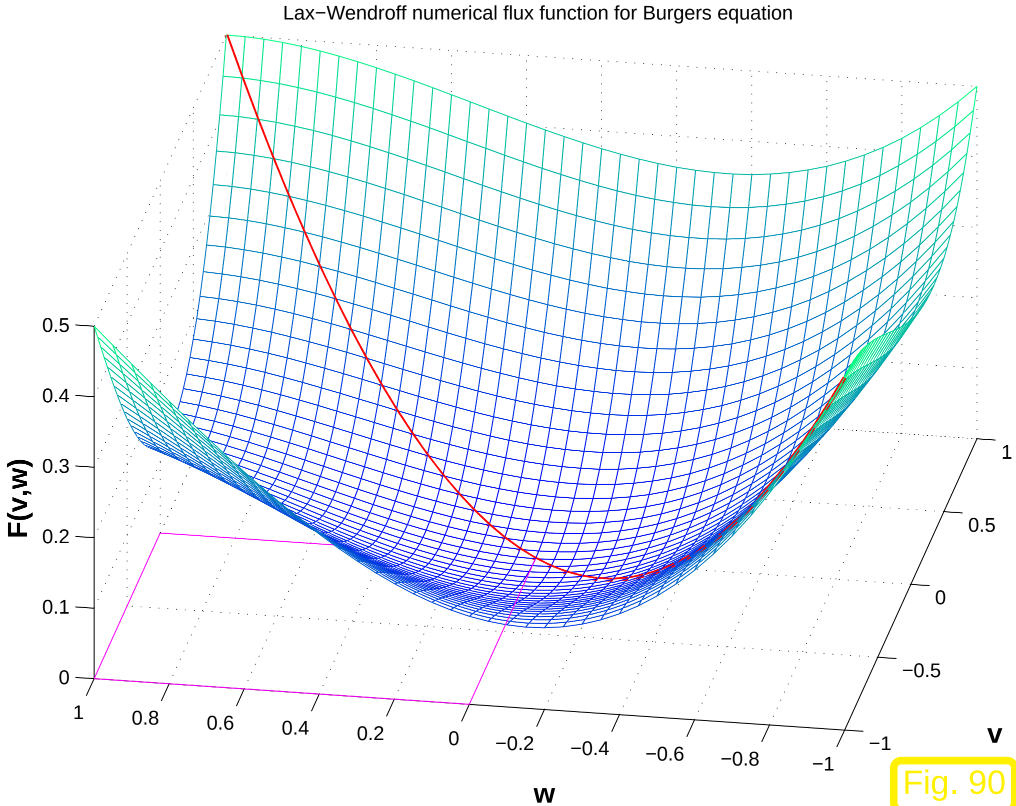


Fig. 90

► Lax-Wendroff numerical flux function:

$$\boxed{F_{\mathrm{LW}}(v,w) = \tfrac{1}{2}(f(v)+f(w)) - \frac{\gamma}{2}\left(f'\left(\tfrac{1}{2}(v+w)\right)\right)^2 (w-v)} \ . \qquad (3.2.25)$$

Lax-Wendroff flux **=** centered flux **+** weighted diffusive flux

► general non-linear Lax-Wendroff-scheme:

$$\mu_j^{(k)} = \mu_j^{(k-1)} - \tfrac{1}{2}\gamma\left(f(\mu_{j+1}^{(k-1)}) - f(\mu_{j-1}^{(k-1)})\right) + \tfrac{1}{2}\gamma^2\Big(f'(\tfrac{1}{2}(\mu_{j+1}^{(k-1)} + \mu_j^{(k-1)}))^2(\mu_{j+1}^{(k-1)} - \mu_j^{(k-1)}) - f'(\tfrac{1}{2}(\mu_j^{(k-1)} + \mu_{j-1}^{(k-1)}))^2(\mu_j^{(k-1)} - \mu_{j-1}^{(k-1)})\Big) \quad (3.2.26)$$

Practical version: replace $f'(\frac{1}{2}(v+w)) \rightarrow \dfrac{f(w)-f(v)}{w-v}$ (still 2nd-order):

$$\widetilde{F}_{\mathrm{LW}}(v,w) := \tfrac{1}{2}(f(v)+f(w)) - \frac{\gamma}{2}\left(\frac{f(w)-f(v)}{w-v}\right)^2 (w-v)\,. \quad (3.2.27)$$

Example 65 (Convergence of Lax-Wendroff-scheme (3.2.26))*.*

- Cauchy problem for Burgers equation (2.1.7)
- initial data u_0 as in Ex. 48 ➢ $0 \le u(x,t) \le 1$ a.e. in $\mathbb{R}\times]0,T[$
- Lax-Wendroff 3-point FDM (3.2.26) with $\gamma = 1$ ➢ CFL-condition satisfied

Monitored: (algebraic) convergence in norms $\max_k \left\|\vec{\mu}^{(k)}\right\|_{l^2(\mathbb{Z})}$, $\max_k \left\|\vec{\mu}^{(k)}\right\|_{l^1(\mathbb{Z})}$ for different u_0 from (4.2.3)-(4.2.5).
("exact" solution by high resolution method, $\rightarrow$ Sect. 3.3 on very fine grid)

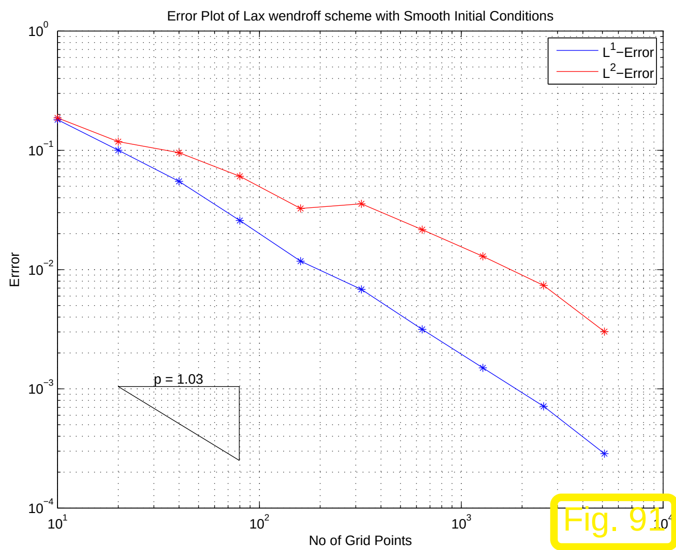


Fig. 91

smooth u_0 from (4.2.3)

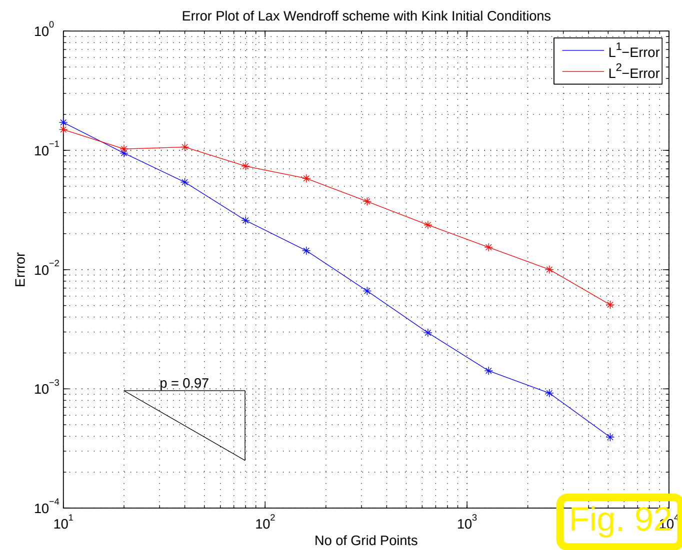


Fig. 92

merely C^0 initial data, (4.2.4)

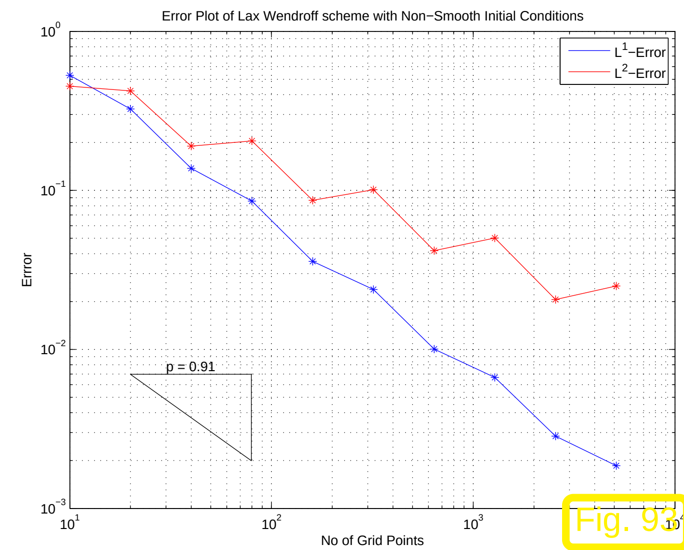
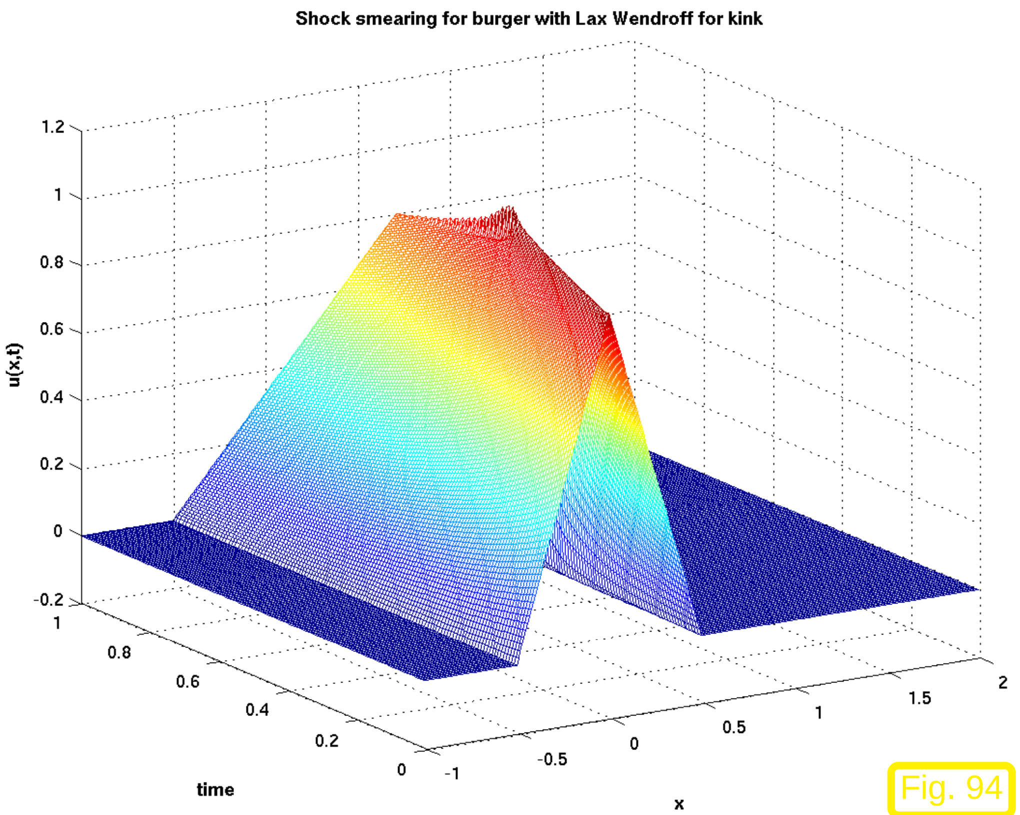


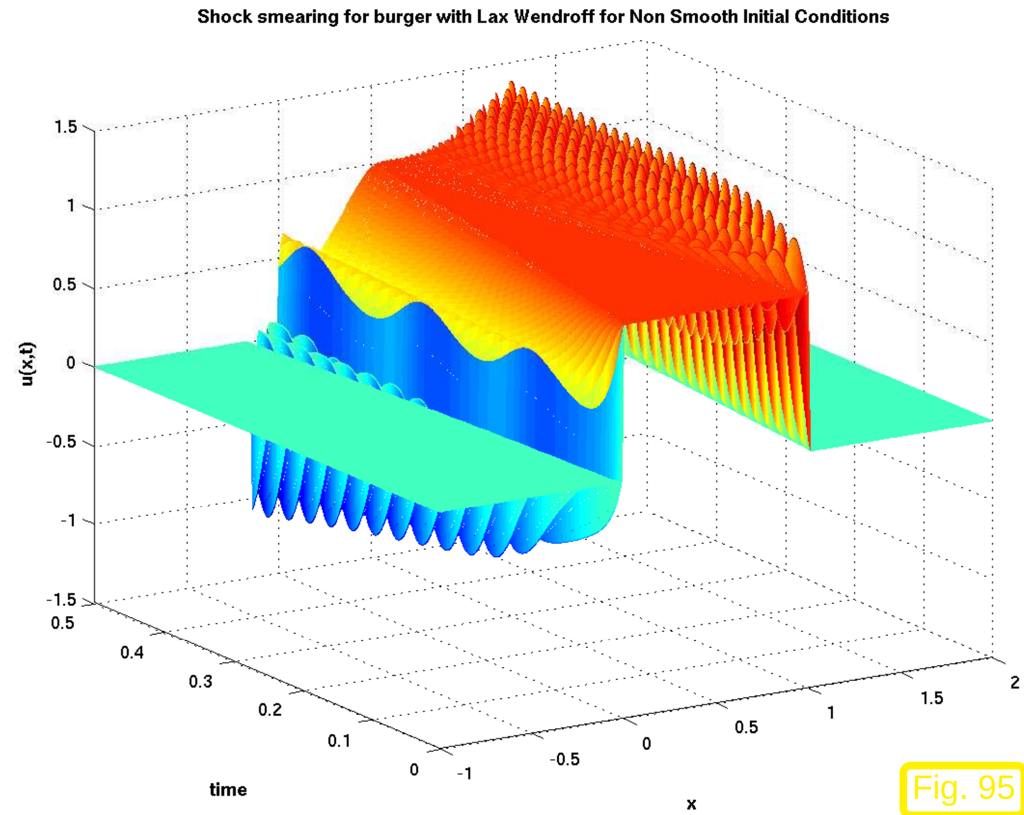
Fig. 93

discontinuous initial data (4.2.5)

Observation: breakdown of smooth solutions ➤ 2nd-order convergence lost (even for smooth u_0)

Monitored: discrete evolutions for non-smooth u_0 from (4.2.4) (merely C^0), (4.2.5) (discontinuous) for $\Delta x = 10^{-2}$, ➤ movie burger_lw_box.avi





$\mu_j^{(k)}$ for u_0 = "saw tooth", $\gamma = 0.8$ $\mu_j^{(k)}$ for u_0 = box function, $\gamma = 0.1$

Observation: Trailing oscillations near kinks/discontinuities of solution **!**

◇

Analysis: examine modified equation (→ Def. 3.2.4) for Lax-Wendroff-scheme

Lax-Wendroff-scheme (3.1.12) for constant advection (2.1.6) is *3rd-order consistent* with

$$\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial x} = -\tfrac{1}{6}v(\Delta x)^2(1-(v\gamma)^2)\frac{\partial^3 u}{\partial x^3} \qquad (3.2.28)$$

Effect of this term **?**

Technique ($\rightarrow$ Sect. 1.3.1): dispersion analysis using plane waves $u(x,t) = e^{i(kx-\omega t)}$

➨ dispersion relation for (3.2.28):

$$-i\omega + ivk = \tfrac{1}{6}v(\Delta x)^2(1-(v\gamma)^2)ik^3 \quad \Rightarrow \quad \omega(k) = vk(1-\tfrac{1}{6}(\Delta x)^2(1-(v\gamma)^2)k^2)\ .$$

▶ group velocity: $c_g = \dfrac{d\omega(k)}{dk} = v(1-\tfrac{1}{2}(\Delta x)^2(1-(v\gamma)^2)k^2)$. (3.2.29)

➤ (3.2.28) is dispersive ($\rightarrow$ Def. 1.3.3): If $|\gamma v| \leq 1$ (CFL-condition) & $|k\Delta x| \leq \sqrt{2}$ (aliasing)
$\Rightarrow$ higher (spatial) frequencies travel more slowly !

Modified equation for 2nd-order FDM are non-diffusive, but dispersive
➤ $\mu_j^{(k)}$ feature spurious oscillations near shocks

Example 66 (Dispersion for Lax-Wendroff scheme).

- constant advection (2.1.6), $v = 1$, on $\Omega =]-\frac{1}{2}, \frac{1}{2}[$ + periodic boundary conditions
- linear Lax-Wendroff FDM (3.1.12), equidistant space-time grid, $\Delta x = 0.01$, $\Delta t = 0.008$

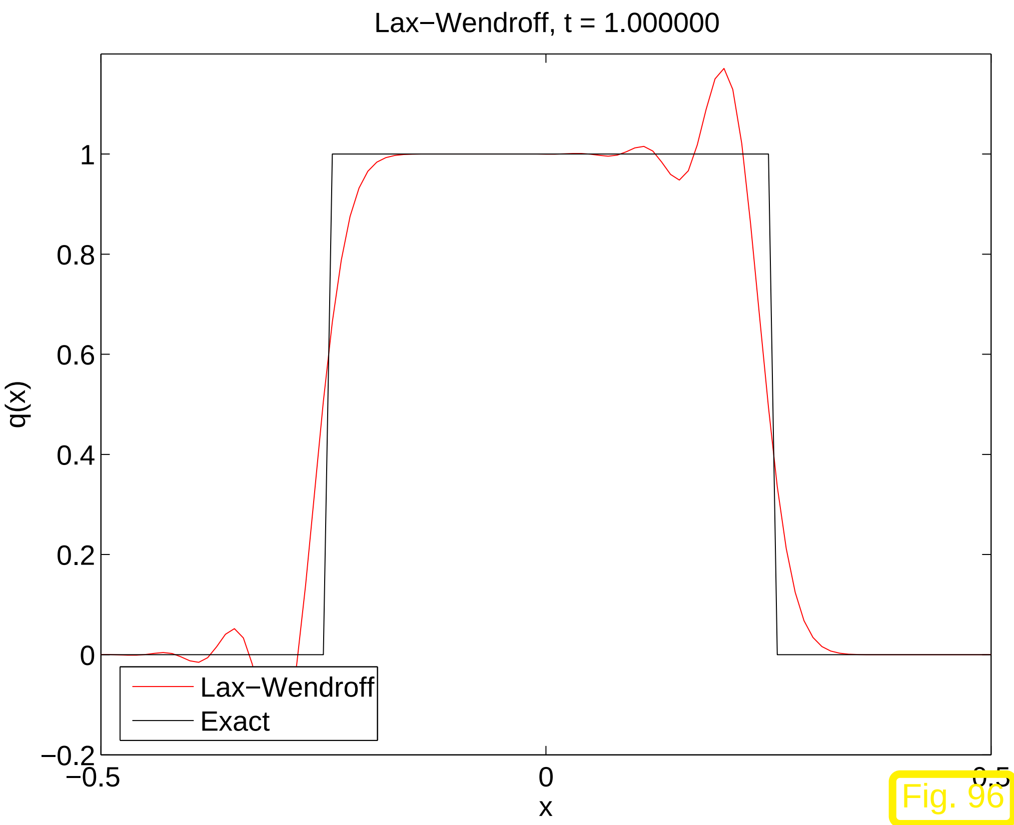


Fig. 96

$$u_0 = \chi_{]-1/4,1/4[}$$

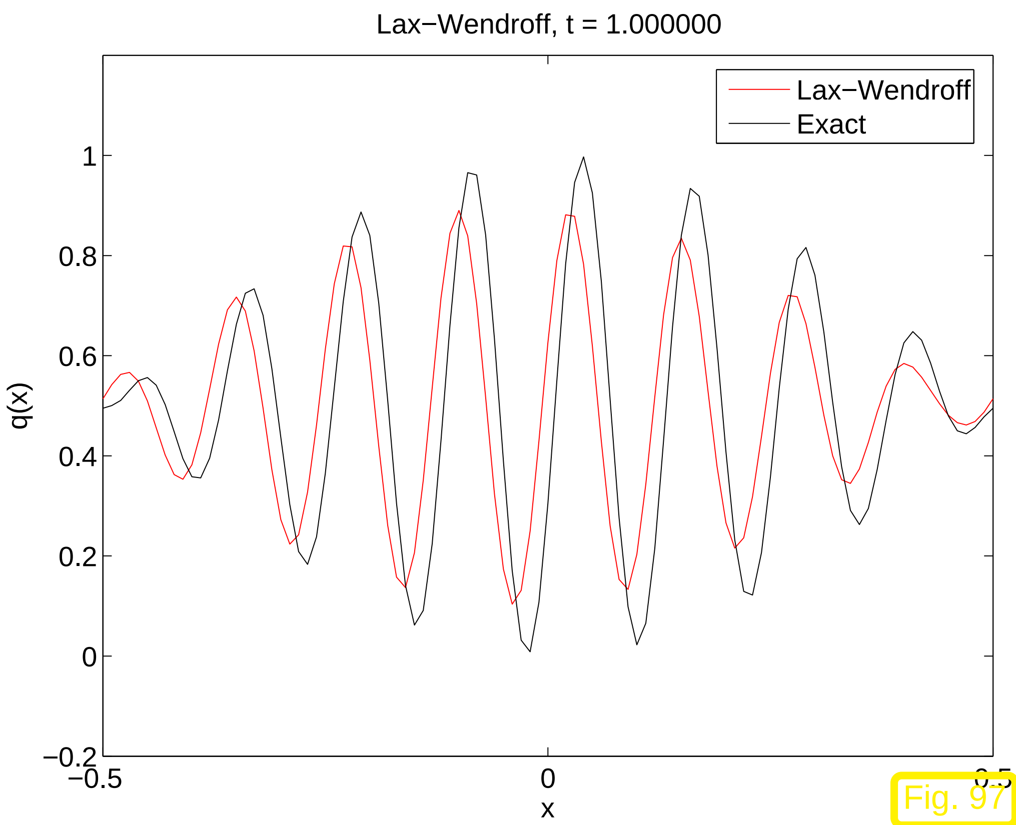


Fig. 97

$$u_0(x) = 0.5\cos(\pi/2x)\sin(8\pi x) + 0.5$$

$\diamond$

3.2.4 Conservation property

Example 67 (“Dishonest” scheme).

- Cauchy problem (2.2.1) with *strictly* convex f, $f'(u) \geq 0$ for $u \geq 0$, $f'(0) = 0$
- $u_0 \geq 0$ ➢ $u(x,t) \geq 0$ a.e. in $\mathbb{R}\times]0,T[$ ➢ only propagation in $+x$-direction
- Non-standard upwind method

$$\mu_j^{(k)} = \mu_j^{(k-1)} - \gamma f'(\mu_j^{(k-1)})(\mu_j^{(k-1)} - \mu_{j-1}^{(k-1)}) \ . \tag{3.2.30}$$

➤ 1st-order consistent ($\rightarrow$ Lemma 3.1.11) **&** (CFL assumed) monotone ($\rightarrow$ Def. 3.1.14)

Thm. 3.1.10 $\stackrel{?}{\Rightarrow}$ scheme (3.2.30) convergent (in $l^p(\mathbb{Z})$-norm, $1 \leq p \leq \infty$)

Yet: $\vec{\mu}^{(0)}$ from Riemann problem:
$$\mu_j^{(0)} = \begin{cases} 1 & \text{, if } j < 0; \ , \\ 0 & \text{, if } j \geq 0 \ . \end{cases}$$

$$\longleftrightarrow \qquad u_0(x) = 1 \text{ for } x < x_{-1/2}, \quad u_0(x) = 0 \text{ for } x > x_{-1/2}$$

Entropy solution (for this u_0) = travelling shock ($\rightarrow$ Lemma 2.4.3), speed $\dot{s} = f(1) > 0$	$\lhd\rhd$	Numerical solution: $\vec{\mu}^{(k)} = \vec{\mu}^{(0)}$ for all k !

➤ 3-point FDM (3.2.30) "converges" to wrong solution !

$\diamond$

◗ Consider explicit, time-invariant, translation-invariant FDM in conservation form ($\rightarrow$ Def. 3.2.1) with consistent ($\rightarrow$ Def. 3.2.2) numerical flux function F (for (2.2.1))

Assume: equidistant tensor product grid, ratio $\gamma := \Delta t/\Delta x$ fixed

Initial data "constant at $\pm\infty$": $\mu_{-j}^{(0)} = u_l$, $\mu_j^{(0)} = u_r$ for large j

$$\Delta x \sum_{j\in\mathbb{Z}} \mu_j^{(k)} - \Delta x \sum_{j\in\mathbb{Z}} \mu_j^{(k-1)} = \Delta t\,(F(u_l,\dots,u_l) - F(u_r,\dots,u_r)) = \Delta t(f(u_l) - f(u_r)) \,, \tag{3.2.31}$$

$$\overset{\text{Def. 3.1.18}}{\longleftrightarrow} \int_{-\infty}^{\infty} u(x,t+\Delta t)\,\mathrm{d}x - \int_{-\infty}^{\infty} u(x,t)\,\mathrm{d}x = \Delta t(f(u_l) - f(u_r)) \,.$$

Situation: discrete solution $\vec{\mu}^{(k)}$ decreasing & supposed to approximate a shock

approximate site of shock (at time $t = t_k$):

$$x_*^{(k)} \in \mathbb{R}:$$

$$\int_{-\infty}^{x_*^{(k)}} u_l - \mathcal{C}\vec{\mu}^{(k)}\,\mathrm{d}x = \int_{x_*^{(k)}}^{\infty} \mathcal{C}\vec{\mu}^{(k)} - u_r\,\mathrm{d}x$$

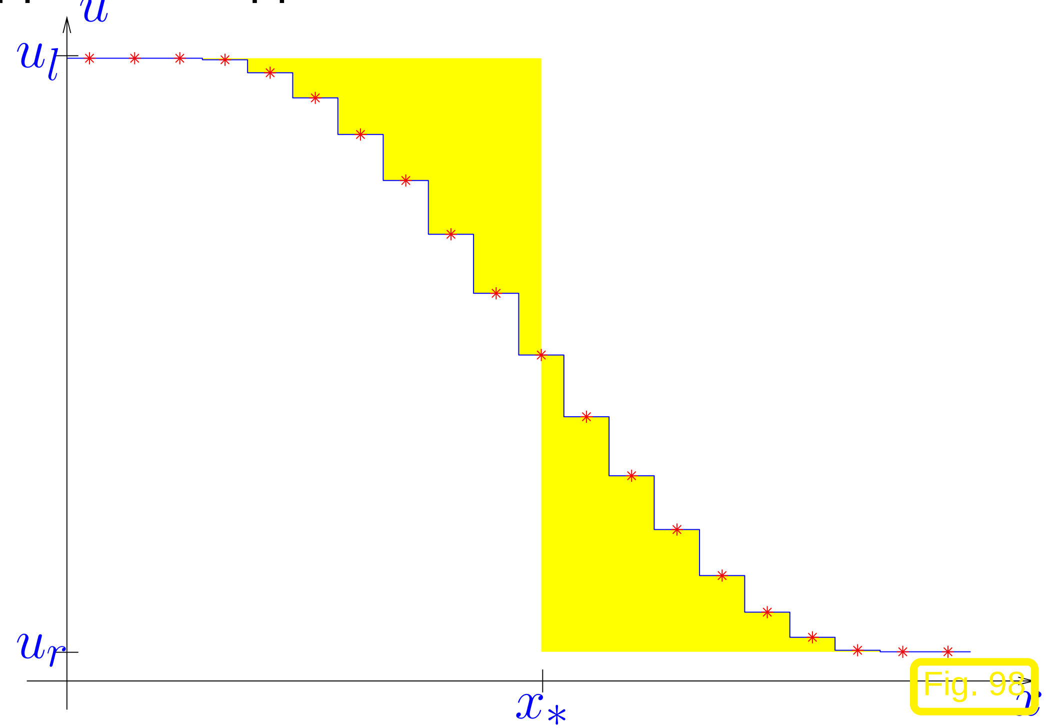


Fig. 98

► $$\int_{-R}^{R} \mathcal{C}\vec{\mu}^{(k)}\,\mathrm{d}x = (x_*^{(k)} + R)u_l + (R - x_*^{(k)})u_r$$ (R large, $R \in x_{\cdot+1/2}$) .

$$\overset{(3.2.31)}{\Longrightarrow} \quad \frac{x_*^{(k)} - x_*^{(k-1)}}{\Delta t} = \frac{1}{u_l - u_r} \sum_{j\in\mathbb{Z}} \left(\mu_j^{(k)} - \mu_j^{(k-1)}\right) = \frac{f(u_l) - f(u_r)}{u_l - u_r} \overset{\text{Thm. 2.3.2}}{=} \dot{s}\ .$$

FVM yield correct “discrete shock speed” (not liable to effect of Ex. 67)

Setting:
- sequence of meshwidths $\tau_l \in \mathbb{R}$, $l \in \mathbb{N}$, $\lim_{l\to\infty} \tau_l = 0$
- sequence of equidistant space-time meshes $\mathcal{M}_l$, $\gamma := \dfrac{\Delta t_l}{\Delta x_l}$ fixed, $\Delta x_l = \tau_l$
- $u_l := \mathcal{C}\vec{\mu}^{(\cdot)} \in L^\infty(\mathbb{R}\times]0,T[)$, $\vec{\mu}^{(\cdot)}$ generated by consistent FVM (→ Def. 3.2.1) for (2.2.1), $\vec{\mu}^{(0)}$ from (3.2.3)

Theorem 3.2.6 (Lax-Wendroff theorem). → *[31, Thm. 12.1], [29, Thm. 2.3.1]*

In the above setting we assume

(i) $\exists u \in L^\infty(\mathbb{R}\times]0,T[)$: $\lim_{l\to\infty} \|u_l - u\|_{L^1(K)} = 0 \quad \forall$ *compact* $K \subset \mathbb{R}\times]0,T[$

(ii) $\exists C > 0$: $TV_{\mathbb{R}}(u_l(\cdot,t)) \leq C \quad \forall t \in]0,T[$.

Then u *is a weak solution (→ Def. 2.3.1) of the Cauchy problem* (2.2.1).

Sketch of proof. (details → proof of Thm. 2.3.1 in [29]) Pick $\Phi \in C_0^\infty(\mathbb{R} \times [0,T[)$

✎ notation: $\Phi_j^{(k)} := \Phi(x_k, t_k)$, $(x_j, t_k) \in \mathcal{M}_l$ (index l suppressed)

From conservation form by *summation by parts* $\left(\sum_{i=1}^{n} a_i(b_i - b_{i-1}) = a_n b_n - a_1 b_0 - \sum_{i=1}^{n-1}(a_{i+1} - a_i) b_i\right)$

$$-\sum_{j\in\mathbb{Z}} \Phi_j^{(0)} \mu_j^{(0)} - \sum_{k=1}^{M} \sum_{j\in\mathbb{Z}} (\Phi_j^{(k)} - \Phi_j^{(k-1)}) \mu_j^{(k)} = \gamma \sum_{k=1}^{M} \sum_{j\in\mathbb{Z}} (\Phi_{j+1}^{(k)} - \Phi_j^{(k)}) f_{j-1/2}^{(k)}$$

$$\boxed{\Delta x \Delta t \sum_{k=1}^{M} \sum_{j\in\mathbb{Z}} \left(\frac{\Phi_j^{(k)} - \Phi_j^{(k-1)}}{\Delta t}\right) \mu_j^{(k)}} + \boxed{\Delta x \Delta t \sum_{k=1}^{M} \sum_{j\in\mathbb{Z}} \left(\frac{\Phi_{j+1}^{(k)} - \Phi_j^{(k)}}{\Delta x}\right) f_{j-1/2}^{(k)}} = \boxed{= -\Delta x \sum_{j\in\mathbb{Z}} \Phi_j^{(0)} \mu_j^{(0)}}\ .$$

❶

$$\xrightarrow{l\to\infty} \int_0^T \int_{-\infty}^{\infty} \frac{\partial \Phi}{\partial t}(x,t) u(x,t) \,\mathrm{d}x\mathrm{d}t$$

❷

$$\xrightarrow{l\to\infty} -\int_{-\infty}^{\infty} \Phi(x,0) u_0(x) \,\mathrm{d}x$$

$$\xrightarrow{l\to\infty} \int_0^T \int_{-\infty}^{\infty} \frac{\partial \Phi}{\partial x}(x,t) f(u(x,t)) \,\mathrm{d}x\mathrm{d}t$$

❶: uses L^1-convergence of u_l

❷: requires $TV_{\mathbb{R}}(u_l(\cdot,t)) \leq C$ for $\lim_{\Delta x\to 0} \int_{\mathbb{R}} |u_l(x,+\Delta x,t) - u_l(x,t)| \,\mathrm{d}x = 0$ $\square$

Finite difference methods in conservation form do not lie !

("An algorithm may fail, but it must not lie" — B. Parlett)

3.2.5 Stability

➜ apply results of Sect. 3.1.3.2 to FDM in conservation form ($\longrightarrow$ Def. 3.2.1)

Focus: 3-point finite volume methods on equidistant grids

$$\mu_j^{(k)} = \mu_j^{(k-1)} - \gamma(F(\mu_j^{(k-1)}, \mu_{j+1}^{(k-1)}) - F(\mu_{j-1}^{(k-1)}, \mu_j^{(k-1)})) \; . \qquad (3.2.2)$$

Assume: numerical flux function $F : \mathbb{R}^2 \mapsto \mathbb{R}$ smooth

Lemma 3.2.7 (Monotone 3-point FVM). $\longrightarrow$ *[29, Def. 2.3.]*

A 3-point finite volume method (3.2.2) *with* $F \in C^1$ *induces a monotone discrete evolution (*$\longrightarrow$ *Def. 3.1.14), if*

$$\partial_l F(v,w) \geq 0 \;, \quad \partial_r F(v,w) \leq 0 \;, \quad 1 - \gamma(\partial_l F - \partial_r F) \geq 0 \; .$$

Theorem 3.2.8 (Order barrier for monotone FDM in conservation form)**.**
A monotone finite difference method in conservation form (→ Def. 3.2.1) for (2.2.1) *with* C^1 *numerical flux function is at most consistent of order 1.*

▶ Thm. 3.2.3 ➤ Godunov's method (→ Sect. 3.2.2) is only 1st-order consistent with (2.2.1)

Survey:

stability properties of consistent finite difference methods in conservation form

→ Thm. 3.1.19,
→ Lemma 3.1.21,
→ Thm. 3.2.8

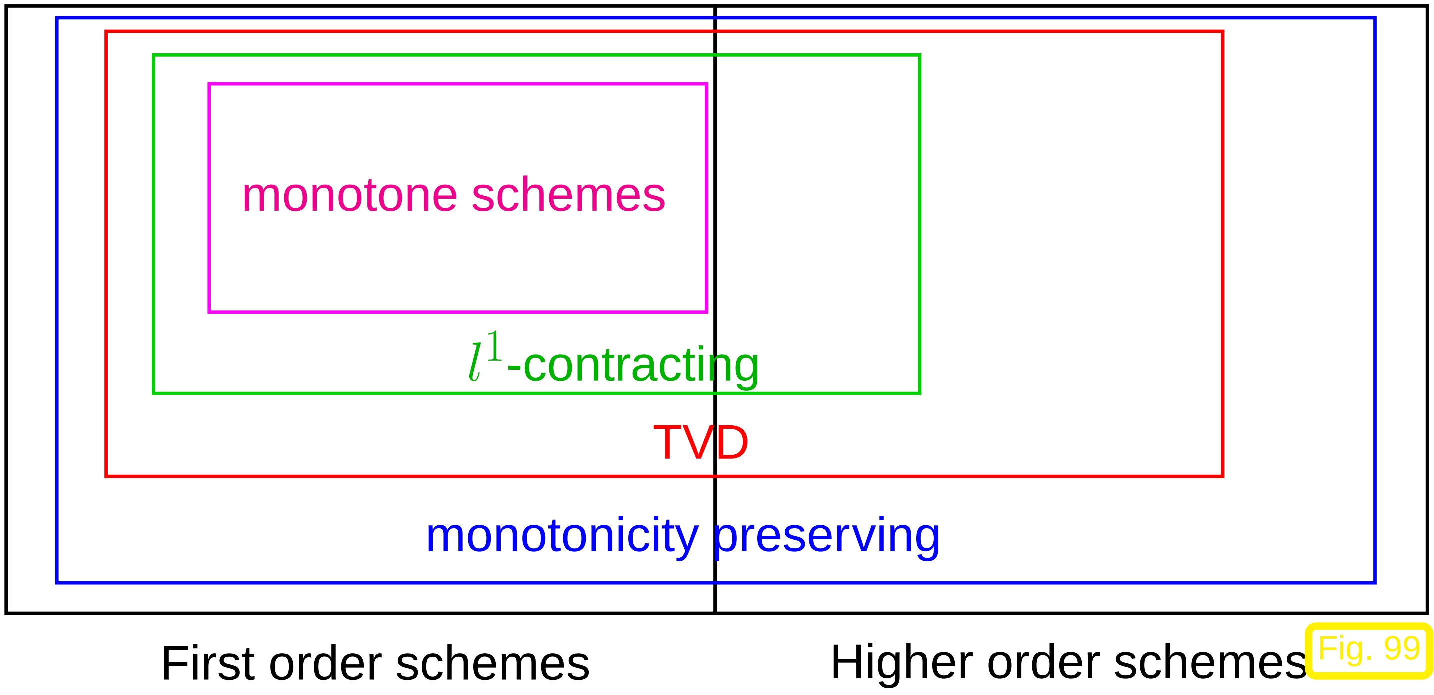


Fig. 99

Thm. 3.1.25 ▶ for linear FVM: all notions of stability coincide !

Remark 68. Thm. 3.1.25 & Thm. 3.2.8 ➤

even for *linear* advection (2.1.6): only a non-linear FVM to achieve 2nd-order **and** monotonicity preservation ($\longrightarrow$ Def. 3.1.24), *cf.* oscillations in Lax-Wendroff evolutions $\longrightarrow$ Ex. 65

△

Remark 69 (Order barrier for TVD 3-point FVM).

A TVD ($\longrightarrow$ Def. 3.1.20) 3-point finite difference method in conservation form (3.2.2) for (2.2.1) is at most first-order consistent, [15, Thm. 3.7], [37, Sect. 2].

△

Lemma 3.2.9 (l^1-stability of TVD FVM)**.**
A TVD ($\longrightarrow$ Def. 3.1.20) finite difference method in conservation form ($\longrightarrow$ Def. 3.2.1) with Lipschitz-continuous numerical flux function is linearly ($\longrightarrow$ Thm. 3.1.10) $l^1(\mathbb{Z})$-stable.

Terminology: Numerical flux function F is Lipschitz-continuous, if

$$\exists L > 0: \quad |F(u_{-m_l+1},\dots,u_{m_r}) - F(\bar{u}_{-m_l+1},\dots,\bar{u}_{m_r})| \leq L \sum_{l=-m_l+1}^{m_r} |u_l - \bar{u}_l| \tag{3.2.32}$$

for sufficiently small $|u_l - \bar{u}_l|$.

3.2.6 Convergence

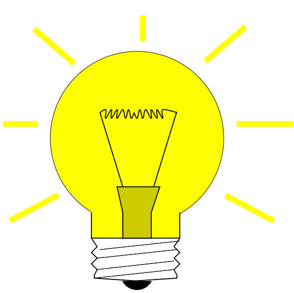
For *non-linear* scalar conservation laws:

possible breakdown of classical solution ($\longrightarrow$ Thm. 2.2.4)

- ▷ blow-up of spatial derivatives
- ▷ no control of truncation errors ($\longrightarrow$ Def. 3.1.6)
- ▷ Thm. 3.1.10 cannot be applied **!**
- ▷ convergence of FDM/FVM for (2.2.1) and relevant classes of solutions **?**

Put up with very weak notions of convergence (weaker than Def. 3.1.5) **:** convergence of *sub-sequences* ($\leftrightarrow$ compactness arguments)

Recall: Topological space V compact $:\Leftrightarrow$ every sequence in V has convergent subsequence



Idea: Consider family of grids ➢ family of discrete evolutions

- ▷ family of discrete solutions u_l, *cf.* Thm. 3.2.6
- ▷ if $\{u_l\} \subset$ compact set ➢ $\exists$ convergent subsequence

refers to same (norm-)topology

Recall: compact embeddings of function spaces $\to$ [27, Def. 2.11.2], Ω bounded:

- $\{v \in L^p(\Omega)\colon \|v\|_{L^q(\Omega)} \leq 1\}$ is compact subset of $L^p(\Omega)$ for $q > p$,
- $\{v \in W^{m-1,p}(\Omega)\colon \|v\|_{W^{m,p}(\Omega)} \leq 1\}$, $p \geq 1$, is compact subset of $W^{m-1,p}(\Omega)$

  $\to$ embedding theorem [27, Thm. 4.2.13] for Sobolev spaces [27, Def. 4.2.1]

► $\Omega \subset \mathbb{R}^d$ bounded, $(f_l)_{l\in\mathbb{N}} \subset L^1(\Omega)$:

$$\|f_l\|_{W^{1,1}(\Omega)} = \int_\Omega |f_l| \, \mathrm{d}\boldsymbol{x} + \int_\Omega |\,\mathbf{grad}\, f_l| \, \mathrm{d}\boldsymbol{x} \leq C \quad \forall l \in \mathbb{N}$$
$$\Rightarrow \quad \exists \{i_1, i_2, \ldots\} \subset \mathbb{N},\ f \in L^1(\Omega)\colon \quad \lim_{k\to\infty} f_{i_k} = f \ .$$

Note: $TV_\Omega(f) = \int_\Omega |\,\mathbf{grad}\, f| \, \mathrm{d}\boldsymbol{x}$ for $f \in W^{1,1}(\Omega)$: $\|f_l\|_{W^{1,1}(\Omega)} = \|f\|_{L^1(\Omega)} + TV_\Omega(f)$

Theorem 3.2.10 (Compactness in BV_{loc}).
For $\Omega \subset \mathbb{R}^d$ *(not necessarily bounded) let* $(f_l)_{l\in\mathbb{Z}} \subset BV_{\mathrm{loc}}(\Omega)$ *satisfy*

$$\forall K \subset \Omega,\ K \text{ compact}: \quad \exists C > 0: \quad \|f_l\|_{L^1(K)} \leq C \quad \wedge \quad TV_K(f) \leq C \quad \forall l \in \mathbb{N}\,.$$

Then $\exists \{i_1, i_2, \ldots\} \subset \mathbb{N}$, $f \in L^1_{\mathrm{loc}}(\Omega)$ *such that* $\lim\limits_{k\to\infty} f_{i_k} = f$ *in* $L^1_{\mathrm{loc}}(\Omega)$.

Proof. by Arzela-Ascoli selection theorem & mollifier techniques □

Idea: use this compactness result on $\widetilde{\Omega} = \mathbb{R}\times]0,T[$ **!**

For equidistant infinite space-time tensor product grid $\mathcal{M}$ (spatial meshwidth Δx, timestep Δt), grid function $\vec{\mu}^{\cdot} : \mathcal{M} \mapsto \mathbb{R}$, $\mu_j^{(k)} \neq 0$ for finitely many $(j,k) \in \mathbb{Z} \times \{0,\ldots,M\}$:

$$TV_{\mathcal{M}}(\vec{\mu}^{\cdot}) = TV_{\mathbb{R}\times]0,T[}(\mathcal{C}\vec{\mu}^{\cdot}) = \sum_{k=1}^{M} \sum_{j\in\mathbb{Z}} \Delta t |\mu_j^{(k-1)} - \mu_{j-1}^{(k-1)}| + \Delta x |\mu_j^{(k)} - \mu_j^{(k-1)}|\,. \tag{3.2.33}$$

Lemma 3.2.11 (TVD FVM is TV-stable in space-time)**.**
Let $(\vec{\mu}^{(k)})_{k=0}^{M}$ *be generated by a TVD (*$\longrightarrow$ *Def. 3.1.20) finite difference scheme in conservation form (*$\longrightarrow$ *Def. 3.2.1) on equidistant grid with Lipschitz-continuous numerical flux function* F*, i.e.,* (3.2.32) *holds with some* $L > 0$*.*

$$\Longrightarrow \quad TV_{\mathcal{M}}(\vec{\mu}^{(\cdot)}) \leq ((m_l + m_r)L + 1)T \cdot TV_{\Delta x}(\vec{\mu}^{(0)}) \quad \forall \mu_0 \in C^0(\mathcal{G}_{\Delta x}), \sharp\{\mu_j^{(0)} \neq 0\} < \infty \,.$$

Proof. see proof of Lemma 3.2.9, use (3.2.33) $\square$

Setting:
- sequence of meshwidths $\tau_l \in \mathbb{R}$, $l \in \mathbb{N}$, $\lim_{l\to\infty} \tau_l = 0$,
- sequence of equidistant space-time meshes $\mathcal{M}_l$, $\gamma := \dfrac{\Delta t_l}{\Delta x_l}$ fixed, $\Delta x_l = \tau_l$,
- $u_l := \mathcal{C}\vec{\mu}^{(\cdot)} \in L^\infty(\mathbb{R}\times]0,T[)$, $\vec{\mu}^{(\cdot)}$ generated by FDM ($\longrightarrow$ Def. 3.1.1) on $\mathcal{M}_l$ for Cauchy problem (2.2.1),
- $\vec{\mu}^{(0)}$ from cell averaging (3.2.3).

Theorem 3.2.12 (Convergence of TVD finite volume methods). → *[29, Thm. 2.3.9]*

In the above setting we assume that

(i) the finite difference methods are in conservation form (→ Def. 3.2.1) with a Lipschitz-continuous numerical flux function F that is consistent (→ Def. 3.2.2) with the flux function f,

(ii) the finite difference methods are TVD (→ Def. 3.1.20),

(iii) initial data $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ satisfy $TV_{\mathbb{R}}(u_0) < \infty$.

Then, possibly after selecting a sub-sequence,

$$u_l \to u \quad \text{for } l \to \infty \quad \text{in } L^1_{\mathrm{loc}}(\mathbb{R}\times]0,T[)\ , \quad u \text{ is weak solution of (2.2.1)}\ .$$

Theorem 3.2.13 (Convergence to weak solutions).

Let $\mathcal{W} \subset L^1(\mathbb{R}\times]0,T[) \cap L^\infty(\mathbb{R}\times]0,T[)$ be the set of weak solutions of (2.2.1). Under the assumptions of Thm. 3.2.12

$$\forall K \subset \mathbb{R}\times]0,T[,\ K \text{ compact}: \quad \lim_{l\to\infty} \inf_{u\in\mathcal{W}} \|u_l - u\|_{L^1(K)} = 0\ .$$

3.2.7 Discrete entropy solutions

Thm. 3.2.12: convergence to entropy solution ($\rightarrow$ Sect. 2.5.2) of Cauchy problem **?**

Example 70 (FVM can converge to expansion shock).

- Cauchy problem (2.2.1) for Burgers equation (2.1.7), i.e., $f(u) = \frac{1}{2}u^2$
- $u_0(x) = 1$ for $x > 0$, $u_0(x) = -1$ for $x < 0$

➤ entropy solution **=** rarefaction wave ($\rightarrow$ Lemma 2.4.4)

- FVM: Roe upwinding (3.2.6) on equidistant grid, $x_j = (j + \frac{1}{2})\Delta x$, $\Delta x > 0$, CFL-condition satisfied

➤ $$\mu_j^{(0)} = \begin{cases} -1 & \text{for } j < 0 \,, \\ 1 & \text{for } j \geq 0 \,. \end{cases}$$

▶ $\mu_j^{(k)} = \mu_j^{(0)}$ for all k ➤ for $\Delta x \rightarrow 0$, convergence to entropy violating expansion shock **!**

▶ finite volume method may converge to entropy violating weak solutions **!**

◇

Question: How to tell that a scheme guarantees convergence to entropy solution **?** ($\leftrightarrow$ "does not lie", *cf.* Sect. 3.2.4)

Remember: entropy inequalities ($\longrightarrow$ Def. 2.5.3) satisfied by entropy solution of (2.2.1): for any pair (η, ψ) of entropy functions ($\longrightarrow$ Def. 2.5.2)

$$\int_{x_0}^{x_1} \eta(u(x,t_1)) - \eta(u(x,t_0))\,\mathrm{d}x + \int_{t_0}^{t_1} \psi(u(x_1,t)) - \psi(u(x_0,t))\,\mathrm{d}t \leq 0 \tag{3.2.34}$$

for almost all $x_0 < x_1$, $0 < t_0 < t_1 < T$, whenever u is entropy solution of (2.2.1).

Definition 3.2.14 (Entropy consistency).
A finite difference solution $\vec{\mu}^{(\cdot)}$ of (2.2.1) *on an equidistant grid is entropy consistent with a pair (η, ψ) of entropy function (→ Def. 2.5.2), if there is a numerical entropy flux function $\Psi : \mathbb{R}^{m_l+m_r} \mapsto \mathbb{R}$ consistent with the entropy flux ψ, that is,*

$$\exists C > 0,\, \delta > 0\colon \quad |\Psi(\mu_{-m_l+1}, \dots, \mu_{m_r}) - \psi(u)| \leq C \sum_{l=-m_l}^{m_r} |\mu_l - u|$$

for all $\mu_{-m_l+1}, \dots, \mu_{m_r}, u$: $\quad |\mu_l - u| \leq \delta$, such that the discrete entropy inequality

$$\eta(\mu_j^{(k)}) \leq \eta(\mu_j^{(k-1)}) - \gamma(\psi_{j+1/2}^{(k-1)} - \psi_{j-1/2}^{(k-1)}) \quad \forall j \in \mathbb{Z},\, k = 1, \dots, M\ , \tag{3.2.35}$$

holds, where $\psi_{j+1/2}^{(k)} := \Psi(\mu_{j-m_l+1}^{(k-1)}, \dots, \mu_{j+m_r}^{(k-1)})$.

Definition 3.2.15 (Discrete entropy condition).
A finite difference method (on an equidistant grid) for (2.2.1) *satisfies the discrete entropy condition, if it is entropy consistent (→ Def. 3.2.14) with **any** pair of entropy functions (→ Def. 2.5.2) for* (2.2.1).

Theorem 3.2.16 (Convergence to entropy solutions)**.**
Let the assumptions of the Lax-Wendroff theorem, Thm. 3.2.6, be satisfied. If the solutions $\vec{\mu}^{(\cdot)}$ *of all discrete evolutions satisfy the discrete entropy condition (*$\longrightarrow$ *Def. 3.2.15), then* u *will be an entropy solution of* (2.2.1)*.*

Proof. analoguous to that of Thm. 3.2.6 $\square$

By uniqueness of the entropy solution, Thm. 2.5.4:

Theorem 3.2.17 (Strong convergence theorem)**.**
In addition to the assumptions of Thm. 3.2.12 (TVD, conservation form, consistent with (2.2.1)*), let a finite volume method satisfy the discrete entropy condition.*
Then $u_l \longrightarrow u$ *for* $l \longrightarrow \infty$ *in* $L^1_{\mathrm{loc}(\mathbb{R}\times]0,T[)}$*, where* u *is the entropy solution of the Cauchy problem* (2.2.1)*.*

✌ $\boxed{\text{Discrete entropy condition holds for Godunov's method} \quad (\longrightarrow \text{Sect. 3.2.2})}$

Tool: Jensen's inequality: if $\eta : \mathbb{R} \mapsto \mathbb{R}$ *convex*, $\int_\Omega 1 \,\mathrm{d}\boldsymbol{x} = 1$, then

$$\eta\left(\int_\Omega g \,\mathrm{d}\boldsymbol{x}\right) \leq \int_\Omega \eta(g) \,\mathrm{d}\boldsymbol{x} \tag{3.2.36}$$

for measurable $g : \Omega \mapsto \mathbb{R}$.

Thm. 3.2.16 ▶ Godunov solutions converge to entropy solutions.

Theorem 3.2.18 (Monotone FVM are entropy consistent). *[15, Thm. 4.2], [29, Thm. 2.3.19]*

FDM for (2.2.1) *monotone (→ Def. 3.1.14)* *consistent (→ Def. 3.2.2)* *in conservation form (→ Def. 3.2.1)* $\Longrightarrow$ *discrete entropy condition (→ Def. 3.2.15)*

Tool for the proof: Kruzkov pair of non-smooth entropy functions for $\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u) = 0$:

$$\eta_c(u) = |u - c| \quad , \quad \psi_c(u) = \mathrm{sgn}(u - c)(f(u) - f(c)) \; , \quad c \in \mathbb{R} \; . \tag{3.2.37}$$

➢ in the sense of distributions $\psi_c' = \eta_c' \cdot f'$

Significance of Kruzkov entropies:

- finite *positive combinations* of Kruzkov entropies approximate convex functions in $W^{1,1}_{\mathrm{loc}}(\mathbb{R})$ (modulo linear modification) [29, Lemma 2.1.18]

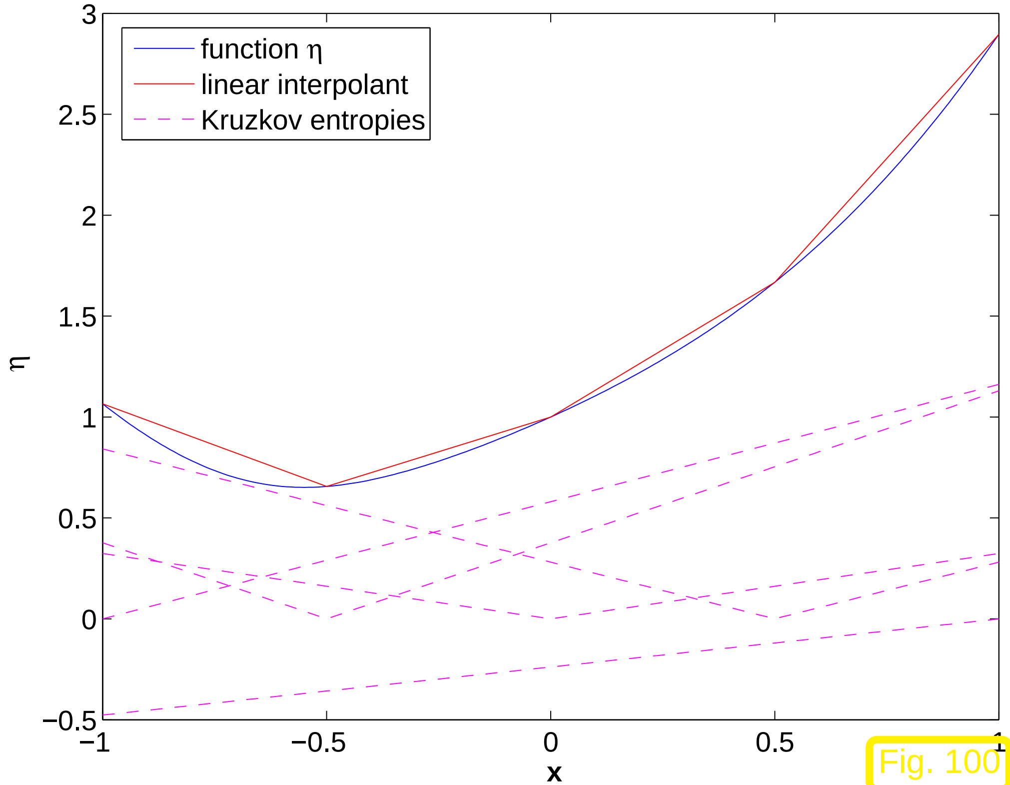


Fig. 100

- FDM entropy consistent ($\longrightarrow$ Def. 3.2.14, (3.2.35)) with entropy pairs (η,ψ), $(\bar{\eta},\bar{\psi})$

$\Rightarrow$ entropy consistent with any convex combination (of (η,ψ), $(\bar{\eta},\bar{\psi})$) !

FDM entropy consistent with all Kruzkov pairs of non-smooth entropy functions
$\Rightarrow$ FDM satisfies discrete entropy condition

Monotone & consistent FVM converge !

A more general class of FVM satisfying the discrete entropy condition ($\rightarrow$ Def. 3.2.15):

Definition 3.2.19 (E-schemes). $\rightarrow$ *[36], [31, Sect. 12.7], [15, Sect. 4.2]*

A 3-point finite difference method in conservation form (3.2.2) *($\rightarrow$ Def. 3.2.1) for* (2.2.1) *is an E-scheme, if*

$$\operatorname{sgn}(w-v)(F(v,w)-f(u)) \leq 0 \quad \forall u \in [\min\{v,w\}, \max\{v,w\}] \;.$$

relationship with Godunov scheme (3.2.15): for a 3-point FDM in conservation form

$$\begin{aligned} F(v,w) &\leq F_{\mathrm{GD}}(v,w) \quad \text{, if } v \leq w \;, \\ F(v,w) &\geq F_{\mathrm{GD}}(v,w) \quad \text{, if } v > w \;, \end{aligned} \quad \Leftrightarrow \quad \text{FVM is an E-scheme}$$

Lax-Friedrichs scheme (3.1.29) & Engquist-Osher scheme (3.2.7) are E-schemes

Lemma 3.2.20 (TVD property of E-schemes).

(3.2.2) *E-scheme ($\rightarrow$ Def. 3.2.19)*
$|\gamma(|\partial_l F(v,w)| + |\partial_r F(v,w)|) \leq 1 \;\forall$*possible* v,w $\Rightarrow$ (3.2.2) *TVD ($\rightarrow$ Def. 3.1.20)*

Proof. convert to incremental form (3.1.30) and Thm. 3.1.22

Theorem 3.2.21 (Order barrier for E-schemes). $\longrightarrow$ *[36, Lemma 2.1], cf. Thm. 3.2.8*

E-schemes are at most first order consistent

Lemma 3.2.22 (Monotone schemes as E-schemes).
A consistent ($\longrightarrow$ Def. 3.2.2) monotone ($\longrightarrow$ Def. 3.1.14) 3-point scheme in conservation form (3.2.2) *is an E-scheme.*

Proof. (3.2.2) monotone ➤ $F(v,w)$ non-decreasing in v, non-increasing in w

$$\begin{aligned} v<u<w \quad &\Rightarrow \quad F(v,w)-F(u,u)\leq 0\;, \\ w<u<v \quad &\Rightarrow \quad F(v,w)-F(u,u)\geq 0\;. \end{aligned}$$

Lemma 3.2.23 (Discrete entropy condition for E-schemes). *[43, Sect. 5]*
E-schemes ($\longrightarrow$ Def. 3.2.19) for (2.2.1) *satisfy the discrete entropy condition ($\longrightarrow$ Def. 3.2.15) under the tightened CFL-condition*

$$\gamma\left|\frac{f(v)-2F(v,w)+f(w)}{\ldots}\right|\leq\frac{1}{2}\;.$$

Heuristics. Consider *semi-discrete* equation for $\vec{\mu} = \vec{\mu}(t)$, $0 \le t \le T$, $\vec{\mu}(0) = \vec{\mu}^{(0)}$

$$\frac{d}{dt}\vec{\mu} = -\frac{1}{\Delta x}(F(\mu_j(t), \mu_{j+1}(t)) - F(\mu_{j-1}(t), \mu_j(t))) \; . \tag{3.2.38}$$

For any pair (η, ψ) of entropy functions:

$$\overset{\eta'(\mu_j)\cdot(3.2.38)}{\Longrightarrow} \quad \Delta x \frac{d}{dt}\eta(\mu_j(t)) = -\eta'(\mu_j(t))(F(\mu_j(t), \mu_{j+1}(t)) - F(\mu_{j-1}(t), \mu_j(t)))$$

numerical entropy flux function: $\boxed{\Psi(v, w) := \eta'(w)(F(v, w) - f(w)) + \psi(w) - \psi(v)}$

$$\begin{aligned}
&\Delta x \frac{d}{dt}\eta(\mu_j) + \Psi(\mu_j, \mu_{j+1}) - \Psi(\mu_{j-1}, \mu_j) \\
&\quad = F(\mu_j, \mu_{j+1})(\eta'(\mu_{j+1}) - \eta'(\mu_j)) + (\psi(\mu_{j+1}) - \psi(\mu_j)) - \eta'(\mu_{j+1})f(\mu_{j+1}) + \eta'(\mu_j)f(\mu_j) \\
&\quad = \int_{\mu_j}^{\mu_{j+1}} \underbrace{\eta''(\tau)}_{\ge 0} \underbrace{(F(\mu_j, \mu_{j+1}) - f(\tau))}_{\le 0 \,\leftarrow\, \text{E-scheme !}} \,\mathrm{d}\tau \le 0 \; .
\end{aligned}$$

Heuristics: integrate over $[t_{k-1}, t_k]$ & (partially) freeze time:

$$\eta(\mu_j^{(k)}) - \eta(\mu_j^{(k-1)}) + \gamma(\Psi(\mu_j^{(k-1)}, \mu_{j+1}^{(k-1)}) - \Psi(\mu_{j-1}^{(k-1)}, \mu_j^{(k-1)})) \le 0 \; .$$

▶ discrete entropy condition ($\longrightarrow$ Def. 3.2.15): Ψ consistent with ψ □

Thm. 3.2.17 $\Longrightarrow$ Consistent 3-point E-schemes converge to the entropy solution, if $\gamma(|\partial_l F| + |\partial_r F|) \leq 1$

PS: Bad news from [37]: another order barrier, *cf.* Thm. 3.2.8

A finite difference method for (2.2.1) in conservation form ($\longrightarrow$ Def. 3.2.1) that satisfies the discrete entropy condition ($\longrightarrow$ Def. 3.2.15) is at most first-order consistent.

3.2.8 A priori error estimate

Thm. 3.2.17: convergence, but how fast **?** ($\longrightarrow$ "no rate")

Setting:

- Cauchy problem (2.2.1), initial data $u_0 \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$, final time $T > 0$, entropy solution $u \in L^1(\mathbb{R}\times]0,T[) \cap L^\infty(\mathbb{R}\times]0,T[)$

- Sequence of equidistant meshes $\mathcal{M}_M$, $M \in \mathbb{N}$, spatial meshwidth $\Delta x = \Delta x_M$, timesteps $\boxed{\Delta t = \Delta t_M = T/M}$, fixed ratio $\gamma = \Delta t/\Delta x$.

- Finite volume discrete evolutions ($\rightarrow$ Def. 3.2.1) on $\mathcal{M}_M$, $\vec{\mu}^{(0)}$ from (3.2.3)

➤ solution grid functions $\vec{\mu}_M^{(\cdot)} : \mathcal{M}_M \mapsto \mathbb{R}$ $\leftrightarrow$ approximate solutions $u_M := \mathcal{C}\vec{\mu}_M^{(\cdot)}$

Theorem 3.2.24 (A priori error estimate for monotone FVM). *[15, Thm. A.1]*
If the FDM is monotone *($\rightarrow$ Def. 3.1.14) and $\sqrt{\Delta t} \leq T$, then there is $C > 0$ independent of Δt, u_0 (➾ notation $C \neq C(\Delta t, \mu_0)$) such that*

$$\|u(\cdot,T) - u_M(\cdot,T)\|_{L^1(\mathbb{R})} \leq \|u(\cdot,0) - u_M(\cdot,0)\|_{L^1(\mathbb{R})} + C\,T \cdot TV_{\mathbb{R}}(u_0)\,\sqrt{\Delta t}\;.$$

Proof. Idea: use Kruzkov pairs (η_c, ψ_c) of non-smooth entropy functions (3.2.37), *parameterized by* u/u_M *!*

For $v, w \in L^\infty(\mathbb{R}\times]0,T[) \cap L^1(\mathbb{R}\times]0,T[)$ define for $\Phi \in C_0^\infty(\mathbb{R}^4)$

$$J(v,w,\Phi) := \int_{-\infty}^{\infty}\int_0^T\int_{-\infty}^{\infty}\int_0^T \eta_{w(x,t)}(v(y,s))\frac{\partial\Phi}{\partial s}(x,t,y,s) + \psi_{w(x,t)}(u(y,s))\frac{\partial\Phi}{\partial y}(x,t,y,s)\,\mathrm{d}s\mathrm{d}y\,\mathrm{d}t\mathrm{d}x$$

$$+ \int_{-\infty}^{\infty}\int_0^T\int_{-\infty}^{\infty} \eta_{w(x,t)}(u(y,0))\Phi(x,t,y,0) - \eta_{w(x,t)}(u(y,T))\Phi(x,t,y,T)\,\mathrm{d}y\,\mathrm{d}t\mathrm{d}x\ .$$

Special choice: $\boxed{\Phi(x,t,y,s) = \varphi(x-y)\varphi(t-s)}$, $\varphi \in C_0^\infty(\mathbb{R})$, $\varphi(x) = \varphi(-x)$

➤ use $\frac{\partial\Phi}{\partial s} = -\frac{\partial\Phi}{\partial t}$, $\frac{\partial\Phi}{\partial x} = -\frac{\partial\Phi}{\partial y}$, $\Phi(x,t,y,s) = \Phi(y,s,x,t)$ & swap $x \leftrightarrow y$,$s \leftrightarrow t$

$$\int\limits_{-\infty}^{\infty}\int\limits_{0}^{T}\int\limits_{-\infty}^{\infty} |u(y,T) - u_M(x,t)|\Phi(x,t,y,T)\,\mathrm{d}y\,\mathrm{d}t\mathrm{d}x$$
$$+ \int\limits_{-\infty}^{\infty}\int\limits_{0}^{T}\int\limits_{-\infty}^{\infty} |u(y,s) - u_M(x,T)|\Phi(x,T,y,s)\,\mathrm{d}x\,\mathrm{d}s\mathrm{d}y$$
$$= -J(u,u_M,\Phi) - J(u_M,u,\Phi) + \int\limits_{-\infty}^{\infty}\int\limits_{0}^{T}\int\limits_{-\infty}^{\infty} |u(y,0) - u_M(x,0)|\Phi(x,t,y,0)\,\mathrm{d}y\,\mathrm{d}t\mathrm{d}x$$
$$+ \int\limits_{-\infty}^{\infty}\int\limits_{0}^{T}\int\limits_{-\infty}^{\infty} |u(y,s) - u_M(x,0)|\Phi(x,0,y,s)\,\mathrm{d}x\,\mathrm{d}s\mathrm{d}y$$

use mollifier

(↔ "approximate convolution unit")

$$\varphi_\epsilon(x) = \epsilon^{-1}\varphi_1(x/\epsilon)\ ,\epsilon > 0,\ x \in \mathbb{R}\ ,$$

$$\varphi_1(x) = \begin{cases} \exp(-1/(1-x^2)) & \text{, if } |x| < 1\ , \\ 0 & \text{elsewhere.} \end{cases}$$

$$\int_{\mathbb{R}} \varphi_\epsilon(x)\,\mathrm{d}x = 1\ .$$

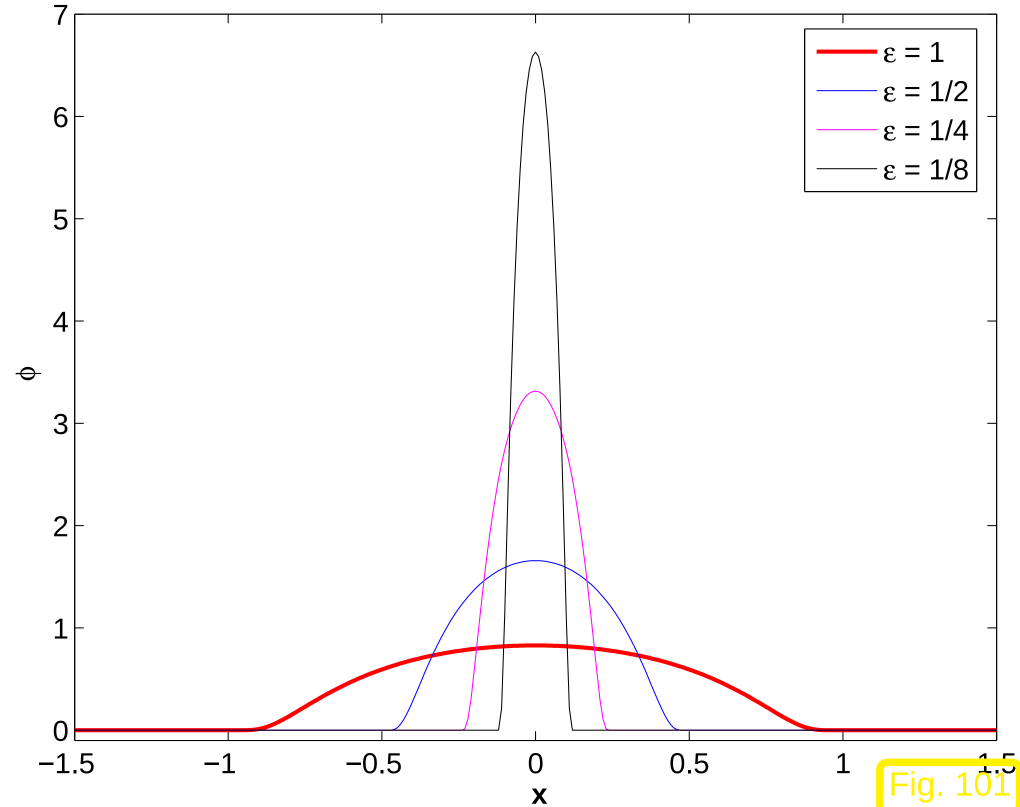


Fig. 101

▶ uniformly $\lim_{\epsilon\to 0}\int_{\mathbb{R}} g(y)\varphi_\epsilon(x-y)\,\mathrm{d}y = g(x)$ for $g \in C^0(\mathbb{R})$.

use $\varphi = \varphi_\epsilon$ for small ϵ:

$$\int_{-\infty}^{\infty}\int_{0}^{T}\int_{-\infty}^{\infty} |u(y,T) - u_M(x,t)|\Phi(x,t,y,T)\,\mathrm{d}y\,\mathrm{d}t\mathrm{d}x$$

$$+ \int_{-\infty}^{\infty}\int_{0}^{T}\int_{-\infty}^{\infty} |u(y,s) - u_M(x,T)|\Phi(x,T,y,s)\,\mathrm{d}x\,\mathrm{d}s\mathrm{d}y$$

$$= 2\int_{-\infty}^{\infty} |u(x,T) - u_M(x,T)|\,\mathrm{d}x + O(\epsilon)\ .$$

requires: ($TV_{\mathbb{R}\times]0,T[}(u)$, $TV_{\mathbb{R}\times]0,T[}(u_M)$, $\|u\|_{L^\infty(\mathbb{R}\times]0,T[)}$, $\|u_M\|_{L^\infty(\mathbb{R}\times]0,T[)}$ $\rightarrow$ constant in "$O(\epsilon)$")

- $TV_{\mathbb{R}\times]0,T[}(u)$ bounded, uniform boundedness $TV_{\mathbb{R}\times]0,T[}(u_M) \leq C$ $\rightarrow$ Lemma 3.2.11
- $\|u\|_{L^\infty(\mathbb{R}\times]0,T[)} \leq C$ bounded, uniform boundedness $\|u_M\|_{L^\infty(\mathbb{R}\times]0,T[)} \leq C$ $\rightarrow$ Lemma 3.1.15

$$\|u(\cdot,T) - u_M(\cdot,T)\|_{L^1(\mathbb{R})} \leq \cancel{-J(u,u_M,\Phi)} - J(u_M,u,\Phi) + \|u(\cdot,0) - u_M(\cdot,0)\|_{L^1(\mathbb{R})} + O(\epsilon)\ . \tag{3.2.39}$$

➤ by weak entropy inequality ($\rightarrow$ Def. 2.5.3) for u: $J(u,u_M,\Phi) \geq 0$

Next lemma ([15, Lemma A.1]) uses discrete entropy inequality for Kruzkov entropies, *cf.* proof of Thm. 3.2.18

Lemma 3.2.25.

$$\exists C \neq C(u_0, \Delta t)\colon \quad J(u_M, u, \Phi) \leq C\, T \cdot TV_{\mathbb{R}}(u_0)\, \Delta t\; \|\varphi\|_{W^{1,1}(\mathbb{R})} \;.$$

► choose $\epsilon = \sqrt{\Delta t}$ for mollifier ➤ $\|\varphi\|_{W^{1,1}(\mathbb{R})} \approx (\Delta t)^{-1/2}$ □

Remark 71. Thm. 3.2.24 (partly) explains observed convergence of FVM for non-smooth solutions → Ex. 50

△

3.2.9 Numerical viscosity

Recall: viscous modification of finite volume method (→ Rem. 61)

New schemes (→ Lax-Friedrichs scheme (3.1.29)) through viscous modification of centered scheme (3.1.17)

(3.2.9), (3.2.25) ➡ numerical flux function in viscous form

$$F(v,w) = \tfrac{1}{2}(f(v)+f(w)) - \tfrac{1}{2\gamma}Q(v,w)(w-v)\ , \Updownarrow \tag{3.2.40}$$

$$Q(v,w) = \gamma\frac{f(w) - 2F(v,w) + f(v)}{w-v}\ , \quad v \neq w\ . \tag{3.2.41}$$

centered flux Q = numerical viscosity control function diffusive flux

$Q(v,w) > 0$ $\hat{=}$ "numerical viscosity" ($\rightarrow$ compare: no viscosity in conservation law, *cf.* Sect. 2.5.1)

cf. viscous form, Def. 3.1.16

$$\mu_j^{(k)} = \underbrace{\mu_j^{(k-1)} - \tfrac{1}{2}\gamma\left(f(\mu_{j+1}^{(k-1)}) - f(\mu_{j-1}^{(k-1)})\right)}_{\text{centered scheme (3.1.17)}} + \tfrac{1}{2}Q(\mu_j^{(k-1)},\mu_{j+1}^{(k-1)})(u_{j+1}^{(k-1)} - \mu_j^{(k-1)})$$

$$- \tfrac{1}{2}Q(\mu_{j-1}^{(k-1)},\mu_j^{(k-1)})(u_j^{(k-1)} - \mu_{j-1}^{(k-1)})\ , \quad j \in \mathbb{Z}\ .$$

Thm. 3.1.23:

$$\boxed{\gamma\left|\frac{f(w)-f(v)}{w-v}\right| \leq Q(v,w) \leq 1 \quad \Rightarrow \quad \text{TVD}}$$

(3.2.9) ➤ Lax-Friedrichs scheme: $Q(v,w) = 1$

(3.2.25) ➤ Lax-Wendroff scheme: $Q(v,w) = (\gamma f'(\tfrac{1}{2}(v+w)))^2$

Lemma 3.2.5: Diffusivity of 1st-order FVM with flux in viscous form (3.2.40)

$$(3.2.19) \quad \Rightarrow \quad b(u,\gamma) = \frac{1}{2\gamma^2}\Big(Q(u,u) - (\gamma f'(u))^2\Big) \; . \tag{3.2.42}$$

- Lax-Wendroff scheme has *minimal* numerical viscosity required for stability, *cf.* Sect. **??**
- $Q(u,u) = (\gamma f'(u))^2$ necessary for 2nd-order consistency ($\rightarrow$ Def. 3.1.7), [37, Sect. 3]

Example 72 (Numerical viscosity for 3-point finite volume methods).

Assume: $Q(v,w)$ can be extended to a Lipschitz-continuous function $Q : \mathbb{R}^2 \mapsto \mathbb{R}$

- Burgers equation (2.1.7): $f(u) = \frac{1}{2}u^2$
- Equidistant space-time tensor product mesh, $\gamma := \Delta t / \Delta x = 1$

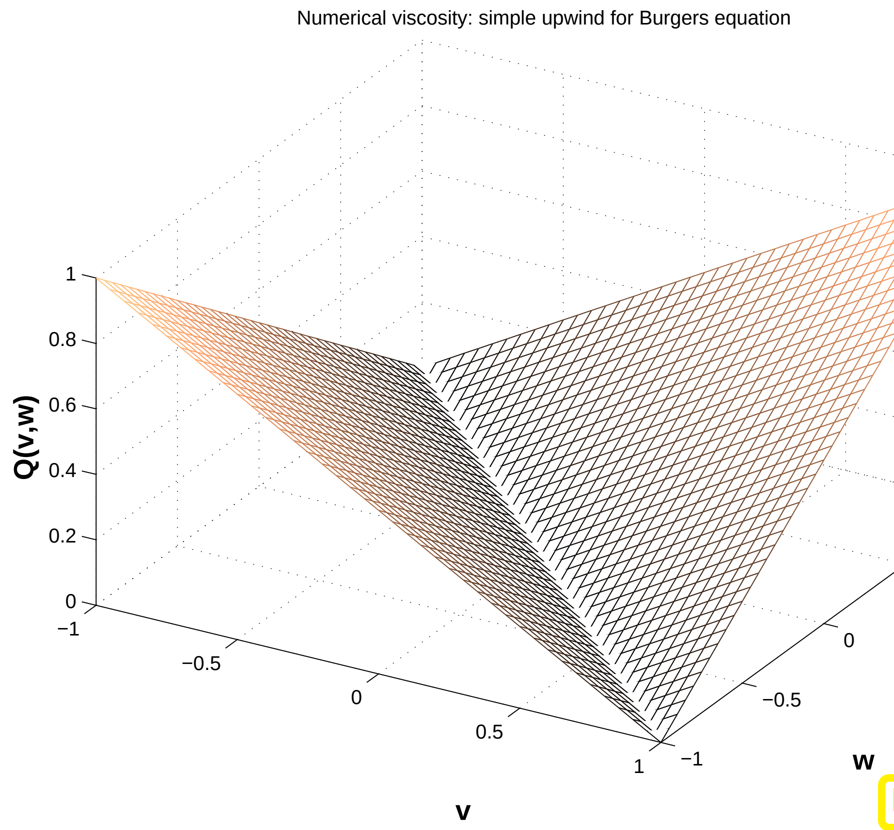


Fig. 102

Simple upwinding (3.2.6)

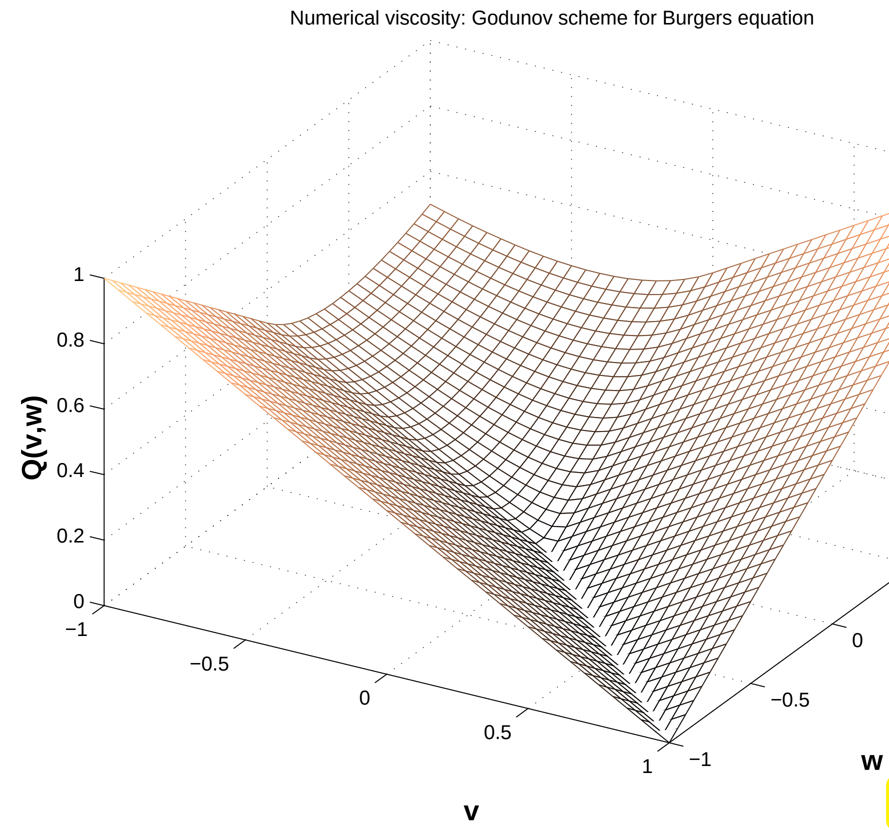


Fig. 103

Godunov scheme (3.2.17)

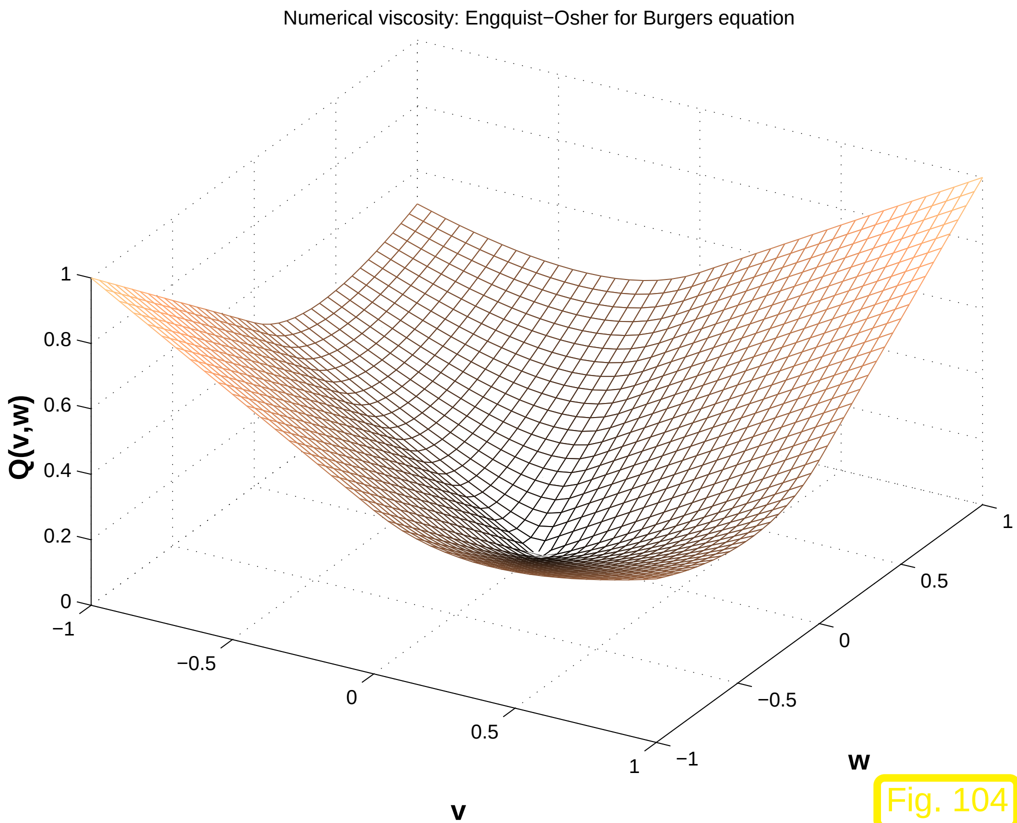


Fig. 104

Engquist-Osher (3.2.7)

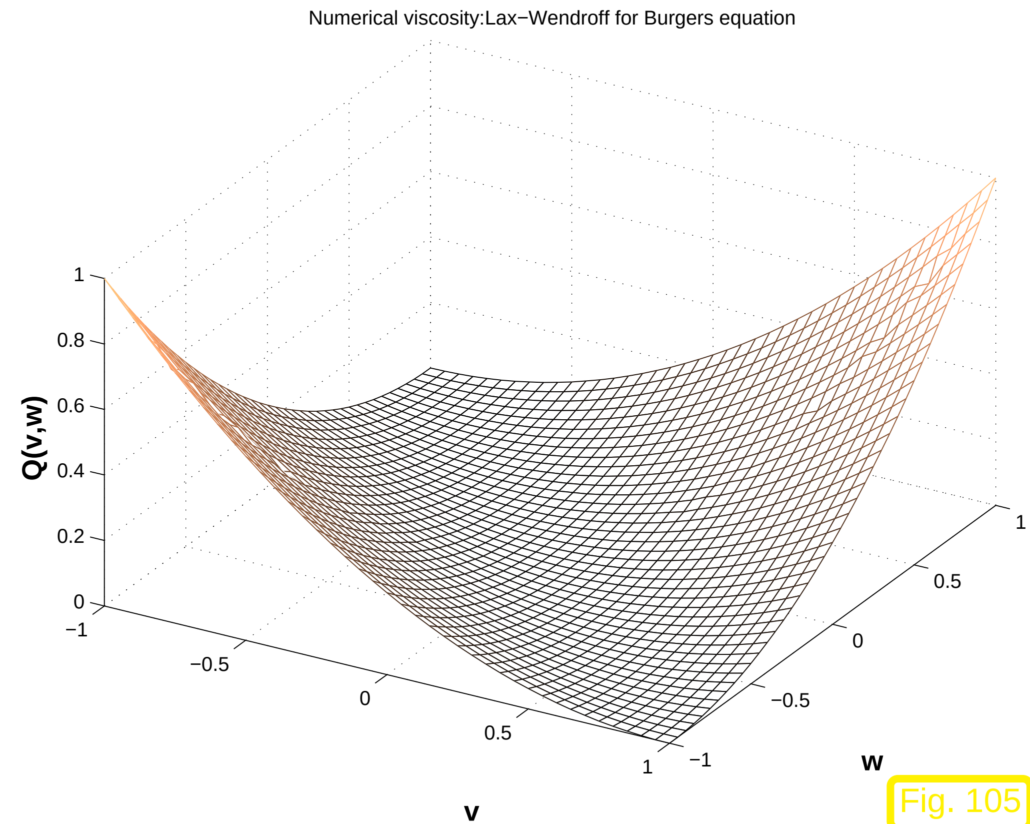


Fig. 105

Lax-Wendroff (3.2.25)

Godunov's method has the least numerical viscosity among all E-schemes ($\longrightarrow$ Def. 3.2.19) $\diamond$

Entropy fix

Numerical viscosity for simple upwinding (3.2.6):

$$F_{\mathrm{uw}} \text{ from (3.2.5)} \quad \stackrel{(3.2.41)}{\Rightarrow} \quad Q_{\mathrm{uw}}(v,w) = \begin{cases} \gamma \left|\frac{f(w)-f(v)}{w-v}\right| & \text{, if } v \neq w \,, \\ f'(v) & \text{, if } v = w \,. \end{cases} \tag{3.2.43}$$

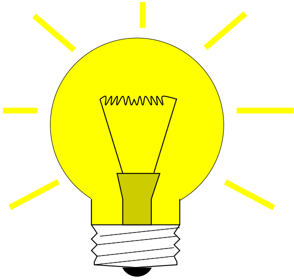
$$\stackrel{(3.2.42)}{\blacktriangleright} \quad b(u,\gamma) = \frac{1}{2\gamma^2}\left(|f'(u)| - (\gamma f'(u))^2\right): \quad f'(u) = 0 \quad \Rightarrow \quad b(u,\gamma) = 0 \,. \tag{3.2.44}$$

"Too little" numerical viscosity for $u \approx u^*$, $f'(u^*) = 0$

Ex. 70 $\leftrightarrow$ Simple upwinding for Cauchy problem (2.2.1) with convex flux function $f \in C^2(\mathbb{R})$, $f(u) = f(-u)$ ➤ $f'(0) = 0$

➥ danger of convergence to entropy violating solutions !

Idea: Entropy fix



☞ slightly enhance $Q(v,w)$ for $w, v \approx u^*$

$$\text{for (3.2.43):} \quad \widetilde{Q}_{\mathrm{uw}}(v,w) = \gamma m_\epsilon\left(\frac{f(w)-f(v)}{w-v}\right) \,, \tag{3.2.45}$$

with $m_\epsilon(\xi) > \min\{|\xi|, \epsilon\}$ everywhere.

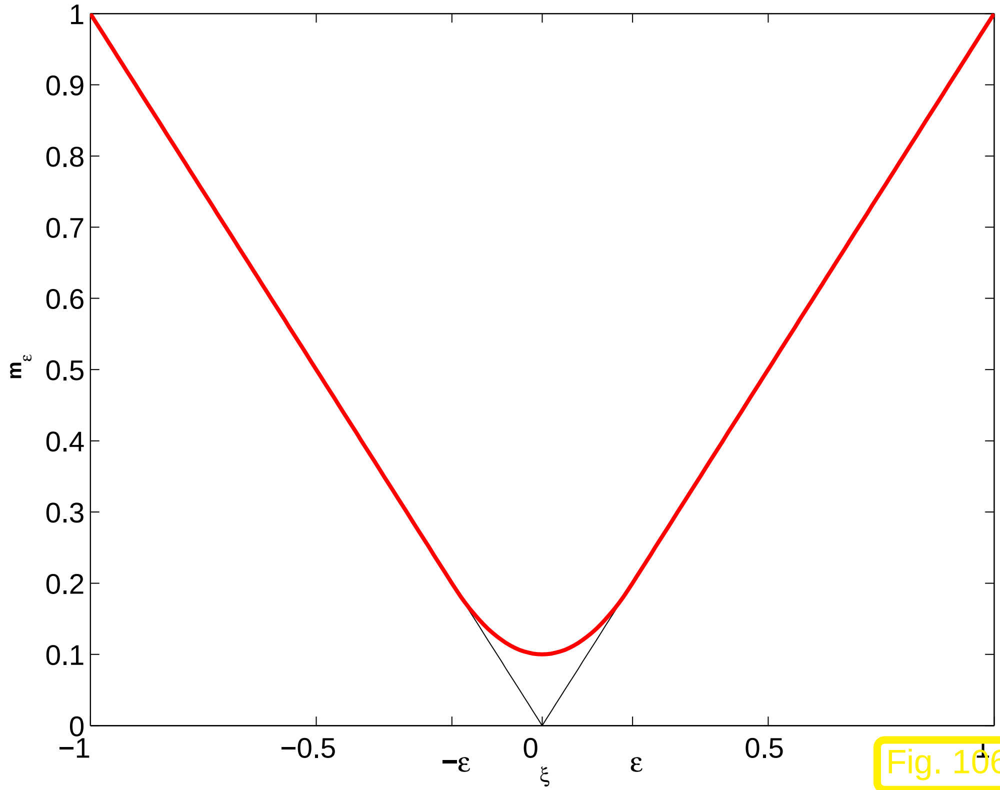


Fig. 106

$$m_\epsilon(\xi) = \begin{cases} \frac{\xi^2}{4\epsilon} + \epsilon & \text{, if } |\xi| < 2\epsilon \ , \\ |\xi| & \text{, if } |\xi| > 2\epsilon \ . \end{cases}$$

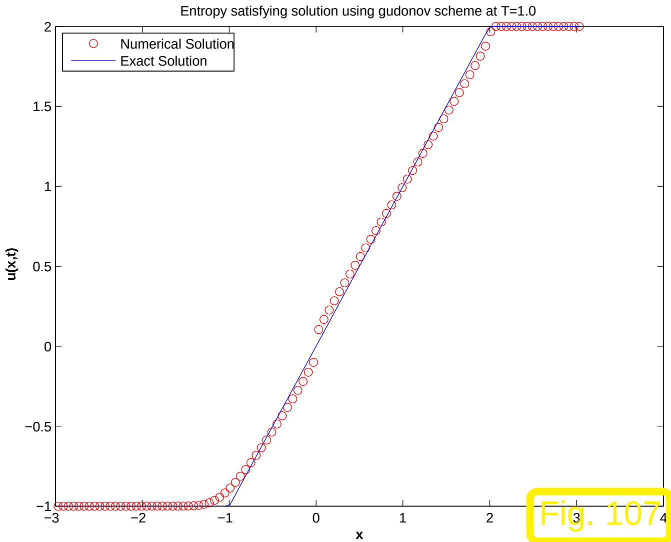
Practice: strength of "entropy fix" $\sim$ mesh resolution:

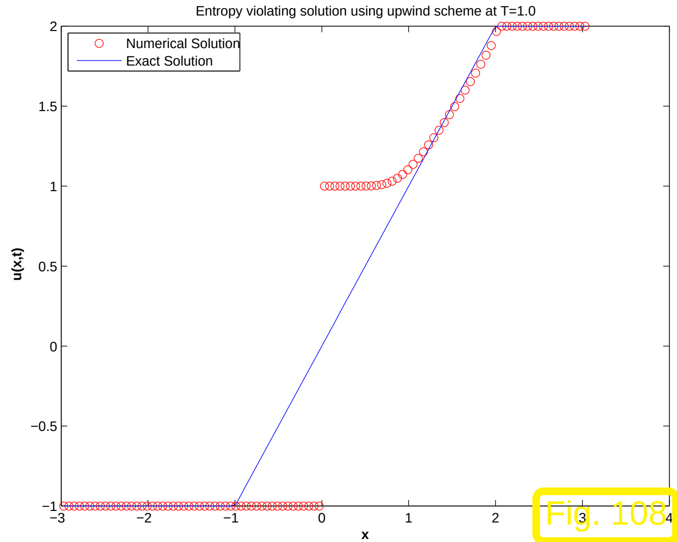
$$\epsilon \sim \Delta x \ .$$

Example 73 (Entropy fix for Burgers equation).

- Cauchy problem for Burgers equation of Ex. 70 (rarefaction)
- comparison: Godunov scheme ($\longrightarrow$ Sect. 3.2.2), simple upwinding (3.2.6) **+** entropy fix (3.2.45)
- equidistant space-time mesh, $\Delta x = 0.06$, $\gamma = 1$

➨ movies: burger_upwind.avi, burger_godunov.avi, burger_upwind_efix.avi





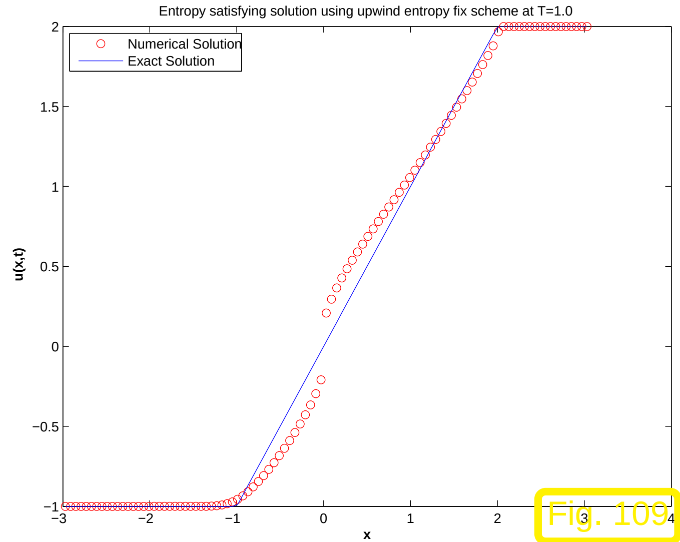


Fig. 107 Godunov's method

Fig. 108 simple upwinding

Fig. 109 upwind **+** entropy fix

Observation: Entropy improves convergence to rarefaction solution, though remnants of (spurious) expansion shock ◇

3.3 High resolution methods

▷ Thm. 3.2.21, Thm. 3.2.8: ➤ E-schemes/monotone FVM at most 1st-order consistent

▷ Rem. 69 ➤ TVD 3-point FVM are at most first order consistent

▷ Rem. 68 ➤ linear advection: only non-linear methods can be 2nd-order & TVD

▷ Sect. 3.2.3 ➤ 1st-order monotone/TVD FVM diffusive (➔ shock smearing)

Goal:	construct (formally) 2nd-order TVD finite volume methods

3.3.1 Limiters

Focus: finite difference method in conservation form (⟶ Def. 3.2.1)

$$\mu_j^{(k)} = \mu_j^{(k-1)} - \gamma(F(\mu_{j-m_l+1}^{(k-1)}, \dots, \mu_{j+m_r}^{(k-1)}) - F(\mu_{j-m_l}^{(k-1)}, \dots, \mu_{j+m_r-1}^{(k-1)})) ,$$

consistent with $\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u) = 0$, on equidistant infinite space-time grid $\mathcal{M} = \mathcal{G}_{\Delta x} \times \mathcal{G}_{\Delta t}$, $\gamma := {}^{\Delta t}/_{\Delta x}$ fixed.

3.3.1.1 Linear reconstruction

Godunov's method, Sect. 3.2.2: piecewise constant reconstruction $\Rightarrow$ only 1st-order consistent

Cor. 2.6.2, 2.6.3 ➢ for the REA-algorithm with *exact* **E**volve:
($u : \mathbb{R} \mapsto \mathbb{R}$ sufficiently smooth)

$$\begin{aligned} \|u - w_0(\mathsf{R}u)\|_{L^\infty(\mathbb{R})} &= O((\Delta x)^q)\ , \\ TV_{\mathbb{R}}(u - w_0(\mathsf{R}u)) &= O((\Delta x)^q)\ , \end{aligned} \quad \stackrel{*}{\Longrightarrow} \quad \text{REA-evolution order } q \text{ consistent w.r.t } \|\cdot\|_{l^1(\mathbb{Z})}\ .$$

$\mathsf{R} \hat{=}$ cell averaging operator, Sect. 3.1

$*$: analoguous conclusion *not valid* for $L^\infty(\mathbb{R})$-norm **!** (Cor. 2.6.2 "too weak")

Recall: interpolation/approximation error estimates for piecewise polynomials, *cf.* [27, Sect. 4.2.5].

Idea: 2nd-order consistency through REA-algorithm ($\rightarrow$ Sect. 3.2.2) with piecewise linear reconstruction:

➡ given $\vec{\mu}^{(k-1)}$ obtain $\vec{\mu}^{(k)}$ in 3 steps:

① **R**econstruct: find $w_0 = w_0(\vec{\mu}^{(k-1)})$, p.w. linear on grid cells with (suitable) slopes $\sigma_j^{(k-1)}$

$$w_0(x) = \mu_j^{(k-1)} + \sigma_j^{(k-1)}(x - x_j) \quad \text{for} \quad x_{j-1/2} < x < x_{j+1/2} \,. \tag{3.3.2}$$

② **E**volve: solve the Cauchy problem

$$\frac{\partial w}{\partial t} + \frac{\partial}{\partial x} f(w) = 0 \quad \text{in } \mathbb{R}\times]0,\Delta t[\,, \quad w(x,0) = w_0(x) \,,\, x \in \mathbb{R} \,. \tag{3.2.12}$$

③ **A**verage: get $\vec{\mu}^{(k)}$ from cell avarages: $\mu_j^{(k)} := \dfrac{1}{\Delta x}\displaystyle\int_{x_{j-1/2}}^{x_{j+1/2}} w(x,\Delta t)\,\mathrm{d}x$ (3.2.13)

Obvious: preservation of cell averages: $\displaystyle\int_{x_{j-1/2}}^{x_{j+1/2}} w_0(x)\,\mathrm{d}x = \mu_j^{(k-1)}$!

Special case: constant scalar advection (2.1.6) $\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial x} = 0$ ► $w(x,\Delta t) = w_0(x - v\Delta t)$

$$\overset{v>0}{\blacktriangleright} \quad \mu_j^{(k)} = v\gamma(\mu_{j-1}^{(k-1)} + \tfrac{1}{2}(\Delta x - v\Delta t)\sigma_{j-1}^{(k-1)}) + (1 - v\gamma)(\mu_j^{(k-1)} - \tfrac{1}{2}v\Delta t\sigma_j^{(k-1)}) \tag{3.3.3}$$
$$= \underbrace{\mu_j^{(k-1)} - v\gamma(\mu_j^{(k-1)} - \mu_{j-1}^{(k-1)})}_{\text{upwind (3.1.26)}} - \underbrace{\tfrac{1}{2}v\gamma(\Delta x - v\Delta t)(\sigma_j^{(k-1)} - \sigma_{j-1}^{(k-1)})}_{\text{correction}}$$

How to choose the slopes $\sigma_j^{(k-1)}$ **?**

"Downwind slope"

$$\sigma_j^{(k-1)} = \frac{1}{\Delta x}(\mu_{j+1}^{(k-1)} - \mu_j^{(k-1)}) \tag{3.3.4}$$

(3.3.3)

$\Updownarrow$

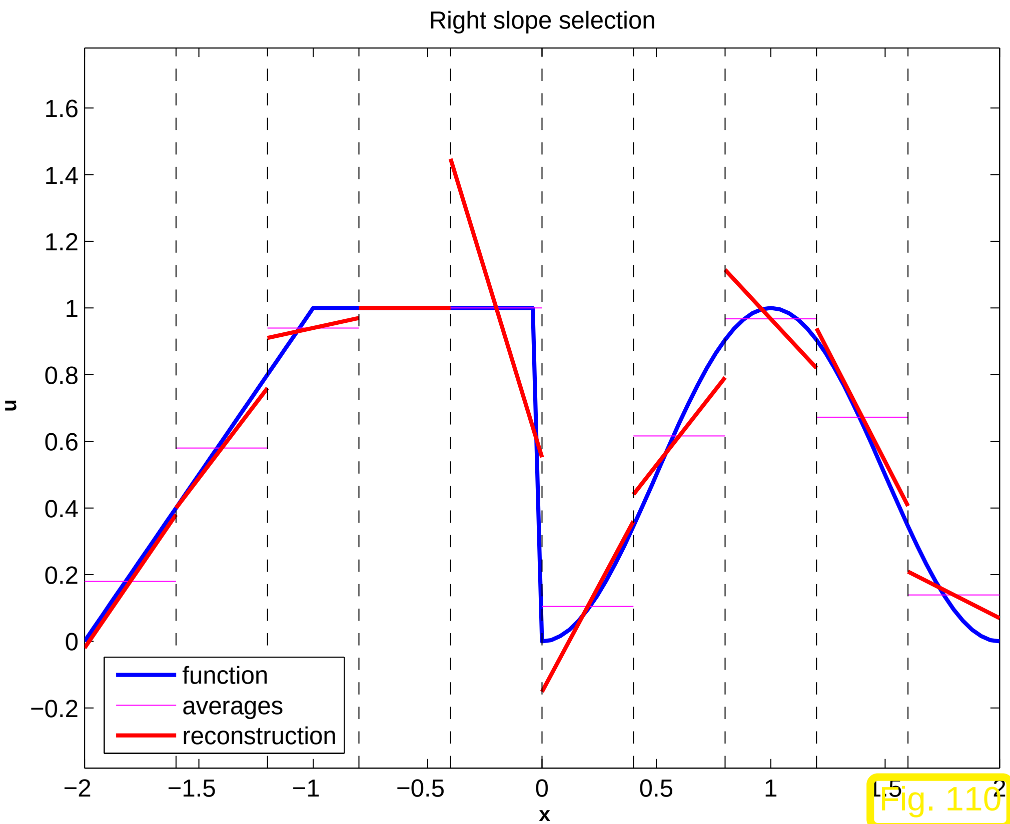


Fig. 110

$$\mu_j^{(k)} = \mu_j^{(k-1)} - v\gamma(\mu_j^{(k-1)} - \mu_{j-1}^{(k-1)}) - \tfrac{1}{2}v\gamma(1-v\gamma)(\mu_{j+1}^{(k-1)} - 2\mu_j^{(k-1)} + \mu_{j-1}^{(k-1)}) \;, \qquad (3.3.5)$$

= Lax-Wendroff scheme (3.1.12) for linear advection **!**

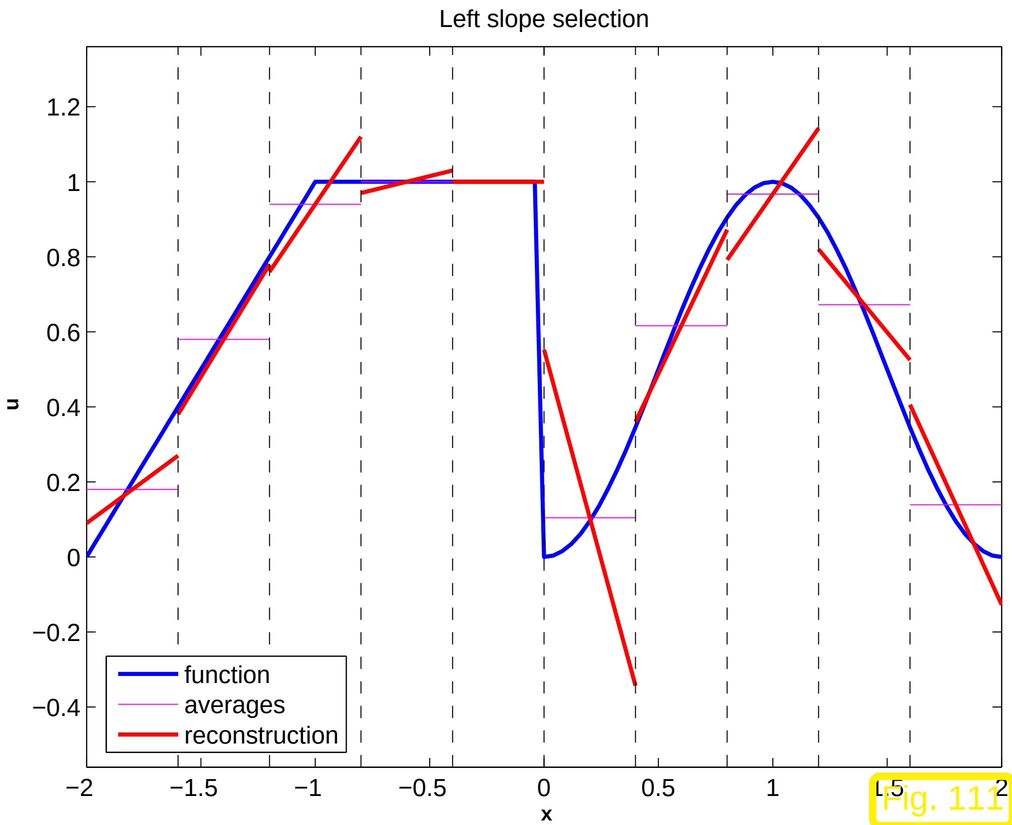


Fig. 111

“Upwind slope”

$$\sigma_j^{(k-1)} = \frac{1}{\Delta x}(\mu_j^{(k-1)} - \mu_{j-1}^{(k-1)})$$

(3.3.3)

$\Updownarrow$

$$\mu_j^{(k)} = \mu_j^{(k-1)} - v\gamma(\mu_j^{(k-1)} - \mu_{j-1}^{(k-1)}) - \tfrac{1}{2}v\gamma(1-v\gamma)(\mu_j^{(k-1)} - 2\mu_{j-1}^{(k-1)} + \mu_{j-2}^{(k-1)}) \;, \qquad (3.3.6)$$

= Beam-Warming scheme for linear advection

"Centered slope"

$$\sigma_j^{(k-1)} = \frac{1}{2\Delta x}(\mu_{j+1}^{(k-1)} - \mu_{j-1}^{(k-1)})$$

(3.3.3)

$\Updownarrow$

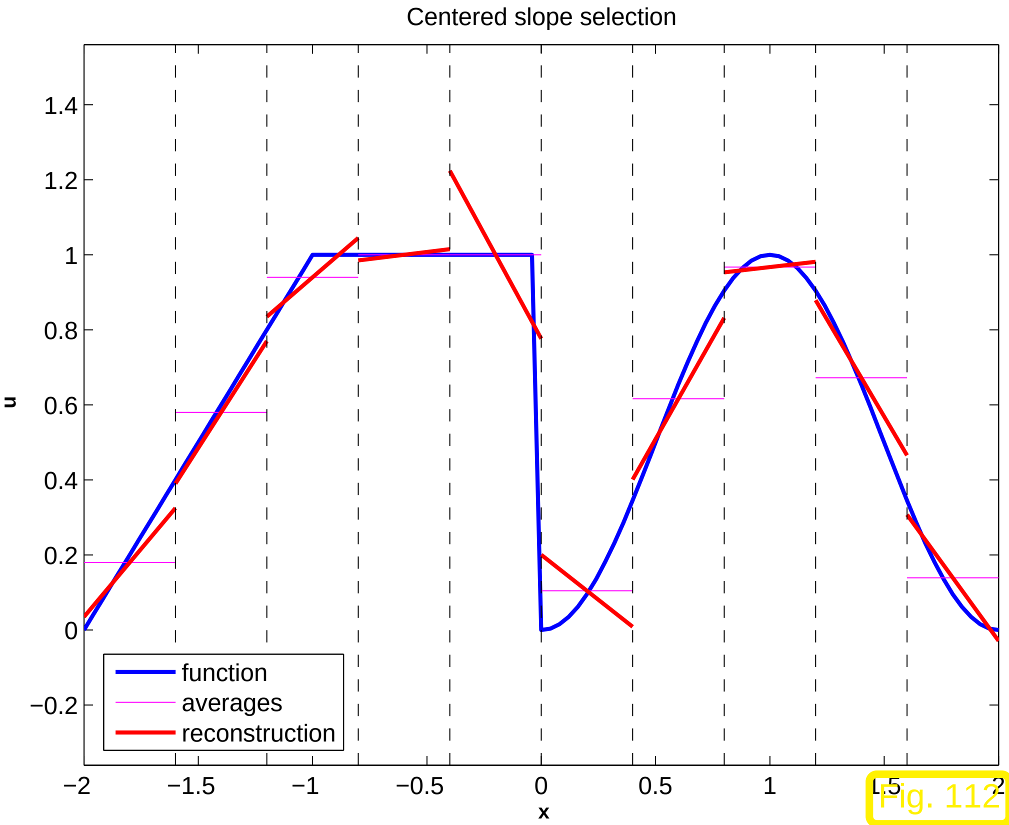


Fig. 112

$$\mu_j^{(k)} = \mu_j^{(k-1)} - \tfrac{1}{4}v\gamma(\mu_{j+1}^{(k-1)} + 3\mu_j^{(k-1)} - 5\mu_{j-1}^{(k-1)} + \mu_{j-2}^{(k-1)}) - \tfrac{1}{4}(v\gamma)^2(\mu_{j+1}^{(k-1)} - \mu_j^{(k-1)} - \mu_{j-1}^{(k-1)} - \mu_{j-2}^{(k-1)}) \quad (3.3.7)$$

= Fromm's scheme for linear advection

For all choices of slopes:

$\|u - w_0\|_{L^\infty(\mathbb{R})} = O((\Delta x)^2)$,
if w_0 reconstructed from cell averages of smooth u

The Lax-Wendroff (3.3.5), Beam-Warming (3.3.6), and Fromm scheme (3.3.7) are 2nd-order consistent with (2.1.6)

Ex. 65 ➪ Lax-Wendroff introduces *oscillations* near discontinuities: another explanation

For "downwind slope" (3.3.4) $\leftrightarrow$ Lax-Wendroff scheme (3.3.5):

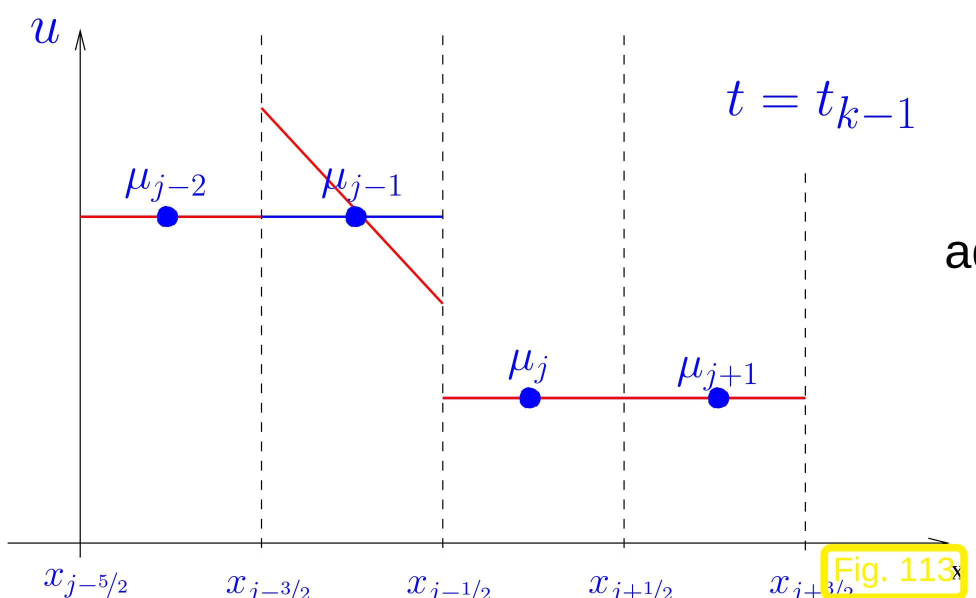


Fig. 113

advect $\Longrightarrow$

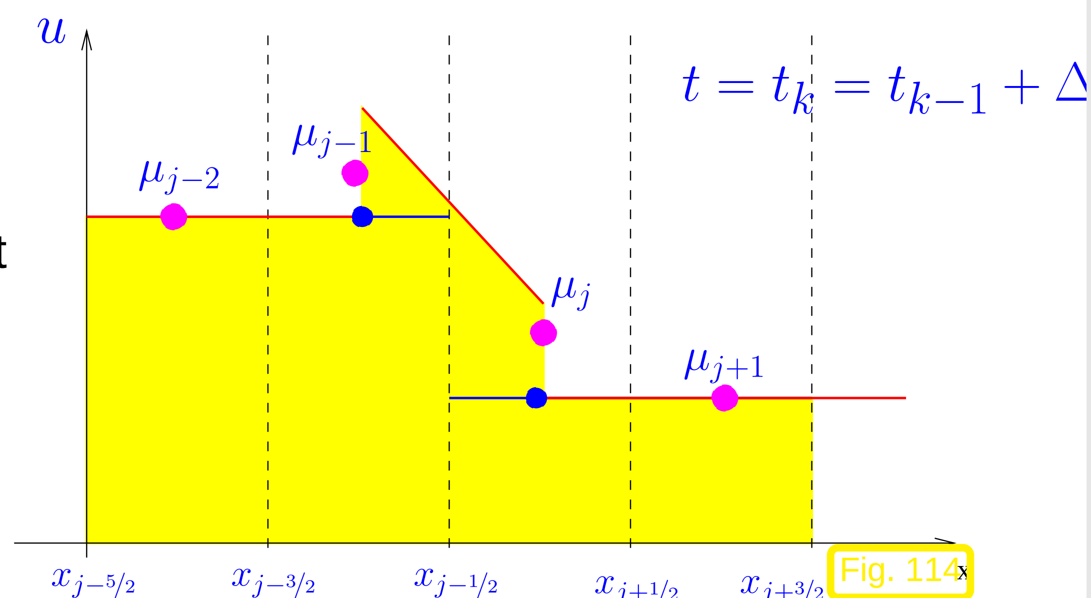


Fig. 114

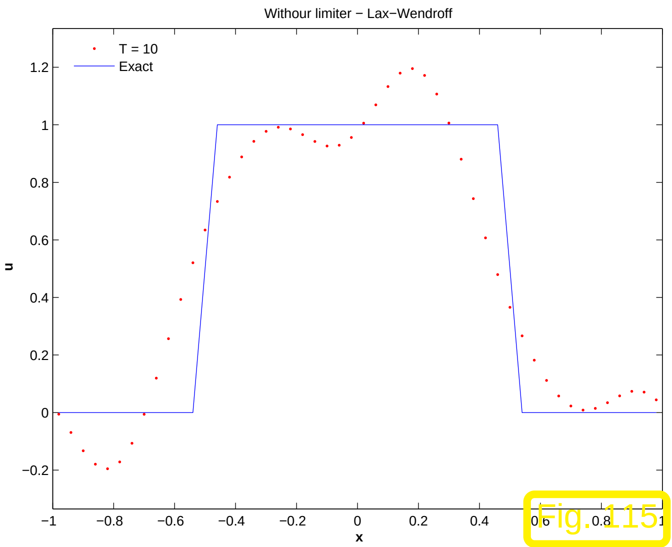
(● $\hat{=}\ \mu_j^{(k-1)}$, — $\hat{=}$ p.w. linear reconstruction, ● $\hat{=}\ \mu_j^{(k)}$)

➤ oscillations trailing shock (as in Ex. 65)

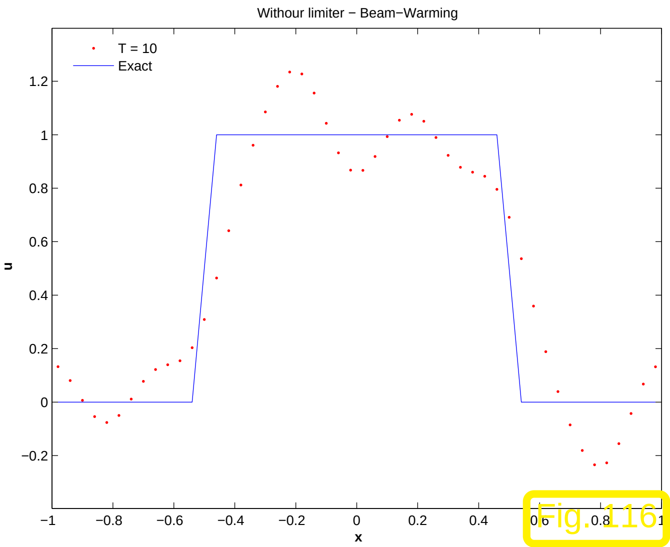
Example 74 (2nd-order schemes for linear advection).

- linear advection (2.1.6), $v = 1$, $u_0 = \chi_{[-1/2,1/2]}$, $T = 2$ $\Omega =]-1,1[$ + periodic boundary conditions

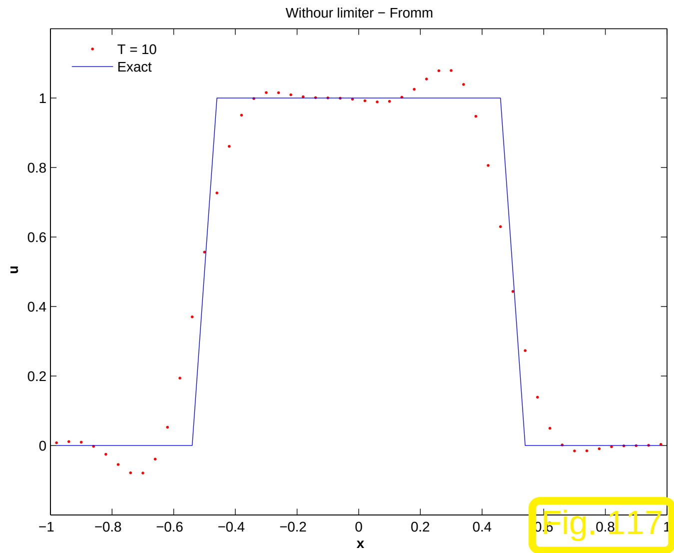
- linear FVM: Lax-Wendroff, Bream-Warming, and Fromm scheme on equidistant mesh, $\Delta x = 0.04$, $\Delta t = 0.033$



Lax-Wendroff scheme (3.3.5)



Beam-Warming scheme (3.3.6)



Fromm's scheme (3.3.7)

Observation: ☞ Lax-Wendroff: oscillations trailing discontinuity

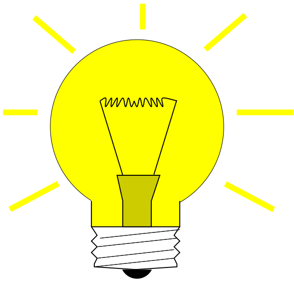
☞ Beam-Warming: oscillations ahead of discontinuity

☞ Fromm: oscillations on both sides of discontinuity ◇

3.3.1.2 Slope limiting

Recall ($\rightarrow$ Sect. 3.1.3.2): TVD-property ($\rightarrow$ Def. 3.1.20) ➥ no oscillations can arise

Note: REA-steps (exact) **E**volve & **A**verage are TVD ($\rightarrow$ Thm. 2.6.8)



Idea: ensure TVD **R**econstruction (3.3.2) !

☞ monotonicity preserving reconstruction without overshoots **!**

➨ notation: $\mathcal{P}_1(\mathcal{G}_{\Delta x}) \hat{=}$ space of cell-p.w. linear (discontinuous) functions $\mathbb{R} \mapsto \mathbb{R}$

Definition 3.3.1 (Monotonicity preserving linear interpolation)**.**
An operator $\mathsf{I} : C^0(\mathcal{G}_{\Delta x}) \mapsto \mathcal{P}_1(\mathcal{G}_{\Delta x})$ *is a monotonicity preserving linear interpolation, if*

$$(\mathsf{I}\vec{\mu})(x_j) = \mu_j \quad \wedge \quad \begin{array}{l} \mu_j \leq \mu_{j+1} \Rightarrow \mathsf{I}\vec{\mu} \textit{ non-decreasing in }]x_j, x_{j+1}[\ , \\ \mu_j \geq \mu_{j+1} \Rightarrow \mathsf{I}\vec{\mu} \textit{ non-increasing in }]x_j, x_{j+1}[\ . \end{array}$$

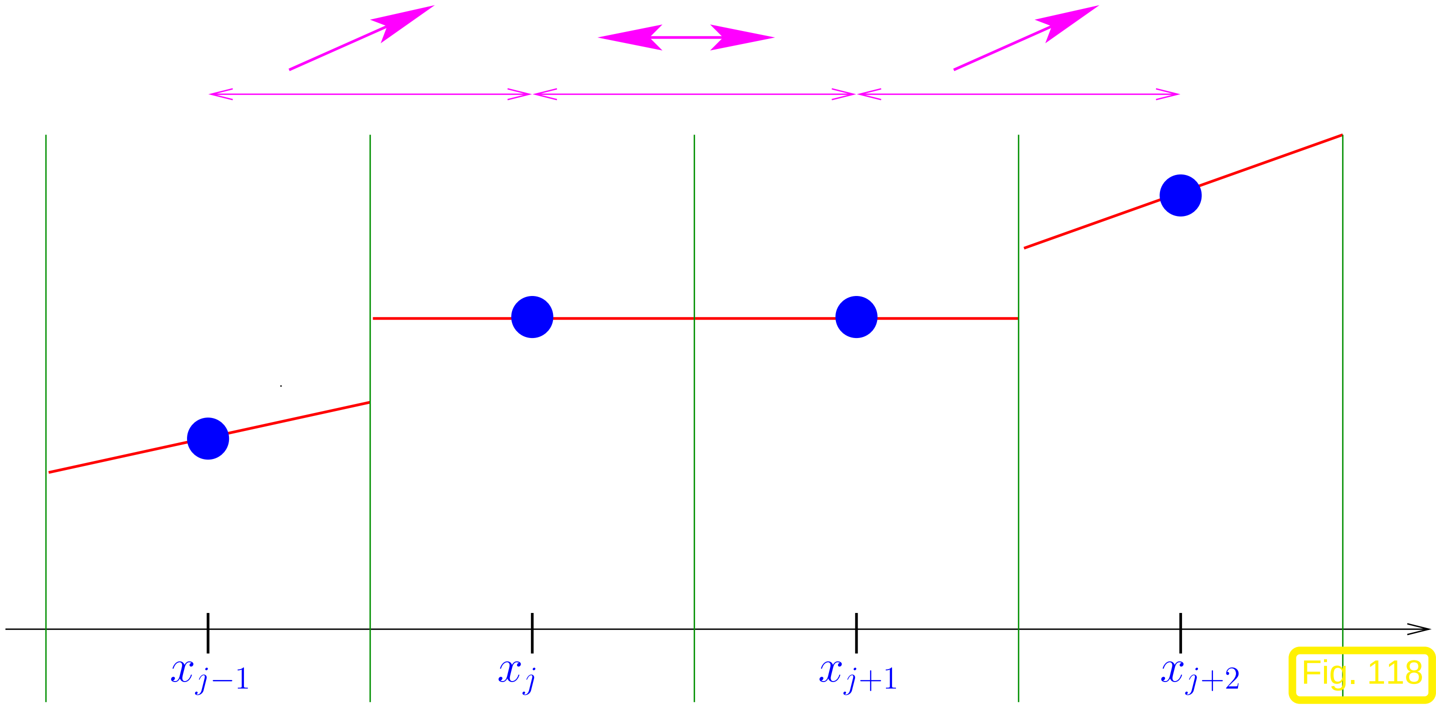


Fig. 118

Monotonicity preserving linear interpolants:

- constant at plateaus
- constant at (local) extrema

Lemma 3.3.2 (Monotonicity preserving linear interpolation is TVD)**.**
For a monotonicity preserving linear interpolation operator ($\longrightarrow$ *Def. 3.3.1)*

$$TV_{\mathbb{R}}(\mathsf{I}\vec{\mu}) = TV_{\Delta x}(\vec{\mu}) \quad \forall \vec{\mu} \in C^0(\mathcal{G}_{\Delta x}), TV_{\Delta x}(\vec{\mu}) < \infty \ .$$

Definition 3.3.3 (Minmod interpolation).

The minmod interpolation $\mathsf{I}_{\mathrm{mm}} : C^0(\mathcal{G}_{\Delta x}) \mapsto \mathcal{P}_1(\mathcal{G}_{\Delta x})$ *is defined by*

$$(\mathsf{I}_{\mathrm{mm}}\vec{\mu})(x) = \mu_j + \sigma_j(x - x_j)$$

for $\quad x_{j-1/2} < x < x_{j+1/2}\ , j \in \mathbb{Z}\ ,$

$$\sigma_j := \frac{1}{\Delta x}\,\mathrm{minmod}(\mu_{j+1} - \mu_j, \mu_j - \mu_{j-1})\ ,$$

$$\mathrm{minmod}(v,w) := \begin{cases} v & , vw > 0, |v| < |w|\ , \\ w & , vw > 0, |w| < |v|\ , \\ 0 & , vw \leq 0\ . \end{cases}$$

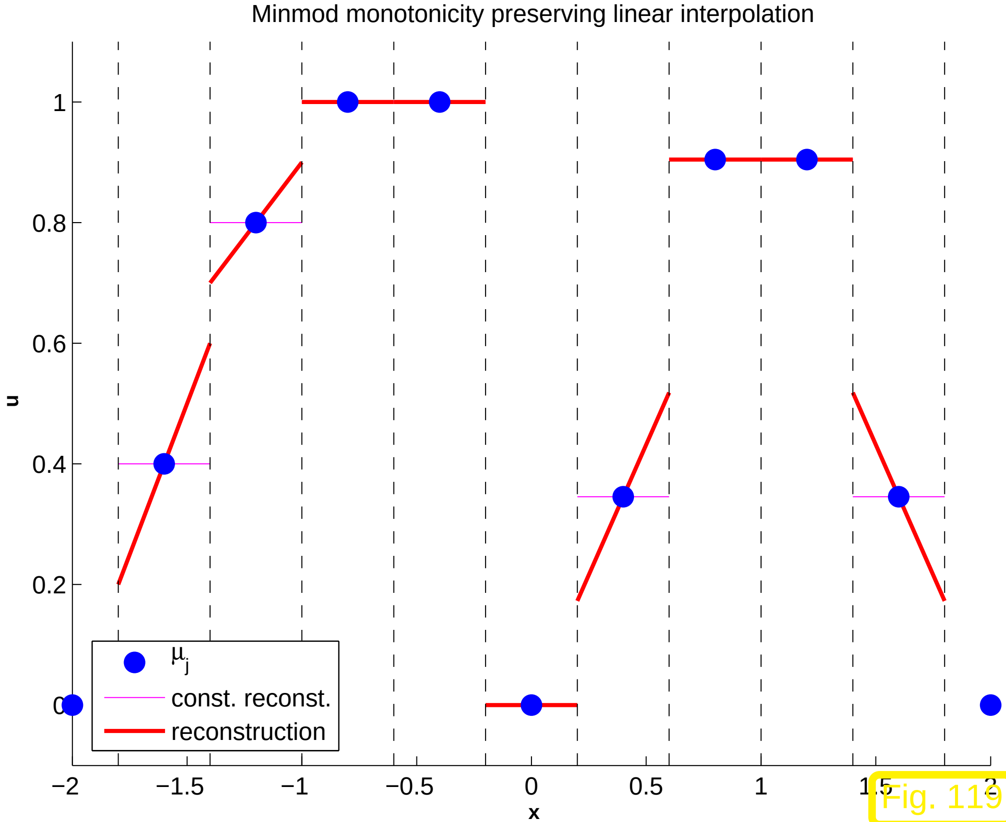


Fig. 119

Convention: use average at cell boundaries $(\mathsf{I}_{\mathrm{mm}}\vec{\mu})(x_{j+1/2}) = \frac{1}{2}(\mu_j + \mu_{j+1}) + \frac{1}{4}(\sigma_j - \sigma_{j+1})\Delta x$

Lemma 3.3.4 (Monotonocity preservation of minmod interpolation).

Minmod interpolation ($\longrightarrow$ *Def. 3.3.3) is monotonicity preserving (*$\longrightarrow$ *Def. 3.3.1)*

Terminology: effect of minmod-function in I_{mm}: slope limiting: minmod = slope limiter

Lemma 3.3.5 (Approximation by minmod interpolation). $\longrightarrow$ *[21, Thm. 109.3]*

$$u \in W^{2,\infty}(\mathbb{R}) \quad \Rightarrow \quad \exists C > 0\colon \; |u(x) - (\mathsf{I}_{\mathrm{mm}}\mathsf{R}u)(x)| \leq C(\Delta x)^2 \quad \forall \Delta x > 0 \,.$$

Example 75 (Accuracy of piecewise linear reconstruction).

- C^1-function $u(x) = 1 - cos(2\pi(x + \chi_{[1/2,3/2]}\cos^2(\pi x)))$ for $0 \leq x \leq 2$, $u \equiv 0$ elsewhere
- $w_0 \hat{=}$ p.w. linear interpolant of cell avarages of u on equidistant grid, downwind slope (3.3.4) & minmod slope ($\longrightarrow$ Def. 3.3.3)

Recorded: norms of approximation error $\|u - w_0\|_{L^1(\mathbb{R})}$ and $\|u - w_0\|_{L^\infty(\mathbb{R})}$ for $\Delta x \in \{\frac{1}{5}, \frac{1}{10}, \frac{1}{20}, \frac{1}{40}, \frac{1}{80}, \frac{1}{160}, \frac{1}{320}\}$, preasymptotic algebraic decay rates of errors

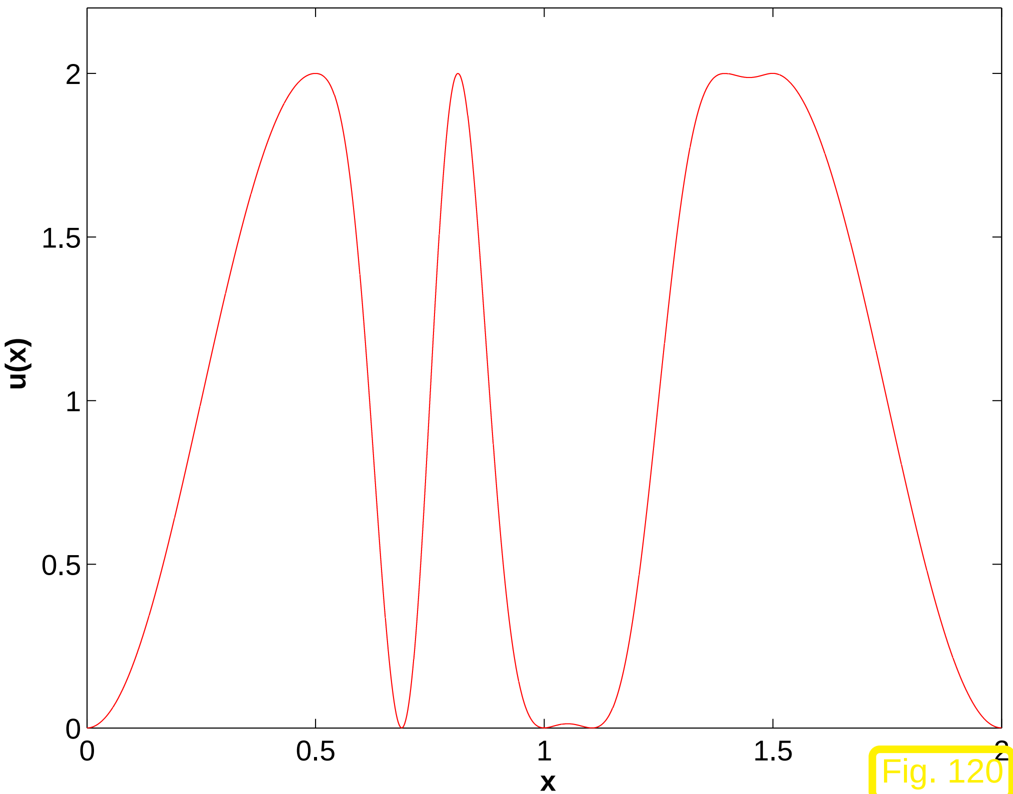


Fig. 120

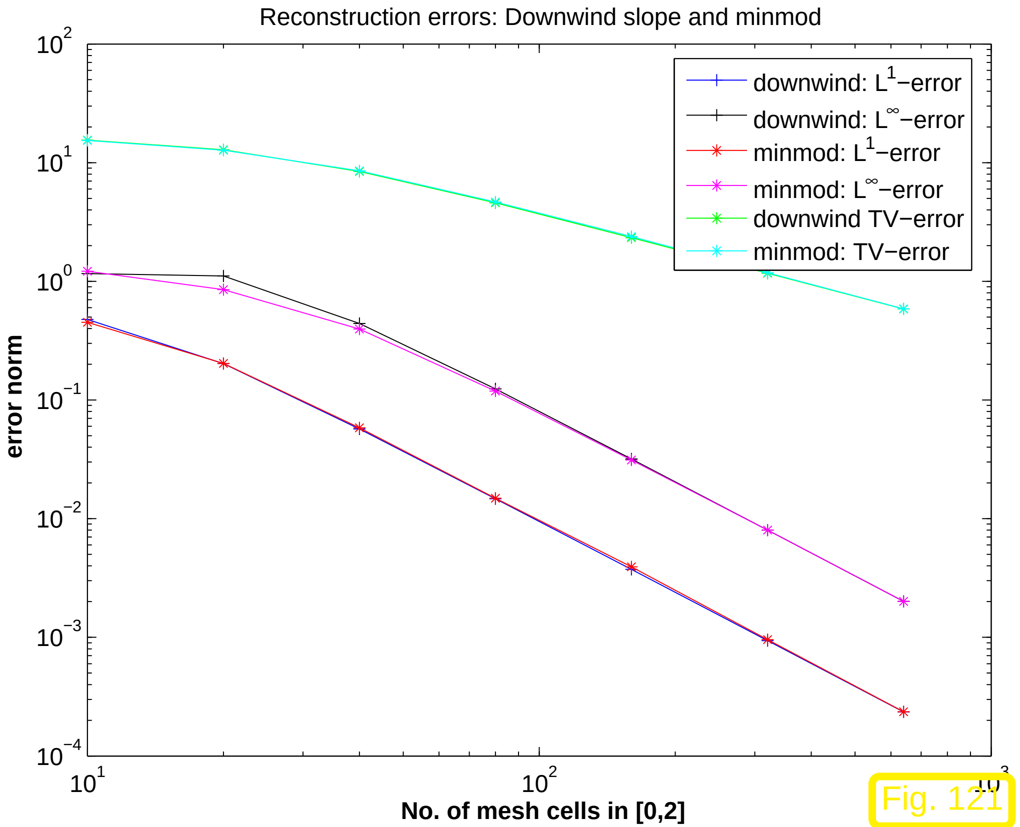


Fig. 121

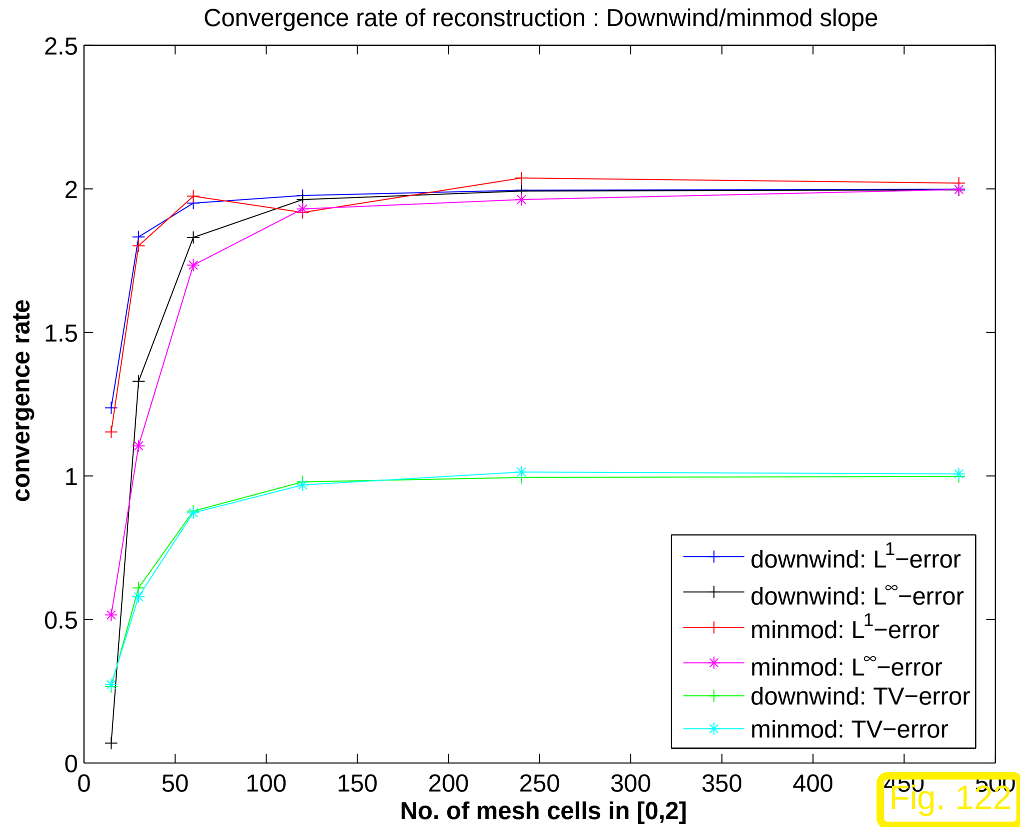


Fig. 122

◇

➤ REA-algorithm with minmod reconstruction (→ Def. 3.3.3) for linear advection ($v > 0$):

$$\begin{aligned}\mu_j^{(k)} =& \mu_j - v\gamma(\mu_j^{(k-1)} - \mu_{j-1}^{(k-1)}) \\ &- \tfrac{1}{2}v\gamma(1-v\gamma)\Big(\mathrm{minmod}(\mu_j^{(k-1)} - \mu_{j-1}^{(k-1)}, \mu_{j+1}^{(k-1)} - \mu_j^{(k-1)}) \\ &- \mathrm{minmod}(\mu_{j-1}^{(k-1)} - \mu_{j-2}^{(k-1)}, \mu_j^{(k-1)} - \mu_{j-1}^{(k-1)})\Big) \ .\end{aligned} \tag{3.3.8}$$

➤ 2nd-order consistent with $\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial x} = 0$ for smooth *strictly monotone* u

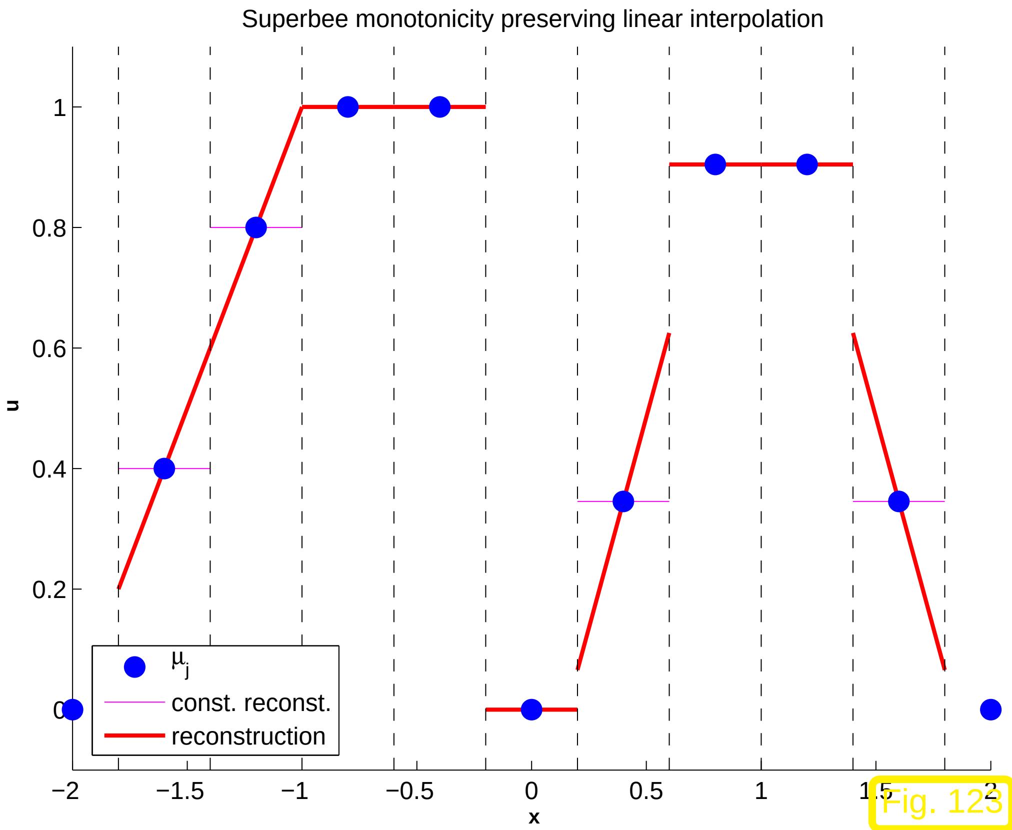
Remark 76 (Other monotonicity preserving linear interpolation schemes). $\longrightarrow$ [31, Sect. 6.9]

- Superbee reconstrunction:

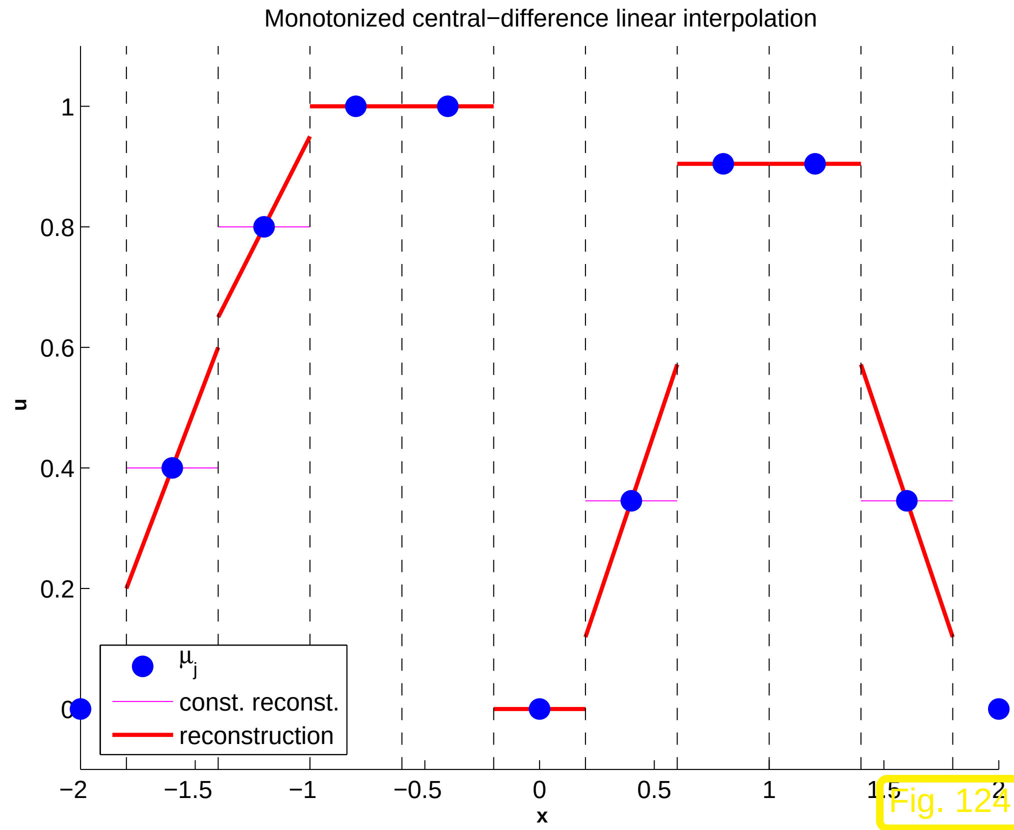
$$\sigma_j = \frac{1}{\Delta x}\,\mathrm{maxmod}(\mathrm{minmod}(\mu_{j+1}-\mu_j, 2(\mu_j-\mu_{j-1})), \mathrm{minmod}(2(\mu_{j+1}-\mu_j), \mu_j-\mu_{j-1}) \;.$$

- Monotonized central differencing (MC):

$$\sigma_j = \frac{1}{\Delta x}\,\mathrm{minmod}(\frac{\mu_{j+1}-\mu_{j-1}}{2}, 2(\mu_j-\mu_{j-1}), 2(\mu_{j+1}-\mu_j)) \;.$$



superbee reconstruction



MC slope limiting

$\triangle$

Remark 77. Averaging step in REA-algorithm has smoothing effect: slightly TVD-violating reconstructions can be accommodated

$\triangle$

3.3.1.3 Flux limiting

Issue: How to do **E**volve for piecewise linear w_0 and general f **?**

❶ special case: constant scalar linear advection (2.1.6) $\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial x} = 0$

REA-algorithm in conservation form ($\longrightarrow$ Def. 3.2.1), *cf.* (3.2.14):

$$\text{numerical flux} \qquad f_{j+1/2} = \frac{1}{\Delta t}\int_{t_{k-1}}^{t_k} v w_0(x_{j+1/2} - vt)\,\mathrm{d}t \ .$$

$$\blacktriangleright \qquad f_{j+1/2} = \begin{cases} v\mu_j^{(k-1)} + \frac{1}{2}v(\Delta x - v\Delta t)\sigma_j^{(k-1)} & \text{, if } v > 0; \ , \\ v\mu_{j+1}^{(k-1)} - \frac{1}{2}v(\Delta x - v\Delta t)\sigma_j^{(k-1)} & \text{, if } v < 0; \ , \end{cases}$$

➨ notation (increments): for $\vec{\mu} \in C^0(\mathcal{G}_{\Delta x})$ write $\Delta\mu_{j+1/2} := \mu_{j+1} - \mu_j$, $j \in \mathbb{Z}$

$$\boxed{f_{j+1/2} = \underbrace{F_{\mathrm{uw}}(\mu_j^{(k-1)}, \mu_{j+1}^{(k-1)})}_{\text{upwind flux (3.2.5)}} + \underbrace{\tfrac{1}{2}|v|(1 - |v|\gamma)\phi_{j+1/2}^{(k-1)} \Delta\mu_{j+1/2}^{(k-1)}}_{\text{anti-diffusive flux}}}\,, \tag{3.3.9}$$

$$\phi_{j+1/2}^{(k-1)} = \begin{cases} \Delta x \cdot \dfrac{\sigma_j^{(k-1)}}{\Delta\mu_{j+1/2}^{(k-1)}} & \text{, if } v > 0\,, \\ \Delta x \cdot \dfrac{\sigma_{j+1}^{(k-1)}}{\Delta\mu_{j+1/2}^{(k-1)}} & \text{, if } v < 0\,. \end{cases}$$

➤ $\phi_{j+1/2}^{(k-1)} \sim$ “strength of antidiffusive flux” (which is necessary for 2nd-order consistency) **!**

Recall (Sect. 3.3.1.1): Lax-Wendroff-scheme (3.3.5): $\phi_{j+1/2}^{(k-1)} = 1$

Beam-Warming-scheme (3.3.6): $\phi_{j+1/2}^{(k-1)} = \dfrac{\Delta\mu_{j-1/2}^{(k-1)}}{\Delta\mu_{j+1/2}^{(k-1)}}$

▶ numerical flux for REA-algorithm with minmod reconstruction (3.3.8):

$$f_{j+1/2} = F_{\mathrm{uw}}(\mu_j^{(k-1)}, \mu_{j+1}^{(k-1)}) + \tfrac{1}{2}|v|(1-|v|\gamma)\,\mathrm{minmod}(1, \theta_{j+1/2}^{(k-1)})\Delta\mu_{j+1/2}^{(k-1)}\,, \tag{3.3.10}$$

$$\theta_{j+1/2}^{(k-1)} := \begin{cases} \Delta\mu_{j-1/2}^{(k-1)} : \Delta\mu_{j+1/2}^{(k-1)} & \text{, if } v > 0\,, \\ \Delta\mu_{j+3/2}^{(k-1)} : \Delta\mu_{j+1/2}^{(k-1)} & \text{, if } v < 0\,. \end{cases} \tag{3.3.11}$$

Rationale:

$\theta_{j+1/2}^{(k-1)} \approx 1$ where approximate solution varies "smoothly" in space (w.r.t. Δx)
➜ "switch on 2nd-order Lax-Wendroff"

$\theta_{j+1/2}^{(k-1)} \ll 1$ upwind of a discontinuity
➜ "switch off 2nd-order Lax-Wendroff"

$\theta_{j+1/2}^{(k-1)} < 0$ when $\vec{\mu}$ oscillating at j
➜ switch to diffusive upwinding

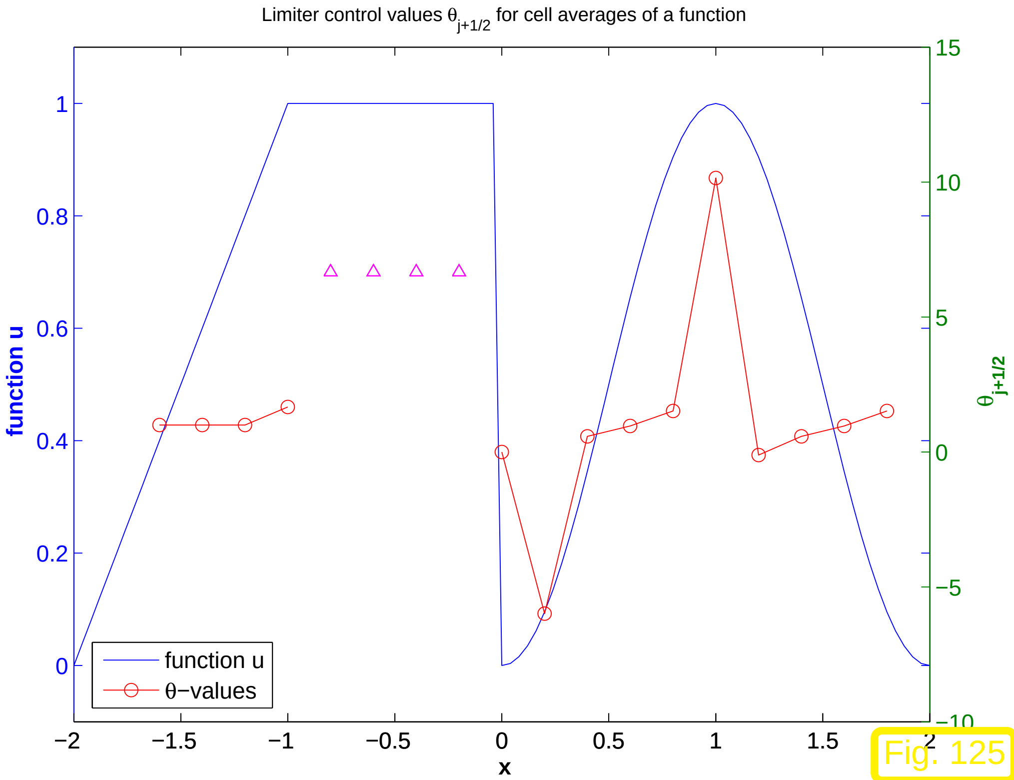


Fig. 125

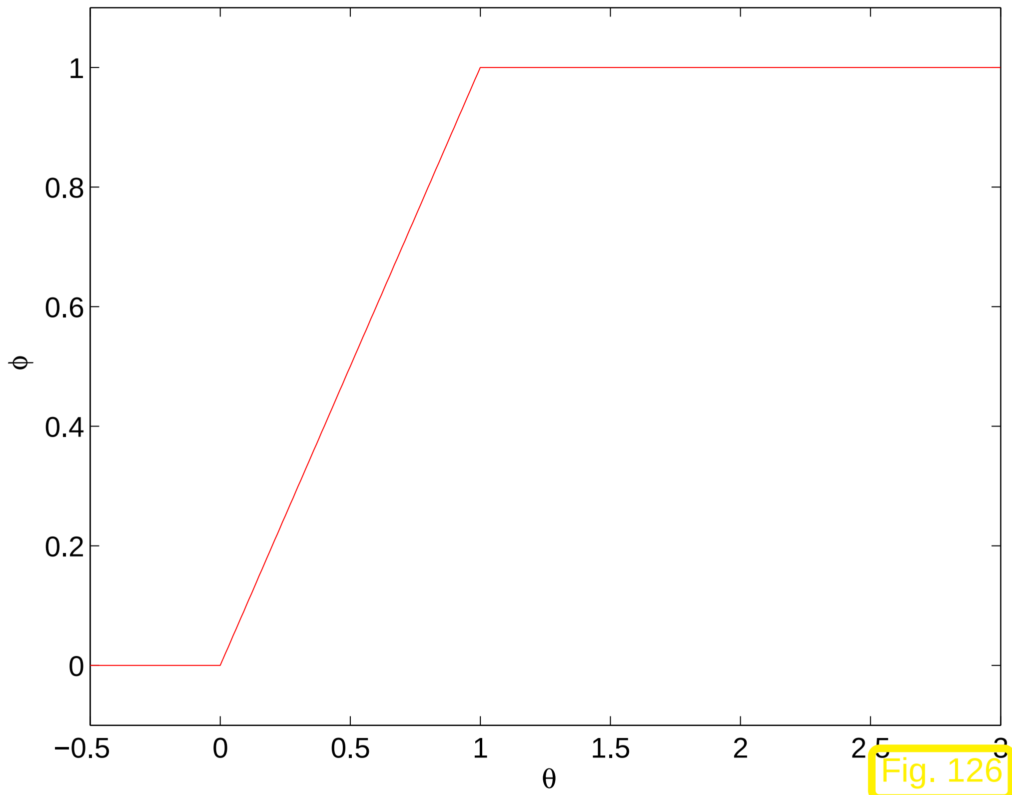


Fig. 126

$\triangleleft$ desired behavior $\phi_{j+1/2} = \phi_{j+1/2}(\theta_{j+1/2})$

$$\theta_{j+1/2}^{(k-1)} \approx 1 \rightarrow \phi_{j+1/2} = 1$$

$$\theta_{j+1/2}^{(k-1)} \ll 1 \rightarrow \phi_{j+1/2} = 0$$

$$\theta_{j+1/2}^{(k-1)} < 0 \rightarrow \phi_{j+1/2} = 0$$

$$\theta_{j+1/2}^{(k-1)} \gg 1 \quad \textbf{?}$$

(3.3.10) motivates: flux limited FDM for constant linear advection

$$\mu_j^{(k)} = \mathsf{H}_{\mathrm{uw}}(\mu_{j-1}^{(k-1)}, \mu_j^{(k-1)}, \mu_{j+1}^{(k-1)}) \\ - \tfrac{1}{2}|v\gamma|(1-|v\gamma|)\big(\varphi(\theta_{j+1/2}^{(k-1)})(\mu_{j+1}^{(k-1)} - \mu_j^{(k-1)}) - \varphi(\theta_{j-1/2}^{(k-1)})(\mu_j^{(k-1)} - \mu_{j-1}^{(k-1)})\big) \;, \quad \textbf{(3.3.12)}$$

with flux limiter function $\varphi : \mathbb{R} \mapsto \mathbb{R}$

$$\Updownarrow$$

Flux limited finite volume method with numerical flux, *cf.* (3.3.10)

$$f_{j+1/2} = F_{\mathrm{uw}}(\mu_j^{(k-1)}, \mu_{j+1}^{(k-1)}) + \tfrac{1}{2}|v|(1-\gamma|v|)\varphi(\theta_{j+1/2}^{(k-1)})(\mu_{j+1}^{(k-1)} - \mu_j^{(k-1)}) \ . \qquad (3.3.13)$$

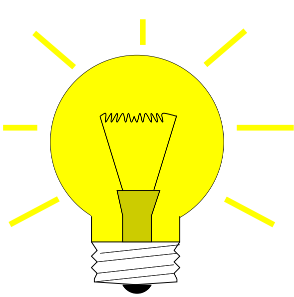
Theorem 3.3.6 (Order of flux limited schemes for linear advection)**.**
Let u be a smooth solution of (2.1.6)*. If the flux limiter function φ has the representation*

$$\varphi(\theta) = 1 - \phi(\theta) + \phi(\theta)\theta \quad \textit{with} \quad \phi \textit{ Lipschitz continuous}, \quad 0 \leq \phi \leq 1 \ ,$$

then the local truncation error (→ Def. 3.1.6) for (3.3.13) *in (x,t) is of order $(\Delta t)^2$, provided that $\frac{\partial u}{\partial x}(x,t) \neq 0$.*

Proof. by (tedious) Taylor expansion, see [29, Lemma 2.5.6].

❷ general scalar conservation law (2.2.1): $\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u) = 0$



Idea: rewrite "practical" Lax-Wendroff flux (3.2.27)

$$\widetilde{F}_{\mathrm{LW}}(v,w) = \underbrace{F_{\mathrm{uw}}(v,w)}_{\text{simple upwind flux (3.2.5)}} + \underbrace{\tfrac{1}{2}|\dot{s}|(1-\gamma|\dot{s}|)(w-v)}_{\text{anti-diffusive flux}} \ , \quad \dot{s} := \frac{f(w)-f(v)}{w-v} \ . \qquad (3.3.14)$$

$\longleftrightarrow$ (3.3.13).

Numerical flux for general flux limited FVM:

$$\boxed{f_{j+1/2} := F_{\mathsf{GD}}(\mu_j^{(k-1)}, \mu_{j+1}^{(k-1)}) + \tfrac{1}{2}|\dot{s}|(1-\gamma|\dot{s}|)\varphi(\theta_{j+1/2}^{(k-1)})(w-v)} \;, \tag{3.3.15}$$

$$\dot{s} := \frac{f(\mu_{j+1}^{(k-1)}) - f(\mu_j^{(k-1)})}{\mu_{j+1}^{(k-1)} - \mu_j^{(k-1)}}$$

(3.3.11) ➤ $$\theta_{j+1/2}^{(k-1)} := \begin{cases} \Delta\mu_{j-1/2}^{(k-1)} : \Delta\mu_{j+1/2}^{(k-1)} & \text{, if } \dot{s} > 0 \;, \\ \Delta\mu_{j+3/2}^{(k-1)} : \Delta\mu_{j+1/2}^{(k-1)} & \text{, if } \dot{s} < 0 \;. \end{cases} \quad \text{(downwind slope!)} \tag{3.3.16}$$

3.3.1.4 TVD limiters

For simplicity: focus on scalar constant linear advection (2.1.6) $\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial x} = 0$, $v > 0$

Sect. 3.3.1.3, (3.3.12) ➥ flux limited FDM in conservation form

$$\mu_j^{(k)} = \mu_j^{(k-1)} - \gamma v(\mu_j^{(k-1)} - \mu_{j-1}^{(k-1)}) - \tfrac{1}{2}|v\gamma|(1-|v\gamma|)\big(\varphi(\theta_{j+1/2}^{(k-1)})\Delta\mu_{j+1/2}^{(k-1)} - \varphi(\theta_{j-1/2}^{(k-1)})\Delta\mu_{j-1/2}^{(k-1)}\big) \;. \tag{3.3.17}$$

Theorem 3.3.7 (TVD flux limited FVM)**.**
If $\gamma v \leq 1$ *(CFL-condition) and*

$$\varphi(\theta) = 0 \quad \textit{for} \quad \theta \leq 0 \quad \wedge \quad 0 \leq \max\left\{\frac{\varphi(\theta)}{\theta}, \varphi(\theta)\right\} \leq 2 \quad \textit{for} \quad \theta > 0 \;,$$

then the discrete evolution (3.3.17) *is TVD (→ Def. 3.1.20).*

Proof. Idea: put (3.3.17) into (the right) incremental form (3.1.30) & Thm. 3.1.22

(3.3.17) = (3.1.30) with

$$c_{j-1/2} = \gamma v + \tfrac{1}{2}(1-\gamma v)\gamma v \, \frac{\left(\varphi(\theta^{(k-1)}_{j+1/2})\Delta\mu^{(k-1)}_{j+1/2} - \varphi(\theta^{(k-1)}_{j-1/2})\Delta\mu^{(k-1)}_{j-1/2}\right)}{\mu^{(k-1)}_j - \mu^{(k-1)}_{j-1}} \;,$$

$$d_{j+1/2} = 0 \;.$$

$$\blacktriangleright \qquad 0 \leq c_{j-1/2} = \gamma v + \tfrac{1}{2}(1-\gamma v)\gamma v \left(\frac{\varphi(\theta^{(k-1)}_{j+1/2})}{\theta^{(k-1)}_{j+1/2}} - \varphi(\theta^{(k-1)}_{j+1/2}) \right) \leq 1 \;. \qquad \square$$

Sufficient condition for assertion of Thm. 3.3.7

$$0 \leq \varphi(\theta) \leq 2\theta \;, \quad \text{if } 0 < \theta < 1 \;,$$
$$0 \leq \varphi(\theta) \leq 2 \;, \quad \text{if } 1 \leq \theta \;.$$

— $\hat{=}$ TVD region

— $\hat{=}$ "2nd-order region", Thm. 3.3.6

(only neighborhood of 1 relevant)

▷

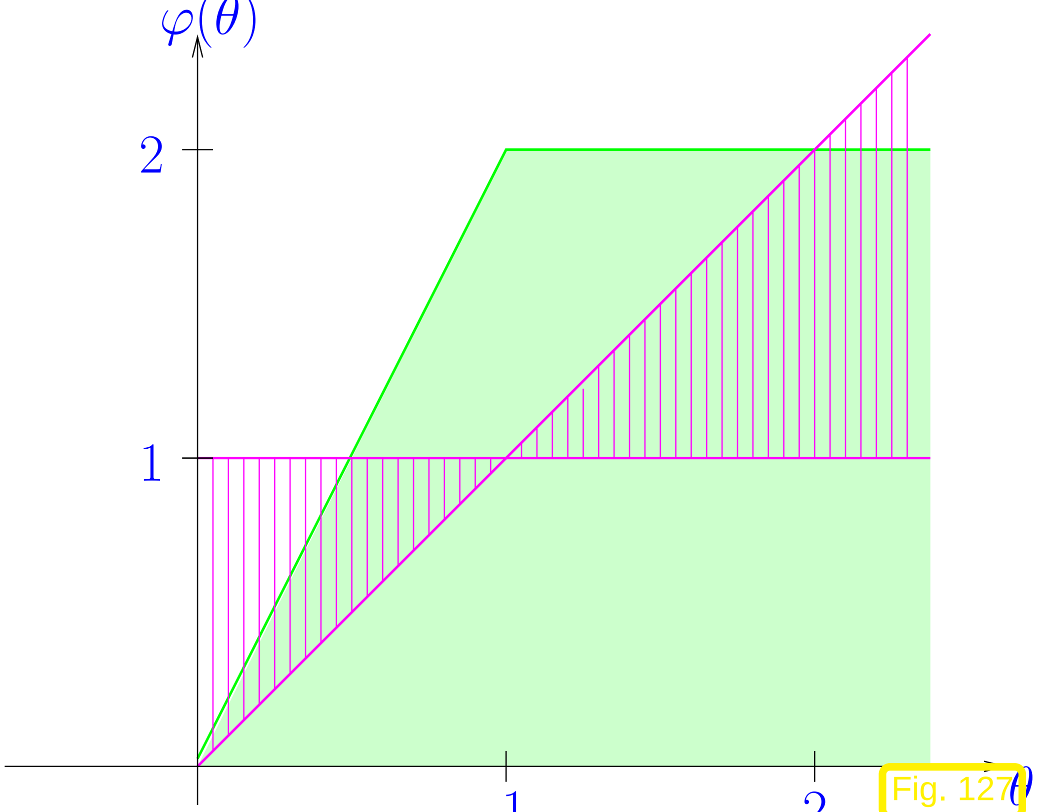


Fig. 127

Popular flux limiter functions:

$$\text{minmod:}\quad \varphi(\theta) = \max\{0, \min\{\theta, 1\}\}\ ,$$
$$\text{superbee:}\quad \varphi(\theta) = \max\{0, \min\{2\theta, 1\}, \min\{\theta, 2\}\}$$
$$\text{van Leer:}\quad \varphi(\theta) = \frac{|\theta| + \theta}{1 + |\theta|}\ ,$$
$$\text{van Albada:}\quad \varphi(\theta) = \max\left\{0, \frac{r^2 + r}{1 + r^2}\right\}\ ,$$
$$\text{MC:}\quad \varphi(\theta) = \max\{0, \min\{1, 2\theta\}, \min\{2, \theta\}\}\ .$$

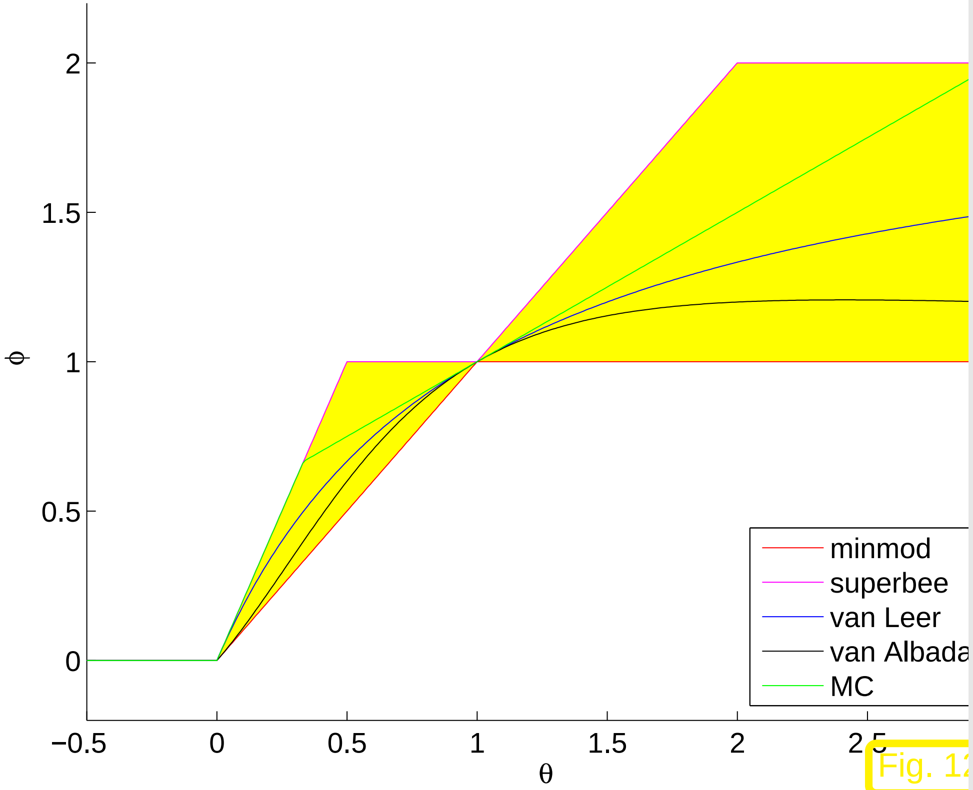


Fig. 12

Example 78 (Flux limited FVM for linear advection).

➠ same setting as Ex. 74.

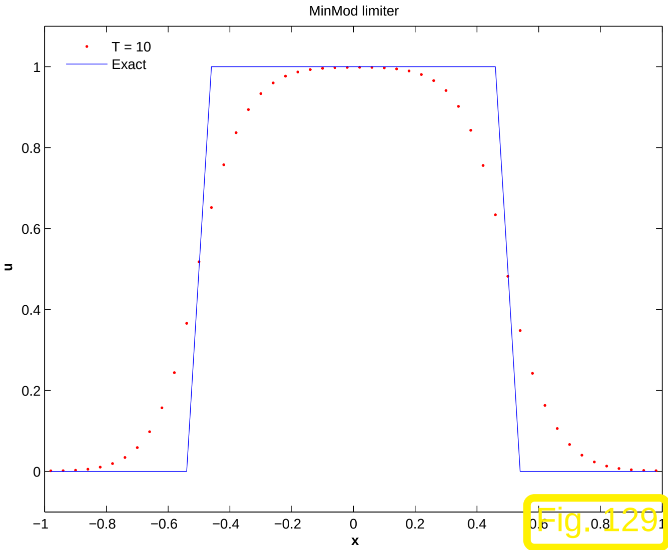


Fig. 129

minmod limiter

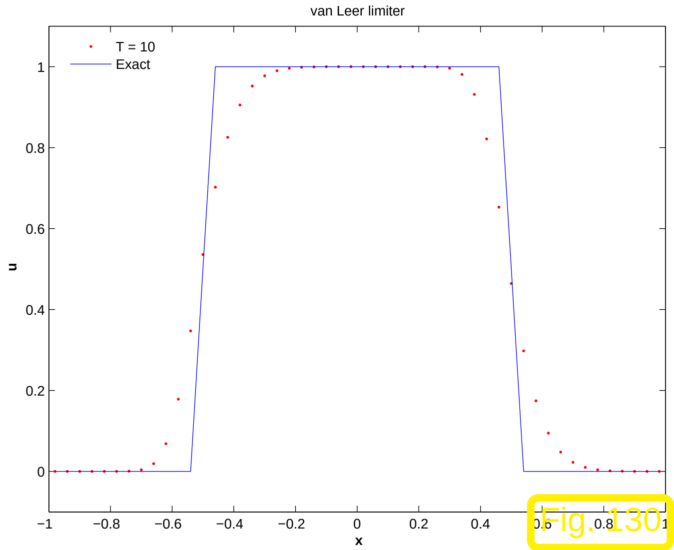


Fig. 130

van Leer limiter

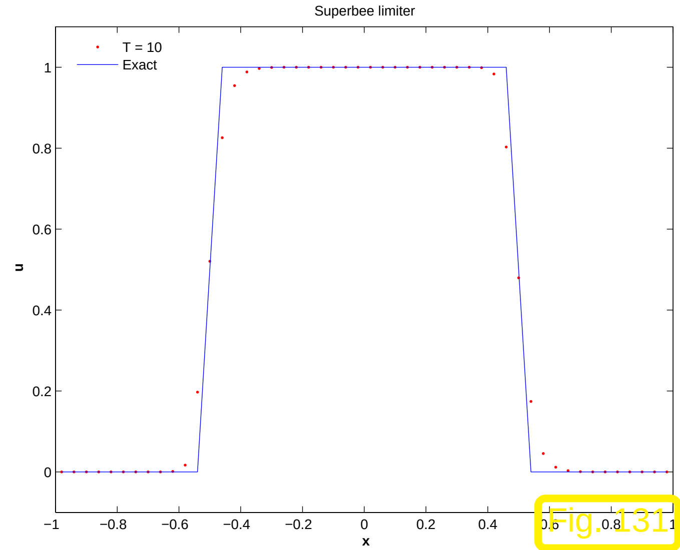


Fig. 131

superbee limiter

Observation: oscillations completely avoided ! ($\leftrightarrow$ Ex. 74)

now: $T = 10$, $\gamma = 0.8$, smooth initial data $u_0(x) = \chi_{]-1/2,1/2[}\cos^2(\pi x)$

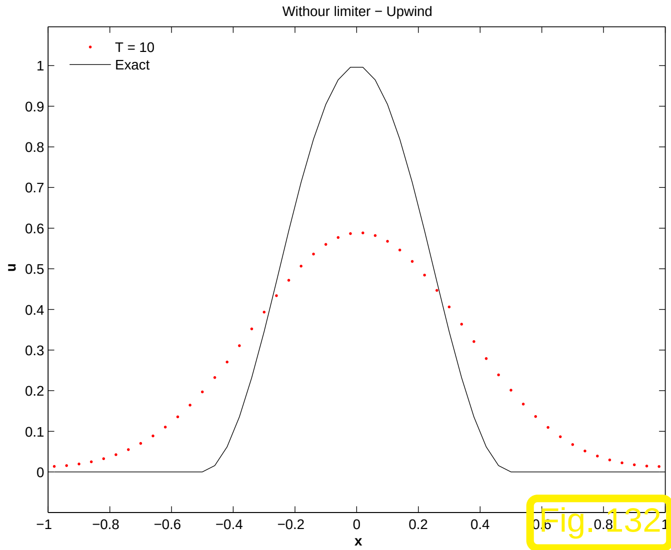


Fig. 132

upwind FDM

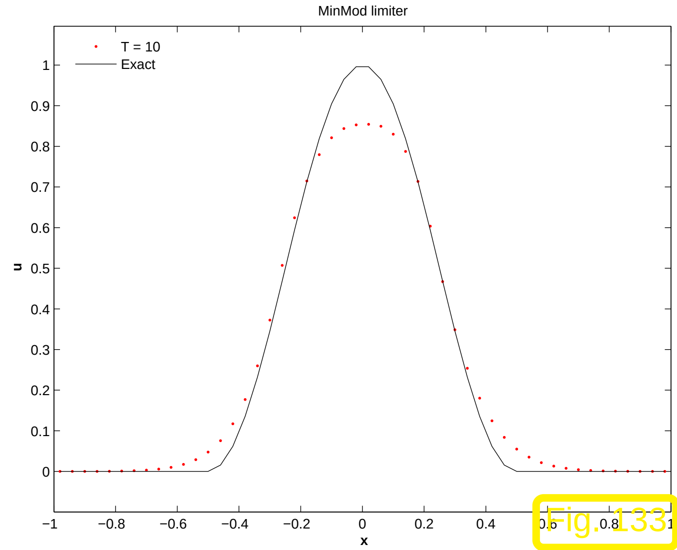


Fig. 133

minmod limiter

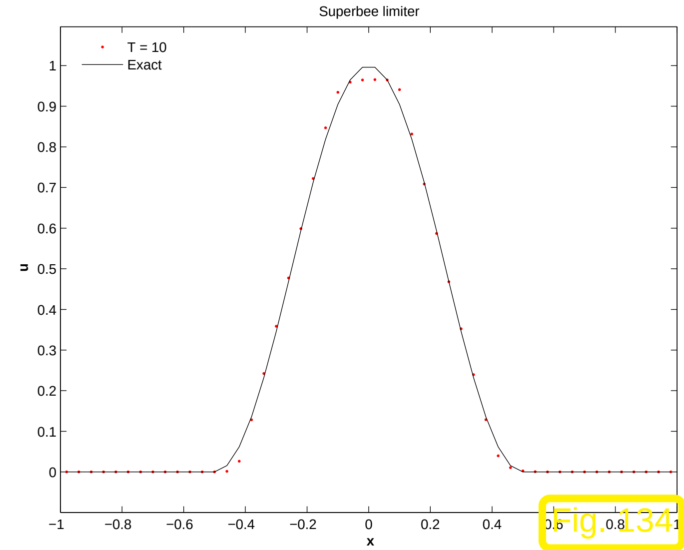


Fig. 134

superbee limiter

Observation: diffusivity: upwind $>$ minmod $>$ superbee $\diamond$

limited schemes: convergence to entropy solution not guaranteed
($\longrightarrow$ use "entropy fix", Sect. 3.2.9)

Example 79 (Convergence of flux limited schemes).

- Cauchy problem for linear advection $\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0$ on $\Omega =]-1,1[$ **+** periodic b.c., $T = 1$,
- smooth initial data $u_0(x) = \sin(\pi x)^4$

- TVD flux limited finite volume methods (3.3.17) on equidistant meshes, $\gamma := \frac{\Delta t}{\Delta x} = 1$

Monitored: error norms $\left\| \vec{\mu}^{(M)} - \mathsf{R}u(\cdot, T) \right\|_{l^\infty(\mathcal{G}_{\Delta x})}$, $\left\| \vec{\mu}^{(M)} - \mathsf{R}u(\cdot, T) \right\|_{l^1(\mathcal{G}_{\Delta x})}$ at final time for different resolutions $\Delta x \in \{\frac{1}{20}, \frac{1}{40}, \frac{1}{80} \frac{1}{160}, \frac{1}{320}\}$
approximate algebraic convergence rates: $\frac{1}{\log 2} (\log \|\text{error}(2\Delta x)\| - \log \|\text{error}(\Delta x)\|)$

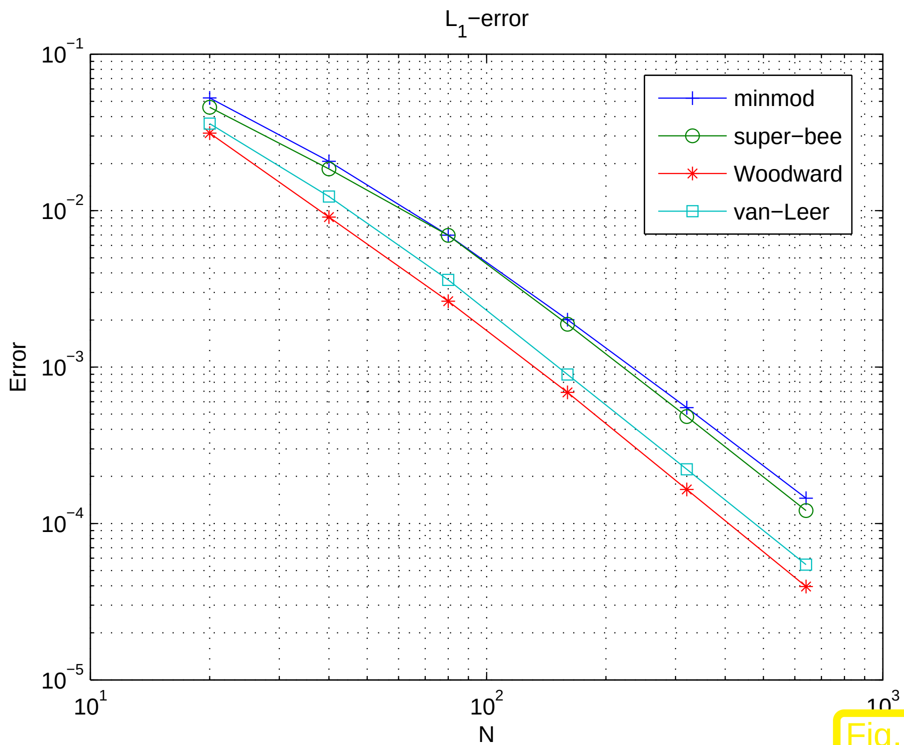


Fig. 135

convergence history, l^1-norm

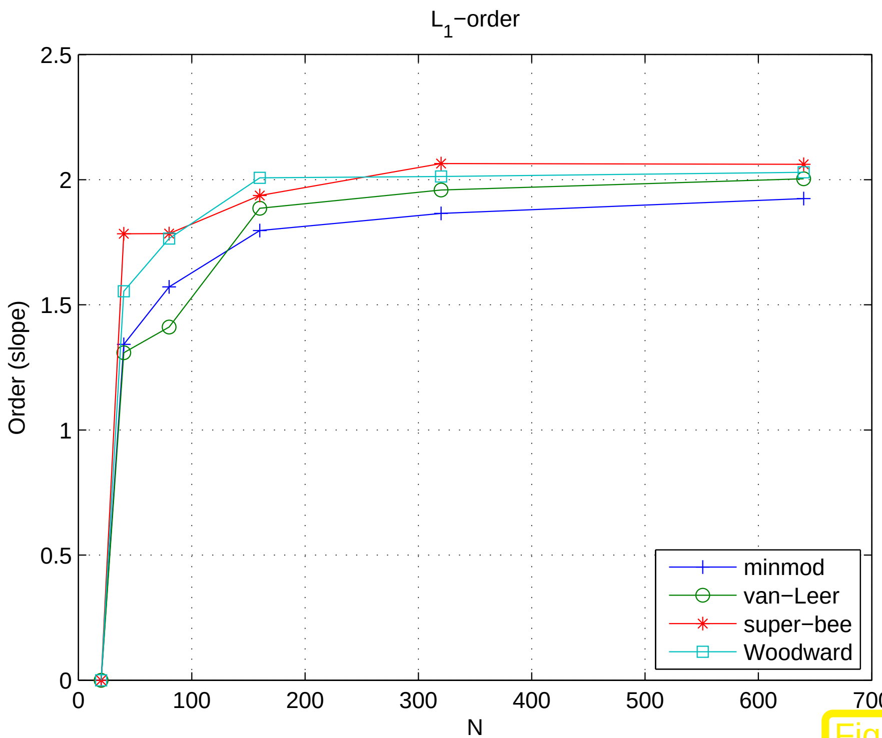


Fig. 136

approximate rates, l^1-norm

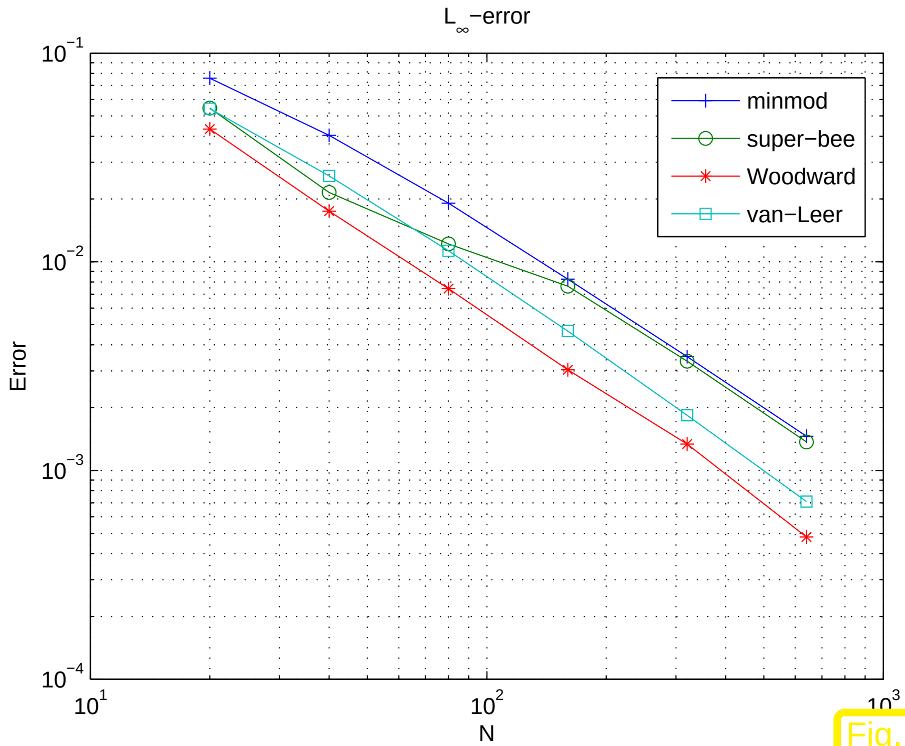


Fig. 137

convergence history, l^∞-norm

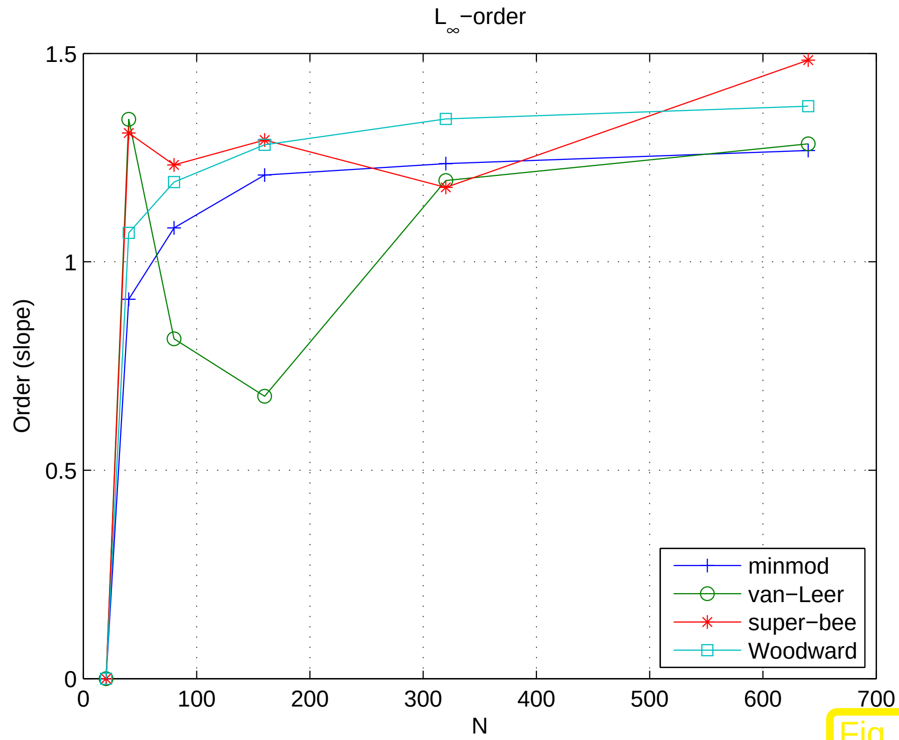


Fig. 138

approximate rates, l^∞-norm

Observation:
- 2nd-order convergence in l^1-norm, *cf.* Lemma 3.3.5
- slower convergence in l^∞-norm ($\rightarrow$ impact of extrema, *cf.* Thm. 3.3.6) $\diamond$

Remark 80 (Local order barrier for TVD FVM).

[37, Sect. 3]: if $\frac{\partial u}{\partial x}(x,t) = 0$ and $f(u(x,t)) \neq 0$, then the local truncation error in (x,t) of a TVD finite volume scheme ($\rightarrow$ Def. 3.2.1) is at most of first order in Δx.

$\triangle$

3.3.2 Central schemes

REA-algorithm ($\longrightarrow$ Sect, 3.2.2) without solving local Riemann problems (3.2.12) **?**

Idea: staggered spatial grids

k even: $\mu_j^{(k)} \approx \dfrac{1}{\Delta x}\displaystyle\int_{x_{j-1/2}}^{x_{j+1/2}} u(x,t_k)\,\mathrm{d}x$, k odd: $\mu_j^{(k)} \approx \dfrac{1}{\Delta x}\displaystyle\int_{x_{j-1}}^{x_j} u(x,t_k)\,\mathrm{d}x$.

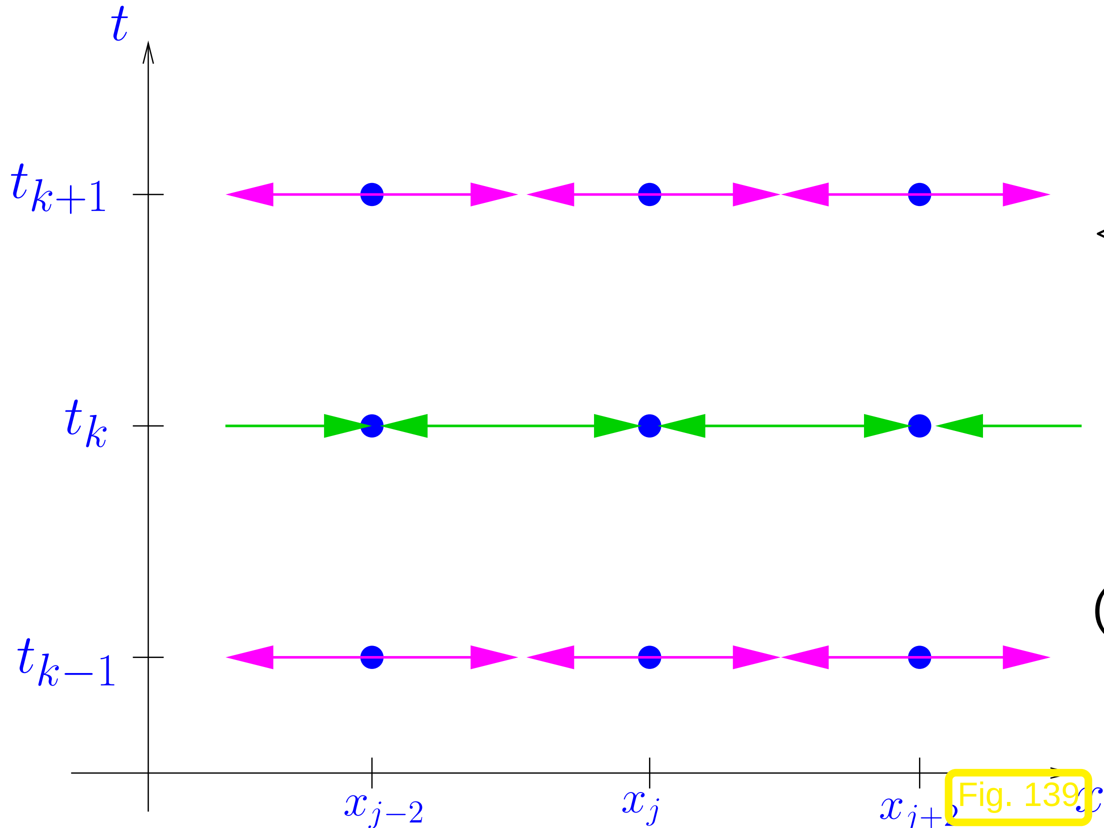


Fig. 139

$\triangleleft$ staggered spatial grids:

$\longleftrightarrow$ $\hat{=}$ cells for even k: grid $\mathcal{G}_{\Delta x}$

$\longleftrightarrow$ $\hat{=}$ cells for odd k: grid $\hat{\mathcal{G}}_{\Delta x}$

$$x_j \leftrightarrow x_{j+1/2}$$

(Uniform meshwidth Δx, timestep Δt assumed)

REA-algorithm based on staggered grids:

➡ given $\vec{\mu}^{(k-1)}$ obtain $\vec{\mu}^{(k)}$ in 3 steps:

① **R**econstruct: $w_0 \hat{=}$ p.w. polynomial on $\mathcal{G}_{\Delta x}$ (k odd)/ $\hat{\mathcal{G}}_{\Delta x}$ (k even) with cell avarages $\mu_j^{(k-1)}$

② **E**volve: solve the Cauchy problem

$$\frac{\partial w}{\partial t} + \frac{\partial}{\partial x} f(w) = 0 \quad \text{in } \mathbb{R}\times]0,\Delta t[\ , \quad w(x,0) = w_0(x)\ ,\ x \in \mathbb{R}\ . \tag{3.2.12}$$

③ **A**verage: $\vec{\mu}^{(k)} \leftarrow$ cell avarages:

$$\begin{aligned} \mu_j^{(k)} &:= \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} w(x,\Delta t)\,\mathrm{d}x,\ k \text{ even}, \\ \mu_j^{(k)} &:= \frac{1}{\Delta x}\int_{x_{j-1}}^{x_j} w(x,\Delta t)\,\mathrm{d}x,\ k \text{ odd}. \end{aligned} \tag{3.3.18}$$

w.l.o.g. (symmetry of $\mathcal{G}_{\Delta x}$, $\hat{\mathcal{G}}_{\Delta x}$) assume k odd ➤ averaging on $\hat{\mathcal{G}}_{\Delta x}$

(2.3.3) for $\widetilde{V} =]x_{j-1}, x_j[\times]t_{k-1}, t_k[$: for weak solution u of (2.2.1)

$$\int_{x_{j-1}}^{x_j} u(x,t_k)\,\mathrm{d}x = \int_{x_{j-1}}^{x_j} u(x,t_{k-1})\,\mathrm{d}x - \int_{t_{k-1}}^{t_k} f(u(x_j,t)) - f(u(x_{j-1},t))\,\mathrm{d}t$$

❶ piecewise constant reconstruction: $w_0 := \mathcal{C}\vec{\mu}^{(k-1)}$

➢ Godunov's method on staggered grids:

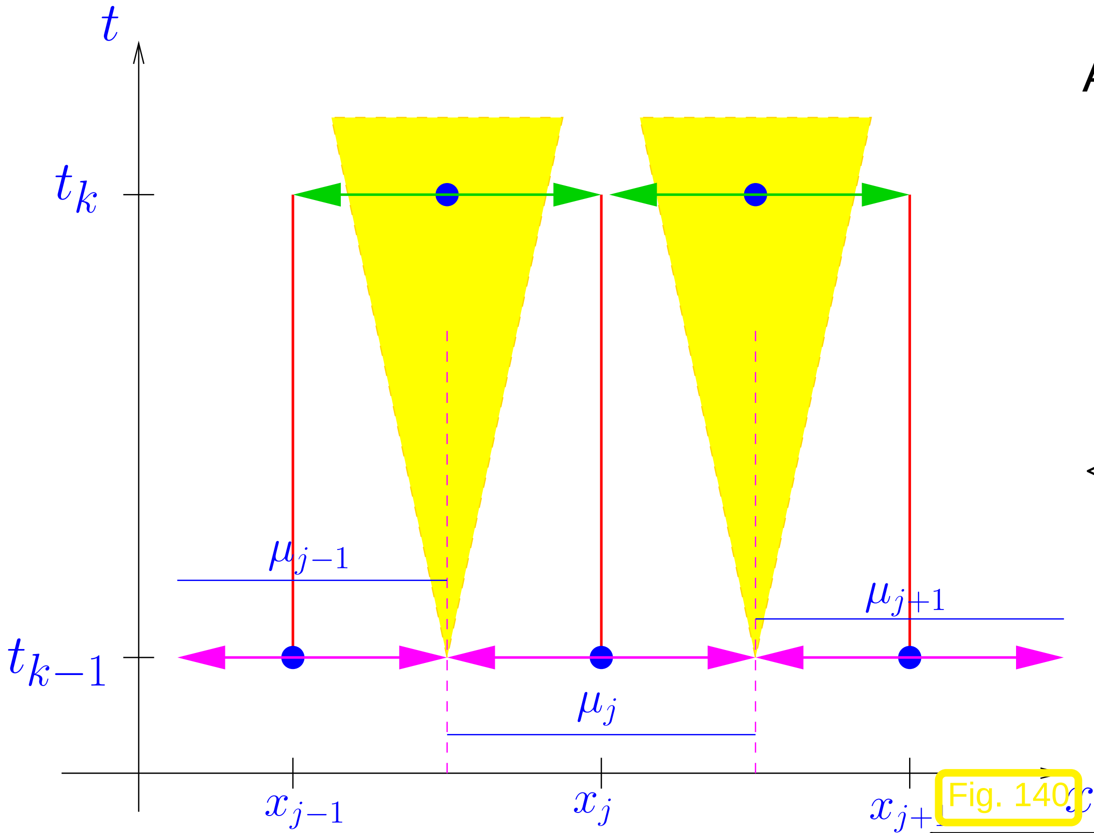


Fig. 140

Assume: "CFL/2"-condition:

$$\max_u |\gamma f'(u)| \leq \frac{1}{2}\,. \qquad (3.3.19)$$

discontinuities at $x_{j+1/2}$, $j \in \mathbb{Z}$ do not influence $w(x_j,t)$, $0 \leq t \leq \Delta t$!

◁ $\hat{=}$ maximal domain of influence of jumps at $x_{j+1/2}$

| $\hat{=}$ flux evaluation on this line

$\longleftrightarrow$ $\hat{=}$ cells for even k: grid $\mathcal{G}_{\Delta x}$

$\longleftrightarrow$ $\hat{=}$ cells for odd k: grid $\hat{\mathcal{G}}_{\Delta x}$

$$w(x_j,t) = \mu_j^{(k-1)} \quad \forall j \in \mathbb{Z} \quad \Rightarrow \quad \boxed{\mu_j^{(k)} = \tfrac{1}{2}(\mu_j^{(k-1)} + \mu_{j-1}^{(k-1)}) - \gamma\big(f(\mu_j^{(k-1)}) - f(\mu_{j-1}^{(k-1)})\big)} \ . \tag{3.3.20}$$

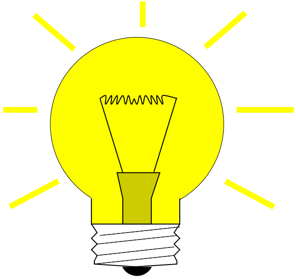
= Lax-Friedrichs scheme (3.1.29) on (even,even)/(odd,odd) space-time gridpoints **!**

highly diffusive, *cf.* Ex. 64

➩ try to counter numerical viscosity by higher order consistency **!**

❷ piecewise linear TVD reconstruction (3.3.2) $\longrightarrow$ Sect. 3.3.1.1:

▶ REA-algorithm of Sect. 3.3.1.1 with **A**verage step according to (3.3.18)



Idea: approximate **E**volve: (linearization $\longrightarrow$ local advection equation)

on cell $[x_{j-1/2}, x_{j+1/2}]$: replace

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u) = 0 \quad \boldsymbol{\rightarrow} \quad \frac{\partial u}{\partial t} + f'(\mu_j^{(k-1)})\frac{\partial u}{\partial x} = 0$$

(3.3.19) ▶ $$w(x_j,t) = w_0(x - f'(\mu_j^{(k-1)})t) \ , \quad 0 \leq t \leq \Delta t \ .$$

$$\mu_j^{(k)} = \tfrac{1}{2}(\mu_j^{(k-1)} + \mu_{j-1}^{(k-1)}) + \tfrac{1}{8}\Delta x(\sigma_{j-1}^{(k-1)} - \sigma_j^{(k-1)})$$
$$- \frac{1}{\Delta x}\int\limits_{t_{k-1}}^{t_k} f(\mu_j^{(k-1)} - \sigma_j^{(k-1)} f'(\mu_j^{(k-1)})t) - f(\mu_{j-1}^{(k-1)} - \sigma_{j-1}^{(k-1)} f'(\mu_{j-1}^{(k-1)})t)\,\mathrm{d}t \;. \quad \textbf{(3.3.21)}$$

Another approximation [35]: midpoint quadrature rule $\int_{t_{k-1}}^{t_k} g(t)\,\mathrm{d}t \approx \Delta t g(t_{k-1} + \frac{1}{2}\Delta t)$

$$\mu_j^{(k)} = \tfrac{1}{2}(\mu_j^{(k-1)} + \mu_{j-1}^{(k-1)}) + \tfrac{1}{8}\Delta x(\sigma_{j-1}^{(k-1)} - \sigma_j^{(k-1)})$$
$$- \gamma\big(f(\mu_j^{(k-1)} - \tfrac{1}{2}\sigma_j^{(k-1)} f'(\mu_j^{(k-1)})\Delta t) - f(\mu_{j-1}^{(k-1)} - \tfrac{1}{2}\sigma_{j-1}^{(k-1)} f'(\mu_{j-1}^{(k-1)})\Delta t)\big) \;. \quad \textbf{(3.3.22)}$$

Lemma 3.3.8 (Consistency of central scheme)**.** $\longrightarrow$ *[35]*
For a smooth solution u *of* (2.2.1) *and fixed* $\gamma := \Delta t/\Delta x$*, the local truncation error (*$\longrightarrow$ *Def. 3.1.6) for* (3.3.22) *in* (x_j, t_k) *is* $O((\Delta x)^2)$*, provided that* $\sigma_j^{(k)} = \frac{\partial u}{\partial x}(x_j, t_k) + O(\Delta x)$*.*

Assumptions of Lemma 3.3.8 hold for *slope limited p.w. linear reconstructions* of Sect. 3.3.1.2, e.g.

$$\sigma_j^{(k-1)} = \frac{1}{\Delta x}\,\mathrm{minmod}(\mu_{j+1}^{(k-1)} - \mu_j^{(k-1)}, \mu_j^{(k-1)} - \mu_{j-1}^{(k-1)}) \;. \quad \textbf{(3.3.23)}$$

Rewrite (3.3.22) in “staggered conservation form”, *cf.* Def. 3.2.1:

$$\begin{aligned} \mu_j^{(k)} &= \tfrac{1}{2}(\mu_j^{(k-1)} + \mu_{j-1}^{(k-1)}) - \gamma(f_j - f_{j-1}) \;, \\ f_j &:= \tfrac{1}{8\gamma}\Delta x \sigma_j^{(k-1)} + f(\mu_j^{(k-1)} - \tfrac{1}{2}\sigma_j^{(k-1)} f'(\mu_j^{(k-1)})\Delta t) \;. \end{aligned} \tag{3.3.24}$$

Lemma 3.3.9 (TVD criterion for staggered conservation form)**.**
The discrete evolution (3.3.24) *is TVD (⟶ Def. 3.1.20), if it satisfies (the “generalized CFL-condition”)*

$$\gamma \left| \frac{f_j - f_{j-1}}{\mu_j^{(k-1)} - \mu_{j-1}^{(k-1)}} \right| \leq \frac{1}{2} \quad \forall j \in \mathbb{Z} \;.$$

Proof. convert (3.3.24) in incremental form (3.1.30) and apply Thm. 3.1.22 □

▶ TVD-property under strengthened CFL-condition [35, Cor. 3.3]:

$$\gamma \max_{j\in\mathbb{Z}} |f'(\mu_j^{(k)})| \leq \tfrac{1}{2}(\sqrt{7}-2) \approx 0.32 \quad \Rightarrow \quad \text{(3.3.24) with (3.3.23) is TVD}$$

Example 81 (Convergence of central scheme for advection).

- constant linear advection (2.1.6), $v = 1$
- central scheme (3.3.22), minmod reconstruction (3.3.23), equidistant mesh, fixed $\gamma = \frac{1}{6}$

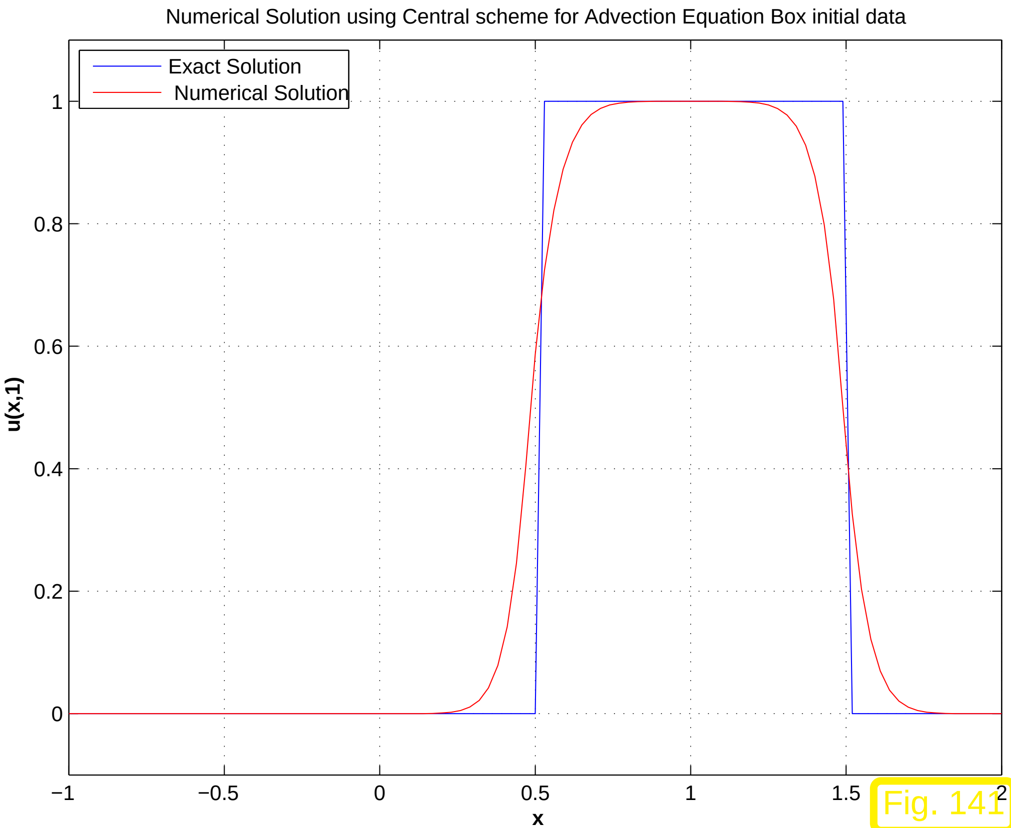


Fig. 141

$u_0 = \chi_{]-0.5,0.5[}(x)$ ("box function"), $\Delta x = \frac{3}{100}$

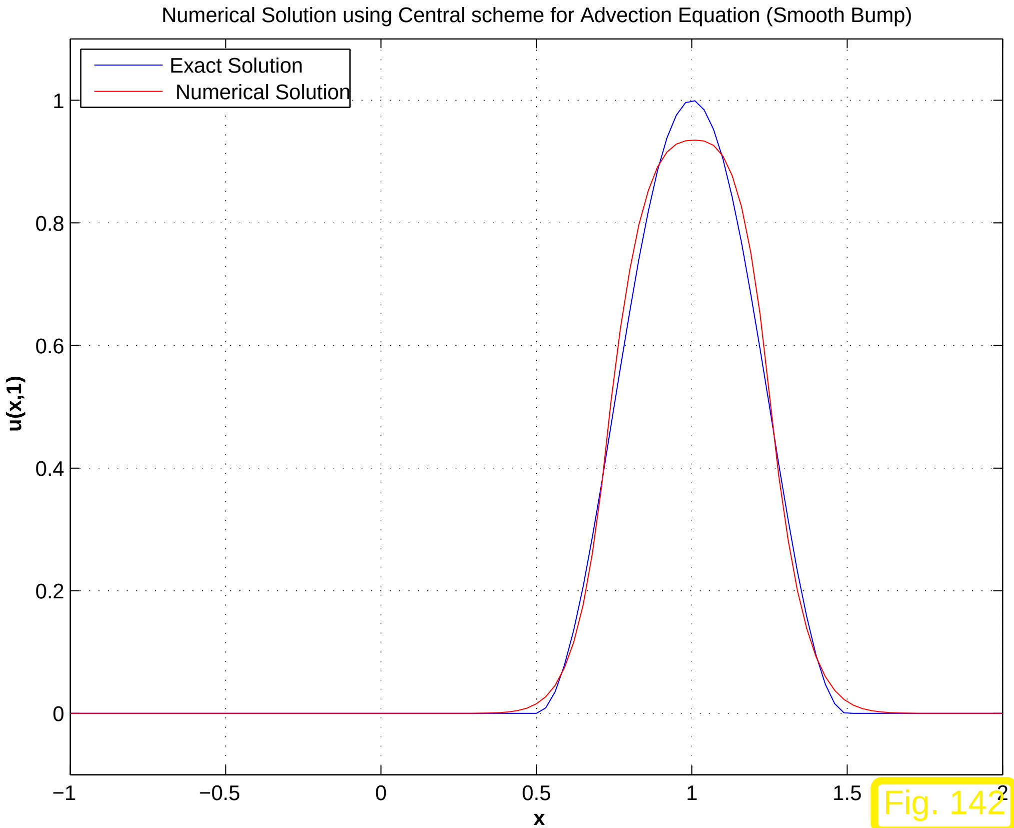


Fig. 142

$u_0 = C^1$-"bump" (4.2.3), $\Delta x = \frac{3}{100}$

Recorded: discretization error (+ rate) for $T = 1$, $l^1(\mathbb{Z})$-norm, $l^2(\mathbb{Z})$-norm, and $l^\infty(\mathbb{Z})$-norm:

❶ $u_0 = \chi_{]-0.5,0.5[}(x)$ ("box function")

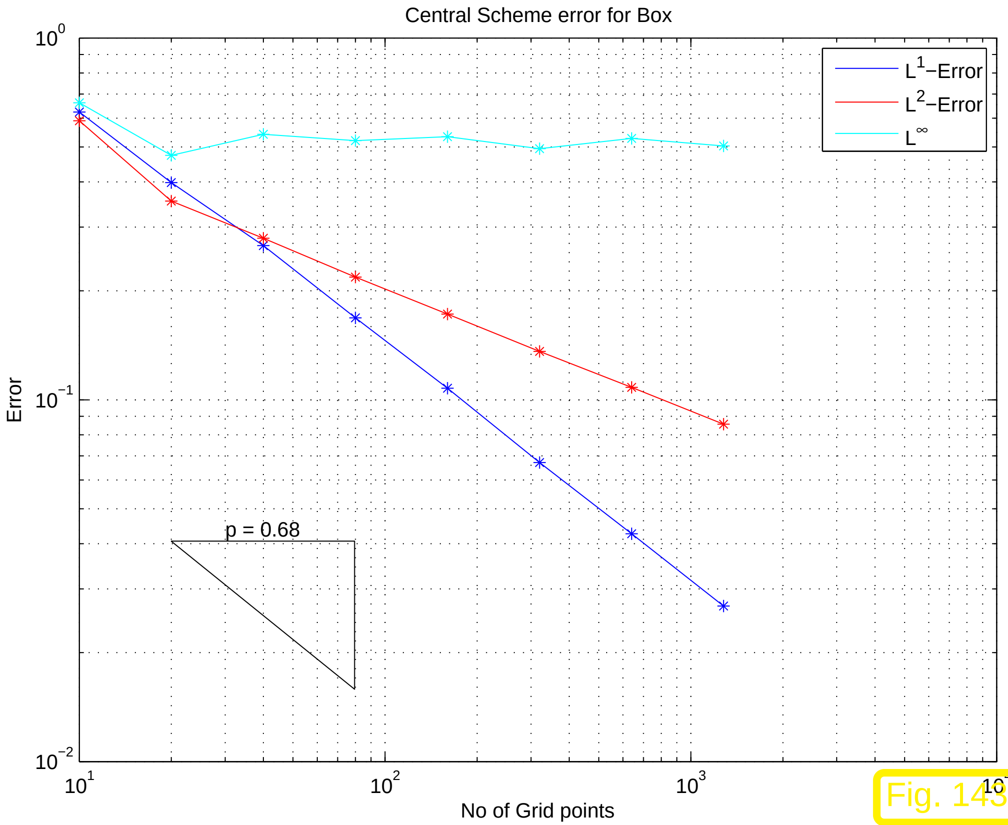


Fig. 143

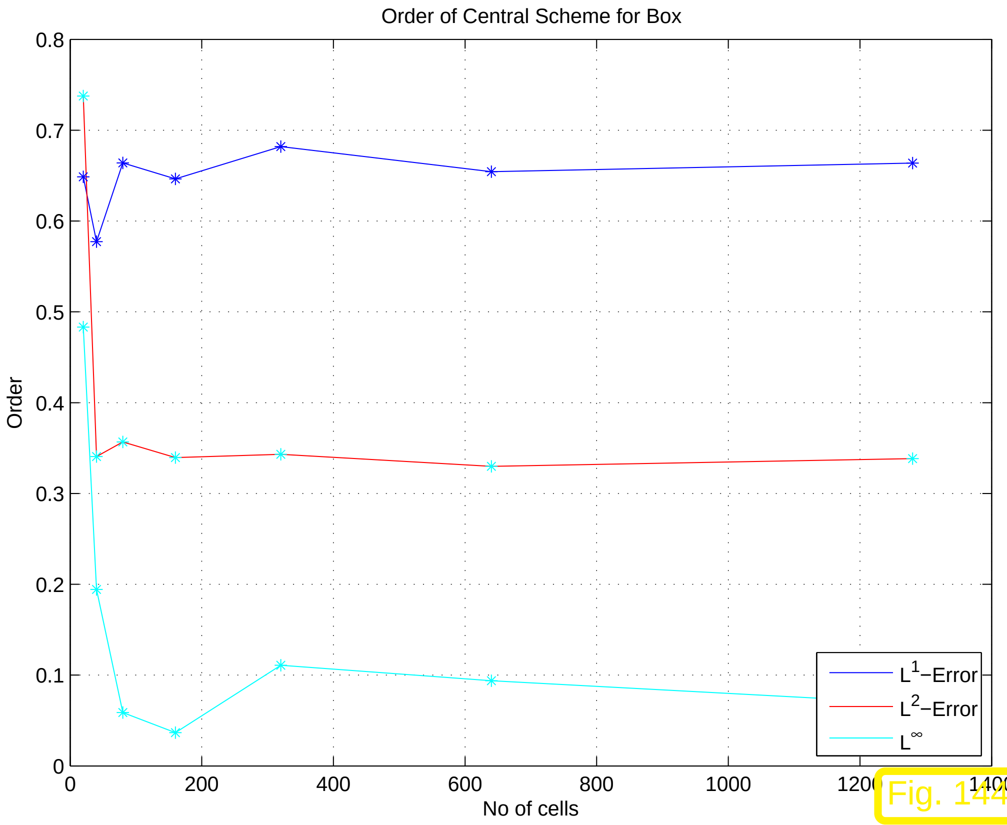


Fig. 144

❷ u_0 from (4.2.3) ("bump function")

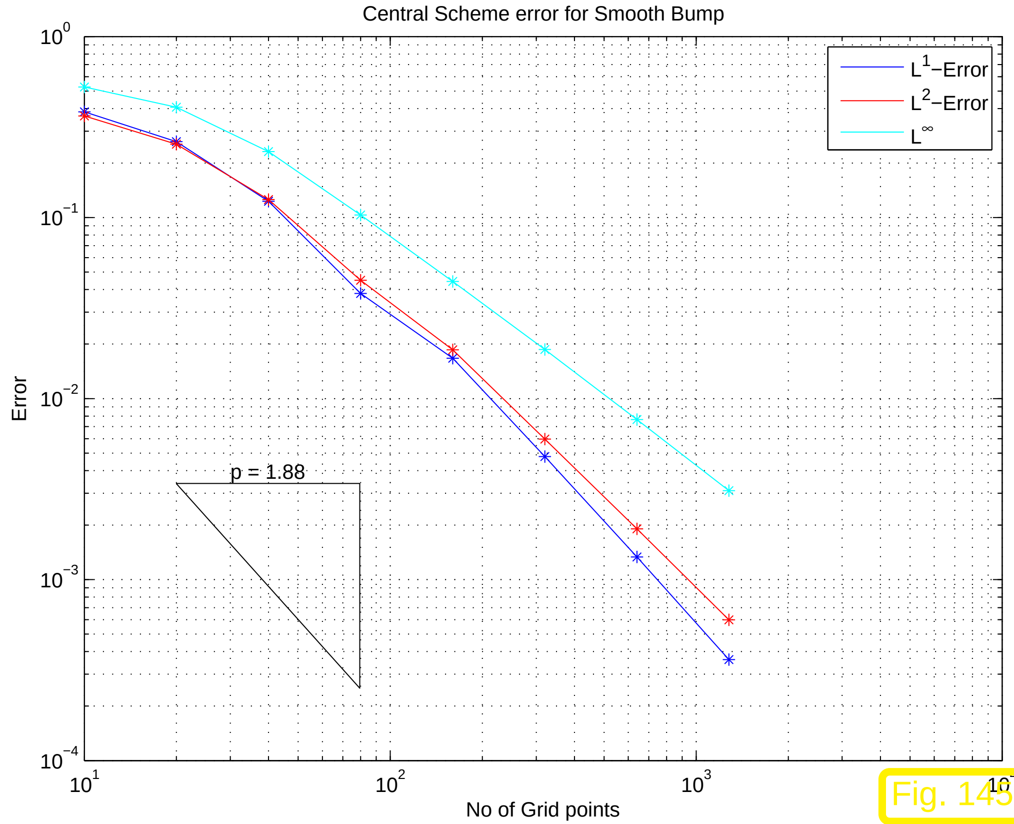


Fig. 145

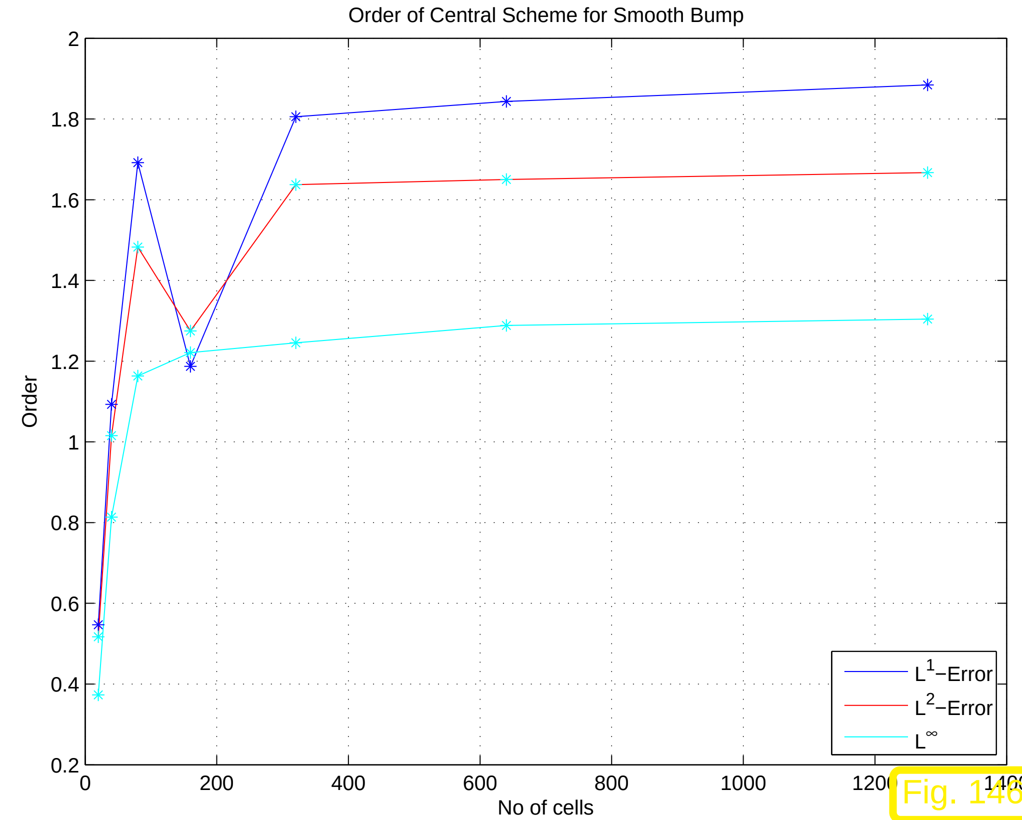


Fig. 146

Observation: ☞ 2nd-order algebraic convergence for smooth u in l^1/l^2-norm, worse for l^∞-norm (impact of spatial extrema, *cf.* Ex. 79)

☞ discontinuous u ➤ reduced convergence rate (for all norms) ◇

Example 82 (Central scheme for Burgers equation).

- Cauchy problem for Burgers equation (2.1.7), $u_0(x) = -0.5 + \chi_{]0,1[}(x)$
- central scheme (3.3.22), minmod reconstruction (3.3.23), equidistant mesh $\gamma = \frac{1}{6}$, $\Delta x = \frac{3}{100}$

solution for $T = 1$ ▷

movie ➨ burger_movie_box.avi

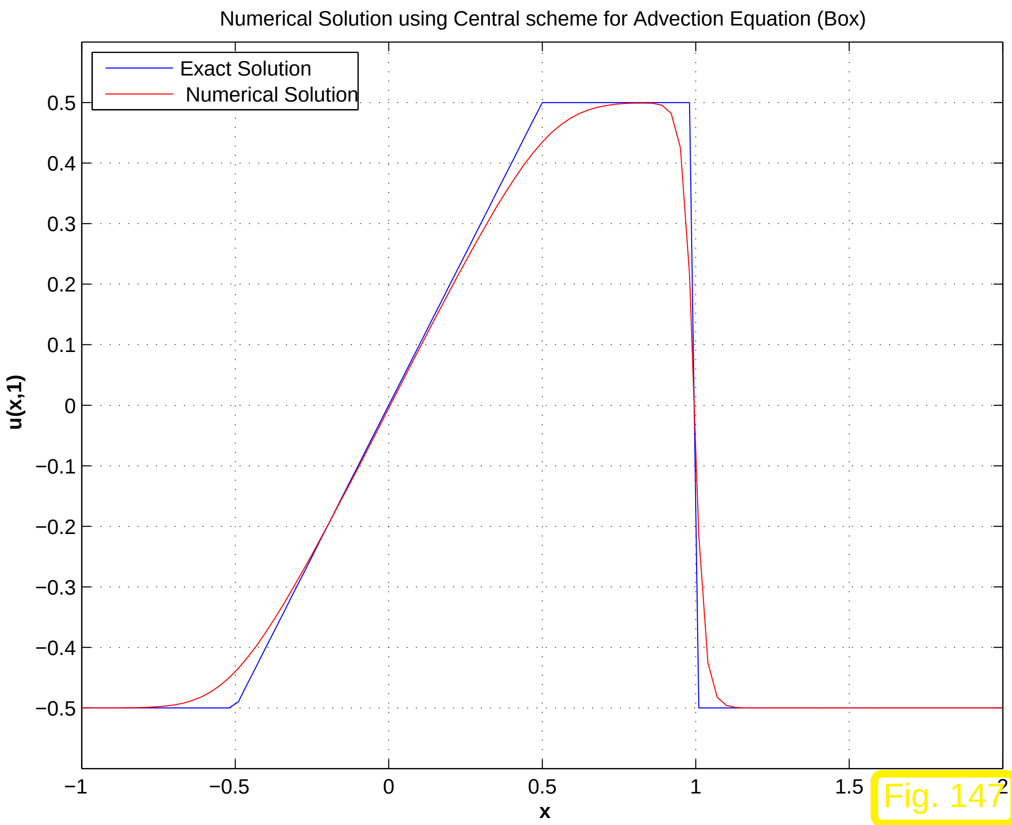


Fig. 147

Observation: moderately diffusive, no "entropy glitch" $\leftrightarrow$ Ex. 73

◇

3.3.3 Method of lines

$\leftrightarrow$ method of lines for wave equation, Sect. 1.6

Spatial semi-discretization of Cauchy problem (2.2.1) ➡ 1st-oder “ODE”

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u) = 0 \quad ➡ \quad \begin{array}{c} \frac{d}{dt}\vec{\mu}(t) + \mathcal{L}_{\Delta x}(\vec{\mu}(t)) = 0 \;, \\ \vec{\mu}(0) \quad \text{from} \quad u_0 \;. \end{array} \tag{3.3.25}$$

$\mathcal{L}_{\Delta x} : C^0(\mathcal{G}_{\Delta x}) \mapsto C^0(\mathcal{G}_{\Delta x})$ $\hat{=}$ “difference operator” approximating $\frac{\partial}{\partial x} f(\cdot)$

Definition 3.3.10 (Consistency of spatial semi-discretization). *cf. Def. 3.1.7*
A semi-discretization $\frac{d}{dt}\vec{\mu}(t) + \mathcal{L}_{\Delta x}(\vec{\mu}(t)) = 0$ *on equidistant spatial grids is* *consistent* *with* (2.2.1)*, if for a solution* u

$$\Big\| \underbrace{\mathcal{L}_{\Delta x}(\mathsf{R}u(\cdot,t)) - \mathsf{R}\Big(\frac{\partial}{\partial x} f(u)(\cdot,t)\Big)}_{\text{“spatial truncation error”} \to \text{Def. 3.1.6}} \Big\|_{\Delta x} \to 0 \quad \textit{for } \Delta x \to 0 \;, \quad \forall t \in]0,T[\;,$$

where R *is a suitable restriction operator onto* $C^0(\mathcal{G}_{\Delta x})$*, cf. Sect. 3.1.1.*
It is consistent of order $q \in \mathbb{N}$:⇔

$$\exists C > 0\colon \quad \Big\| \mathcal{L}_{\Delta x}(\mathsf{R}u(\cdot,t)) - \mathsf{R}\Big(\frac{\partial}{\partial x} f(u)(\cdot,t)\Big) \Big\|_{\Delta x} \leq C(\Delta x)^q$$

for all sufficiently small Δx*,* $t \in]0,T[$*.*

$\mathcal{L}_{\Delta x}$ **=** translation invariant finite difference operator, if, *cf.* Def. 3.1.1, 3.1.3 ($m_l, m_r \in \mathbb{N}$)

$$(\mathcal{L}_{\Delta x}(\vec{\mu}))_j = \mathsf{L}(\mu_{j-m_l}, \dots, \mu_{j+m_r}) \;, \quad j \in \mathbb{Z} \;. \tag{3.3.26}$$

☞ check consistency by means of Taylor expansion, see Sect. 3.1.2 (**smooth** u required)

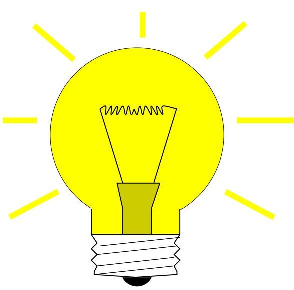
3.3.3.1 Finite volume semi-discretization

Standard finite volume interpretation, cf. Sect. 3.2: $\mu_j(t) \approx \frac{1}{\Delta x} \int\limits_{x_{j-1/2}}^{x_{j+1/2}} u(x,t)\,\mathrm{d}x$

(➢ below $\mathsf{R} \mathrel{\hat{=}}$ cell averaging operator)

$$\overset{(2.3.3)}{\blacktriangleright} \qquad \frac{d}{dt}\mu_j(t) = -\frac{1}{\Delta x}\Big(f(u(x_{j+1/2},t)) - f(u(x_{j-1/2},t))\Big)\ ,$$

on equidistant spatial grid $\mathcal{G}_{\Delta x}$, meshwidth $\Delta x > 0$.



Idea: approximation $f(u(x_{j+1/2},t)) \approx F(\mu_{j-m_l+1}(t),\dots,\mu_{j+m_r}(t))$

with consistent (→ Def. 3.2.2) numerical flux function $F : \mathbb{R}^{m_l+m_r} \mapsto \mathbb{R}$

➪ All F from Sect. 3.2 eligible, unless dependent on Δt **!**

e.g., Godunov flux F_{GD} (3.2.17), local Lax-Friedrichs flux (3.2.10), Enquist-Osher flux (3.2.7)

$$F = F(v,w) \quad \blacktriangleright \quad \mathsf{L}(\mu_{j-1},\mu_j,\mu_{j+1}) = -\frac{1}{\Delta x}\Big(F(\mu_j,\mu_{j+1}) - F(\mu_{j-1},\mu_j)\Big)\ . \tag{3.3.27}$$

► spatially semi-discrete finite volume scheme:

$$\boxed{\frac{d}{dt}\mu_j(t) = -\frac{1}{\Delta x}\Big(F(\mu_j(t),\mu_{j+1}(t)) - F(\mu_{j-1}(t),\mu_j(t))\Big)}\ .$$

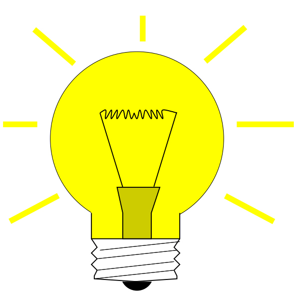
Assume: f, F continuously differentiable, u classical solution ($\rightarrow$ Def. 2.2.1) of (2.2.1)

F consistent with f ($\rightarrow$ Def. 3.2.2) $\Rightarrow$ L from (3.3.27) 1st-order consistent ($\rightarrow$ Def. 3.3.10)

3.3.3.2 Higher order reconstruction

Taylor expansion ➜ (3.3.27) only 1st-order consistent (in space), because cell avarages directly plugged into F ("$\frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} u(x)\,\mathrm{dx} - u(x) = O(\Delta x)$")

Borrow idea of Sect. 3.3.1.1: linear reconstruction



$$\mathsf{L}(\dots,\mu_j,\dots) = -\frac{1}{\Delta x}(F(w^-_{j+1/2}, w^+_{j+1/2}) - F(w^-_{j-1/2}, w^+_{j-1/2}))\;, \tag{3.3.28}$$

where $w^{\pm}_{j+1/2} := \lim_{\epsilon\rightarrow 0} w_0(x_{j+1/2} \pm \epsilon)$, w_0 p.w. linear on cells of $\mathcal{G}_{\Delta x}$, see (3.3.2), w_0 locally reconstructed from μ_j

semi-discrete evolution:

$$\frac{d}{dt}\mu_j(t) = -\frac{1}{\Delta x}(F(w^-_{j+1/2}, w^+_{j+1/2}) - F(w^-_{j-1/2}, w^+_{j-1/2}))\;. \tag{3.3.29}$$

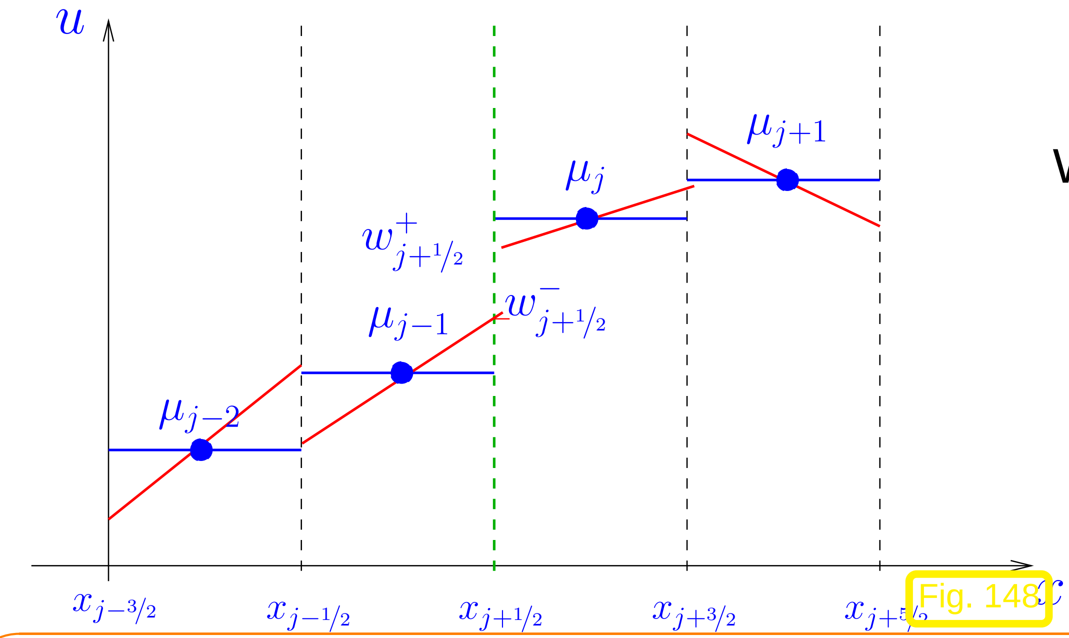


Fig. 148

With cell slopes σ_j, *cf.* (3.3.2):

$$\begin{aligned} w^-_{j+1/2} &= \mu_j + \tfrac{1}{2}\sigma_j \Delta x \;, \\ w^+_{j+1/2} &= \mu_{j+1} - \tfrac{1}{2}\sigma_{j+1} \Delta x \;. \end{aligned}$$

Lemma 3.3.11 (2nd-order consistent semi-discrete FV schemes)**.**
f, F *smooth,* $F(u,u) = f(u)$, u *smooth solution of* (2.2.1)*,*

$$|w^\pm_{j+1/2} - u(x_{j+1/2})| = O((\Delta x)^2) \;,$$
$$|w^+_{j+1/2} - w^+_{j-1/2} - (w^-_{j+1/2} - w^-_{j-1/2})| = O((\Delta x)^3)$$
$$\Rightarrow \qquad \frac{1}{\Delta x}\left(F(w^-_{j+1/2}, w^+_{j+1/2}) - F(w^-_{j-1/2}, w^+_{j-1/2})\right) = \frac{\partial}{\partial x} f(u)_{|x=x_j} + O((\Delta x)^2) \;.$$

Proof. Taylor expansions around $(u(x_j,t),t)$ and (x_j,t), see [29, Lemma 2.5.15] $\square$

Example 83 (Linear extrapolation).

Assumptions of Lemma 3.3.11 met for

$$w^-_{j+1/2} = \mu_j + \tfrac{1}{2}(\mu_j - \mu_{j-1}) \ ,$$
$$w^+_{j+1/2} = \mu_{j+1} - \tfrac{1}{2}(\mu_{j+2} - \mu_{j+1}) \ .$$

Problem ($\longrightarrow$ Fig. 113):

"overshooting" values $w^{\pm}_{j+1/2}$

Remedy **?**

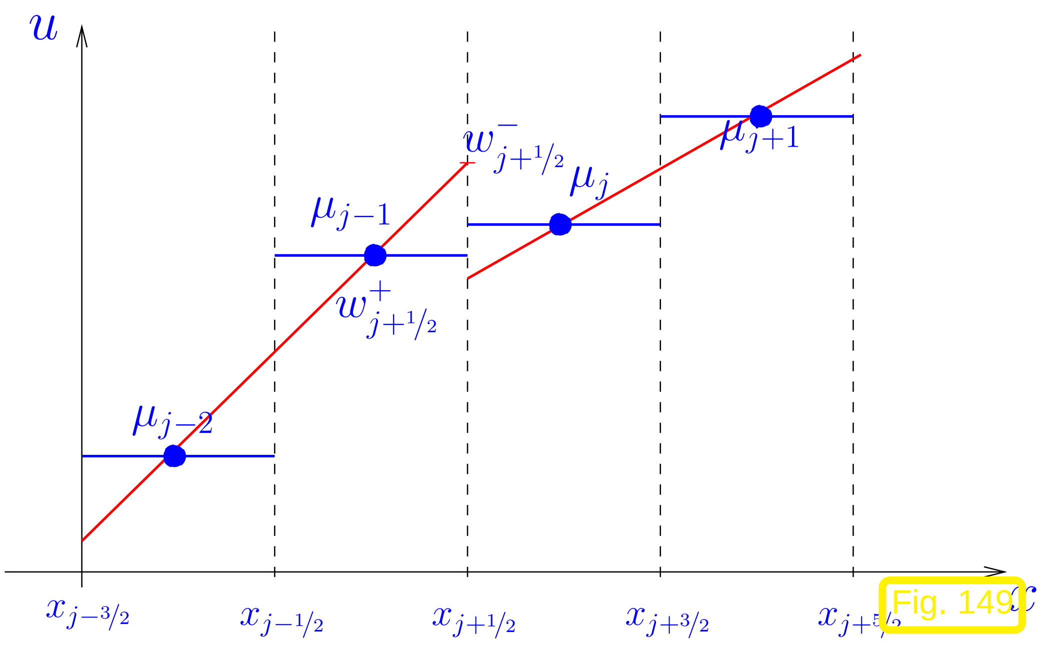


Fig. 149

$\diamond$

Limited reconstruction ($\longrightarrow$ Sect. 3.3.1.2), e.g., $\quad \sigma_j = \dfrac{1}{\Delta x}\,\mathrm{minmod}(\mu_{j+1} - \mu_j, \mu_j - \mu_{j-1})$.

Terminology: MUSCL (monontone upstream centered) schemes [29, Sect. 2.5]

Lemma 3.3.12 (TVD property of semi-discrete evolution)**.**

- F *non-decreasing in the first argument, non-increasing in the second argument, cf. Lemma 3.2.7,*
- $w_0 = w_0(\vec{\mu})$ *by local* piecewise linear *reconstruction, satisfies* $TV_{\mathbb{R}}(w_0) \leq TV_{\Delta x}(\vec{\mu})$*: TVD-reconstruction,*
- $\vec{\mu}(0)$ *has finitely many local extrema.*

Then $TV_{\Delta x}\vec{\mu}(t)$ *is non-increasing for solution* $\vec{\mu}(t)$ *of* (3.3.29)*.*

Idea of proof:

No new (local) extrema can arise !

Adjacent values cannot "overtake":

local maximum: cannot move up

local minmum: cannot move down

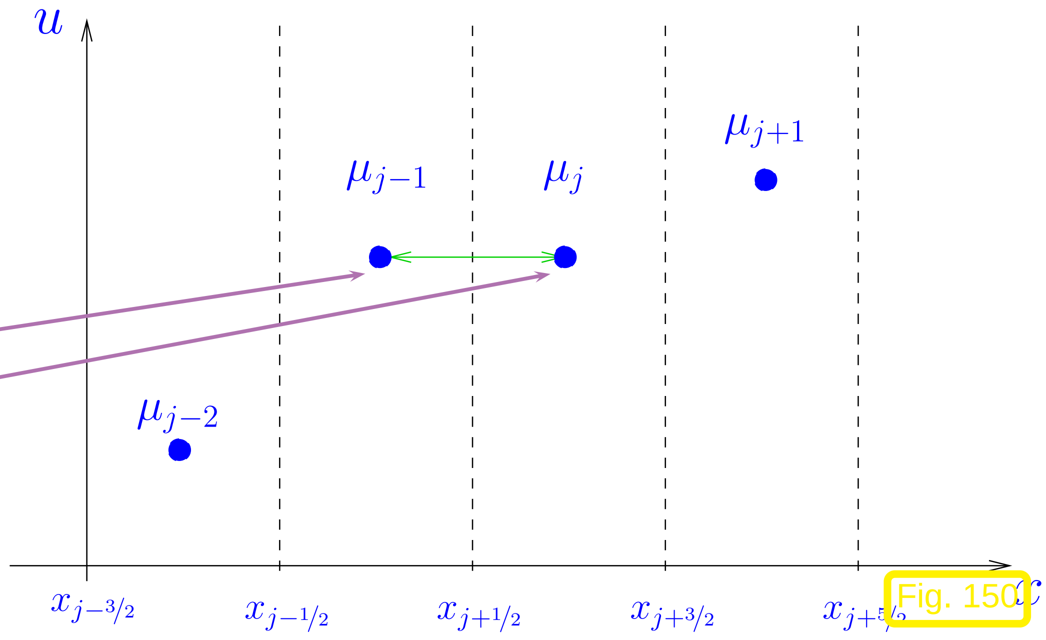


Fig. 150

Determine $w^{\pm}_{j+1/2}(t)$ in (3.3.29) through TVD p.w. linear interpolation of $\vec{\mu}(t) \longrightarrow$ Sect. 3.3.1.2

For smooth, monotone solutions of (2.2.1):

Slope limited TVD reconstructions of Sect. 3.3.1.2 (minmod $\longrightarrow$ Def. 3.3.3, superbee (76), MC (76)) yield 2nd-order consistent ($\longrightarrow$ Def. 3.3.10) spatially semi-discrete evolutions.

▶ General formula for slope limited p.w. linear reconstruction

$$w^{-}_{j+1/2} = u^{(k-1)}_j + \tfrac{1}{2}\varphi(\theta^{(k-1)}_{j+1/2})(\mu^{(k-1)}_j - \mu^{(k-1)}_{j-1}) \;,$$
$$w^{+}_{j-1/2} = u^{(k-1)}_j - \tfrac{1}{2}\varphi(\theta^{(k-1)}_{j+1/2})(\mu^{(k-1)}_j - \mu^{(k-1)}_{j-1}) \;,$$

with $\theta^{(k-1)}_{j+1/2}$ from (3.3.16), flux limiter function $\varphi : \mathbb{R} \mapsto \mathbb{R}$.

Remark 84 (Other higher order reconstructions).

- piecewise quadratic reconstruction [34]
- logarithmic reconstruction [4],
- rational reconstruction [32]

▷ Problems:
oscillations (TVD-property **?**)
"large stencils"

△

3.3.3.3 ENO-methods

➜ instance of a special recipe for higher order reconstruction *with "minimal" oscillations*

Setting:
- Cauchy problem (2.2.1) for 1D scalar conservation law $\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0$
- Equidistant spatial grid $\mathcal{G}_{\Delta x}$, meshwidth $\Delta x > 0$

Consider: spatially semi-discrete evolution

$$\frac{d}{dt}\mu_j(t) = -\frac{1}{\Delta x}\left(F(w^-_{j+1/2}, w^+_{j+1/2}) - F(w^-_{j-1/2}, w^+_{j-1/2})\right) , \tag{3.3.29}$$

$w^{\pm}_{j+1/2} := \lim_{\epsilon\to 0} w_0(x_{j+1/2} \pm \epsilon)$, $w_0 \hat{=}$ reconstruction of $u(\cdot,t)$ from cell averages $\vec{\mu}(t)$

Assume: $\vec{\mu}$ exact cell averages: $\mu_j = \frac{1}{\Delta x}\int\limits_{x_{j-1/2}}^{x+1/2} u(x)\,\mathrm{d}x$ for $u \in L^1(\mathbb{R}) \cap BV_{\mathrm{loc}}(\mathbb{R})$

Goal: algorithm for finding $w_0 = w_0(\vec{\mu}) \in \mathcal{P}_r(\mathcal{G}_{\Delta x})$, degree $r \geq 1$, with

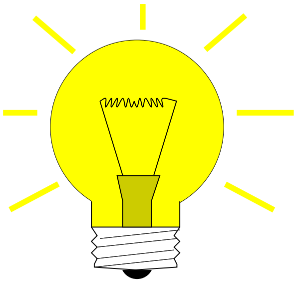
high order approximation: $\boxed{\|u - w_0\|_{L^\infty(\mathbb{R})} = O((\Delta x)^{r+1})}$ for smooth u, $\Delta x \to 0$, (3.3.30)

TVB-property: $\boxed{TV_{\mathbb{R}}(w_0) \leq TV_{\mathbb{R}}(u) + O((\Delta x)^{r+1})}$. (3.3.31)

TVB $\leftrightarrow$ total variation bounded (replaces TVD, which restricts order of approximation to 2)

Now: fix degree $r \geq 0$ and position index $j \in \mathbb{Z}$ ➢ consider single cell $]x_{j-1/2}, x_{j+1/2}[$:

Idea: • match cell avarages



$$p_{j-l}^{j-l+r} \in \mathcal{P}_r(\mathbb{R})\text{:} \quad \frac{1}{\Delta x}\int_{x_{j+i-1/2}}^{x_{j+i+1/2}} p_{j-l}^{j-l+r}(x)\,\mathrm{d}x = \mu_{j+i} \quad \begin{array}{l} \forall i = -l,\dots,-l+r\ , \\ l = 0,\dots,r\ . \end{array}$$

[Terminology: index set $\{j-l,\dots,j-l+r\} \hat{=}$ "stencil" of reconstruction.]

• select "least oscillatory" p_{j-l}^{j-l+r} ➤ provides $w_{0|]x_{j-1/2},x_{j+1/2}[}$.

Example 85 (Reconstruction by average matching polynimials).

• cell averages $\mu_i = 1$ for $i < j$, $\mu_j = \frac{1}{2}$, $\mu_i = 0$ for $i > 1$

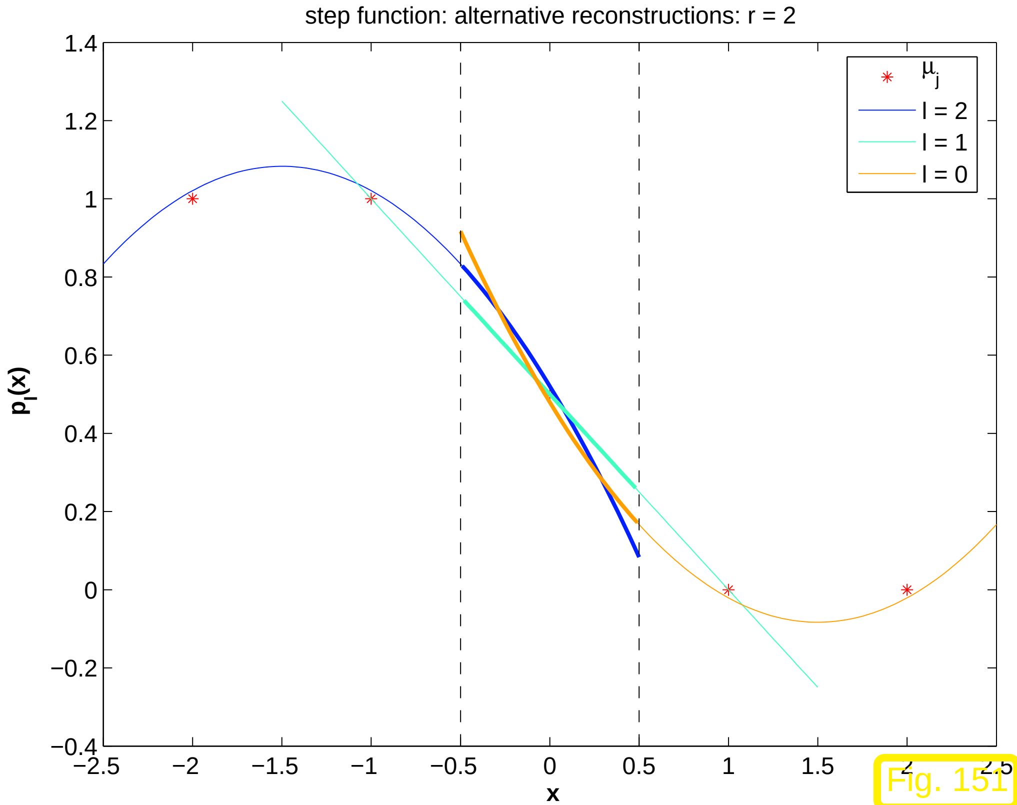


Fig. 151

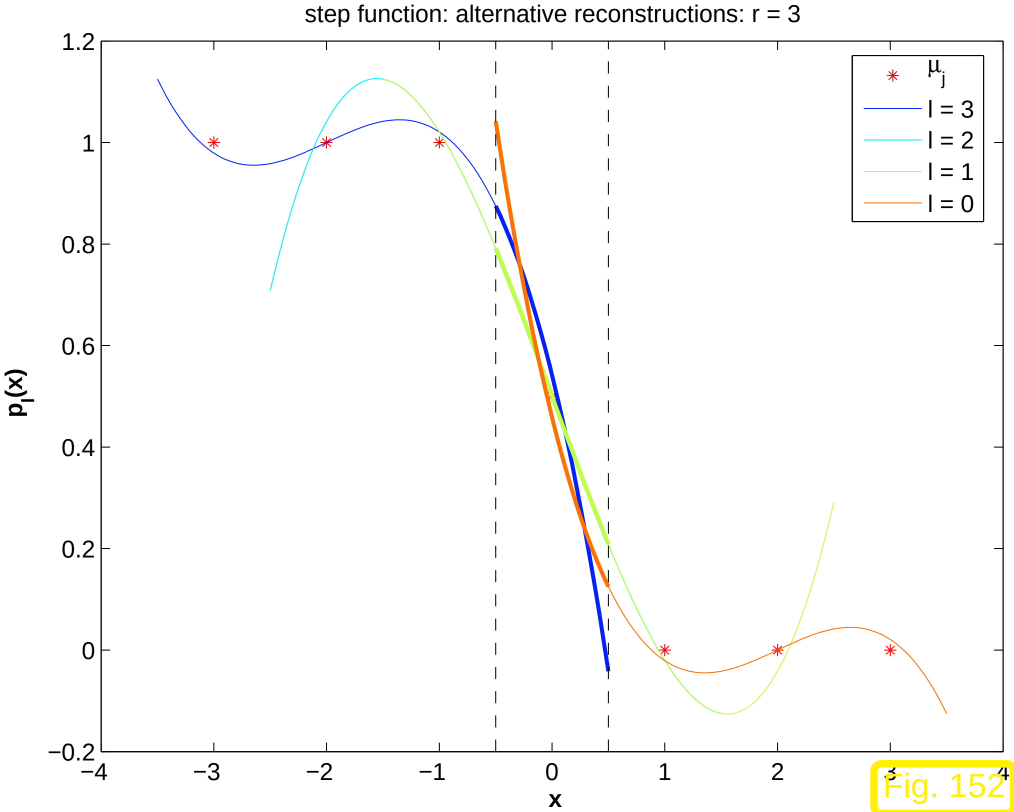


Fig. 152

- "trapezoidal function", $\mu_i = 1 - (j - i - 1)/10$ for $i < j$, $\mu_i = 1 - (i - j)/10$ for $i \geq j$

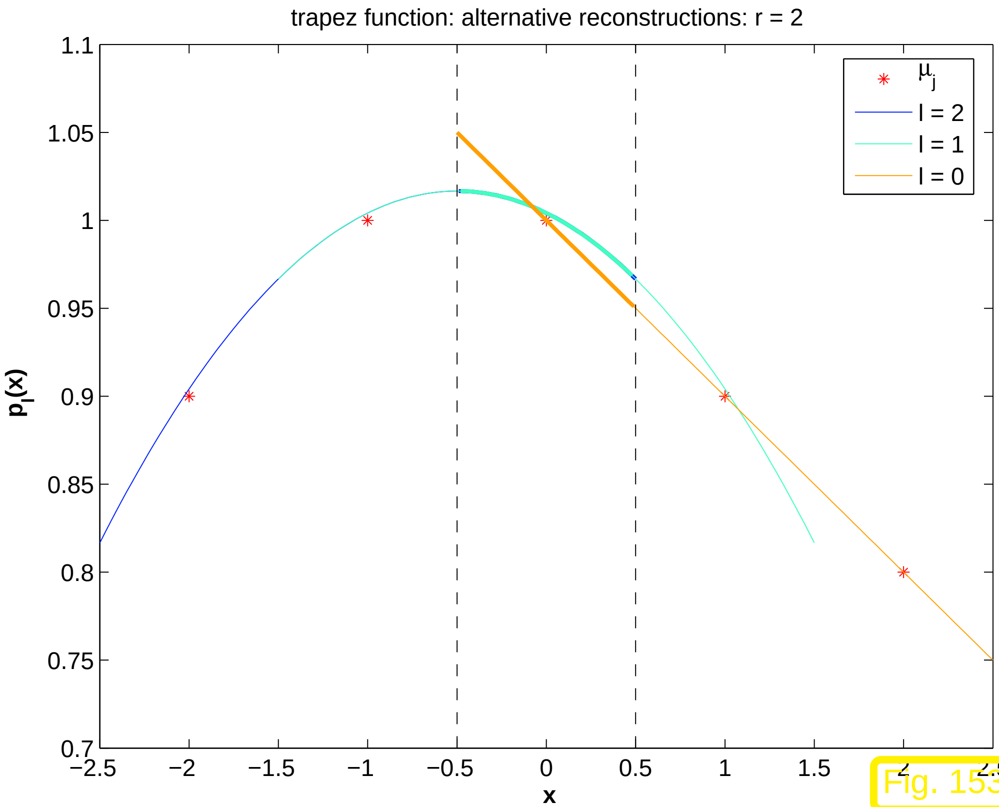


Fig. 153

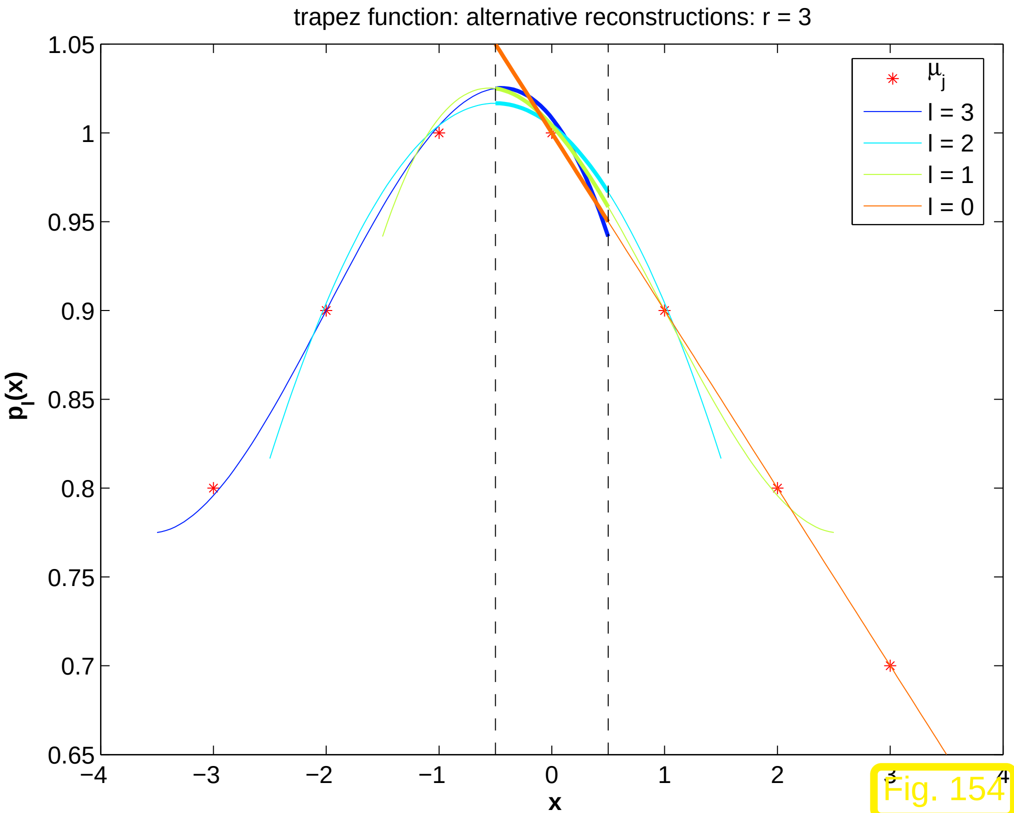


Fig. 154

◇

ENO (essentially non-oscillatory) approach:

construct ENO-stencil $S_{r,j} := \{j-l,\dots,j-l+r\}$ ($\leftrightarrow$ find l) through binary decision tree:
$S_{0,j} = \{j\}$ and assume that $S_{r-1,j} = \{j_1,\dots,j_r\}$ already found

$$\blacktriangleright \quad S_{r,j} = \begin{cases} S_{r-1,j} \cup \{j_1 - 1\} & \text{, if} \quad |C^{j_r}_{j_1-1}| < |C^{j_r+1}_{j_1}| \,, \\ S_{r-1,j} \cup \{j_r + 1\} & \text{, if} \quad |C^{j_r}_{j_1-1}| \geq |C^{j_r+1}_{j_1}| \,, \end{cases}$$

where $C^n_i \mathrel{\hat{=}}$ leading coefficient of average matching polynomial $p^n_i \in \mathcal{P}_{n-i}(\mathbb{R})$.

Note: average matching polynomial p^n_i by interpolating primitive of u **!**

$$p^n_i = q' \quad \text{with} \qquad q \in \mathcal{P}_{n-i+1}(\mathbb{R}) \,, \quad q(x_{j+1/2}) = \sum_{k=-\infty}^{j} \mu_k \,, \quad j = i-1,\dots,n \,. \tag{3.3.32}$$

Practical ENO-implementation (on equidistant grid): comparison of divided differences

Recall: given $(x_j, \mu_j) \in \mathbb{R}^2$, $j \in \mathbb{Z}$: divided difference $[x_i,\dots,x_k]\vec{\mu}$ = leading coefficient of polynomial (degree $k-i+1$) interpolating (x_j,μ_j), $i \leq j \leq k$.

Important: recursion formula for divided differences [10, Lemma 7.11]:

$$[x_i,\dots,x_k]\vec{\mu} = \frac{[x_{i+1},\dots,x_k]\vec{\mu} - [x_i,\dots,x_{k-1}]\vec{\mu}}{x_k - x_i}\;. \tag{3.3.33}$$

recursive computation of degree r ENO stencil for j-th grid cell.
(`dd(mu)` computes divided differences for nodal values `mu` on equidistant grid)

▷

MATLAB-CODE selection of ENO stencil

```
function stn = enostn(mu,j,r)
stn = [j,j];
if (r > 0)
 for k=1:r
  ddl = dd(mu(stn(1)-1,stn(2)));
  ddr = dd(mu(stn(1),stn(2)+1));
  if (abs(ddl) < abs(ddr))
   stn(1) = stn(1)-1;
  else
   stn(2) = stn(2)+1;
end end end
```

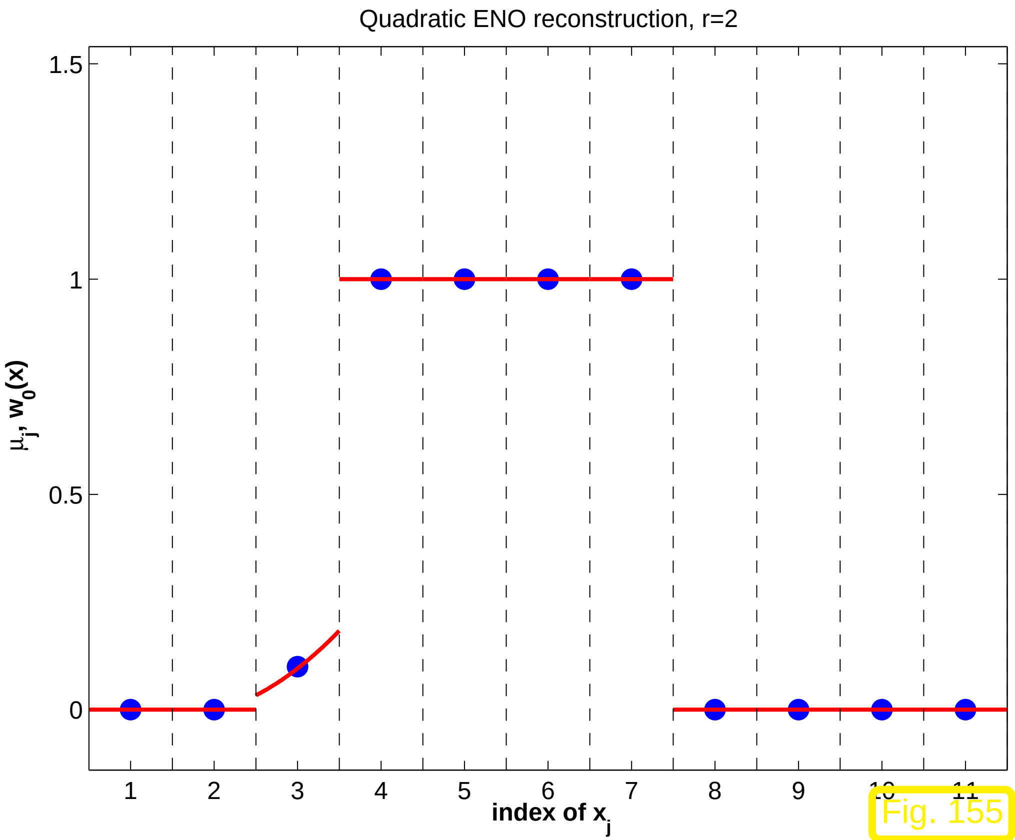
Once, ENO-stencil is found: due to linearity of mapping $\vec{\mu} \mapsto w^{\pm}_{j+1/2}$

$$w^{+}_{j-1/2} = \sum_{k=j-l}^{j-l+r} c^{-}_{jk}\mu_k\;,\quad w^{-}_{j+1/2} = \sum_{k=j-l}^{j-l+r} c^{+}_{jk}\mu_k\;.$$

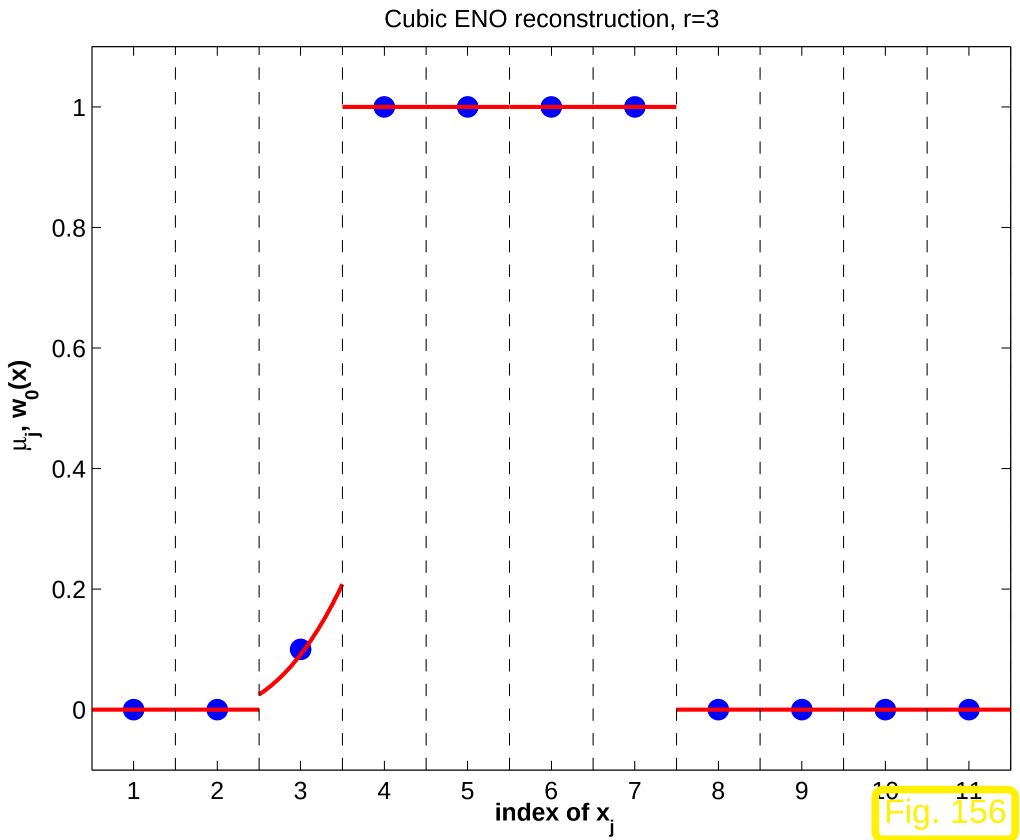
On equidistant mesh: precompute lookup-table for $c_{jk} = c_k(l)$, see [41, 42]

Example 86 (ENO reconstruction).

Here: $\vec{\mu} \hat{=}$ periodic grid function, period = 11



p.w. quadratic ENO reconstruction, $r = 2$



p.w. cubic ENO reconstruction, $r = 3$

Observation: ☞ TVD resolution of (isolated) discontinuities

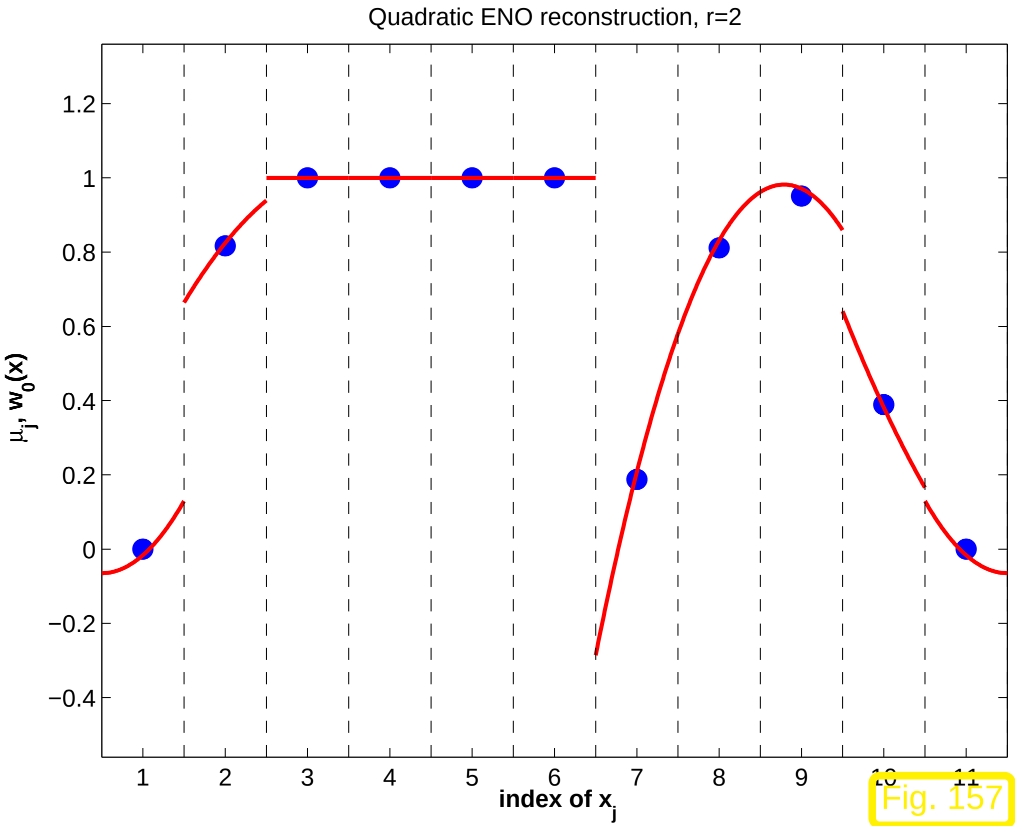


Fig. 157

p.w. quadratic ENO reconstruction, $r = 2$

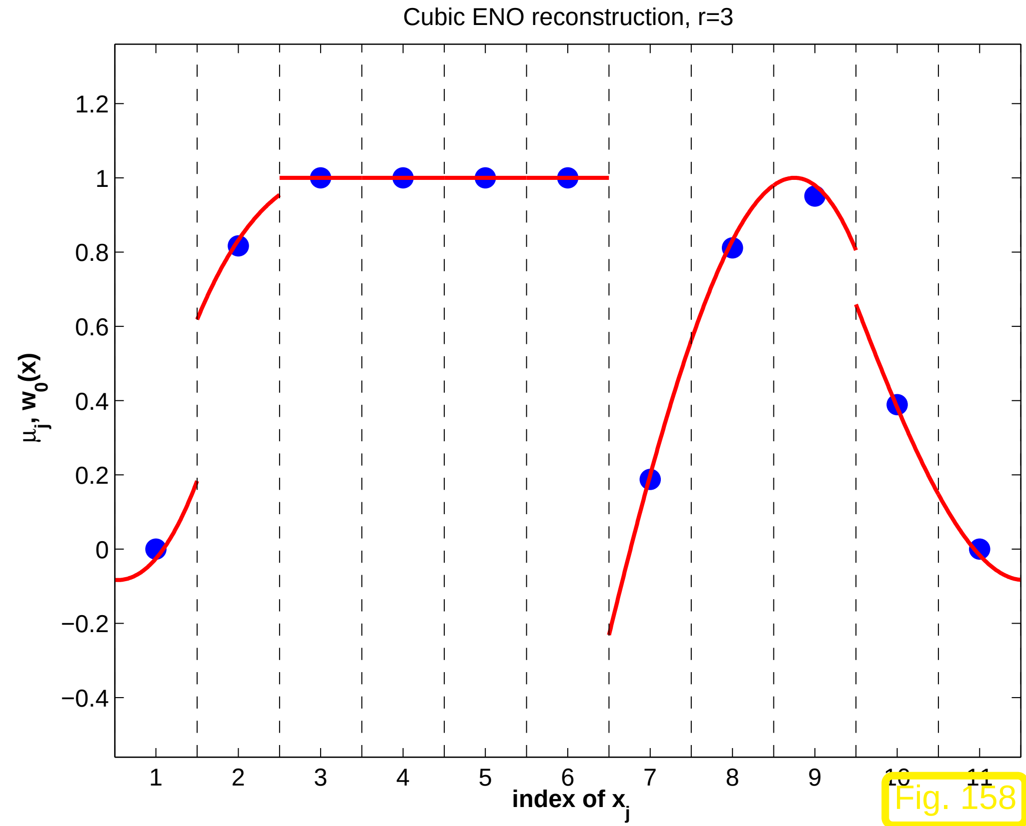


Fig. 158

p.w. cubic ENO reconstruction, $r = 3$

Observation: ☞ small overshoots at extrema of $\vec{\mu}$ ($\rightarrow 0$ as $\Delta x \rightarrow 0$!) ◇

Example 87 (TVB-property of ENO reconstruction).

- $\vec{\mu} \leftarrow$ sampling of 1-periodic function on equidistant grids, $\Delta x \in \{\frac{1}{10}, \frac{1}{20}, \frac{1}{40}, \frac{1}{80}, \frac{1}{160}, \frac{1}{320}, \frac{1}{640}\}$
- $w_0 \leftarrow$ degree r, $r = 2, 3, 4$, ENO-reconstruction based on $\vec{\mu}$

Measured: ratios $TV_{\mathbb{R}}(w_0) : TV_{\Delta x}(\vec{\mu})$ on different grids

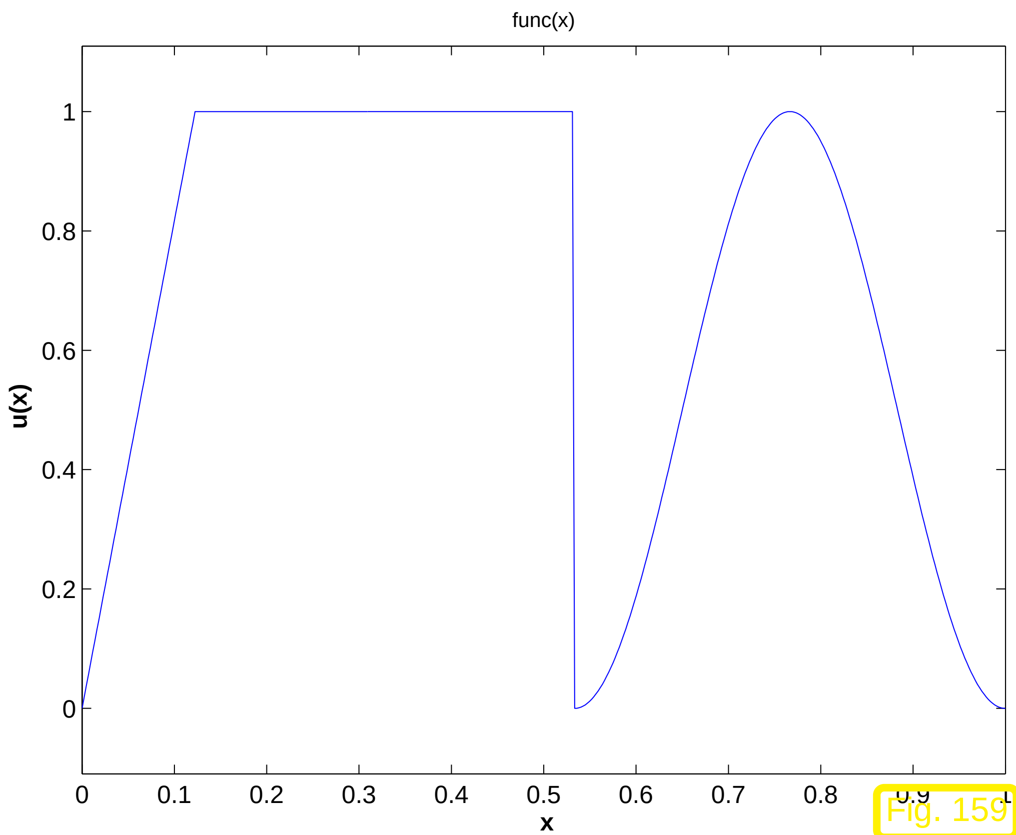


Fig. 159

sampled function: $\mu_j = u(\Delta x j)$, $j \in \mathbb{Z}$

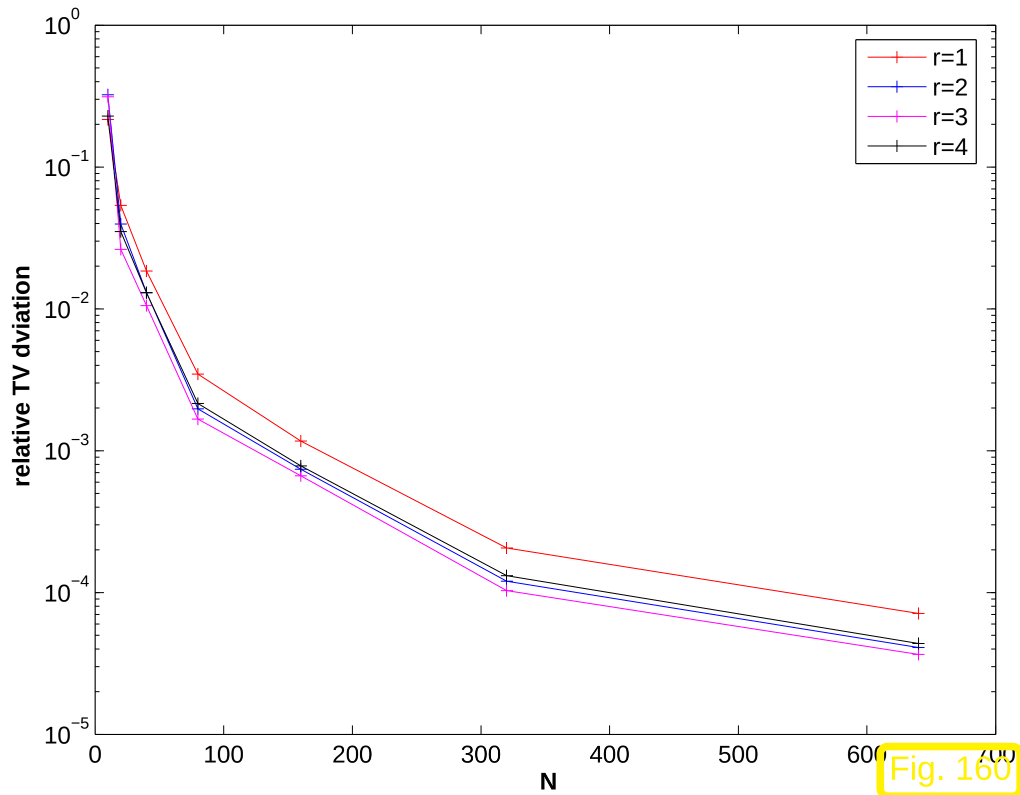


Fig. 160

Relative TV-increase for ENO reconstruction

Observation: in this case: ENO-reconstruction is TVB in the sense of (3.3.31)

◇

Remark 88 (Weighted essentially non-oscillatory schemes (WENO)).

Extension of ENO idea ➤ WENO: use suitable convex combinations of local polynomial reconstructions [41, Sect. 2].

$\triangle$

3.3.3.4 Strong Stability Preserving (SSP) timestepping

MOL: spatial semidiscretization (3.3.25) **+** timestepping $\Rightarrow$ numerical method for (2.2.1)

Simplest choice: explicit Euler timestepping for (3.3.25)

$$\vec{\mu}^{(k)} = \vec{\mu}^{(k-1)} + \Delta t\, \mathcal{L}_{\Delta x}(\vec{\mu}^{(k-1)}) \;, \quad k = 1, \ldots, M := T/\Delta t \;. \tag{3.3.34}$$

Note: explicit Euler (3.3.34) **+** semi-discrete FV (3.3.3.1) **=** 3-point FVM (3.2.2)

Example 89 (Necessity of higher order timestepping).

- constant linear advection (2.1.6), $v = 1$, “bump” initial data (4.2.3)
- spatial semi-discretization: quadratic ENO reconstruction ($\rightarrow$ Sect. 3.3.3.3), equidistant grid
- explicit Euler timestepping (3.3.34) with fixed timestep $\Delta t = \Delta x$, $\gamma := \Delta t/\Delta x$ constant.

  Alternative: 2nd-order Heun method (3.3.41) (see below)

Monitored: l^1-norm of discretization error at $T = 1$ for $\Delta x \in \{\frac{1}{10}, \frac{1}{20}, \frac{1}{40}, \frac{1}{80}, \frac{1}{160}, \frac{1}{320}, \frac{1}{640}\}$ **+** approximate convergence rates, *cf.* Ex. 79

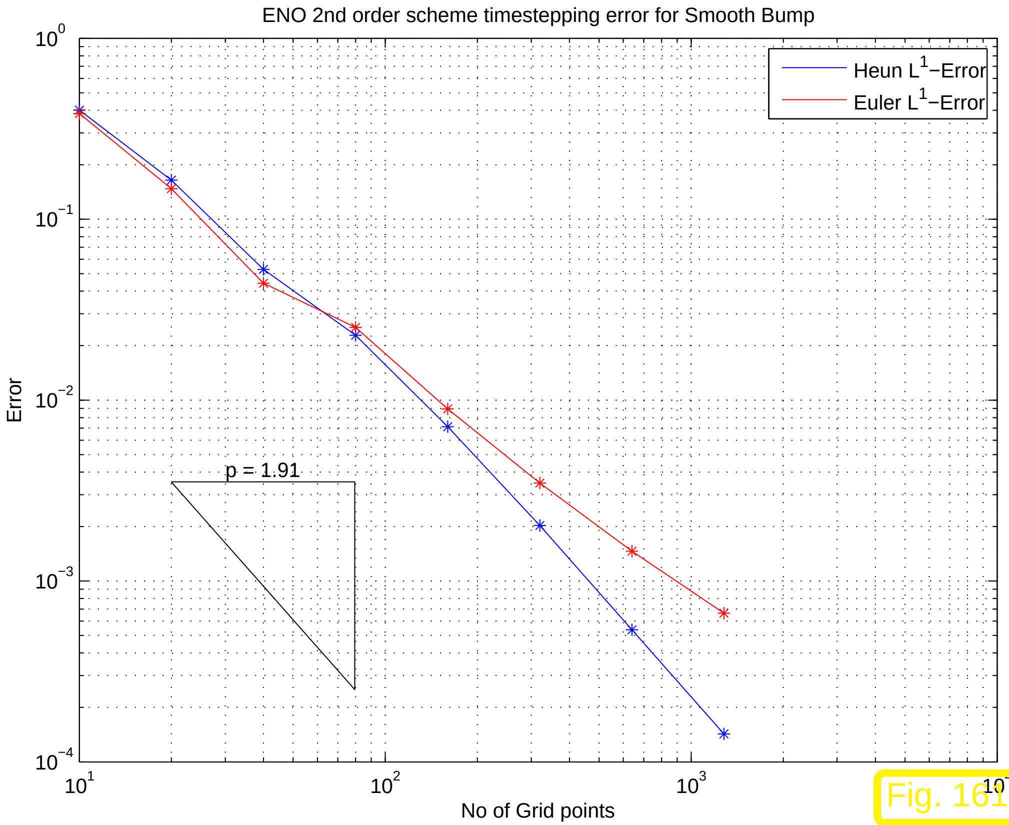


Fig. 161

$L^1(bbR)$-errors

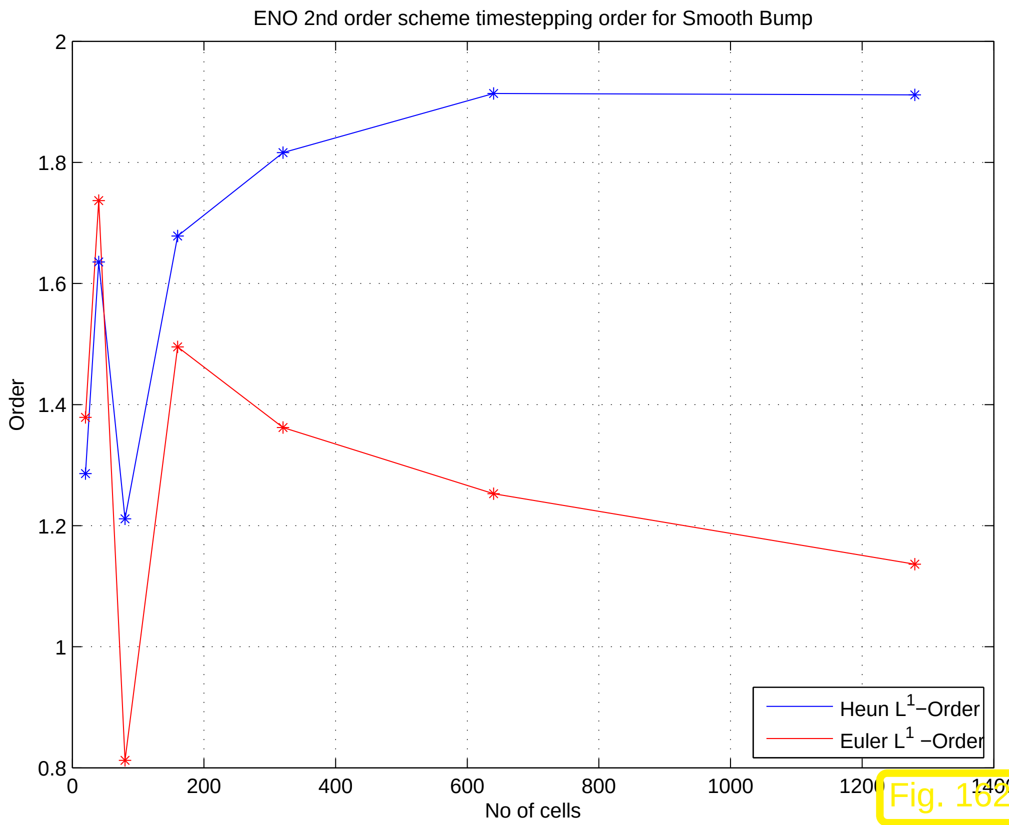


Fig. 162

Approximate order of convergence

◇

Guideline for method of lines (→ Sect. 3.3.3):

Order of temporal discretization has to match order of spatial discretization (→ Def. 3.3.10)

Focus: **Explicit single step** timestepping methods

Recall from numerical analysis of integrators for ODEs [9]:

Definition 3.3.13 (Explicit single step timestepping method).

An explicit single step timestepping method for the autonomous ordinary differential equation $\frac{d}{dt}\vec{\eta} = L(\vec{\eta})$ *computes the approximation* $\vec{\eta}^{(k)}$ *of* $\vec{\eta}(t_k)$ *at* $t_k = t_{k-1} + \Delta t_k$ *from* $\vec{\eta}^{(k-1)}$ *merely using evaluations of* L.

Definition 3.3.14 (Order of timestepping). *($\rightarrow$ Def. 3.1.7, cf. Def. 3.3.10)*
An explicit single step timestepping method $\vec{\eta}^{(k)} = \mathsf{T}_{\Delta t_k}(\vec{\eta}^{(k-1)})$ *is consistent of order* p, $p \in \mathbb{N}$, *with the ODE* $\frac{d}{dt}\vec{\eta} = L(\vec{\eta})$, *if*

$$\exists C > 0: \quad \|\vec{\eta}(t+\Delta t) - \mathsf{T}_{\Delta t}(\vec{\eta}(t))\| \leq C(\Delta t)^{p+1} \quad \Delta t \rightarrow 0, \quad \textit{uniformly in } t \ ,$$

and any solution $\vec{\eta}(t)$ *of the ODE.*

➜ Explicit Euler timestepping (3.3.34) $=$ 1st-order

Known: scores of explicit single step methods for ODEs [20],
most prominent: Runge-Kutta methods [9, Ch. 4]

Example 90 (Danger of using “standard timestepping methods”).

- Cauchy problem for Burgers equation (2.1.7), $u_0(x) = \begin{cases} 1 & \text{, if } x \leq 0 \ , \\ -1/2 & \text{, if } x > 0 \ , \end{cases}$
- spatial semi-discretization: (3.3.29), $F = F_{\mathsf{GD}}$ (Godunov numerical flux function from (3.2.15)),
- piecewise linear reconstruction: minmod ($\rightarrow$ Def. 3.3.3) slopes

if $\Delta t/\Delta x < \frac{1}{2}$ $\Rightarrow$ explicit Euler step (3.3.34) is TVD !

➤ use local timesteps $\Delta t_k = \dfrac{1}{2\max_j \mu_j^{(k-1)}}$

- Two second-order explicit single step timestepping methods:

$$\vec{\eta}^* = \vec{\mu} + \Delta t\,\mathcal{L}_{\Delta x}(\vec{\mu})\;,\quad \mathsf{T}_{\Delta t}(\vec{\mu}) = \tfrac{1}{2}\vec{\mu} + \tfrac{1}{2}\big(\vec{\eta}^* + \Delta t\,\mathcal{L}_{\Delta x}(\vec{\eta}^*)\big)\;, \tag{3.3.35}$$

$$\vec{\eta}^* = \vec{\mu} - 20\Delta t\,\mathcal{L}_{\Delta x}(\vec{\mu})\;,\quad \mathsf{T}_{\Delta t}(\vec{\mu}) = \vec{\mu} + \tfrac{41}{40}\Delta t\,\mathcal{L}_{\Delta x}(\vec{\mu}) - \tfrac{1}{40}\Delta t\,\mathcal{L}_{\Delta x}(\vec{\eta}^*)\;. \tag{3.3.36}$$

Note: both methods agree for linear $\mathcal{L}_{\Delta x}$ **!**

Displayed: $\vec{\mu}^{(500)}$ for both timestepping schemes for $\Delta x = 0.01$

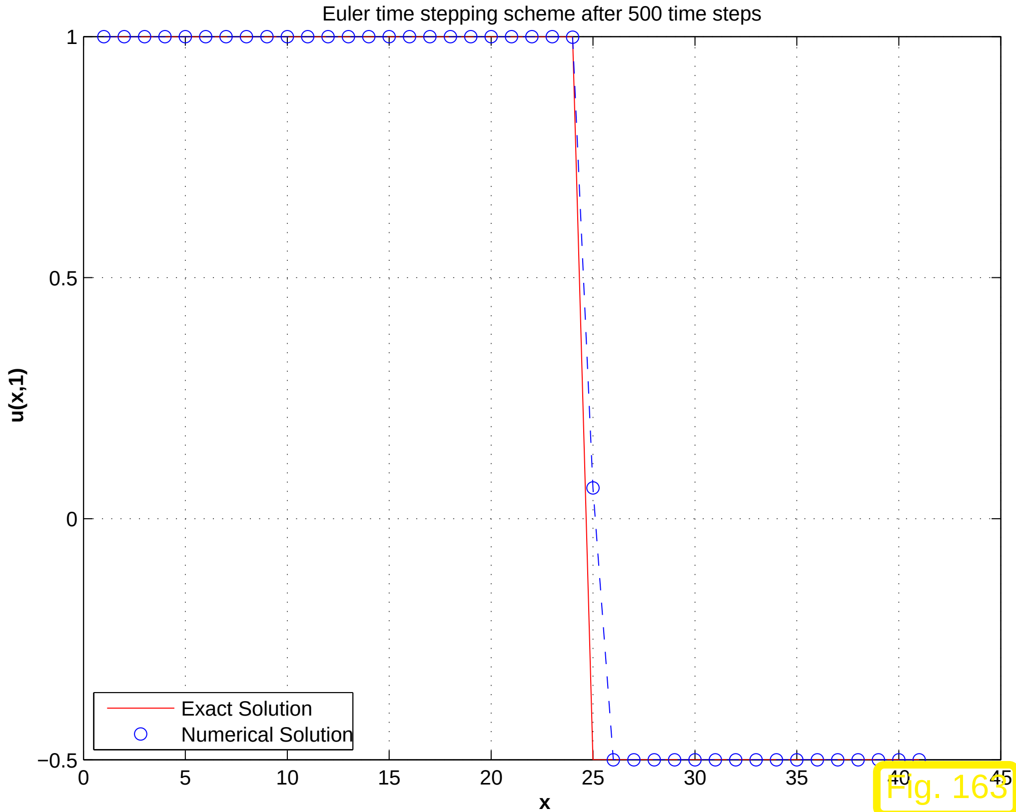


Fig. 163

timestepping (3.3.34)

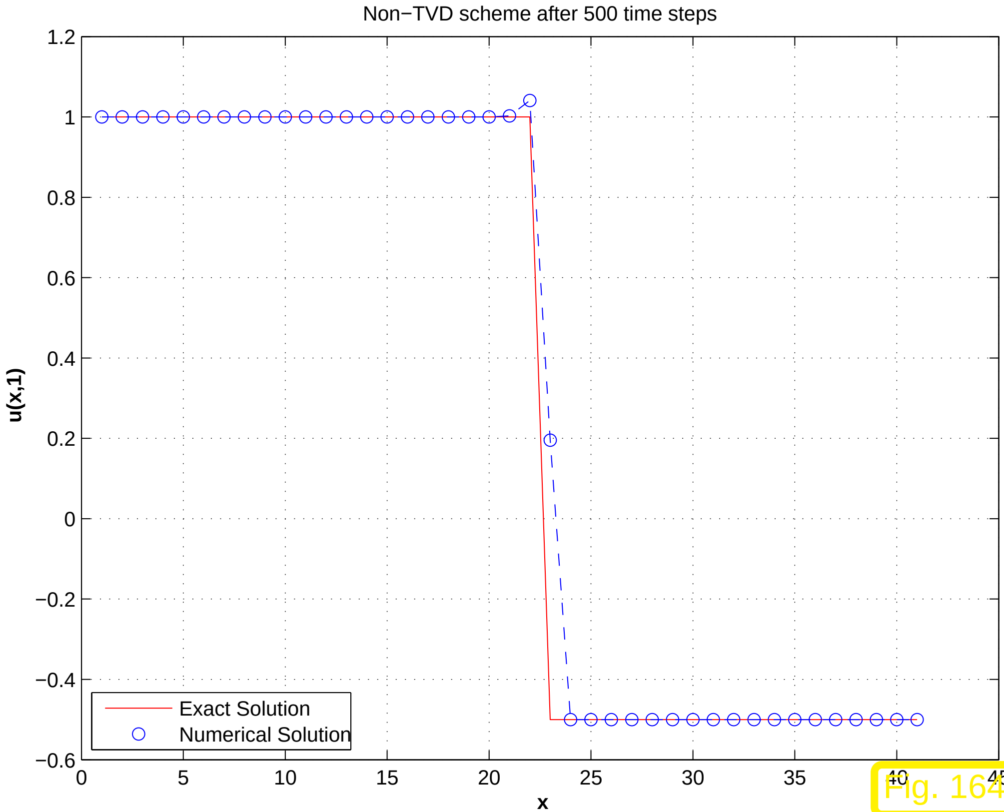


Fig. 164

timestepping (3.3.36)

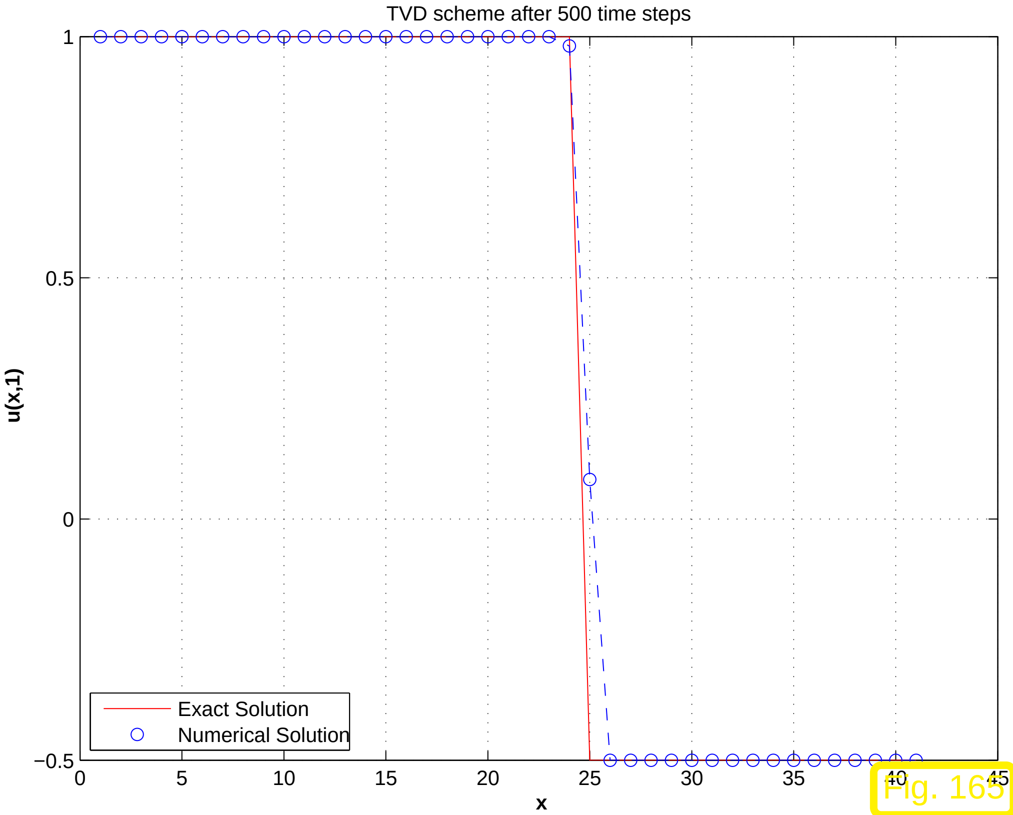


Fig. 165

timestepping (3.3.35)

Observations:

- no spurious oscillations for explicit Euler: TVD-property
- 2nd-order timestepping (3.3.36) ➔ overshoots
  2nd-order timestepping (3.3.35): TVD-property

◇

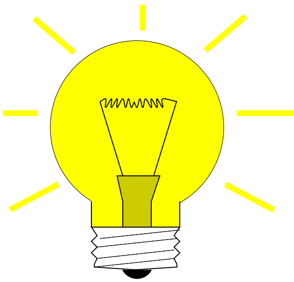
Often known: stability properties (e.g. TVD) known for explicit Euler timestepping (3.3.34)

Definition 3.3.15 (Strong stability preservation (SSP)). *(⟶ [16])*
An explicit timestepping scheme $\vec{\mu}^{(k)} = \mathsf{T}_{\Delta t}(\vec{\mu}^{(k-1)})$ *for* (3.3.25) *is strong stability preserving, if for some (semi-)norm* $\|\cdot\|$ *and* $c > 0$

$$\forall \Delta t \le \Delta t_0 \colon \quad \| \underbrace{\vec{\mu} + \Delta t\, \mathcal{L}_{\Delta x}(\vec{\mu})}_{\textit{explicit Euler step}} \| \le \|\vec{\mu}\| \quad \forall \vec{\mu} \quad \Rightarrow \quad \|\mathsf{T}_{\Delta t}(\vec{\mu})\| \le \|\vec{\mu}\| \quad \boxed{\forall \Delta t \le c\Delta t_0}, \vec{\mu} \;.$$

tighter CFL-condition (⟶ Def. 3.1.4) for higher order timestepping !

Idea: $\mathsf{T}_{\Delta t}$ as convex combination of explicit Euler "microsteps":



$$\vec{\eta}_0 = \vec{\mu} \;, \quad \vec{\eta}_i = \sum_{l=0}^{i-1} \alpha_{il} \left(\vec{\eta}_l + \beta_{il} \Delta t\, \mathcal{L}_{\Delta x}(\vec{\eta}_l) \right) \;, \quad i = 1, \dots, s+1 \;, \tag{3.3.37}$$

$$\mathsf{T}_{\Delta t}(\vec{\mu}) := \vec{\eta}_{s+1} \;,$$

with $\sum_{l=0}^{i-1} \alpha_{il} = 1, \quad \alpha_{il} \ge 0$.

Corollary 3.3.16. $\beta_{il} \ge 0 \quad \Rightarrow \quad$ (3.3.37) *SSP (⟶ Def. 3.3.15) with* $c = \max\limits_{i,l} \beta_{il}^{-1}$

Recall: explicit s-stage, $s \in \mathbb{N}$, Runge-Kutta method for "ODE" $\frac{d}{dt}\vec{\mu}(t) = \mathcal{L}_{\Delta x}(\vec{\mu}(t))$:

$$\vec{\kappa}_i = \mathcal{L}_{\Delta x}(\vec{\mu} + \Delta t \sum_{l=1}^{i-1} a_{il}\,\vec{\kappa}_l)\;, \quad i = 1,\dots,s\;, \quad \mathsf{T}_{\Delta t}(\vec{\mu}) := \vec{\mu} + \Delta t \sum_{l=1}^{s} b_l\,\vec{\kappa}_l\;. \qquad (3.3.38)$$

Runge-Kutta increments   Runge-Kutta coefficients $\in \mathbb{R}$

Short-hand notation für Runge-Kutta methods

Butcher tableau $\triangleright$

$$\begin{array}{c|c} \mathbf{c} & \mathfrak{A} \\ \hline & \mathbf{b}^T \end{array} \quad := \quad \begin{array}{c|ccc} c_1 & a_{11} & \cdots & a_{1s} \\ \vdots & \vdots & & \vdots \\ c_s & a_{s1} & \cdots & a_{ss} \\ \hline & b_1 & \cdots & b_s \end{array}\;. \qquad (3.3.39)$$

(3.3.38) $\Leftrightarrow$

$$\begin{aligned} \vec{\eta}_i &= \vec{\mu} + \Delta t \sum_{l=1}^{i-1} a_{il}\,\mathcal{L}_{\Delta x}(\vec{\eta}_l)\;, \quad i = 1,\dots,s\;, \\ \mathsf{T}_{\Delta t}(\vec{\mu}) &= \vec{\eta}_{s+1} := \vec{\mu} + \Delta t \sum_{l=1}^{s} b_l\,\mathcal{L}_{\Delta x}(\vec{\eta}_l)\;. \end{aligned} \qquad (3.3.40)$$

Choose $\alpha_{il} \geq 0$, $\sum_{l=0}^{i-1} \alpha_{il} = 1$, set $a_{s+1,l} := b_l$:

$$\begin{aligned}
\vec{\eta}_i &= \sum_{l=1}^{i-1} \alpha_{il}\vec{\mu} + \Delta t \sum_{l=1}^{i-1} a_{il}\, \mathcal{L}_{\Delta x}(\vec{\eta}_l) \\
&= \alpha_{i0}\vec{\mu} + \sum_{l=1}^{i-1} \alpha_{il}\Big(\vec{\eta}_l - \Delta t \sum_{k=0}^{l-1} a_{lk}\, \mathcal{L}_{\Delta x}(\vec{\eta}_k)\Big) + \Delta t \sum_{l=1}^{i-1} a_{il}\, \mathcal{L}_{\Delta x}(\vec{\eta}_l) \\
&= \sum_{l=0}^{i-1} \alpha_{il}\Bigg(\vec{\eta}_l + \underbrace{\frac{1}{\alpha_{il}}\Big(a_{il} - \sum_{k=l+1}^{i-1} a_{kl}\alpha_{ik}\Big)}_{=:\beta_{il} \text{ in (3.3.37)}} \Delta t\, \mathcal{L}_{\Delta x}(\vec{\eta}_l)\Bigg)\,, \quad i = 1, \dots, s+1\,.
\end{aligned}$$

☞ 2-stage SSP-Runge-Kutta method for (3.3.25) (Heun method): $\boxed{c = 1}$

$$\begin{aligned}
\vec{\eta}_2 &= \vec{\mu} + \Delta t\, \mathcal{L}_{\Delta x}(\vec{\mu})\,, \\
\vec{\eta}_3 &= \tfrac{1}{2}\vec{\mu} + \tfrac{1}{2}\big(\vec{\eta}_2 + \Delta t\, \mathcal{L}_{\Delta x}(\vec{\eta}_2)\big)\,, \\
\mathsf{T}_{\Delta t}(\vec{\mu}) &= \vec{\eta}_3\,.
\end{aligned}
\quad \Leftrightarrow \quad
\begin{array}{c|cc}
0 & 0 & 0 \\
1 & 1 & 0 \\
\hline
 & 1/2 & 1/2
\end{array}
\tag{3.3.41}$$

MAPLE-computation of order of Heun method:

① `D(y) := x -> L(y(x)); y0 := y(0);`

$$\mathsf{D}(y) := x \mapsto L(y(x))\,; \quad y0 := y(0)$$

②
```
g1 := y0 + h*L(y0); y1 := y0/2 + (g1+h*L(g1))/2;
```
$$g1 := y(0) + hL(y(0))\,; \quad y1 := y(0) + 1/2\,hL(y(0)) + 1/2\,hL(y(0) + hf(y(0)))$$

③
```
taylor(y1-y(h),h=0,4);
```
$$series\left(\left(1/12\left(D^{(2)}\right)(L)(y(0))(f(y(0)))^2 - 1/6\,(\mathsf{D}(L)(y(0)))^2 L(y(0))\right)h^3 + O\left(h^4\right), h, 4\right)$$

➢ Heun method has order 2 (→ Def. 3.3.14)

☞ 3-stage SSP-Runge-Kutta method for (3.3.25): $\boxed{c = 1}$

$$\begin{aligned} \vec{\eta}_2 &= \vec{\mu} + \Delta t\,\mathcal{L}_{\Delta x}(\vec{\mu})\,, \\ \vec{\eta}_3 &= \tfrac{3}{4}\vec{\mu} + \tfrac{1}{4}\left(\vec{\eta}_2 + \Delta t\,\mathcal{L}_{\Delta x}(\vec{\eta}_2)\right)\,, \\ \vec{\eta}_4 &= \tfrac{1}{3}\vec{\mu} + \tfrac{2}{3}\left(\vec{\eta}_3 + \Delta t\,\mathcal{L}_{\Delta x}(\vec{\eta}_3)\right)\,, \\ \mathsf{T}_{\Delta t}(\vec{\mu}) &= \vec{\eta}_4\,. \end{aligned} \quad \Leftrightarrow \quad \begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1/2 & 1/4 & 1/4 & 0 \\ \hline & 1/6 & 1/6 & 2/3 \end{array} \tag{3.3.42}$$

MAPLE-computation of order of 3-stage SSP Runge-Kutta method:

①
```
D(y) := x -> L(y(x)); y0 := y(0);
```
②
```
g1 := y0 + h*L(y0); g2 := 3*y0/4+(g1+h*L(g1))/4;
```
③
```
y1 := y0/3+2*(g2+h*L(g2))/3;
```
④
```
taylor(y1-y(h),h=0,5);
```
$$series\left(-1/24\,(\mathsf{D}(L)(y(0)))^3 L(y(0))\,h^4 + O\left(h^5\right), h, 5\right)$$

➢ 3-stage SSP Runge-Kutta method (3.3.42): order 3 ($\rightarrow$ Def. 3.3.14)

$\nexists$ timestepping (3.3.37) of order > 3 **and** $\beta_{il} \geq 0$

! Remedy: “upwind” & “downwind” spatial semi-discretization of $\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0$:

$$\frac{d}{dt}\vec{\mu}(t) + \mathcal{L}_{\Delta x}(\vec{\mu}(t)) = 0 \quad \text{and} \quad \frac{d}{dt}\vec{\mu}(t) + \widetilde{\mathcal{L}}_{\Delta x}(\vec{\mu}(t)) = 0 \ ,$$

where $\mathcal{L}_{\Delta x}$ and $\widetilde{\mathcal{L}}_{\Delta x}$ are **both** consistent of order q ($\rightarrow$ Def. 3.3.10) with $\frac{\partial f(u)}{\partial x}$ and

$$\forall \Delta t \leq \Delta t_0\colon \quad \|\vec{\mu} + \Delta t\, \mathcal{L}_{\Delta x}(\vec{\mu})\| \leq \|\vec{\mu}\| \quad \wedge \quad \|\vec{\mu} - \Delta t \widetilde{\mathcal{L}}_{\Delta x}(\vec{\mu})\| \leq \|\vec{\mu}\| \quad \forall \vec{\mu}\ . \tag{3.3.43}$$

Example: for linear advection $f(u) = vu$, $v > 0$, equidistant spatial grid

$$\big(\mathcal{L}_{\Delta x}(\vec{\mu})\big)_j = \underbrace{-\frac{v}{\Delta x}(\mu_j - \mu_{j-1})}_{\text{upwind difference, } cf. \text{ (3.1.10)}} \quad , \quad \big(\widetilde{\mathcal{L}}_{\Delta x}(\vec{\mu})\big)_j = \underbrace{-\frac{v}{\Delta x}(\mu_{j+1} - \mu_j)}_{\text{downwind difference, } cf. \text{ (3.1.11)}} \ .$$

General recipe: $\widetilde{\mathcal{L}}_{\Delta x} \leftarrow (-1)\cdot$ discretization of $\frac{\partial}{\partial x}(-f(u))$

☞ 4-stage 4th-order classical Runge-Kutta method: SSP with $\boxed{c = \frac{2}{3}}$ assuming (3.3.43)

$$\begin{aligned}
\vec{\eta}_2 &= \vec{\mu} + \tfrac{1}{2}\mathcal{L}_{\Delta x}(\vec{\mu}) \;, \\
\vec{\eta}_3 &= \tfrac{1}{2}\vec{\mu} - \tfrac{1}{4}\widetilde{\mathcal{L}}_{\Delta x}(\vec{\mu}) + \tfrac{1}{2}\big(\vec{\eta}_2 + \Delta t\mathcal{L}_{\Delta x}(\vec{\mu})\big) \;, \\
\vec{\eta}_4 &= \tfrac{1}{9}\big(\vec{\mu} - \Delta t\widetilde{\mathcal{L}}_{\Delta x}(\vec{\mu})\big) + \tfrac{2}{9}\big(\vec{\eta}_2 - \tfrac{3}{2}\Delta t\widetilde{\mathcal{L}}_{\Delta x}(\vec{\eta}_2)\big) + \tfrac{2}{3}\big(\vec{\eta}_3 + \tfrac{3}{2}\Delta t\mathcal{L}_{\Delta x}(\vec{\eta}_3)\big) \;, \\
\vec{\eta}_5 &= \tfrac{1}{3}\big(\vec{\eta}_2 + \tfrac{1}{2}\Delta t\mathcal{L}_{\Delta x}(\vec{\eta}_2)\big) + \tfrac{1}{3}\vec{\eta}_3 + \tfrac{1}{3}\big(\vec{\eta}_4 + \tfrac{1}{2}\mathcal{L}_{\Delta x}(\vec{\eta}_4)\big) \;,
\end{aligned} \tag{3.3.44}$$

$$\mathsf{T}_{\Delta t}(\vec{\mu}) := \vec{\eta}_5 \;.$$

3.4 Finite volume methods for 2D scalar conservation laws

➨ notation for independent spatial variables $\boldsymbol{x} = (x,y)^T \in \Omega \subset \mathbb{R}^2$

Focus: Cauchy problem ($\Omega = \mathbb{R}^2$) for two-dimensional scalar conservation law

$$\frac{\partial u}{\partial t} + \operatorname{div}_{\boldsymbol{x}} \mathbf{F}(u,\boldsymbol{x}) = \frac{\partial u}{\partial t} + \frac{\partial f_x(u,\boldsymbol{x})}{\partial x} + \frac{\partial f_y(u,\boldsymbol{x})}{\partial y} = 0 \quad \text{in } \mathbb{R}^2\times]0,T[\;, \tag{3.4.1}$$
$$u(x,y,0) = u_0(x,y) \quad \forall (x,y) \in \mathbb{R}^2 \;.$$

Theory (for $\mathbf{F}(u,\boldsymbol{x}) = \mathbf{F}(u)$): uniqueness, existence, $L^1(\mathbb{R}^2)$-, $L^\infty(\mathbb{R}^2)$-, $TV_{\mathbb{R}^2}$-stability of entropy solutions ($\longrightarrow$ Sect. 2.7)

Most important example: (non-constant) linear advection (2.1.4), $\mathbf{F}(u) = u\mathbf{v}(\boldsymbol{x})$

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(v_x(\boldsymbol{x})u) + \frac{\partial}{\partial y}(v_y(\boldsymbol{x})u) = 0 \quad \text{in } \mathbb{R}^2\times]0,T[\ . \tag{3.4.2}$$

Popular test case: "2D" Burgers' equation: $\mathbf{F}(u) = \frac{1}{2}u^2\boldsymbol{d}$, $\boldsymbol{d} \in \mathbb{R}^2$, $|\boldsymbol{d}| = 1$

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(\tfrac{1}{2}u^2 d_1) + \frac{\partial}{\partial y}(\tfrac{1}{2}u^2 d_2) = 0 \quad \text{in } \mathbb{R}^2\times]0,T[\ . \tag{3.4.3}$$

$\longleftrightarrow$ decoupled 1D Cauchy problems for Burgers equation (2.1.7): $\begin{array}{l} x' = d_1 x + d_2 y, \\ y' = d_2 x - d_1 y \end{array}$

$$(3.4.3) \quad \Longleftrightarrow \quad \frac{\partial u}{\partial t} + \frac{\partial}{\partial x'}(\tfrac{1}{2}u^2) = 0 \quad \text{in } \mathbb{R}^2\times]0,T[\ .$$

3.4.1 Operator splitting

$\mathcal{S}(t) : L^\infty(\mathbb{R}^2) \mapsto L^\infty(\mathbb{R}^2) \hat{=}$ evolution operator for Cauchy problem (3.4.1):

$$\mathcal{S}(t)u_0 := u(\cdot, t) \;, \quad u \text{ is entropy solution of (3.4.1).}$$

3.4.1.1 Fractional step semi-discretization

Formal "ODE in function spaces":

$$\text{(3.4.1)} \quad \Leftrightarrow \quad \frac{d}{dt}u = -\mathcal{L}_x u - \mathcal{L}_y u \;, \quad 0 < t < T \;, \quad u(0) = u_0 \;. \tag{3.4.4}$$

spatial differential operators: $\mathcal{L}_x \leftrightarrow \frac{\partial}{\partial x} f_x(u)$, $\mathcal{L}_y \leftrightarrow \frac{\partial}{\partial y} f_y(u)$

Motivation: (3.4.2), constant velocity $\mathbf{v}$:

$$\blacktriangleright \qquad \mathcal{L}_x\, u = v_x \frac{\partial}{\partial x} u \quad , \quad \mathcal{L}_y\, u = v_y \frac{\partial}{\partial y} u \;.$$

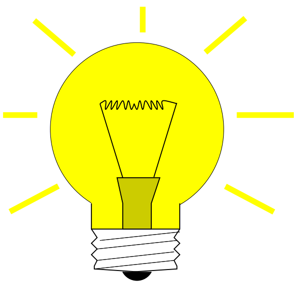
$\mathcal{L}_x$,$\mathcal{L}_y$ linear **&** for smooth u: $\mathcal{L}_x$,$\mathcal{L}_y$ commute

Consider linear commuting operators $\mathsf{A} : V \mapsto V$, $\mathsf{B} : V \mapsto V$, $\dim V < \infty$ and ODE

$$\frac{d}{dt}u = (\mathsf{A}+\mathsf{B})u \;,\; u(0) = u_0 \quad \Rightarrow \quad u(t) = \exp((\mathsf{A}+\mathsf{B})t)u(0) = \exp(\mathsf{A}t)\cdot\exp(\mathsf{B}t)u_0 \;.$$

evolution for $\frac{d}{dt}u = \mathsf{A}u$ evolution for $\frac{d}{dt}u = \mathsf{B}u$

“Algorithm”: first solve $\frac{d}{dt}u = \mathsf{A}u$, $u(0) = u_0$ ➔ u_1, then $\frac{d}{dt}u = \mathsf{B}u$, $u(0) = u_1$.



Idea: over small times (linearization)

(3.4.1) ≈ (3.4.2) with constant velocity

▶ fractional step temporal semidiscretization of (3.4.4)

Given temporal grid $\mathcal{G}_{\Delta t} = \{0 = t_0 < t_1 < \cdots < t_M = T\}$ compute approximation $u_{\Delta t}^{(k)}$ of $u(t_k)$ from approximation $u_{\Delta t}^{(k-1)}$ of $u(t_{k-1})$ by

$$u_{\Delta t}^{(k)} = \big(\mathcal{S}_x(\Delta t_k)\circ\mathcal{S}_y(\Delta t_k)\big)u_{\Delta t}^{(k-1)} \;, k = 1,\dots,M \quad , \quad u_{\Delta t}^{(0)} = u_0 \;, \qquad \text{(3.4.5)}$$

$\mathcal{S}_{x/y}(t) : L^\infty(\mathbb{R}^2) \mapsto L^\infty(\mathbb{R}^2) \mathrel{\hat{=}}$ evolution operator for $\frac{d}{dt}u = -\mathcal{L}_{x/y}\,u$.

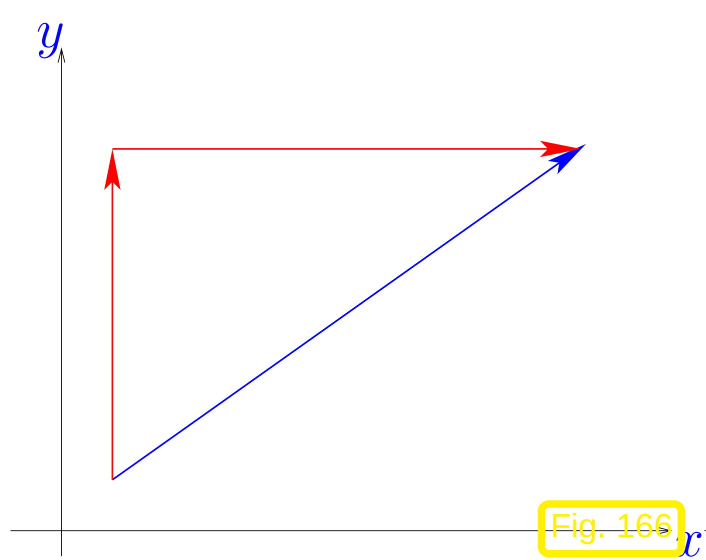
Terminology: (3.4.5) $\leftrightarrow$ fractional step Godunov splitting: $\boxed{\mathcal{S}(\Delta t) \approx \mathcal{S}_x(\Delta t) \circ \mathcal{S}_y(\Delta t)}$

Alternative: fractional step Strang splitting: $\boxed{\mathcal{S}(\Delta t) \approx \mathcal{S}_x(\tfrac{1}{2}\Delta t) \circ \mathcal{S}_y(\Delta t) \circ \mathcal{S}_x(\tfrac{1}{2}\Delta t)}$

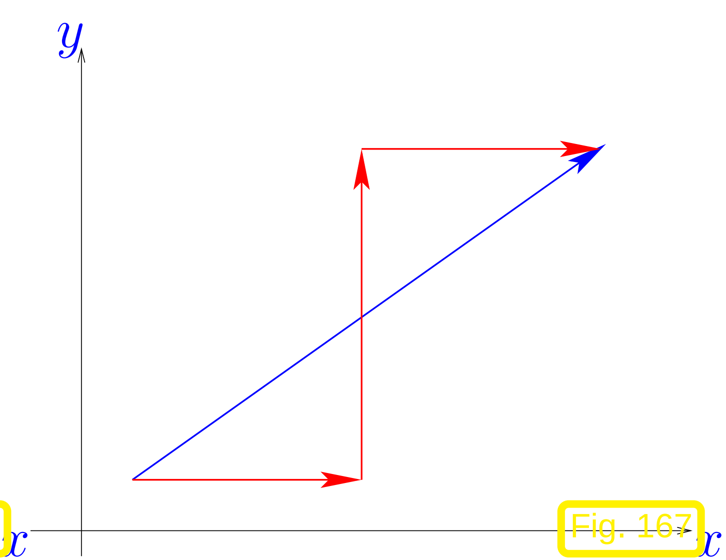
$$u_{\Delta t}^{(k)} = \big(\mathcal{S}_x(\tfrac{1}{2}\Delta t_k) \circ \mathcal{S}_y(\Delta t_k) \circ \mathcal{S}_x(\tfrac{1}{2}\Delta t_k)\big) u_{\Delta t}^{(k-1)} \,, k = 1, \dots, M \quad , \quad u_{\Delta t}^{(0)} = u_0 \,, \tag{3.4.6}$$

Splitting approaches applied to (3.4.4) ➪ dimensional splitting (separation of x/y-directions)

Note:
dimensional splitting exact for constant linear advection (= (3.4.2) with $\mathbf{v}(\boldsymbol{x}) = \mathbf{v}_0$)



Godunov splitting (3.4.5)



Strang splitting (3.4.5)

➨ notation: piecewise constant in time reconstruction: $\mathcal{C}u_{\Delta t}(t) = u^{(k-1)}$ for $t_{k-1} < t \leq t_k$, $k = 1, \dots .M$

Theorem 3.4.1 (Convergence of fractional step temporal semidiscretization). $\longrightarrow$ *[6]*
If $u_0 \in L^\infty(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$ *and* $(\mathcal{G}_{\Delta t,l})_{l\in\mathbb{N}}$ *is a sequence of temporal grids with maximal timestep* $\max_k \Delta t_k \to 0$ *for* $l \to \infty$*, then*

$$\mathcal{C}u^l_{\Delta t} \to u \quad \textit{in} \quad C^0([0,T], L^1_{\text{loc}}(\mathbb{R}^2)) \quad \textit{for} \quad l \to \infty \;,$$

where u *solves* (3.4.1)*, and* $u^l_{\Delta t}$ *is obtained by either* (3.4.5) *of* (3.4.6) *on* $\mathcal{G}^l_{\Delta x}$*.*

Sketch of proof. Show that $\mathcal{C}u^l_{\Delta t}$ is l-uniformly bounded in $L^\infty(\mathbb{R}^2\times]0,T[)$ and $BV_{\text{loc}}(\mathbb{R}^2\times]0,T[)$ and satisfies weak entropy inequality ($\longrightarrow$ Def. 2.5.3). Then use compactness argument ($\longrightarrow$ Thm. 3.2.10) and uniqueness of entropy solution. □

Quantitative convergence estimate, *cf.* Thm. 3.2.24:

Theorem 3.4.2 (Convergence rate of fractional step temporal semidiscretization). $\longrightarrow$ *[45]*
Let $u_0 \in L^\infty(\mathbb{R}^2) \cap BV_{\text{loc}}(\mathbb{R}^2)$ **+** *assumptions/notations of Thm. 3.4.1. Then solutions* $(u^{(k)}_{\Delta t})_{k=0,\dots,M}$ *of* (3.4.5) *or* (3.4.6) *on equidistant temporal grids with timestep* $\Delta t := T/M$ *satisfy*

$$\exists C \neq C(\Delta t): \quad \max_{1\leq k\leq M} \left\| u(\cdot, t_k) - u^{(k)}_{\Delta t} \right\|_{L^1(\mathbb{R}^2)} \leq C\sqrt{\Delta t} \;.$$

Formal view: regard (3.4.5)/(3.4.6) as explicit single step timestepping method ($\longrightarrow$ Def. 3.3.13) for (3.4.4)

What is its order ($\longrightarrow$ Def. 3.3.14) **?**

Abstract: $\mathcal{A}, \mathcal{B} : V \mapsto V$ continuous mappings with uniformly bounded Frechet derivatives ($V =$ Banach space),
$\mathcal{S}_A(\mathcal{S}_B) :]0, T[\times V \mapsto V =$ evolution operator for $\frac{d}{dt}u = \mathcal{A}u$/ $\frac{d}{dt}u = \mathcal{B}u$,
$\mathcal{S} :]0, T[\times V \mapsto V =$ evolution operator for $\frac{d}{dt}u = (\mathcal{A} + \mathcal{B})u$.

Theorem 3.4.3 (Order of fractional step temporal semi-discretizations)**.**

$$\begin{aligned} \|(\mathcal{S}(\Delta t) - \mathcal{S}_A(\Delta t)\mathcal{S}_B(\Delta t))u\| &\leq C(\Delta t)^2 \;, \\ \left\|(\mathcal{S}(\Delta t) - \mathcal{S}_A(\tfrac{1}{2}\Delta t)\mathcal{S}_B(\Delta t)\mathcal{S}_A(\tfrac{1}{2}\Delta t))u\right\| &\leq C(\Delta t)^3 \;, \end{aligned} \qquad \textit{for } \Delta t \to 0 \;,$$

with $C > 0$ independent of Δt and $u \in V$.

Godunov splitting (3.4.5) ➠ first-order consistent
Strang splitting (3.4.6) ➠ second-order consistent

Remark 91. Splitting approach important for constructing integrators for ODEs with special properties, [33] and [19, Sect. II.5].

△

3.4.1.2 Discrete dimensional splitting schemes

Full discretization on infinite space-time tensor product grid: $\mathcal{M} = \mathcal{G}_{\Delta x} \times \mathcal{G}_{\Delta y} \times \mathcal{G}_{\Delta t}$

$$\mathcal{G}_{\Delta x} := \{x_i \in \mathbb{R}\colon x_{i-1} < x_i, i \in \mathbb{Z}\} \quad , \quad \mathcal{G}_{\Delta y} := \{y_j \in \mathbb{R}\colon y_{j-1} < y_j, j \in \mathbb{Z}\} \; .$$

Equidistant case: meshwidths $x_i - x_{i-1} = \Delta x > 0$, $y_j - y_{j-1} = \Delta y > 0 \; \forall j$.

➨ adapt notation for grid functions $\in C^0(\mathcal{G}_{\Delta x} \times \mathcal{G}_{\Delta y}) := \{\mathcal{G}_{\Delta x} \times \mathcal{G}_{\Delta y} \mapsto \mathbb{R}\}$: $\vec{\mu}$, $\vec{\eta}$, etc.
$\vec{\mu} \sim (\mu_{ij})_{i,j\in\mathbb{Z}}$, new: partial grid function $\vec{\mu}_{\cdot,j} := (\mu_{ij})_{i\in\mathbb{Z}} \in C^0(\mathcal{G}_{\Delta x})$, etc.

grid lines and grid cells ▷

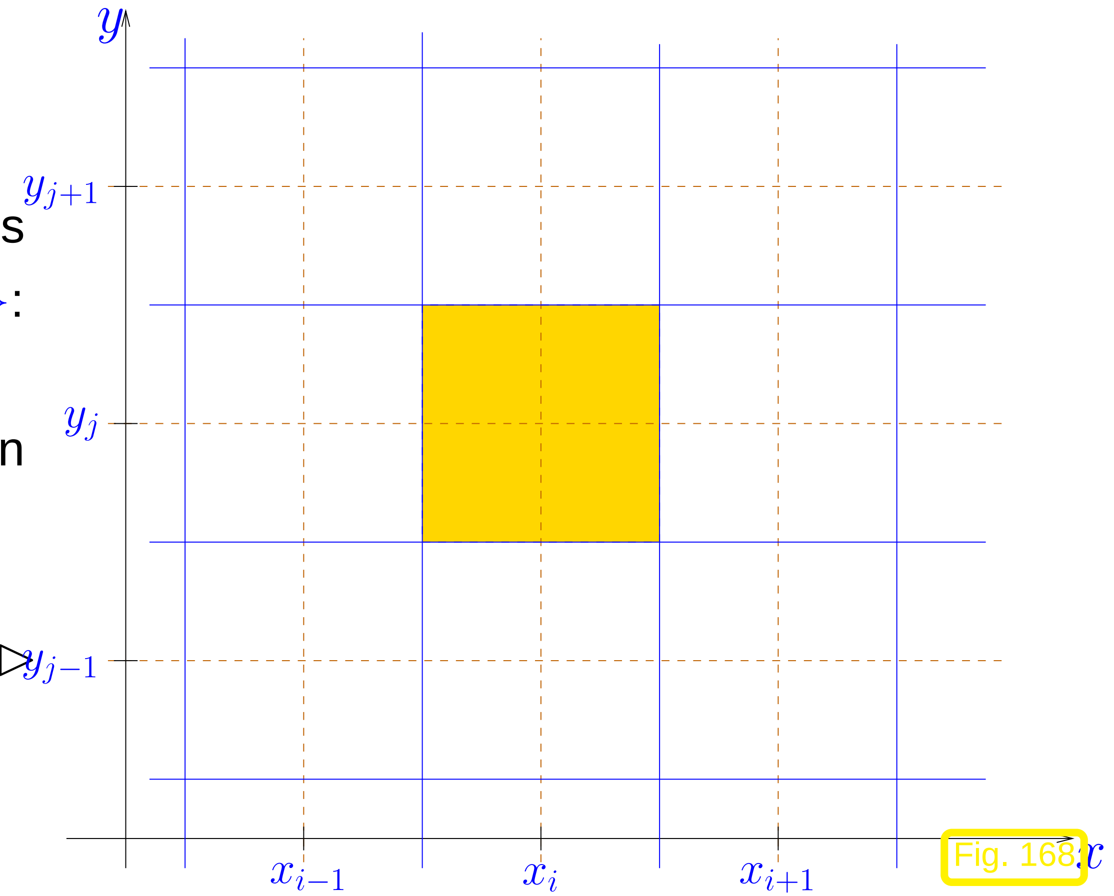


Fig. 168

Interpretation (⟶ Sect. 3.1): FVM for (3.4.1) ➢ approximations $\vec{\mu}^{(k)}$, $k = 1, \dots, M$, of cell averages

$$\mu_{j,i}^{(k)} \approx \frac{1}{\Delta x \Delta y} \int_{x_{i-1/2}}^{x_{i+1/2}} \int_{y_{j-1/2}}^{y_{j+1/2}} u(x, y, t_k)\, \mathrm{d}y \mathrm{d}x \;, \quad i, j \in \mathbb{Z} \;. \tag{3.4.7}$$

Idea: in dimensional splitting approaches (3.4.5)/(3.4.6):

$$\mathcal{S}_x \quad \leftrightarrow \quad \frac{\partial}{\partial t}u(x,y,t) + \frac{\partial}{\partial x}(f_x(u(x,y,t))) = 0 \quad , \quad \mathcal{S}_y \quad \leftrightarrow \quad \frac{\partial}{\partial t}u(x,y,t) + \frac{\partial}{\partial y}(f_y(u(x,y,t))) = 0 \; .$$

$y \mathrel{\hat{=}}$ “parameter” $x \mathrel{\hat{=}}$ “parameter”

▶ given time-invariant discrete evolutions $\mathcal{H}_{x,\Delta t} : C^0(\mathcal{G}_{\Delta x}) \mapsto C^0(\mathcal{G}_{\Delta x})$, $\mathcal{H}_{y,\Delta t} : C^0(\mathcal{G}_{\Delta x}) \mapsto C^0(\mathcal{G}_{\Delta x})$, *for one-dimensional conservation laws*

$$\vec{\mu}^{(k)} = \mathcal{H}_{x,\Delta t}(\vec{\mu}^{(k-1)}) \quad \leftrightarrow \quad \frac{\partial}{\partial t}u(x,y,t) + \frac{\partial}{\partial x}(f_x(u(x,y,t))) = 0 \quad (y \text{ parameter}) \; ,$$
$$\vec{\mu}^{(k)} = \mathcal{H}_{y,\Delta t}(\vec{\mu}^{(k-1)}) \quad \leftrightarrow \quad \frac{\partial}{\partial t}u(x,y,t) + \frac{\partial}{\partial y}(f_y(u(x,y,t))) = 0 \quad (x \text{ parameter}) \; .$$

▼

Godunov splitting (3.4.5) ▶
$$\begin{cases} \vec{\mu}^*_{i,\cdot} = \mathcal{H}_{y,\Delta t}(\vec{\mu}^{(k-1)}_{i,\cdot}) \; , & i \in \mathbb{Z} \; , \\ \vec{\mu}^{(k)}_{\cdot,j} = \mathcal{H}_{x,\Delta t}(\vec{\mu}^*_{\cdot,j}) \; , & j \in \mathbb{Z} \; . \end{cases} \tag{3.4.8}$$

Strang splitting (3.4.6) ▶
$$\begin{cases} \vec{\mu}^1_{\cdot,j} = \mathcal{H}_{x,1/2\Delta t}(\vec{\mu}^{(k-1)}_{\cdot,j}) \; , & j \in \mathbb{Z} \; , \\ \vec{\mu}^2_{i,\cdot} = \mathcal{H}_{y,\Delta t}(\vec{\mu}^1_{i,\cdot}) \; , & i \in \mathbb{Z} \; , \\ \vec{\mu}^{(k)}_{\cdot,j} = \mathcal{H}_{x,1/2\Delta t}(\vec{\mu}^2_{\cdot,j}) \; , & j \in \mathbb{Z} \; . \end{cases} \tag{3.4.9}$$

Consider special case: $\mathcal{H}_{y,\Delta t}$, $\mathcal{H}_{x,\Delta t}$ from finite volume method (→ Def. 3.2.1)

F_x, F_y $\hat{=}$ numerical flux functions consistent with f_x, f_y (→ Def. 3.2.2)

▶ Finite volume fractional step method based on Godunov splitting (3.4.5) (on equidistant mesh)

$$\begin{aligned}
\mu^*_{ji} &= \mu^{(k-1)}_{ji} - \frac{\Delta t}{\Delta y}\left(F_x(\mu_{i,j-m_l+1},\dots,\mu_{i,j+m_r}) - F_x(\mu_{i,j-m_l},\dots,\mu_{i,j+m_r-1})\right) \ , \quad i,j \in \mathbb{Z} \ , \\
\mu^{(k)}_{i,j} &= \mu^*_{i,j} - \frac{\Delta t}{\Delta x}\left(F_y(\mu^*_{i-m_l+1,j},\dots,\mu^*_{i+m_r,j}) - F_y(\mu^*_{i-m_l,j},\dots,\mu^*_{i+m_r-1,j})\right) \ , \quad i,j \in \mathbb{Z} \ .
\end{aligned} \tag{3.4.10}$$

[29, Sect. 3.1]: convergence result analoguous to Sects. 3.2.6, 3.2.7:

Theorem 3.4.4 (Convergence of 2D fractional step FVM)**.**
If F_x *and* F_y *give rise to monotone (→ Def. 3.1.14) FVM, cf. Lemma 3.2.7, and* $\Delta t/\Delta x$, $\Delta t/\Delta y$ *are fixed and sufficiently small, then*

$$u_{\Delta t} \rightarrow u \quad \textit{in} \quad L^1_{\mathrm{loc}}(\mathbb{R}^2 \times]0,T[) \quad \textit{for} \quad \Delta t \rightarrow 0 \ ,$$

where u *solves* (3.4.1) *and* $u_{\Delta t}$ *is the* $\mathcal{M}$*-p.w. constant reconstruction of* $\vec{\mu}^{(k)}$ *obtained by* (3.4.10) *on equidistant space-time mesh with timestep* Δt*.*

Note: Thm. 3.2.24 carries over to 2D ➤ "$O(\sqrt{\Delta t})$-convergence" of monotone schemes

Example 92 (2D dimensionally split FVM).

- Cauchy problem for constant advection (3.4.2), $\mathbf{v} = \binom{1}{1}$
- initial data $u_0(\boldsymbol{x}) = 1 - \cos^2(\pi|\boldsymbol{x} + \binom{1/2}{1/2}|)$ for $|\boldsymbol{x} + \binom{1/2}{1/2}| < \frac{1}{2}$, $u_0(\boldsymbol{x}) = 0$ elsewhere.
- dimensional splitting based on different 1D finite volume methods ($\gamma_x = \gamma_y = 1$):
  1. upwind scheme (3.1.26),
  2. Lax-Friedrichs (3.1.29), see also (3.2.9),
  3. Lax-Wendroff 2nd-order FVM (3.1.12),
  4. minmod-limited high resolution method (3.3.8),
  5. superbee-limited high resolution method (76)

  combined with Godunov splitting (3.4.8)/Strang splitting (3.4.9).

Monitored: l^1 and l^∞-errors at final time $T = 1$ for $\Delta x, \Delta y \in \{\frac{1}{5}, \frac{1}{10}, \frac{1}{20}, \frac{1}{20}, \frac{1}{40}, \frac{1}{80}, \frac{1}{160}, \frac{1}{320}\}$
➤ approximate order of convergence, *cf.* Ex. 79

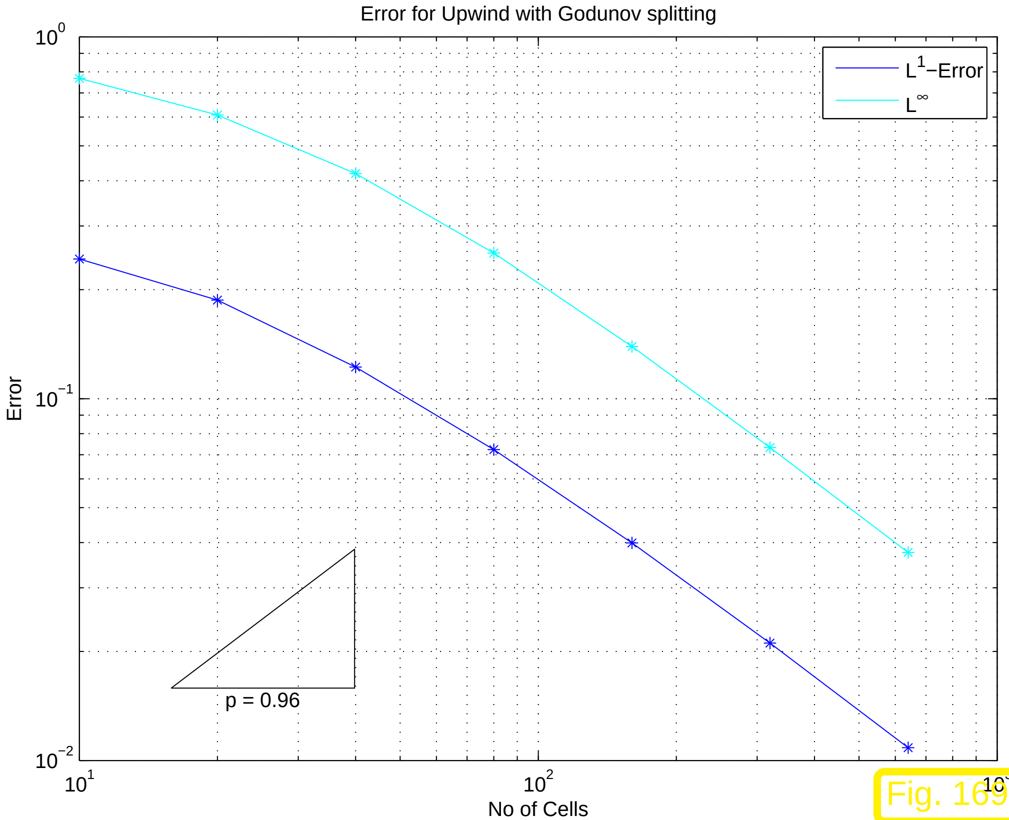


Fig. 169

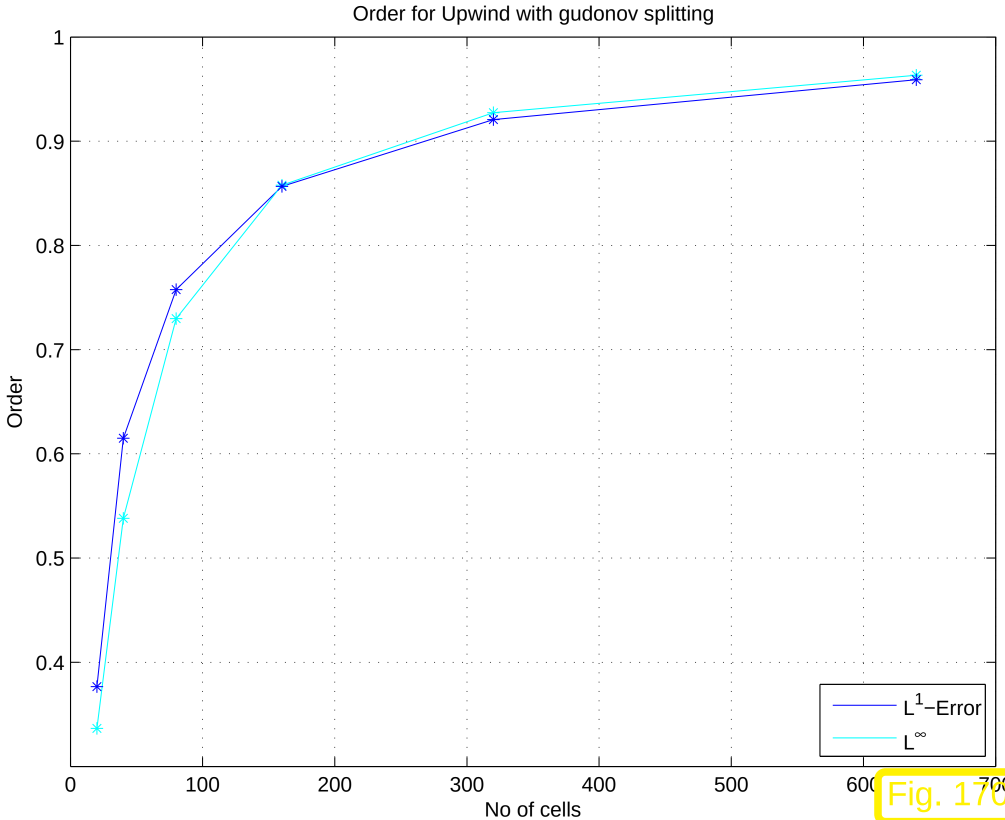


Fig. 170

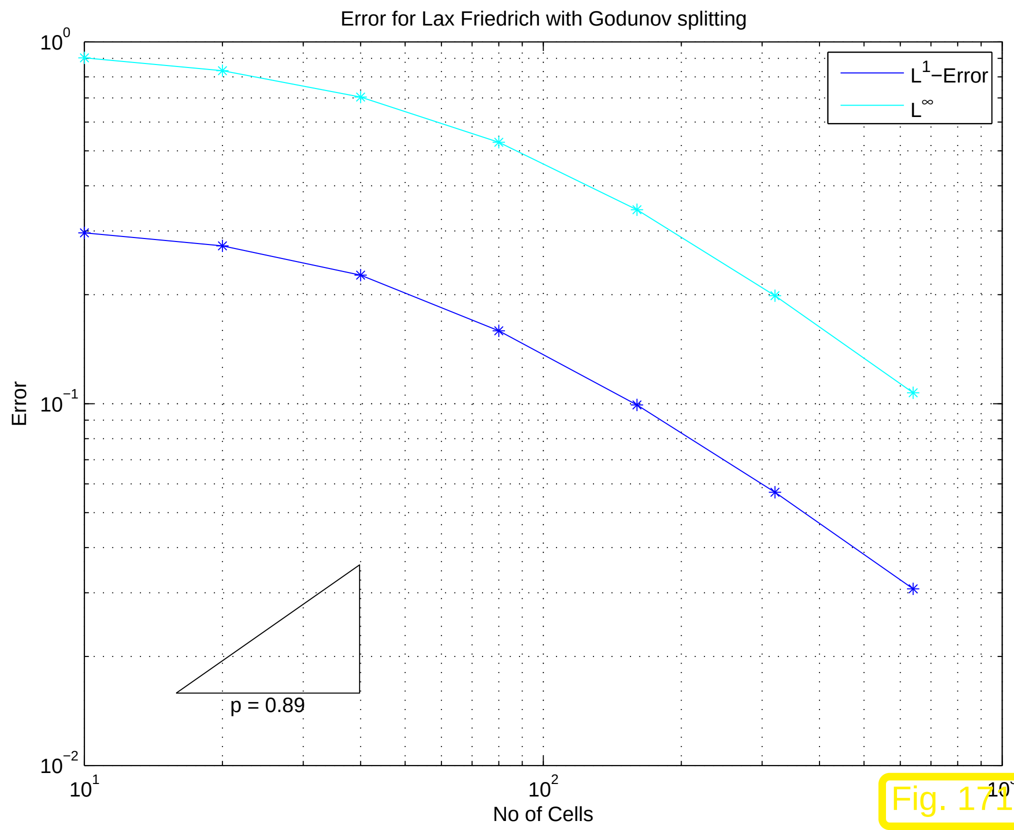


Fig. 171

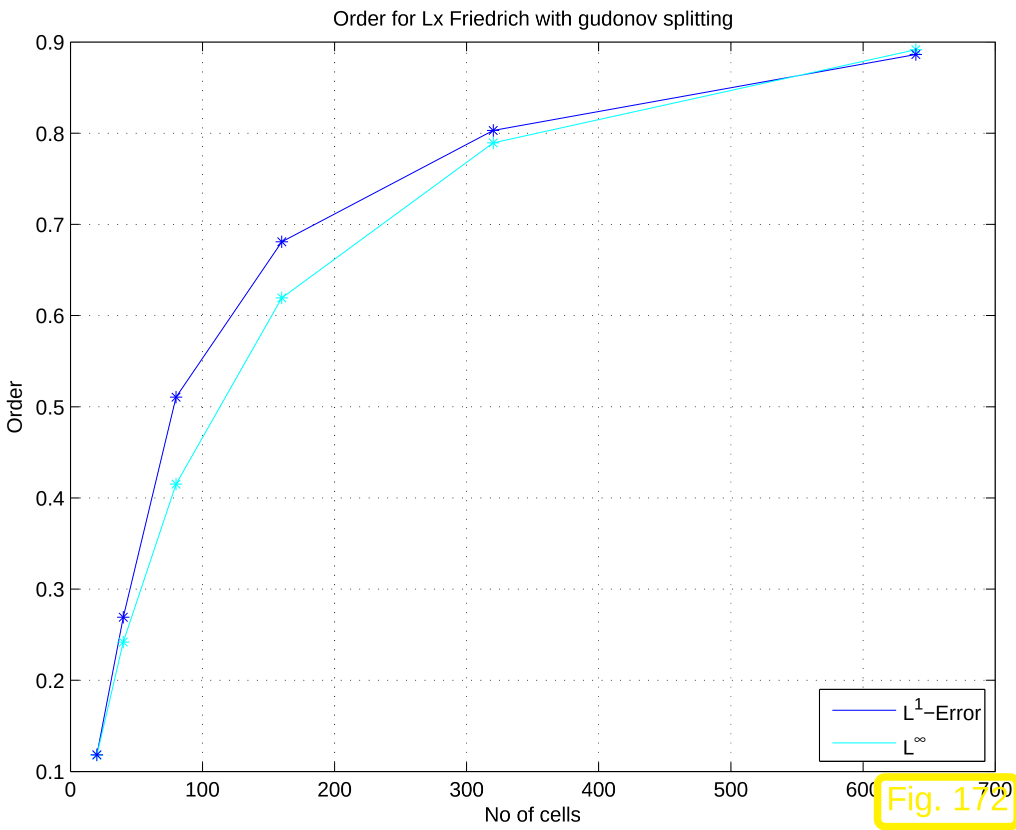


Fig. 172

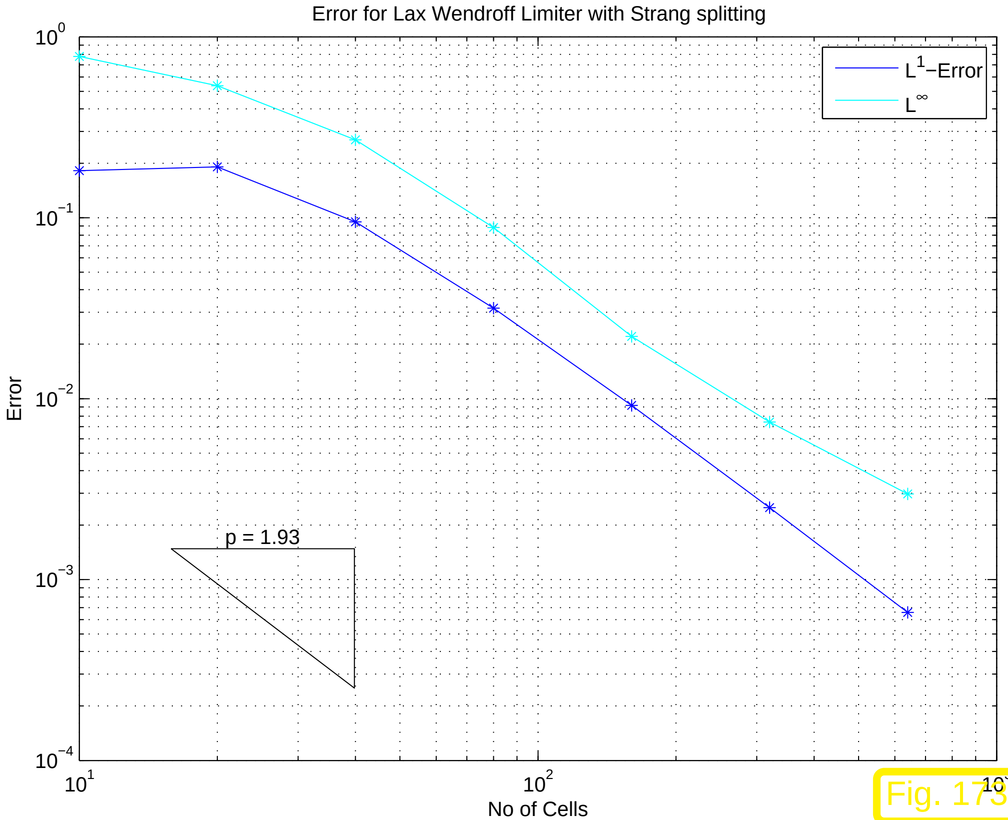


Fig. 173

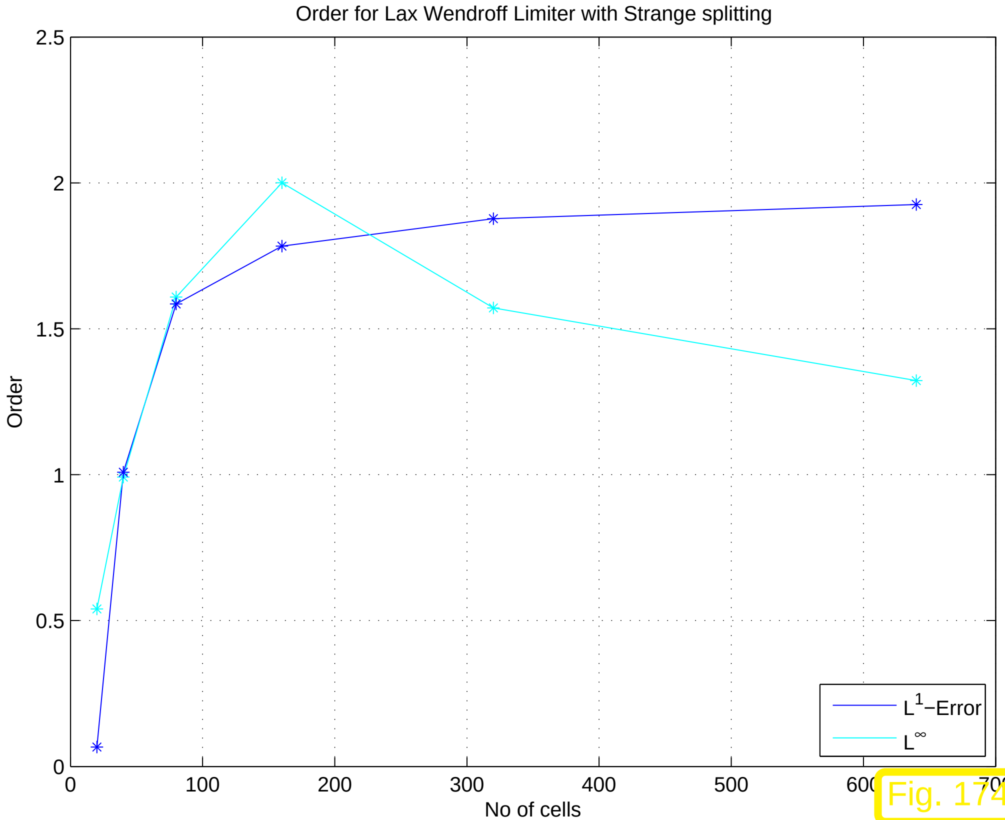


Fig. 174

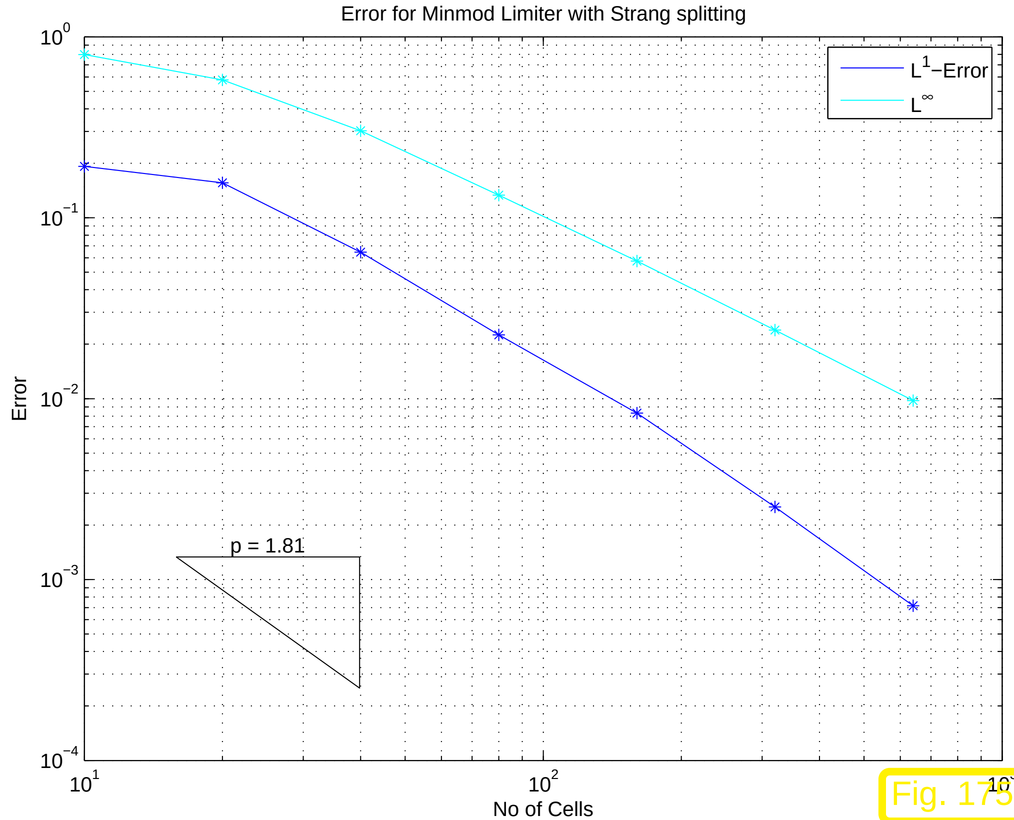


Fig. 175

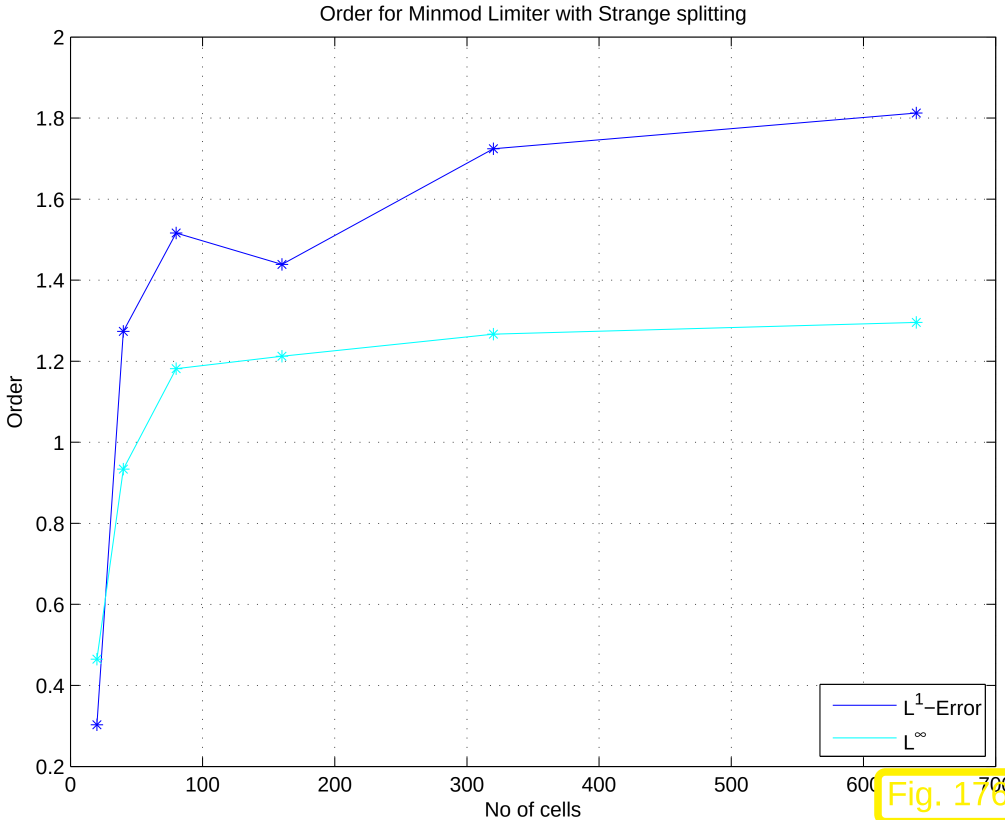


Fig. 176

3.4

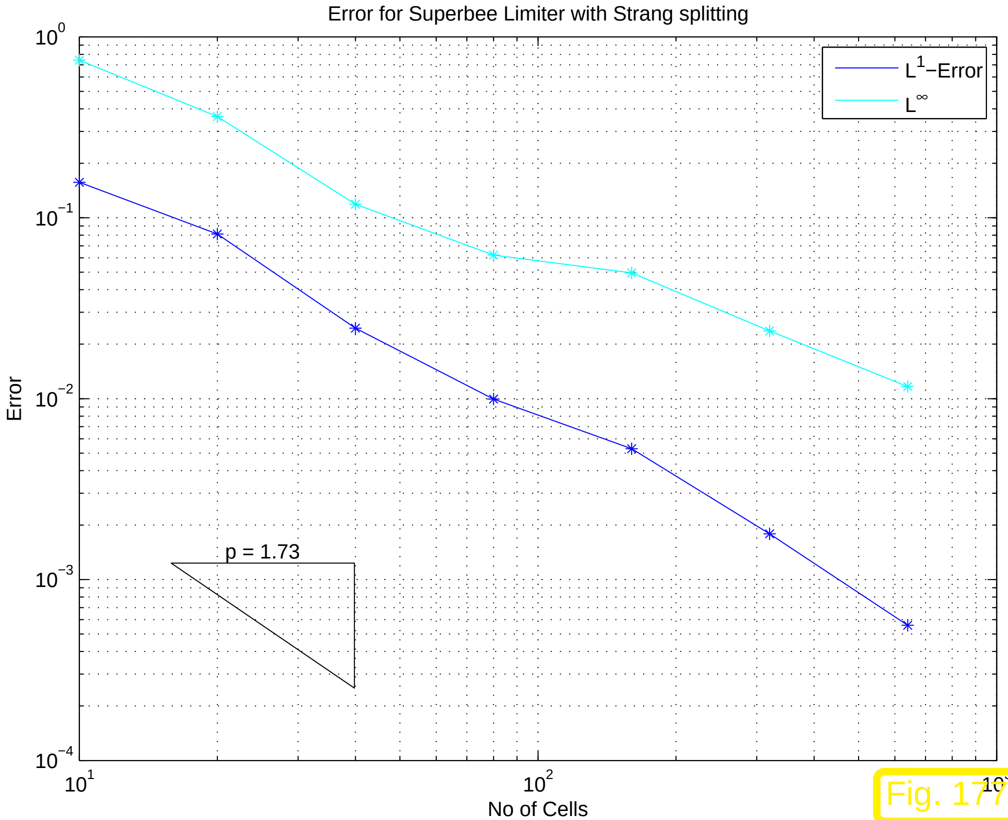


Fig. 177

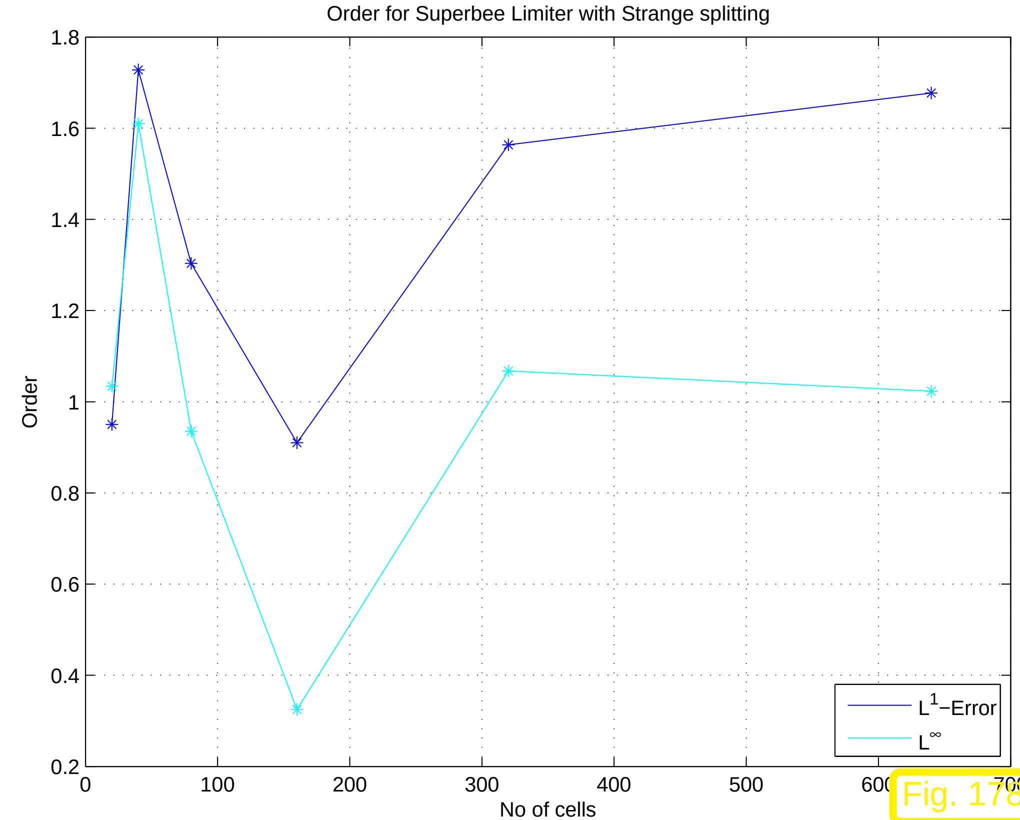


Fig. 178

1st-order 2nd-order	FV (for partial evolutions) **+**	Godunov Strang	splitting	➡	1st-order 2nd-order	scheme

◇

3.4.2 Corner transport upwinding

Given ($\longrightarrow$ Sect. 3.4.1.2): infinite space-time tensor product grid: $\mathcal{M} = \mathcal{G}_{\Delta x} \times \mathcal{G}_{\Delta y} \times \mathcal{G}_{\Delta t}$

$$\mathcal{G}_{\Delta x} := \{x_i \in \mathbb{R}\colon x_{i-1} < x_i, i \in \mathbb{Z}\} \quad , \quad \mathcal{G}_{\Delta y} := \{y_j \in \mathbb{R}\colon y_{j-1} < y_j, j \in \mathbb{Z}\} \ .$$

Focus: equidistant case: meshwidths $x_j - x_{j-1} = \Delta x > 0$, $y_j - y_{j-1} = \Delta y > 0 \ \forall j$, fixed ratios $\gamma_x := \Delta t / \Delta x$, $\gamma_y := \Delta t / \Delta y$.

Goal: update formula for cell averages
$$\mu_{j,i}^{(k)} \approx \frac{1}{\Delta x \Delta y} \int\limits_{x_{j-1/2}}^{x_{j+1/2}} \int\limits_{y_{i-1/2}}^{y_{i+1/2}} u(x, y, t_k)\,\mathrm{d}x\mathrm{d}y \ , \quad j, i \in \mathbb{Z} \ .$$

3.4.2.1 Constant linear advection

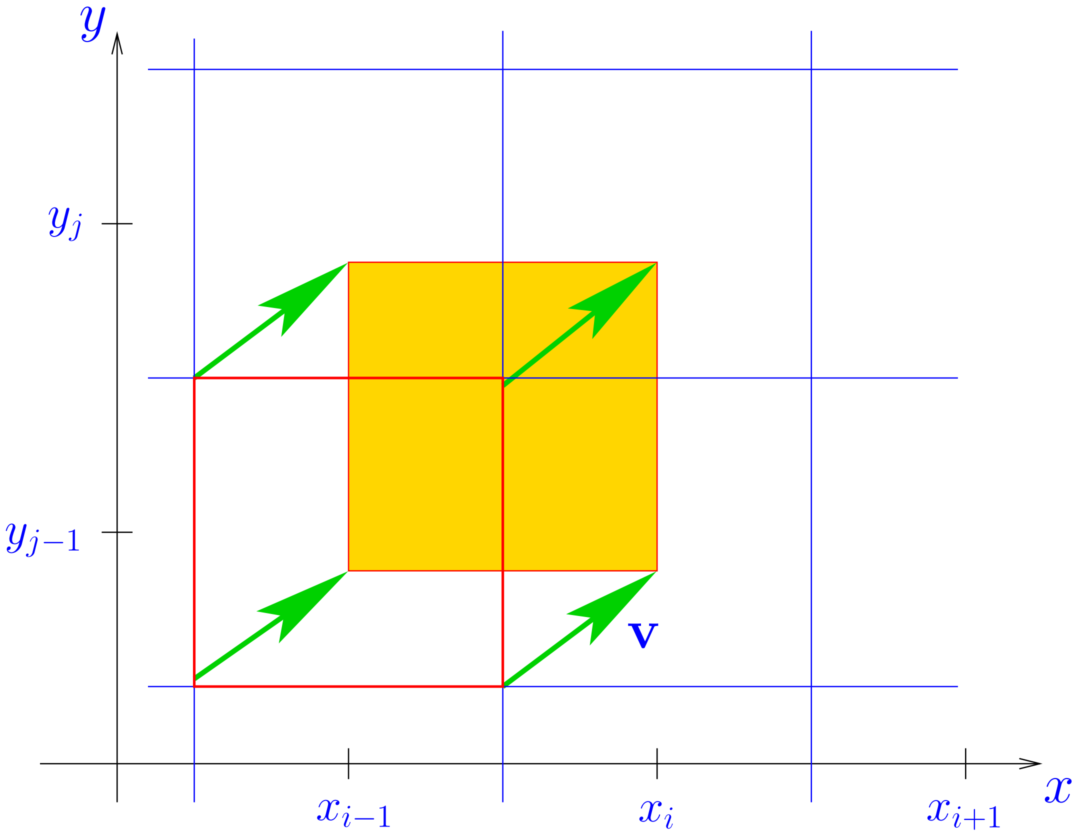
Cauchy problem (3.4.1) with $\mathbf{F}(u) = \mathbf{v}\, u$, $\mathbf{v} = (v_x, v_y)^T \in \mathbb{R}^2$ ($\longrightarrow$ Ex. 29)

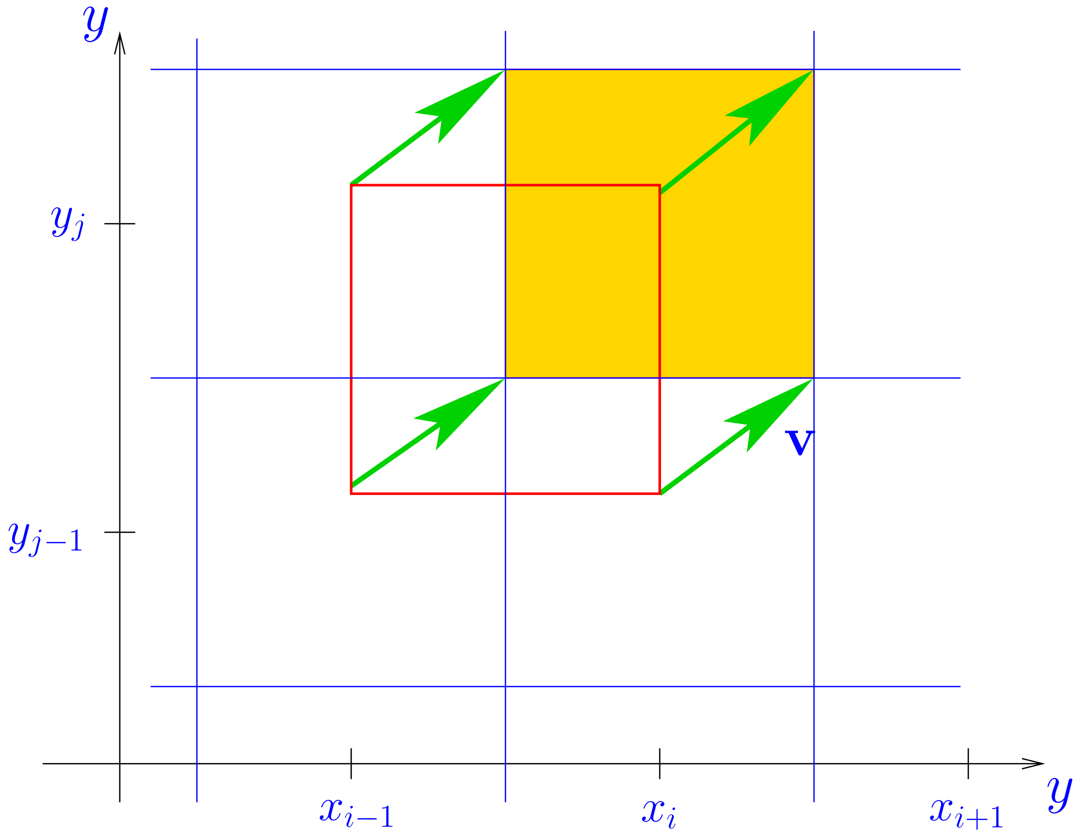
▶ solution $\boxed{u(\boldsymbol{x}, t) = u_0(\boldsymbol{x} - \mathbf{v}t),}$ $\boldsymbol{x} \in \mathbb{R}^2$, $0 \leq t \leq T$.

Approach: REA-algorithm with $\mathcal{G}_{\Delta x} \times \mathcal{G}_{\Delta y}$-constant reconstruction:
($\longrightarrow$ Godunov's method, Sect. 3.2.2)

given $\vec{\mu}^{(k-1)}$ ➢ $w_0(x,y) = \mu_{i,j}^{(k-1)}$ for $\begin{array}{l} x_{i-1/2} < x < x_{i+1/2} \,, \\ y_{j-1/2} < y < y_{j+1/2} \,. \end{array}$

$$\mu_{i,j}^{(k)} = \frac{1}{\Delta x \Delta y} \int\limits_{x_{i-1/2}}^{x_{i+1/2}} \int\limits_{y_{j-1/2}}^{y_{j+1/2}} w_0(x - v_x \Delta t, y - v_y \Delta t) \, \mathrm{d}x \mathrm{d}y \;, \quad i,j \in \mathbb{Z} \,. \tag{3.4.11}$$



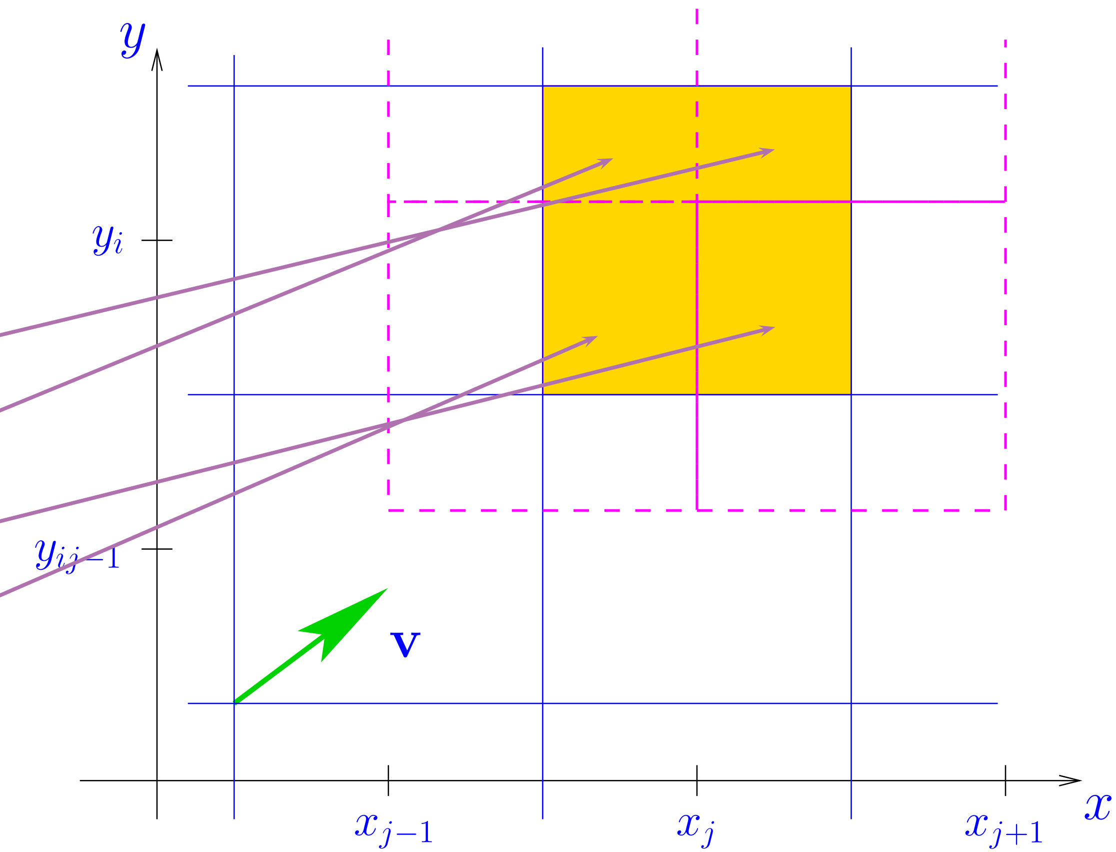


Assume: $\boxed{\text{CFL-condition} \quad \left|v_x \frac{\Delta t}{\Delta x}\right| = \gamma_x v_x \leq 1, \quad \left|v_y \frac{\Delta t}{\Delta y}\right| = \gamma_y v_y \leq 1}$ (→ Def. 3.1.4)

➡ relative shifts in Δt: $c_{x/y} := \gamma_{x/y} v_{x/y}$

For $v_x \geq 0$, $v_y \geq 0$: (3.4.11) $\Rightarrow$

$$\begin{aligned}\mu_{i,j}^{(k)} = \; & (1-c_x)(1-c_y)\,\mu_{i,j}^{(k-1)} + \\ & c_x(1-c_y)\,\mu_{i-1,j}^{(k-1)} + \\ & (1-c_x)c_y\,\mu_{i,j-1}^{(k-1)} + \\ & c_x c_y\,\mu_{i-1,j-1}^{(k-1)} \,.\end{aligned}$$



▶ corner transport upwind (CTU) scheme

$$\mu_{i,j}^{(k)} = \mu_{i,j}^{(k-1)} - \overbrace{\gamma_x v_x(\mu_{i,j}^{(k-1)} - \mu_{i-1,j}^{(k-1)}) - \gamma_y v_y(\mu_{i,j}^{(k-1)} - \mu_{i,j-1}^{(k-1)})}^{\text{upwind finite differences}} + \underbrace{c_x c_y(\mu_{i,j}^{(k-1)} - \mu_{i,j-1}^{(k-1)} - \mu_{i-1,j}^{(k-1)} + \mu_{i-1.j-1}^{(k-1)})}_{\text{corner transport correction}} \qquad (3.4.12)$$

(3.4.12) $\overset{\text{CFL-condition}}{\Rightarrow}$ monotone discrete evolution ($\longrightarrow$ Def. 3.1.14)

3.4.3 Non-constant advection

Cauchy problem: advection of an intensive quantity (no conservation law **!**):

$$\frac{\partial u}{\partial t} + \mathbf{v}(\boldsymbol{x}) \cdot \mathbf{grad}_{\boldsymbol{x}}\, u = \frac{\partial u}{\partial t} + v_x(\boldsymbol{x})\frac{\partial u}{\partial x} + v_y(\boldsymbol{x})\frac{\partial u}{\partial y} = 0 \quad \text{in } \mathbb{R}^2 \times]0,T[\ ,$$
$$u(\boldsymbol{x},0) = u_0(\boldsymbol{x}) \quad \forall \boldsymbol{x} \in \mathbb{R}^2\ . \tag{3.4.13}$$

Cauchy problem: advection of an extensive quantity $\longrightarrow$ Ex. 29, (2.1.4)

$$\frac{\partial u}{\partial t} + \operatorname{div}_{\boldsymbol{x}}(u\,\mathbf{v}) = \frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(v_x(\boldsymbol{x})u) + \frac{\partial}{\partial y}(v_y(\boldsymbol{x})u) = 0 \quad \text{in } \mathbb{R}^2 \times]0,T[\ ,$$
$$u(\boldsymbol{x},0) = u_0(\boldsymbol{x}) \quad \forall \boldsymbol{x} \in \mathbb{R}^2\ . \tag{3.4.14}$$

For (3.4.14) **assume**: incompressible flow: $\operatorname{div} \mathbf{v} = 0$

If $\mathbf{v} \in (C^0(\mathbb{R}^2))^2$, solutions of (3.4.13) and (3.4.14) (for $\operatorname{div} \mathbf{v} = 0$) constant along characteristic curves, *cf.* Def. 2.2.2,

$$\gamma : [0,T] \mapsto \mathbb{R}^2 \colon \quad \frac{d}{d\tau}\gamma(\tau) = \mathbf{v}(\gamma(\tau)) \;, \quad 0 \leq \tau \leq T \;.$$

How to generalize (3.4.12) to (3.4.13), (3.4.14) **?**

Fluctuation splitting form of (3.4.12):

$$\begin{aligned}
\mu_{i,j}^{(k)} = \mu_{i,j}^{(k-1)} & - c_x(1 - \tfrac{1}{2}c_y)(\mu_{i,j}^{(k-1)} - \mu_{i-1,j}^{(k-1)}) \\
& - c_y(1 - \tfrac{1}{2}c_x)(\mu_{i,j}^{(k-1)} - \mu_{i,j-1}^{(k-1)}) \\
& - \tfrac{1}{2}(c_x)(c_y)(\mu_{i-1,j}^{(k-1)} - \mu_{i-1,j-1}^{(k-1)}) \\
& - \tfrac{1}{2}(c_x)(c_y)(\mu_{i,j-1}^{(k-1)} - \mu_{i-1,j-1}^{(k-1)})
\end{aligned}$$

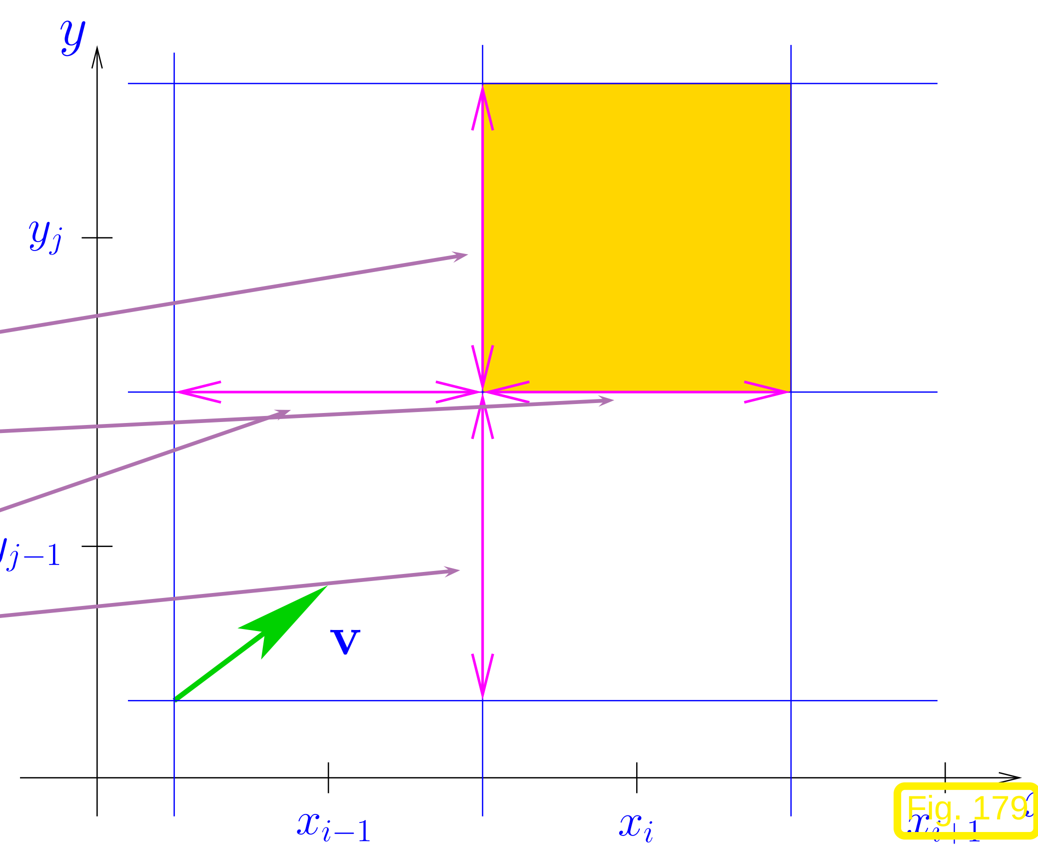


Fig. 179

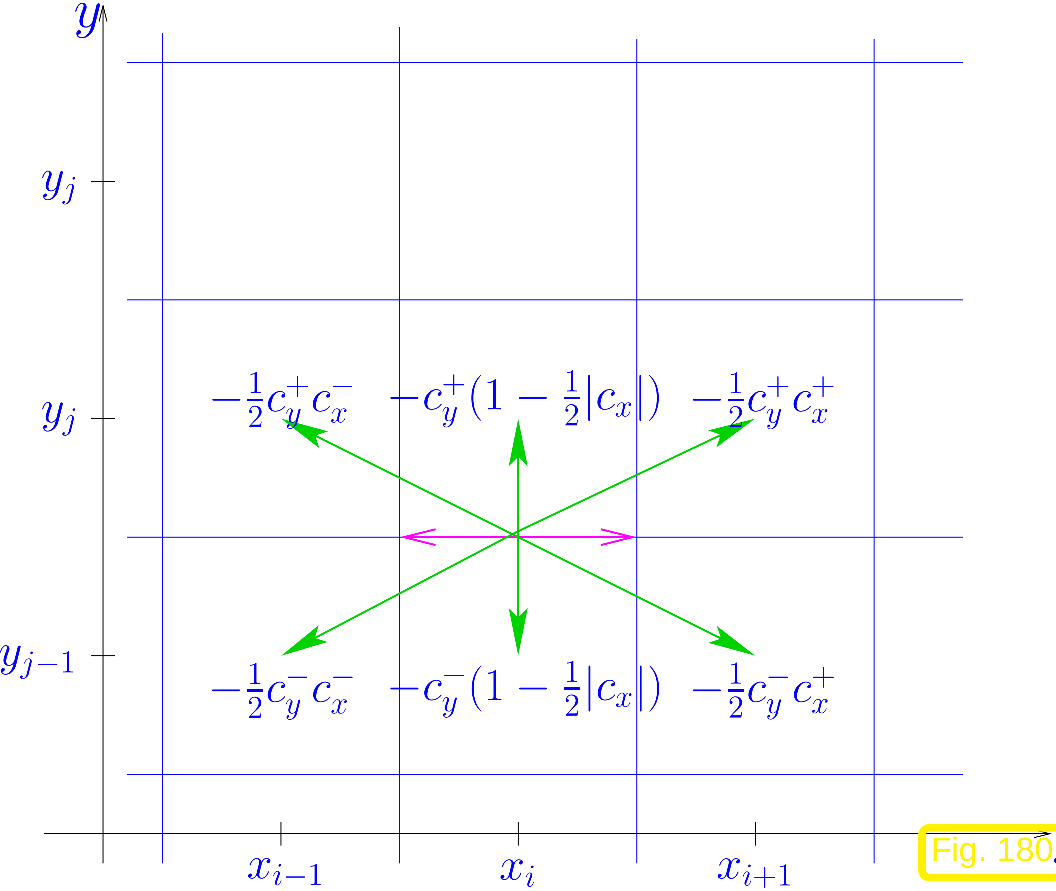


Fig. 180

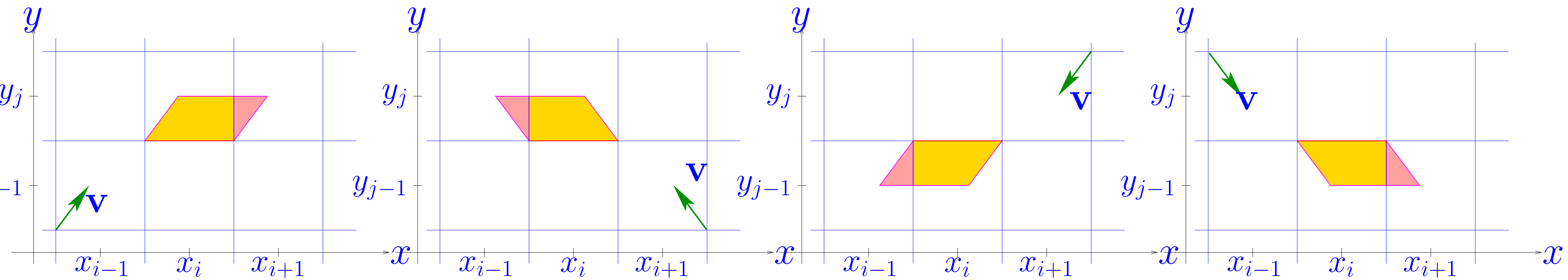
Dual view:

Distribution of edge fluctuations

$$\Delta\mu_{i,j-1/2}^{(k-1)} := \mu_{i,j}^{(k-1)} - \mu_{i,j-1}^{(k-1)} ,$$
$$\Delta\mu_{i+1/2,j}^{(k-1)} := \mu_{i+1,j}^{(k-1)} - \mu_{i,j}^{(k-1)} .$$

➡ $c^+ := \max\{0, c\}$, $c^- := \min\{0, c\}$

fluctuation distribution form

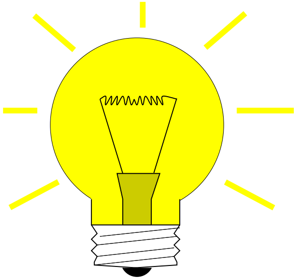


$v_x > 0$, $v_y > 0$ $\quad$ $v_x < 0$, $v_y > 0$ $\quad$ $v_x < 0$, $v_y < 0$ $\quad$ $v_x > 0$, $v_y < 0$

$$
\mu_{i,j}^{(k)} = \overbrace{\mu_{i,j}^{(k-1)} - c_x^+ \Delta\mu_{i-1/2,j} - c_x^- \Delta\mu_{i+1/2,j} - c_y^+ \Delta\mu_{i,j-1/2} - c_y^- \Delta\mu_{i,j+1/2}}^{\begin{aligned} = & -\gamma_x\big(F_{\mathrm{uw}}^x(\mu_{i-1,j}^{(k-1)},\mu_{i,j}^{(k-1)}) - F_{\mathrm{uw}}^x(\mu_{i,j}^{(k-1)},\mu_{i+1,j}^{(k-1)})\big) \\ & -\gamma_y\big(F_{\mathrm{uw}}^y(\mu_{i,j-1}^{(k-1)},\mu_{i,j}^{(k-1)}) - F_{\mathrm{uw}}^y(\mu_{i,j}^{(k-1)},\mu_{i,j+1}^{(k-1)})\big)\end{aligned}}
$$

$$
\begin{aligned}
& - \tfrac{1}{2}c_x^+ c_y^+ \Delta\mu_{i-1,j-1/2}^{(k-1)} - \tfrac{1}{2}c_x^+ c_y^+ \Delta\mu_{i-1/2,j-1}^{(k-1)} + \tfrac{1}{2}c_x^+ c_y^+ \Delta\mu_{i-1/2,j}^{(k-1)} + \tfrac{1}{2}c_x^+ c_y^+ \Delta\mu_{i,j-1/2}^{(k-1)} \\
& - \tfrac{1}{2}c_x^+ c_y^- \Delta\mu_{i-1,j+1/2}^{(k-1)} - \tfrac{1}{2}c_x^+ c_y^- \Delta\mu_{i-1/2,j+1}^{(k-1)} - \tfrac{1}{2}c_x^+ c_y^- \Delta\mu_{i-1/2,j}^{(k-1)} + \tfrac{1}{2}c_x^+ c_y^- \Delta\mu_{i,j+1/2}^{(k-1)} \\
& - \tfrac{1}{2}c_x^- c_y^+ \Delta\mu_{i+1,j-1/2}^{(k-1)} - \tfrac{1}{2}c_x^- c_y^+ \Delta\mu_{i+1/2,j-1}^{(k-1)} + \tfrac{1}{2}c_x^- c_y^+ \Delta\mu_{i+1/2,j}^{(k-1)} - \tfrac{1}{2}c_x^- c_y^+ \Delta\mu_{i,j-1/2}^{(k-1)} \\
& - \tfrac{1}{2}c_x^- c_y^- \Delta\mu_{i+1,j+1/2}^{(k-1)} - \tfrac{1}{2}c_x^- c_y^- \Delta\mu_{i+1/2,j+1}^{(k-1)} - \tfrac{1}{2}c_x^- c_y^- \Delta\mu_{i+1/2,j}^{(k-1)} - \tfrac{1}{2}c_x^- c_y^- \Delta\mu_{i,j+1/2}^{(k-1)} ,
\end{aligned}
\tag{3.4.15}
$$

where F_{uw}^x, F_{uw}^y $\hat{=}$ linear numerical upwind flux functions consistent with f_x/f_y.



Idea: individual “flux distribution velocity” for each edge:

in (3.4.15), e.g.:

$$
\begin{gathered}
c_x^{\pm} c_y^{\pm} \Delta\mu_{i-1,j-1/2} \\
\downarrow \\
(\gamma_x v_x(x_{i-1}, y_{j-1/2})^{\pm})(\gamma_y v_y(x_{i-1}, y_{j-1/2})^{\pm}) \Delta\mu_{i-1,j-1/2} \,.
\end{gathered}
\tag{3.4.16}
$$

CFL-condition ($\rightarrow$ Def. 3.1.4): $\boxed{\gamma_x \max_{\boldsymbol{x}} |v_x(\boldsymbol{x})| \leq 1,\ \gamma_y \max_{\boldsymbol{x}} |v_y(\boldsymbol{x})| \leq 1}$

Example 93 (2D corner transport upwind scheme for circular advection).

- Cauchy problem (3.4.14) with $\mathbf{v}(\boldsymbol{x}) = 2\pi \begin{pmatrix} -y \\ x \end{pmatrix}$

  ➤ rigid rotation $\boldsymbol{\Phi}_t(\boldsymbol{x}) = \begin{pmatrix} \cos(2\pi t) & -\sin(2\pi t) \\ \sin(2\pi t) & \cos(2\pi t) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

- bounded spatial domain $\Omega =]-1,1[^2$ with periodic boundary conditions.

- $u_0(\boldsymbol{x}) = 1$, if $|\boldsymbol{x} - \frac{1}{4}\sqrt{2}\binom{1}{1}| < 0.4$, $u_0(\boldsymbol{x}) = 0$ elsewhere (cylinder), $u_0(\boldsymbol{x}) = \cos^2(\frac{\pi}{0.8}|\boldsymbol{x} - \frac{1}{4}\sqrt{2}\binom{1}{1}|)$, if $|\boldsymbol{x} - \frac{1}{4}\sqrt{2}\binom{1}{1}| < 0.4$, $u_0(\boldsymbol{x}) = 0$ elsewhere (compactly supported smooth bump)

- corner transport upwind discretization (3.4.15) with modification (3.4.16), $2\pi\Delta t = \Delta x = \Delta y$ ($\hat{=}$ CFL-limit), for different meshwidths $\Delta x, \Delta y \in \{\frac{1}{10}, \frac{1}{20}, \frac{1}{40}, \frac{1}{80}, \frac{1}{160}, \frac{1}{320}\}$

Monitored: $\triangleright$ evolution of discrete solutions ➨ movie,
$\triangleright$ l^1-norm $\Delta x \Delta y \sum_i \sum_j |u(x_i, y_j, 1) - \mu_{i,j}^{(M)}|$ of discretization error.

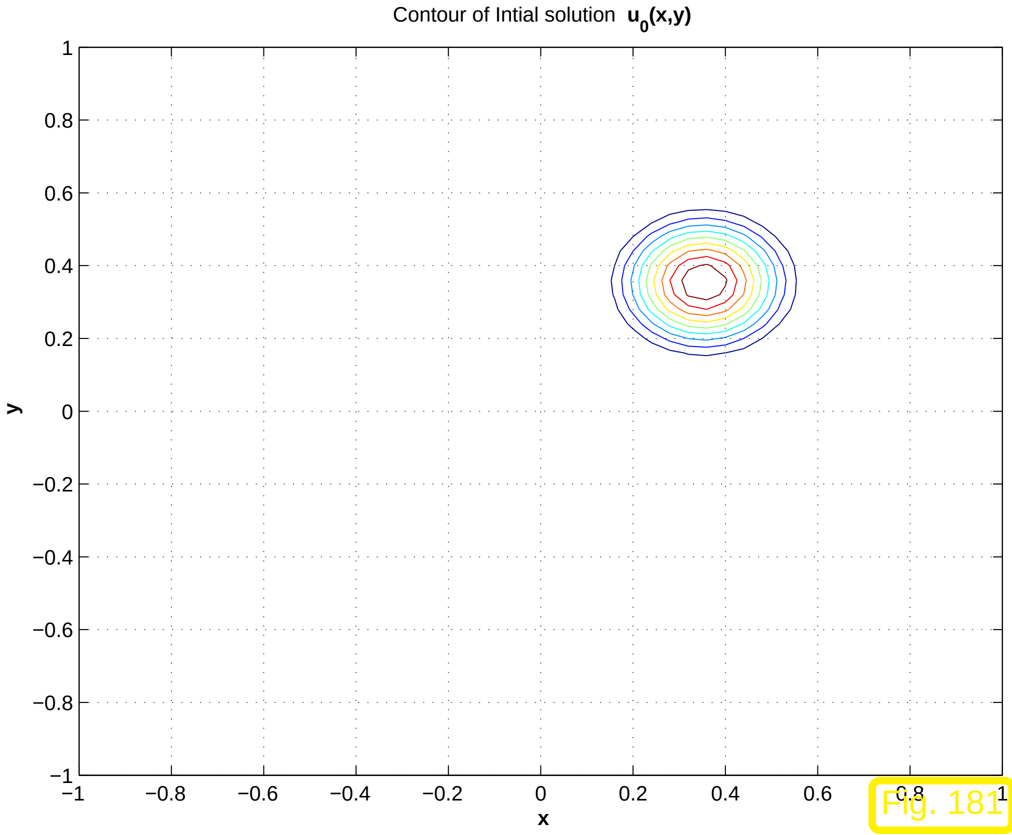


Fig. 181

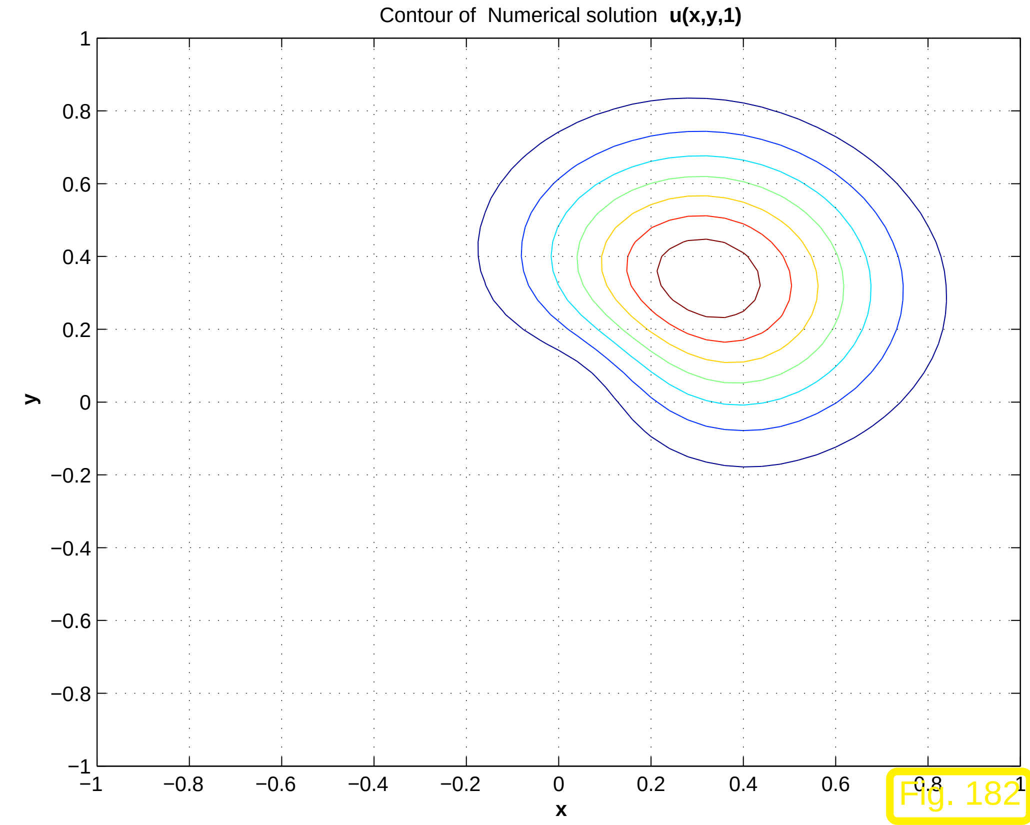


Fig. 182

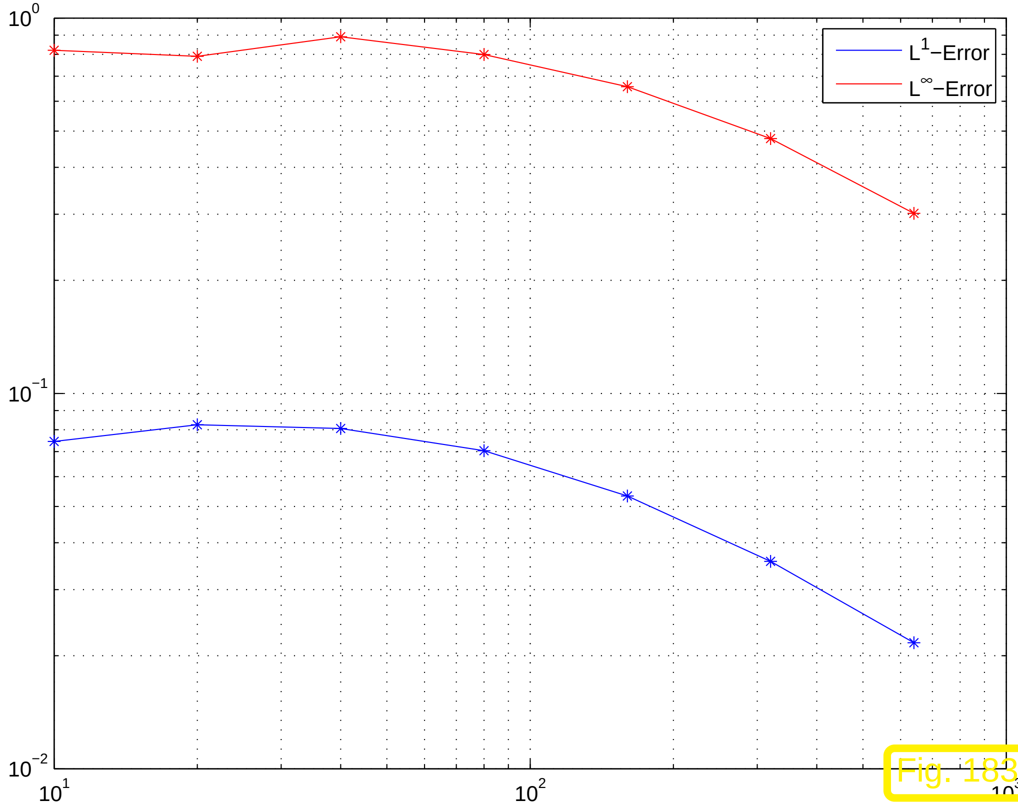


Fig. 183

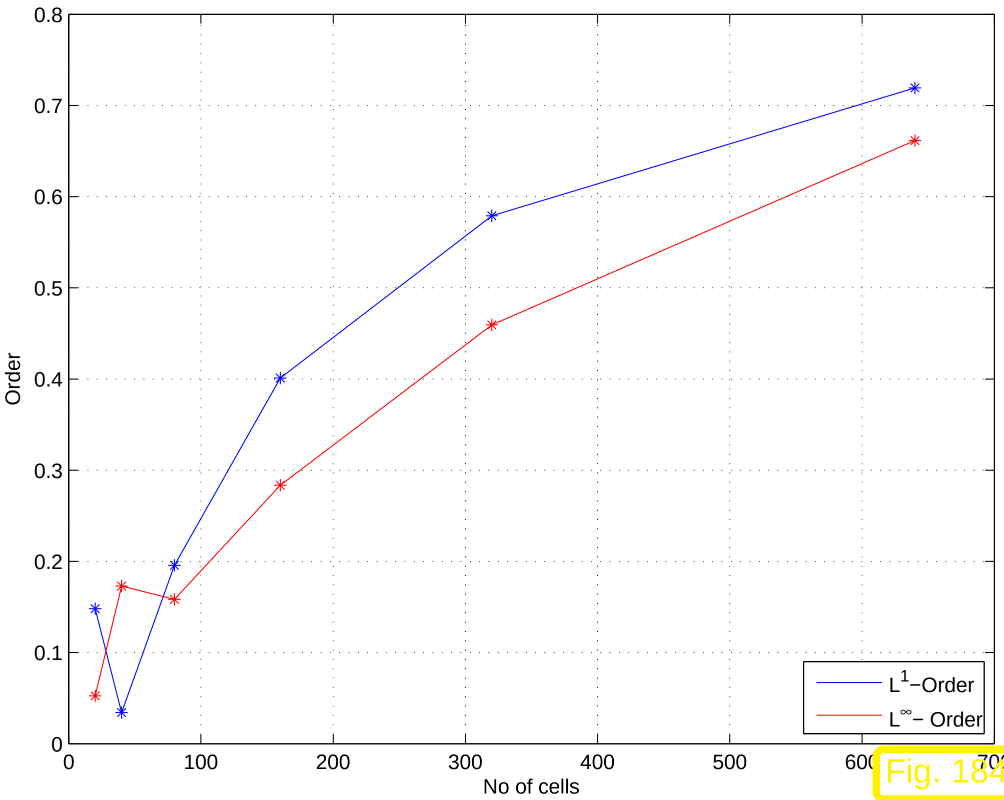


Fig. 184

Observation: 1st-order **&** dissipative ($\rightarrow$ Ex. 64)

$\diamond$

3.4.4 General conservation laws

Idea: generalize (3.4.15) to (3.4.1), general flux function $\mathbf{F} = (f_x, f_y)$ **!**

? How to generalize v_x, v_y to “fluctution distribution velocities” for an edge ?

Idea: use “local Rankine-Hugoniot velocities” (→ Thm. 2.3.2):

$$v_{x/y} \quad \sim \quad \dot{s}^{x/y}_{i+1/2,j} = \begin{cases} \dfrac{f_{x/y}(\mu^{(k-1)}_{i+1,j}) - f_{x/y}(\mu^{(k-1)}_{i,j})}{\mu^{(k-1)}_{i+1,j} - \mu^{(k-1)}_{i,j}} & \text{, if} \quad \mu^{(k-1)}_{i+1,j} \neq \mu^{(k-1)}_{i,j} \,, \\ f'_{x/y}(\mu^{(k-1)}_{i+1,j}) & \text{, if} \quad \mu^{(k-1)}_{i+1,j} = \mu^{(k-1)}_{i,j} \,. \end{cases}$$

for edge $x_{i+1/2} \times [y_{j-1/2}, y_{j+1/2}]$, $i, j \in \mathbb{Z}$ (analoguous for $[x_{i-1/2}, x_{i+1/2}] \times y_{jm+1/2}$).

with numerical flux functions F^x, F^y consistent with f_x, f_y (→ Def. 3.2.2):

$$\begin{aligned} \mu^{(k)}_{i,j} = \mu^{(k-1)}_{i,j} &- \gamma_x (F^x(\mu^{(k-1)}_{i-1,j}, \mu^{(k-1)}_{i,j}) - F^x(\mu^{(k-1)}_{i,j}, \mu^{(k-1)}_{i+1,j})) \\ &- \gamma_y (F^y(\mu^{(k-1)}_{i,j-1}, \mu^{(k-1)}_{i,j}) - F^y(\mu^{(k-1)}_{i,j}, \mu^{(k-1)}_{i,j+1})) \end{aligned} \tag{3.4.17}$$

$+$ corner transport correction, see (3.4.15).$^{(*)}$

$(*)$: corner transport correction as in (3.4.15) with replacement, e.g.

$$c^+_x c^+_y \Delta\mu^{(k-1)}_{i-1,j-1/2} \quad \rightarrow \quad (\dot{s}^x_{i-1,j-1/2})^+ (\dot{s}^y_{i-1,j-1/2})^+ \Delta\mu^{(k-1)}_{i-1,j-1/2} \,. \tag{3.4.18}$$

CFL-condition ($\rightarrow$ Def. 3.1.4)

$$\max_{u}\left\{\gamma_x|f_x'(u)|, \gamma_y|f_y'(u)|\right\} \leq 1$$

Example 94 (CTU scheme for "2D Burgers equation").

- (3.4.3) on torus $\hat{=}\ \Omega =]-1,1[^2$ + periodic boundary conditions, $\boldsymbol{d} = 1/2\sqrt{2}\binom{1}{1}$,
- initial conditions $u_0(\boldsymbol{x}) = \chi_{]0,1/2[^2}(\boldsymbol{x}) - \frac{1}{2}$ (square box),
- Corner transport upwind discretization with $\frac{\Delta t}{\Delta x} = 0.5$ ($\hat{=}$ CFL-limit), and mesh width$\Delta x = \Delta y = \frac{2}{50}$,

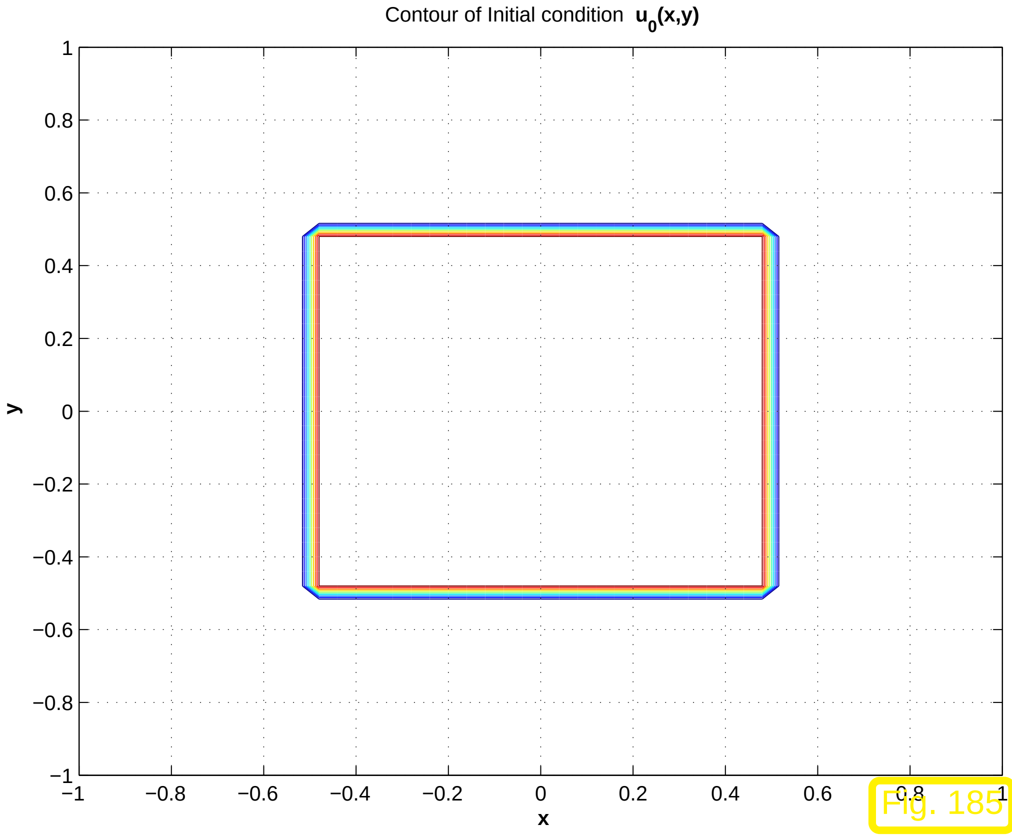


Fig. 185

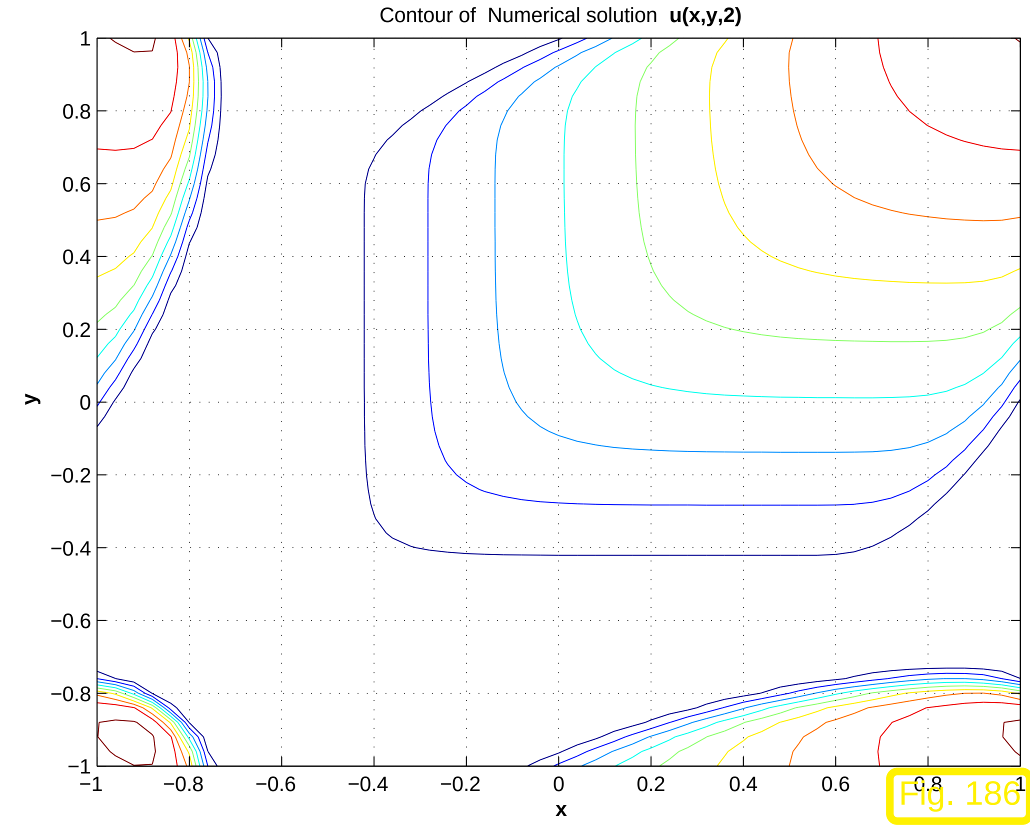


Fig. 186

$\diamond$

Remark 95 (Higher order CTU schemes in 2D).

In (3.4.17): replace $F^x(\mu_{i-1,j}^{(k-1)}, \mu_{i,j}^{(k-1)})$ with numerical fluxes of 1D high resolution methods ($\rightarrow$ Sect. 3.3)

$\triangle$

3.4.5 2D finite volume methods

Given: (infinite) structured/unstructured mesh $\mathcal{M} := \{K\}$ of (polygonal) $\Omega \subset \mathbb{R}^2$ $\rightarrow$ [27, Def. 3.2.1], *cf.* triangulation of Sect. 1.6.1.

Analoguous to (3.2.1): (✑ $\boldsymbol{n}_V \hat{=}$ exterior unit normal at ∂V)

$$\text{(3.4.1)} \quad \Rightarrow \quad \int_V u(\boldsymbol{x}, t_1)\,\mathrm{d}\boldsymbol{x} - \int_V u(\boldsymbol{x}, t_0)\,\mathrm{d}\boldsymbol{x} + \int_{t_0}^{t_1}\int_{\partial V} \mathbf{F}(u, \boldsymbol{x}) \cdot \boldsymbol{n}_V\,\mathrm{d}S(\boldsymbol{x})\mathrm{d}t = 0 \qquad \text{(2.1.2)}$$

$$\Big\Downarrow \quad \leftarrow V = K,\ K \in \mathcal{M}$$

update formula for cell averages $\quad \mu_K^{(k)} = \dfrac{1}{|K|}\displaystyle\int_K u(\boldsymbol{x}, t_k)\,\mathrm{d}\boldsymbol{x}\ , \quad K \in \mathcal{M},\ k = 1, \ldots, M$:

$$\mu_K^{(k)} - \mu_K^{(k-1)} = -\frac{1}{|K|}\int_{t_{k-1}}^{t_k}\left\{\sum_{e \in \mathcal{E}_K}\int_e \mathbf{F}(u, \boldsymbol{x}) \cdot \boldsymbol{n}_K\,\mathrm{d}S\right\}\mathrm{d}t \quad , \quad \mathcal{E}_K := \text{edges of } K\ .$$

► Genuinely 2D conservation form (⟶ Def. 3.2.1) of discrete evolution:

$$\mu_K^{(k)} = \mu_K^{(k-1)} - \frac{\Delta t}{|K|} \sum_{e \in \mathcal{E}_K} |e| \underbrace{f_e^K}_{\text{numerical flux}} \,, \quad f_e^K \approx \frac{1}{\Delta t |e|} \int_{t_{k-1}}^{t_k} \int_e \mathbf{F}(u, \boldsymbol{x}) \cdot \boldsymbol{n}_K \, \mathrm{d}S \,. \tag{3.4.19}$$

As in Def. 3.2.1:
f_e^K obtained from numerical flux function

$$F : \mathbb{R} \times \mathbb{R} \times \mathbb{S}^2 \mapsto \mathbb{R} \,.$$

► $$f_e^K = F(\mu_{K_1}^{(k-1)}, \mu_{K_2}^{(k-1)}, \boldsymbol{n}_K) \,. \tag{3.4.20}$$

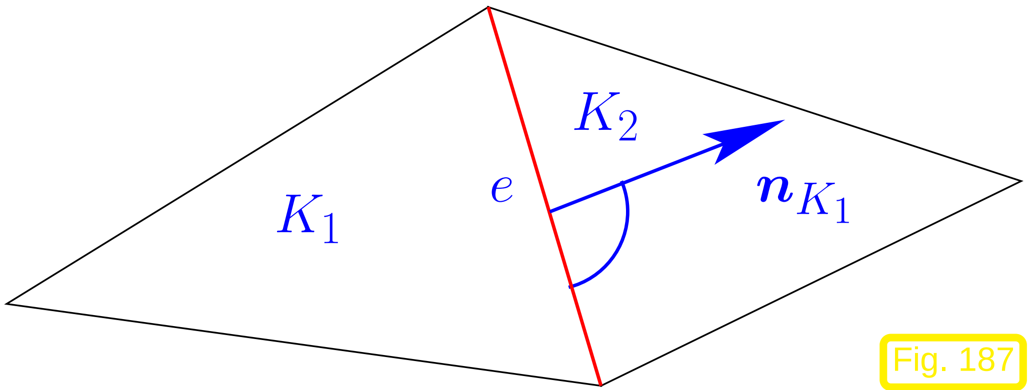


Fig. 187

Analoguous to Sect. 3.2.1 (⟶ Def. 3.2.2) we require

❶ conservation: $F(v, w, \boldsymbol{n}) = -F(w, v, -\boldsymbol{n}) \quad \forall v, w \in \mathbb{R}, \boldsymbol{n} \in \mathbb{R}^2, |\boldsymbol{n}| = 1$

❷ consistency: $F(u, u, \boldsymbol{n}) = \mathbf{F}(u) \cdot \boldsymbol{n} \quad \forall u \in \mathbb{R}, \boldsymbol{n} \in \mathbb{R}^2, |\boldsymbol{n}| = 1$

❸ Lipschitz-continuity: $|F(v, w, \boldsymbol{n}) - F(u, u, \boldsymbol{n})| \leq C(|v - u| + |w - u|)$
for v, w sufficiently close to u

Idea: "projection onto normal direction" $\longrightarrow F$

u solves (3.4.1) ➤ $w(x,t) := u(\boldsymbol{n}x,t)$ satisfies 1D conservation law

$$\frac{\partial w}{\partial t} + \frac{\partial}{\partial x}(\boldsymbol{n}\cdot\mathbf{F}(w)) = 0 \quad \text{in } \mathbb{R}^2\times]0,T[\ . \qquad (3.4.21)$$

$$\boxed{F(\cdot,\cdot,\boldsymbol{n}) \longleftarrow \text{1D numerical flux function consistent with } \boldsymbol{n}\cdot\mathbf{F}(\cdot)}$$

Example: F based on Godunov flux F_{GD} (3.2.17):

$$F(v,w,\boldsymbol{n}) = \begin{cases} \min\limits_{v\le u\le w} \boldsymbol{n}\cdot\mathbf{F}(u) & \text{, if } \quad v < w\ , \\ \max\limits_{w\le u\le v} \boldsymbol{n}\cdot\mathbf{F}(u) & \text{, if } \quad w \le v\ . \end{cases}$$

consistency ✔, if $\mathbf{F}$ Lipschitz-continuous ➥ Lipschitz-continuity ✔,
conservation by direct computation ✔.

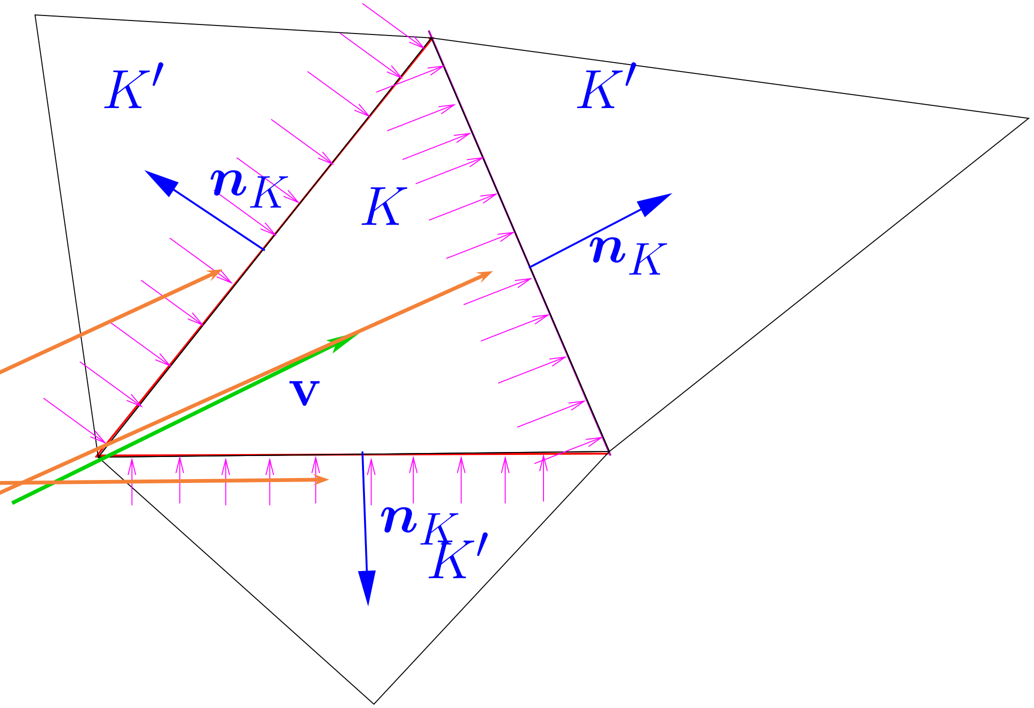
$$\mu_K^{(k)} = \mu_K^{(k-1)} - \frac{\Delta t}{|K|} \sum_{K'\in\mathcal{N}_K} |\bar{K}\cap\bar{K}'|\, F(\mu_K^{(k-1)}, \mu_{K'}^{(k-1)}, \boldsymbol{n}_K)\ . \qquad (3.4.22)$$

➡ mesh neighborhood $\mathcal{N}_K := \{K' \in \mathcal{M}\colon \bar{K}\cap\bar{K}' \neq \emptyset\}$

Special case:

constant linear advection (3.4.2), $\mathbf{v}$ fixed

+ upwind flux (3.2.6)



$$\text{(3.4.5)} \quad \Rightarrow \quad \mu_K^{(k)} = \mu_K^{(k-1)} - \frac{\Delta t}{|K|} \sum_{K' \in \mathcal{N}_K} |\bar{K} \cap \bar{K}'| \Big(\underbrace{(\mathbf{v} \cdot \boldsymbol{n}_K)^+ \mu_K^{(k-1)}}_{\text{outflow from } K} + \underbrace{(\mathbf{v} \cdot \boldsymbol{n}_K)^- \mu_{K'}^{(k-1)}}_{\text{inflow into} K} \Big) . \tag{3.4.23}$$

▶ CFL-condition ($\to$ Def. 3.1.4) for (3.4.23):

$$\max_{K \in \mathcal{M}} \max_{e \in \mathcal{E}_K} \frac{|e|}{|K|} \Delta t \, |\mathbf{v}| \leq \frac{1}{2} \quad \Rightarrow \quad \text{(3.4.23) monotone } (\to \text{ Def. 3.1.14}). \tag{3.4.24}$$

Assuming uniformly bounded shape-regularity measure $\rho_{\mathcal{M}}$ ($\to$ [27, Def. 4.2.21], [27, Sect. 4.2.4])

$$\text{CFL-condition (3.4.24)} \quad \Longleftrightarrow \quad \frac{\Delta t}{h_{\mathcal{M}}} |\mathbf{v}| \leq C \quad \text{for sufficiently small } C = C(\rho_{\mathcal{M}}) > 0$$

4 Galerkin Methods for Scalar Conservation Laws

4.1 Standard Galerkin spatial discretization

4.2 Discontinuous Galerkin (DG) methods

4.2.1 The Runge-Kutta discontinuous Galerkin (RKDG) method

Special case: $d = 1 \leftrightarrow$ 1D scalar conservation law (2.1.5) $\dfrac{\partial u}{\partial t} + \dfrac{\partial}{\partial x} f(u) = 0$, $\Omega = \mathbb{R}$ ((Cauchy problem))

Spatial mesh $\mathcal{M} := \{]x_{j-1/2}, x_{j+1/2}[, j \in \mathbb{Z}\}$ with gridpoints $x_j \in \mathbb{R}$, $x_{j-1} < x_j$, see (3.1.1)

➪ spatially semi-discrete DG evolution: $u_N \in C^1([0,T], V_N)$ satisfies

$$\int_{x_{j-1/2}}^{x_{j+1/2}} \frac{\partial u_N}{\partial t}(x,t)v_N(x) - f(u_N(x,t))v_N'(x)\,\mathrm{d}x + f_{j+1/2}(t) - f_{j-1/2}(t) = 0 \quad \begin{array}{l}\forall v_N \in \mathcal{P}_p(\mathbb{R})\,,\\ \forall j \in \mathbb{Z}\,,\end{array} \tag{4.2.1}$$

with numerical fluxes $f_{j+1/2}(t) := F(u_N(x^-_{j+1/2}, t), u_N(x^+_{j+1/2}, t))$. (4.2.2)

Example 96 (RKDG for 1D linear advection).

- 1D scalar conservation law(2.1.6), $f(u) = cu$, with advection velocity $c = 1$, $T = 1$ ➤ $u(x,t) = u_0(x-t)$
- smooth, non-smooth and discontinuous initial data, supported in $[0,1]$, see Ex. 48

$$u_0(x) = 1 - \cos^2(\pi x)\,,\quad 0 \leq x \leq 1\,,\quad 0 \text{ elsewhere}\,, \tag{4.2.3}$$

$$u_0(x) = 1 - 2 * |x - \tfrac{1}{2}|\,,\quad 0 \leq x \leq 1\,,\quad 0 \text{ elsewhere}\,, \tag{4.2.4}$$

$$u_0(x) = 1\,,\quad 0 \leq x \leq 1\,,\quad 0 \text{ elsewhere}\,. \tag{4.2.5}$$

- RGDK discretization with upwind flux/Lax-Friedrichs (3.2.9) numerical fluxes on equidistant mesh, meshwidth Δx.

Monitored: convergence of RKDG solution w.r.t. to norms $\max_k \left\|\vec{\mu}^{(k)} - \mathsf{R}u(\cdot,t_k)\right\|_{l^2(\mathbb{Z})}$, $\max_k \left\|\vec{\mu}^{(k)} - \mathsf{R}u(\cdot,t_k)\right\|_{l^1(\mathbb{Z})}$, ($\max_k \left\|\vec{\mu}^{(k)} - \mathsf{R}u(\cdot,t_k)\right\|_{l^\infty(\mathbb{Z})}$) for different initial data u_0 and $p = 0, s = 1$, $p = 1, s = 2$, $p = 2, s = 3$ ($s \hat{=}$ no. of stages in SSP-RK timestepping (3.3.37).)

Numerical experiments. Please specify CFL numbers

◇

Example 97 (RKDG for 1D Burger's equation).

- Cauchy problem for Burgers equation (2.1.7)
- "box function" $u_0 = \chi_{]0,1[}$ (4.2.5), *cf.* Ex. 64

◇

Example 98 (P_0 and P_1 DG for circular advection).

- Cauchy problem of Ex. 93
- Spatial discretization: DG with upwind numerical flux function F_{uw}, 2-point Gaussian quadrature for edge flux.
- Timestepping: 2-stage SSP Runge-Kutta method (Heun method) (3.3.41), $\Delta t = \frac{1}{2000}$
- unstructured triangular meshes of spatial domain $\Omega = \{\boldsymbol{x} \in \mathbb{R}^2 \colon |\boldsymbol{x}| < 1\}$

mesh plot

$\diamond$

4.2.2 Stability and convergence

Focus: $d = 1 \hat{=}$ Cauchy problem for 1D scalar conservation law

$$\begin{aligned} \frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u) &= 0 \quad \text{in } \mathbb{R}\times]0,T[\;, \\ u(x,0) &= u_0(x) \quad \text{in } \mathbb{R}\;. \end{aligned} \tag{2.2.1}$$

4.2.2.1 Entropy stabilty

Sect. 2.6.1:

$$\boxed{\text{entropy inequalities } (\longrightarrow \text{ Def. 2.5.3}) \quad \succ \quad \text{stability}}$$

for "semi-norm like" entropies

Focus: quadratic entropy $\leftrightarrow$ pair of entropy functions ($\longrightarrow$ Def. 2.5.2)

$$\eta(w) = \tfrac{1}{2}w^2 \quad , \quad \psi(w) = \int_0^w f(\xi)\xi \,\mathrm{d}\xi = f(w)w - \int_0^w f(\xi)\,\mathrm{d}\xi \,. \tag{4.2.6}$$

Goal: semi-discrete cell entropy inequality, *cf.* Def. 3.2.14, (3.2.35)

$$\frac{d}{dt}\int_{x_{j-1/2}}^{x_{j+1/2}} \eta(u_N(x,t))\,\mathrm{d}x + \psi_{j+1/2} - \psi_{j-1/2} \leq 0, \quad j \in \mathbb{Z} \,, \tag{4.2.7}$$

for spatially *semi-discrete* DG evolution (4.2.1) for (2.2.1)

Here: $\psi_{j+1/2}$, $j \in \mathbb{Z} \mathrel{\hat{=}}$ numerical entropy fluxes

4.2.2.2 Convergence for linear advection

4.2.2.3 CFL condition

RKDG methods: *empiric* CFL numbers for constant scalar linear advection

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0 \ .$$

$$|c|\frac{\Delta t}{\Delta x} \leq \mathrm{CFL} \ .$$

p	0	1	2	3	4	5	6	7	8
$s=1$	1.000	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$
$s=2$	1.000	0.333	$*$	$*$	$*$	$*$	$*$	$*$	$*$
$s=3$	1.256	0.409	0.209	0.130	0.089	0.066	0.051	0.040	0.033
$s=4$	1.392	0.464	0.235	0.145	0.100	0.073	0.056	0.045	0.037
$s=5$	1.608	0.534	0.271	0.167	0.115	0.085	0.065	0.052	0.042
$s=6$	1.776	0.592	0.300	0.185	0.127	0.093	0.072	0.057	0.047
$s=7$	1.977	0.659	0.333	0.206	0.142	0.104	0.080	0.064	0.052
$s=8$	2.156	0.718	0.364	0.225	0.154	0.114	0.087	0.070	0.057
$s=9$	2.350	0.783	0.396	0.245	0.168	0.124	0.095	0.076	0.062
$s=10$	2.534	0.844	0.428	0.264	0.182	0.134	0.103	0.082	0.067
$s=11$	2.725	0.908	0.460	0.284	0.195	0.144	0.111	0.088	0.072
$s=12$	2.911	0.970	0.491	0.303	0.209	0.153	0.118	0.094	0.077

4.2.3 Limiting for RKDG methods

4.3 Streamline upwind Petrov Galerkin methods

5 Systems of Conservation Laws in One Space Dimension

Consider: conservation law (2.1.3) for

spatial dimension $d = 1$ $\leftrightarrow$ 1D
state space dimension $m > 1$ $\leftrightarrow$ system

on space-time rectangle $\Omega\times]0,T[$:

$$\operatorname{div}_{(x,t)}\begin{pmatrix}\mathbf{F}(\mathbf{u})\\ \mathbf{u}\end{pmatrix} = \frac{\partial}{\partial t}\mathbf{u} + \frac{\partial}{\partial x}\mathbf{F}(\mathbf{u}) = \frac{\partial}{\partial t}\begin{pmatrix}u_1\\ \vdots\\ u_m\end{pmatrix} + \frac{\partial}{\partial x}\begin{pmatrix}f_1(u_1,\dots,u_m)\\ \vdots\\ f_m(u_1,\dots,u_m)\end{pmatrix} = 0 \quad \text{in } \Omega\times]0,T[\;, \tag{5.0.1}$$

$m \in \mathbb{N}$, $\mathbf{u} = \mathbf{u}(x,t) : \Omega \subset \mathbb{R}\times]0,T[\mapsto U \subset \mathbb{R}^m$, vector valued flux function $\mathbf{F} : U \subset \mathbb{R}^m \mapsto \mathbb{R}^m$,

$$+ \quad \text{initial conditions:} \quad \mathbf{u}(x,0) = \mathbf{u}_0(x) \quad \text{in } \Omega\;. \tag{5.0.2}$$

Many notions from Ch. 2 (scalar case, $m = 1$) carry over:

- Cauchy problem (⟶ Sect. 2.1): $\Omega = \mathbb{R}$ (➤ no spatial boundary conditions)

- Riemann problem (⟶ Def. 2.4.1) = Cauchy problem for $\mathbf{u}_0(x) = \begin{cases} \mathbf{u}_l \in U & \text{, if } x < 0 \,, \\ \mathbf{u}_r \in U & \text{, if } x \geq 0 \,. \end{cases}$

- Weak solutions **=** solutions in the sense of distributions, *cf.* Def. 2.3.1:

Definition 5.0.1 (Weak solution of Cauchy problem for system of conservation laws)**.**
Given initial data $\mathbf{u}_0 \in (L^\infty(\mathbb{R}))^m$, $\mathbf{u} : \mathbb{R}\times]0,T[\mapsto U \subset \mathbb{R}^m$ *is a weak solution (solution in the sense of distributions) of the Cauchy problem for* (5.0.1)*, if*

$$\mathbf{u} \in (L^\infty(\mathbb{R}\times]0,T[))^m \,, \quad \int_{-\infty}^{\infty} \int_0^T \left\{ \mathbf{u} \cdot \frac{\partial \mathbf{\Phi}}{\partial t} + \mathbf{F}(u) \cdot \frac{\partial \mathbf{\Phi}}{\partial x} \right\} \mathrm{d}t\mathrm{d}x + \int_{-\infty}^{\infty} \mathbf{u}_0(x)\mathbf{\Phi}(x,0)\,\mathrm{d}x = 0 \,,$$

for all $\mathbf{\Phi} \in C_0^\infty(\mathbb{R} \times [0,T[, \mathbb{R}^m)$*.*

5.1 Hyperbolicity

Special case: $\boxed{\text{linear system of conservation laws} \quad \Leftrightarrow \quad \text{(5.0.1) with } \mathbf{F}(\mathbf{u}) = \mathbf{A}\mathbf{u}, \quad \mathbf{A} \in \mathbb{R}^{m,m}}$

▶ Cauchy problem:
$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{A}\frac{\partial \mathbf{u}}{\partial x} = 0 \quad \text{in } \mathbb{R}\times]0,T[\ , \qquad \mathbf{u}(\cdot,0) = \mathbf{u}_0 \in (L^\infty(\mathbb{R}))^m \ . \tag{5.1.1}$$

For (5.1.1) try plane wave solutions, *cf.* Def. 1.3.2:

$$\mathbf{u}(x,t) = \boldsymbol{d}\exp(i(kx-\omega t)) \ , \quad \boldsymbol{d} \in \mathbb{R}^m \ , \quad k,\omega \in \mathbb{C} \ . \tag{5.1.2}$$

Note: $\mathbf{u}_0 \in (L^\infty(\mathbb{R}))^m \quad \Rightarrow \quad k \in \mathbb{R}$

(5.1.2) in (5.1.1) $\Rightarrow$ $(-i\omega + ik\mathbf{A})\boldsymbol{d} = 0 \overset{k\neq 0}{\Longleftrightarrow} \omega/k$ is eigenvalue of $\mathbf{A}$.

$$\omega/k = a + ib \ , \quad a,b \in \mathbb{R} \quad \Rightarrow \quad \mathbf{u} = \boldsymbol{d}\exp(bkt)\exp(ik(x-at)) \ .$$

▶ $\sigma(\mathbf{A}) \not\subset \mathbb{R}$ ➢ (5.1.1) has exponentially growing solutions ($\hat{=}$ ill-posed !)

✎ notation: $\sigma(\mathbf{A}) \hat{=}$ set of eigenvalues (spectrum) of $\mathbf{A} \in \mathbb{R}^{m,m}$

If $\mathbf{u}_0$ = "small perturbation" of constant state $\mathbf{u}^* \in \mathbb{R}^m$ ➢ linearization

▶ Cauchy problem for (5.0.1) $\overset{\approx}{\longleftrightarrow}$ $\dfrac{\partial \widetilde{\mathbf{u}}}{\partial t} + D\mathbf{F}(\mathbf{u}^*)\dfrac{\partial \widetilde{\mathbf{u}}}{\partial x} = 0$, with $\mathbf{u}(x,t) \approx \mathbf{u}^* + \widetilde{\mathbf{u}}(x,t)$ (☞ linear system governs evolution of perturbation).

Definition 5.1.1 ((Strictly) hyperbolic systems of conservation laws)**.**

(5.0.1) *hyperbolic* $:\Leftrightarrow$ $\forall \mathbf{u} \in U\colon \exists \mathbf{R} \in \mathbb{R}^{m,m}\colon$ $\mathbf{R}^{-1} D\mathbf{F}(\mathbf{u})\mathbf{R} = \operatorname{diag}(\lambda_1, \dots, \lambda_m)$, $\lambda_k \in \mathbb{R}$.

(5.0.1) *is strictly hyperbolic, if, in addition,* $D\mathbf{F}(\mathbf{u})$ *has* m *distinct real eigenvalues for all* $\mathbf{u} \in U$*.*

✎ notation: $\sigma(D\mathbf{F}(\mathbf{u})) = \{\lambda_i(\mathbf{u}), i = 1, \dots, m\}$
convention: $\lambda_1(\mathbf{u}) < \lambda_2(\mathbf{u}) < \cdots < \lambda_m(\mathbf{u})$ (in strictly hyperbolic case)

✎ notation: $\mathbf{r}_k = \mathbf{r}_k(\mathbf{u}) \mathrel{\hat{=}}$ eigenvector of $D\mathbf{F}(\mathbf{u}) \leftrightarrow$ eigenvalue $\lambda_k(\mathbf{u})$, $k = 1, \dots, m$
➤ $\mathbf{R} = (\mathbf{r}_1, \dots, \mathbf{r}_m)$ for $\mathbf{R}$ from Def. 5.1.1

Example 99 (1D shallow water equations)*.* $\longrightarrow$ [31, Sect. 13.1]

Inviscid, incompressible fluid flowing in straight shallow long channel (uniform cross-section)

Assume: velocity parallel to channel direction
independent of depth

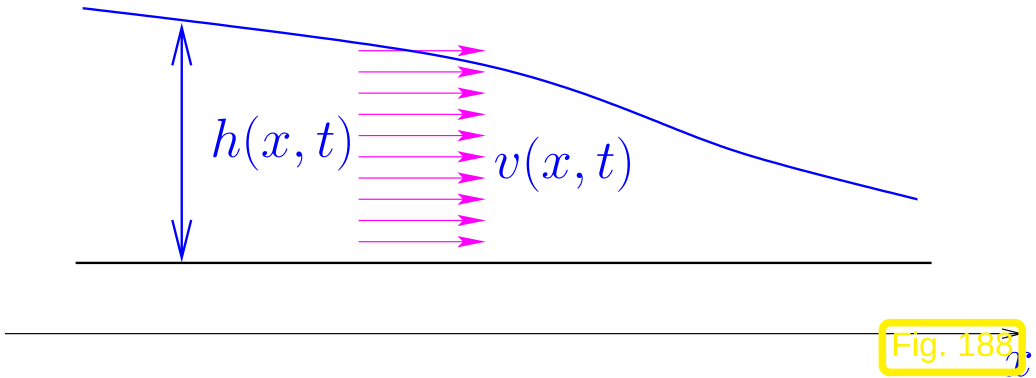


Fig. 188

Physical quantities: $h(x,t)$: height of fluid ($[h] = \mathrm{m}$), ➥ $h \geq 0$

$v(x,t)$: fluid velocity (x-component) ($[v] = \mathrm{ms}^{-1}$)

[31, Sect. 2.6]:

conservation of mass ▶ $$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(vh) = 0 \;, \tag{5.1.3}$$

conservation of momentum ▶ $$\frac{\partial}{\partial t}(hv) + \frac{\partial}{\partial x}(hv^2 + \tfrac{1}{2}gh^2) = 0 \;, \tag{5.1.4}$$

with $g > 0 \hat{=}$ gravity acceleration, $[g] = \mathrm{ms}^{-2}$.

Terminology: h, $hv \hat{=}$ conserved quantities (conservative variables)

(5.1.3)
(5.1.4) $\Leftrightarrow$ $$\underbrace{\text{(5.0.1) with } \mathbf{u} = \begin{pmatrix} h \\ hv \end{pmatrix} \;, \; \mathbf{F}_{\mathrm{sw}}(\mathbf{u}) := \mathbf{F}(\mathbf{u}) = \begin{pmatrix} vh \\ hv^2 + \frac{1}{2}gh^2 \end{pmatrix} = \begin{pmatrix} u_2 \\ u_2^2 u_1^{-1} + \frac{1}{2}gu_1^2 \end{pmatrix}}_{\text{shallow water equations}} \tag{5.1.5}$$

Phase space/state space: $U = \mathbb{R}^+ \times \mathbb{R} \subset \mathbb{R}^2$

$$D\mathbf{F}_{\mathrm{sw}}(\mathbf{u}) = \begin{pmatrix} 0 & 1 \\ -(u_2/u_1)^2 + gu_1 & 2u_2/u_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -v^2 + gh & 2v \end{pmatrix} \;. \tag{5.1.6}$$

▶ eigenvalues λ_1, λ_2 /eigenvectors $\mathbf{r}_1, \mathbf{r}_2 \in \mathbb{R}^2$ of $D\mathbf{F}_{\mathrm{sw}}(\mathbf{u})$:

$$\begin{aligned} \lambda_1 &= v - \sqrt{gh} \quad \leftrightarrow \quad \mathbf{r}_1 = \begin{pmatrix} 1 \\ v - \sqrt{gh} \end{pmatrix} \\ \lambda_2 &= v + \sqrt{gh} \quad \leftrightarrow \quad \mathbf{r}_2 = \begin{pmatrix} 1 \\ v + \sqrt{gh} \end{pmatrix} . \end{aligned} \tag{5.1.7}$$

➤ ($h > 0 \quad \Rightarrow$) Shallow water equations (5.1.5) strictly hyperbolic ($\longrightarrow$ Def. 5.1.1)

◇

5.2 Linear systems

Cauchy problem: $$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{A}\frac{\partial \mathbf{u}}{\partial x} = 0 \quad \text{in } \mathbb{R}\times]0,T[\ , \quad \mathbf{u}(\cdot,0) = \mathbf{u}_0 \in (L^\infty(\mathbb{R}))^m\ . \tag{5.1.1}$$

Assume strict hyperbolicity: $\sigma(\mathbf{A}) = \{\lambda_1 < \lambda_2 < \cdots < \lambda_m\}$

▶ $\mathbf{A} \in \mathbb{R}^m$ can be diagonalized (see Def. 5.1.1)

$\exists \mathbf{R} \in \mathbb{R}^{m,m}$: $\mathbf{R}^{-1}\mathbf{A}\mathbf{R} = \mathrm{diag}(\lambda_1,\ldots,\lambda_m)$, $\mathbf{R} = [\mathbf{r}_1,\ldots,\mathbf{r}_m]$, $\{\mathbf{r}_i\}$ = eigenvectors of $\mathbf{A}$.

➤ diagonalizing (5.1.1): $\mathbf{w}(x,t) = \mathbf{R}^{-1}\mathbf{u}(x,t) \quad \leftrightarrow \quad \mathbf{u}(x,t) = \sum_{k=1}^{m} w_k(x,t)\mathbf{r}_k$

$$(5.1.1) \quad \Leftrightarrow \quad \begin{aligned} &\frac{\partial w_k}{\partial t} + \lambda_k \frac{\partial w_k}{\partial x} = 0 \quad \text{in } \mathbb{R}\times]0,T[\ , \\ &\mathbf{w}(\cdot,0) = \mathbf{R}^{-1}\mathbf{u}_0\ . \end{aligned} \tag{5.2.1}$$

= decoupled constant advection problems (2.1.6)

Ex. 33 ➡ solution of (5.1.1): $\mathbf{u}(x,t) = \sum_{k=1}^{m} (\mathbf{R}^{-1}\mathbf{u}_0)_k(x-\lambda_k t)\,\mathbf{r}_k\ .$ (5.2.2)

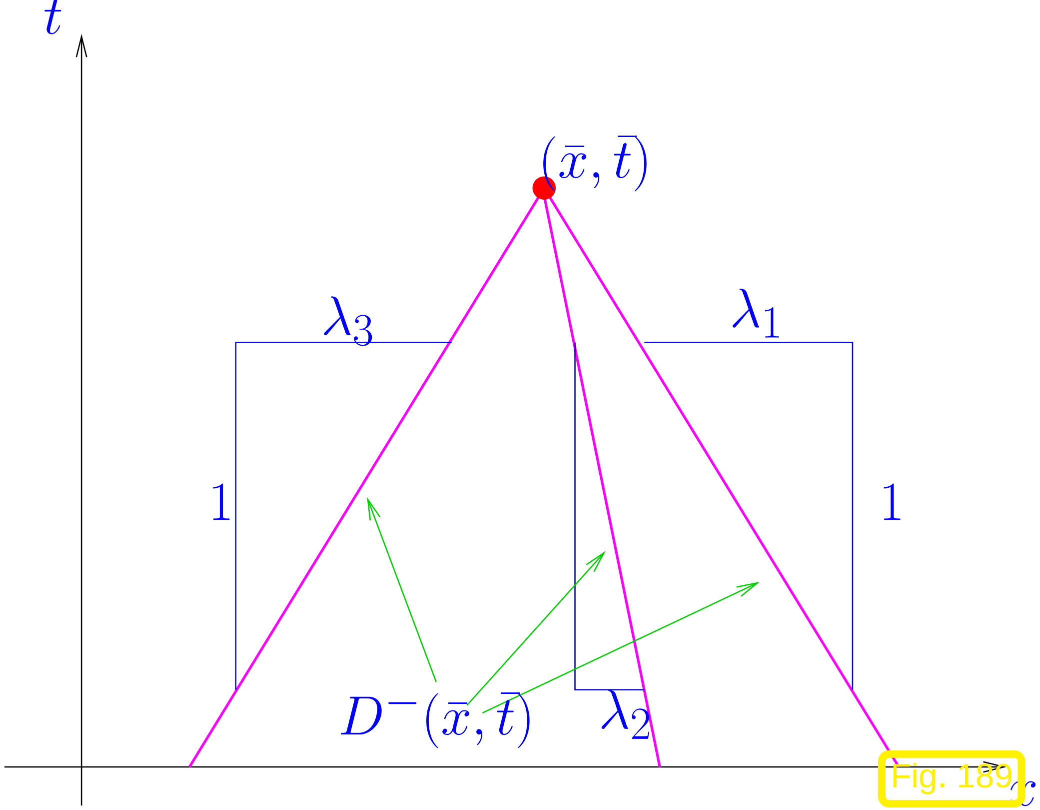
⇨ solution $\mathbf{u}(x,t)$ = superposition of m states $\mathbf{r}_k$ propagating with speeds λ_k:

terminology: $(\mathbf{R}^{-1}\mathbf{u}_0)_k(x-\lambda_k t)\,\mathbf{r}_k$ = k-wave

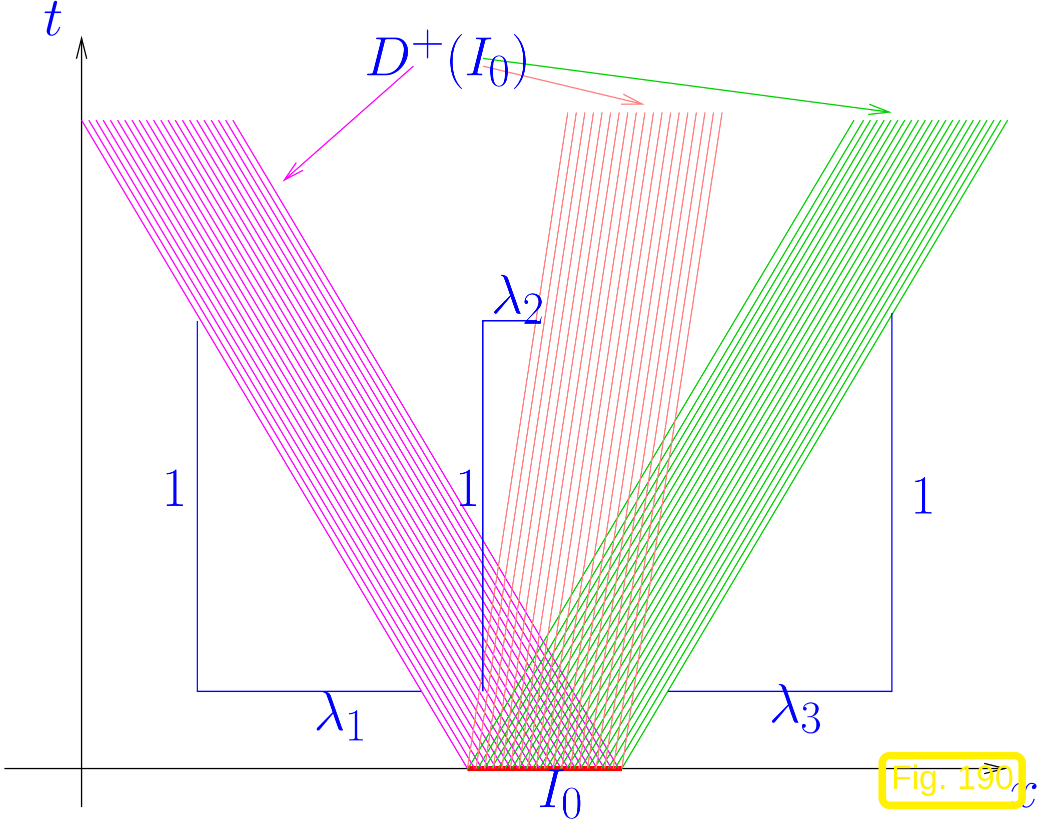
Information propagates along characteristic curves, *cf.* Def. 2.2.2

$$\gamma_k(\tau) = \lambda_k\tau + c\ , \quad 0 \leq \tau \leq T\ , \quad c \in \mathbb{R}\ . \tag{5.2.3}$$

⇨ domains of dependence/influence $\longrightarrow$ Sect. 2.6.2



$m = 3$: domain of dependence of $(\bar{x}, \bar{t}) \in \widetilde{\Omega}$



$m = 3$: domain of influence of $I_0 \subset \mathbb{R}$

Example 100 (1D wave equation as linear hyperbolic system). $\longrightarrow$ [31, Sect. 2.7]

Cauchy problem for 1D wave equation with constant coefficients (1.10.1) ($\longrightarrow$ Def. 1.1.1):

$$c > 0: \quad \frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 \quad , \quad u(x,0) = u_0(x) \; , \quad \frac{\partial u}{\partial t}(x,0) = v_0(x) \; , \quad x \in \mathbb{R} \; . \tag{1.10.1}$$

Secondary unknowns: $w(x,t) = c\dfrac{\partial u}{\partial x}(x,t)$, $v(x,t) = \dfrac{\partial u}{\partial t}(x,t)$, *cf.* (1.12.16)

$$\blacktriangleright \quad \begin{aligned} \frac{\partial v}{\partial t} - c\frac{\partial w}{\partial x} &= 0\,, \\ \frac{\partial w}{\partial t} - c\frac{\partial v}{\partial x} &= 0 \end{aligned} \quad \text{in } \mathbb{R}\times]0,T[\quad , \quad \begin{aligned} w(x,0) &= c\frac{d}{dx}u_0(x)\,, \\ v(x,0) &= v_0(x)\,, \end{aligned} \quad x \in \mathbb{R}\,.$$

$$\text{(1.10.1)} \quad \Rightarrow \quad \frac{\partial}{\partial t}\underbrace{\begin{pmatrix} v \\ w \end{pmatrix}}_{=:\mathbf{u}} + \underbrace{\begin{pmatrix} 0 & -c \\ -c & 0 \end{pmatrix}}_{=:\mathbf{A}}\frac{\partial}{\partial x}\begin{pmatrix} v \\ w \end{pmatrix} = 0 \quad \text{in } \mathbb{R}\times]0,T[\,. \tag{5.2.4}$$

scalar wave equation $\cong$ strictly hyperbolic ($\longrightarrow$ Def. 5.1.1) linear system of conservation laws **!**

Note: conversion (1.10.1) $\longrightarrow$ (5.2.4) is not unique !

(5.2.4): eigenvalues $\lambda_1 = -c$, $\lambda_2 = c$, eigenvectors $\mathbf{r}_1 = \frac{1}{\sqrt{2}}\binom{1}{1}$, $\mathbf{r}_2 = \frac{1}{\sqrt{2}}\binom{1}{-1}$

➢ (5.2.2) $\longleftrightarrow$ D'Alembert solution formula (1.3.3) for (1.10.1)
(answers question in Sect. 1.3.2) $\Diamond$

Remark 101 (Linearized systems of conservation laws).

Linearization, *cf.* reasoning in Sect. 5.1: if $\mathbf{u}_0 = \mathbf{u}^* + \widetilde{\mathbf{u}}_0$ = small perturbation of constant state $\mathbf{u}^* \in \mathbb{R}^m$, $\mathbf{u}(x,t)$ solution of Cauchy problem for (5.0.1), then

$$\mathbf{u}(x,t) = \mathbf{u}^* + \widetilde{\mathbf{u}}(x,t): \quad \begin{aligned} \frac{\partial \widetilde{\mathbf{u}}}{\partial t} + D\mathbf{F}(\mathbf{u}^*)\frac{\partial \widetilde{\mathbf{u}}}{\partial x} = 0 \quad \text{in } \mathbb{R}\times]0,T[\ , \\ \widetilde{\mathbf{u}}(\cdot,0) = \widetilde{\mathbf{u}}_0\ . \end{aligned} \tag{5.2.5}$$

(5.2.5) **=** "acoustic approximation" of non-linear system of conservation laws

(➡ (moduli of) eigenvalues of $D\mathbf{F}(\mathbf{u}^*) \hat{=}$ sound speeds)

$\triangle$

▶ for (5.0.1): small perturbations/information propagate along characteristic curves

Definition 5.2.1 (Characteristic curves for systems of conservation laws). *cf. Def. 2.2.2*

A curve $\Gamma := (\gamma(\tau),\tau) : [0,T] \mapsto \mathbb{R}\times]0,T[$ *in the* (x,t)*-plane is a characteristic curve of the* k*-th family,* $k = 1,\dots,m$*, (*k*-characteristic) for* (5.0.1)*, if*

$$\frac{d}{d\tau}\gamma(\tau) = \lambda_k(\mathbf{u}(\gamma(\tau),\tau))\ , \quad 0 \leq \tau \leq T\ , \tag{5.2.6}$$

where $\mathbf{u}$ *is a (piecewise) classical solution (→ Def. 2.2.1) of* (5.0.1).

Example 102 (Linearized shallow water equations). $\longrightarrow$ Ex. 99

for (5.1.5): state (h^*, v^*) $\leftrightarrow$ evenly flowing fluid (veclocity v^*)of constant depth h^*

propagation of small perturbations $(\widetilde{h}(x,t), \widetilde{v})(x,t)$ ("ripples") governed by, *cf.* (5.1.6),

$$\frac{\partial}{\partial t}\begin{pmatrix} \widetilde{h} \\ \widetilde{h}\widetilde{v} \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -(v^*)^2 + gh^* & 2v^* \end{pmatrix} \frac{\partial}{\partial x}\begin{pmatrix} \widetilde{h} \\ \widetilde{h}\widetilde{v} \end{pmatrix} = 0 \; .$$

➤ ripples travel with velocities $v^* \pm \sqrt{gh^*}$ (velocity $\pm\sqrt{gh^*}$ *relative* too fluid). ◇

Definition 5.2.2 (Symmetric linear hyperbolic systems of conservation laws)**.**
(5.1.1) *is symmetric, if* $\mathbf{A} = \mathbf{A}^T$

Lemma 5.2.3 ("Energy conservation" for symmetric linear hyperbolic systems)**.**
If $\mathbf{A} = \mathbf{A}^T$ *and* $u_0 \in L^2(\mathbb{R})$ *then* $\int_{\mathbb{R}} |\mathbf{u}(x,t)|^2 \,\mathrm{d}x$ *is constant in time for the solution* $\mathbf{u}$ *of* (5.1.1).

Proof. Straight from (5.2.2) □

Extends to the non-linear case:

Definition 5.2.4 (Symmetric one-dimensional system of conservation laws)**.**
(5.0.1) *symmetric* :⇔ $D\mathbf{F}(u) = (D\mathbf{F}(u))^T$ *for all* $\mathbf{u} \in \mathbb{R}^m$

Lemma 5.2.5 (“Energy conservation” for symmetric conservation laws)**.**
If $\mathbf{u}$ *is a compactly supported classical solution of the Cauchy problem* (5.0.1)/(5.0.2) *on* $\mathbb{R} \times [0,T]$ *for a* symmetric *hyperbolic system of conservation laws, and* $\mathbf{F} \in C^1(\mathbb{R}^m, \mathbb{R}^m)$, *then* $\int_{\mathbb{R}} |\mathbf{u}(x,t)|^2 \,\mathrm{d}x$ *is constant in time*

5.2.1 Boundary conditions

Consider:
$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{A}\frac{\partial \mathbf{u}}{\partial x} = 0 \quad \text{in }]a,b[\times]0,T[\;, \quad -\infty < a < b < \infty .$$
$$u(x,0) = u_0(x) \;, \quad x \in I$$

Assume strict hyperbolicity: eigenvalues of $\mathbf{A}$ $\lambda_1 < \lambda_2 < \cdots < \lambda_m$
related eigenvectors $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_m$

Diagonalization, *cf.* (5.2.1): $\mathbf{w}(x,t) = \mathbf{R}^{-1}\mathbf{u}(x,t)$ satisfies

$$\frac{\partial w_k}{\partial t} + \lambda_k \frac{\partial w_k}{\partial x} = 0 \quad \text{in }]a,b[\times]0,T[\;, \quad \mathbf{w}(x,0) = \mathbf{R}^{-1}\mathbf{u}_0(x)\;, \quad x \in]a,b[\;. \tag{5.2.7}$$

$\lambda_k < 0$: k-wave propagating to left ➡ specify $w_k(b,t)$

$\lambda_k = 0$: “stationary wave” ➡ no boundary data

$\lambda_k > 0$: k-wave propagating to right ➡ specify $w_k(a,t)$

$\longleftarrow \quad \mathbf{u} = \mathbf{R}\mathbf{w}$

➡ notation: index sets $\Lambda_- := \{k\colon \lambda_k < 0\}$, $\Lambda_0 := \{k\colon \lambda_k = 0\}$, $\Lambda_+ := \{k\colon \lambda_k > 0\}$.

➡ write $\mathbf{r}_1, \ldots, \mathbf{r}_m \hat{=}$ columns of matrix $\mathbf{R}$, $\mathbf{g}_1, \ldots, \mathbf{g}_m \hat{=}$ rows of matrix $\mathbf{R}^{-1}$

➤ $$\mathbf{R}^x := [\mathbf{r}_j]_{j\in\Lambda_x} \quad , \quad \mathbf{G}^x := [\mathbf{g}_i^T]^T_{i\in\Lambda_x}, \quad x \in \{-,0,+\}$$

at $x = a$ (left boundary) : $\mathbf{R}^+\mathbf{G}^+\mathbf{u}(a,t) = \mathbf{g}_l(t)$, $\mathbf{g}_l(t) \in \mathrm{Span}\,\{\mathbf{r}_k\colon \lambda_k > 0\}$

at $x = b$ (right boundary) : $\mathbf{R}^-\mathbf{G}^-\mathbf{u}(b,t) = \mathbf{g}_r(t)$, $\mathbf{g}_r(t) \in \mathrm{Span}\,\{\mathbf{r}_k\colon \lambda_k < 0\}$

▶ If $\lambda_1 < \cdots < \lambda_j < 0 < \lambda_{j+1} < \cdots < \lambda_m$:
☞ $m - j$ boundary conditions at $x = a$
☞ j boundary conditions at $x = b$

5.3 The Riemann problem

Cf. Def. 2.4.1: Riemann problem = Cauchy problem for (5.0.1) with $\mathbf{u}_0(x) = \begin{cases} \mathbf{u}_l \in \mathbb{R}^m & \text{, if } x < 0 \text{ ,} \\ \mathbf{u}_r \in \mathbb{R}^m & \text{, if } x \geq 0 \text{ .} \end{cases}$

5.3.1 The linear Riemann problem

Consider: Riemann problem for $\dfrac{\partial \mathbf{u}}{\partial t} + \mathbf{A}\dfrac{\partial \mathbf{u}}{\partial x} = 0$ in $\mathbb{R}\times]0,T[$

Assume strict hyperbolicity: eigenvalues of $\mathbf{A}$ $\lambda_1 < \lambda_2 < \cdots < \lambda_m$
related eigenvectors $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_m$

Wave decomposition: $\mathbf{u}_l = \sum_{k=1}^m w_k^l \mathbf{r}_k \quad , \quad \mathbf{u}_r = \sum_{k=1}^m w_k^r \mathbf{r}_k$

Solution of Riemann problem by diagonalization, see (5.2.1): $\longrightarrow$ [31, Ch. 3]

$$\mathbf{u}(x,t) = \sum_{k=1}^m w_k(x,t)\mathbf{r}_k \quad , \quad w_k(x,t) = \begin{cases} w_k^l & \text{, if } x < \lambda_k t \ , \\ w_k^r & \text{, if } x > \lambda_k t \ . \end{cases} \tag{5.3.1}$$

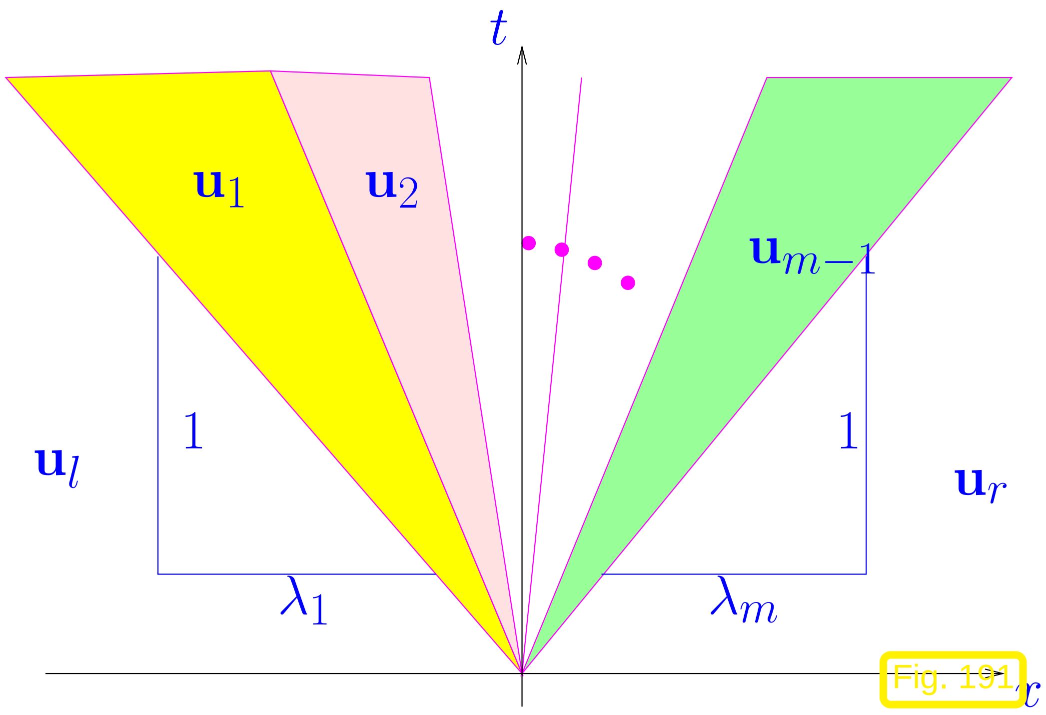


Fig. 191

$\triangleleft$ wave fan

Right and left states connected by $m-1$ intermediate states ($\mathbf{u}_0 := \mathbf{u}_l$, $\mathbf{u}_m := \mathbf{u}_r$)

$$\mathbf{u}_j = \mathbf{u}_l + \sum_{k=1}^{j} (w_k^r - w_k^l)\mathbf{r}_k \ ,$$
$$j = 1, \ldots, m-1 \ .$$

Jumps: $\mathbf{u}_k - \mathbf{u}_{k-1} = (w_k^r - w_k^l)\mathbf{r}_k$ ➤ $\boxed{\mathbf{A}(\mathbf{u}_k - \mathbf{u}_{k-1}) = \lambda_k(\mathbf{u}_k - \mathbf{u}_{k-1})}$, $k = 1, \dots, m$.

(5.3.2)

Parlance: $\mathbf{u}_k - \mathbf{u}_{k-1} \;\hat{=}\;$ k-wave

$m = 5$: solution of Riemann problem for $t = t^*$:

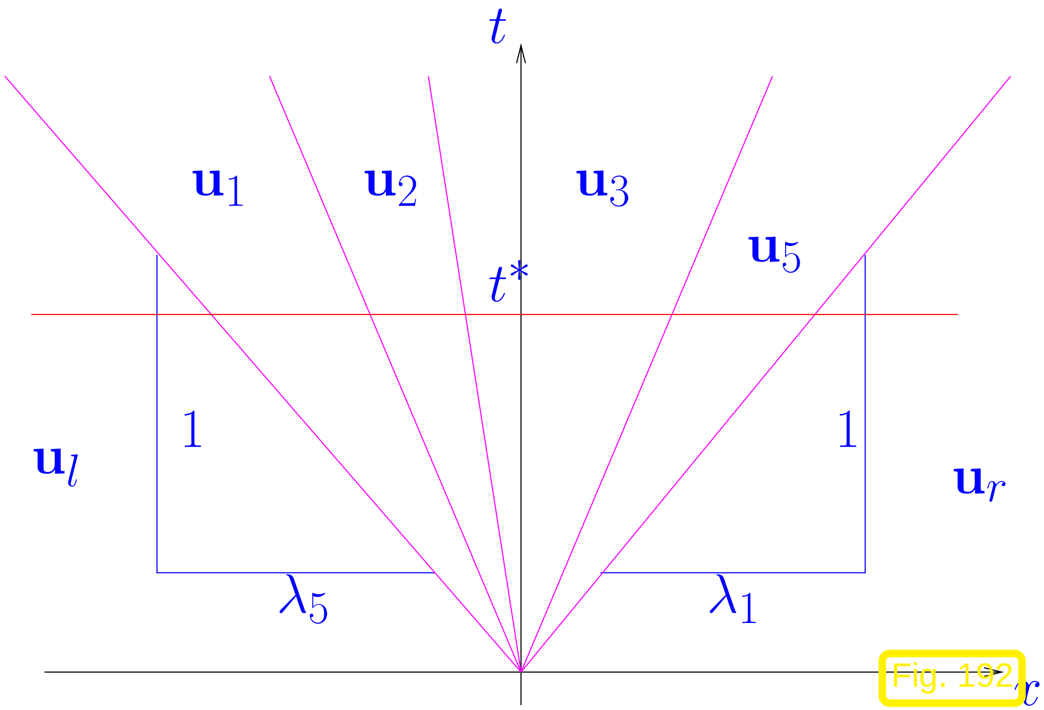


Fig. 192

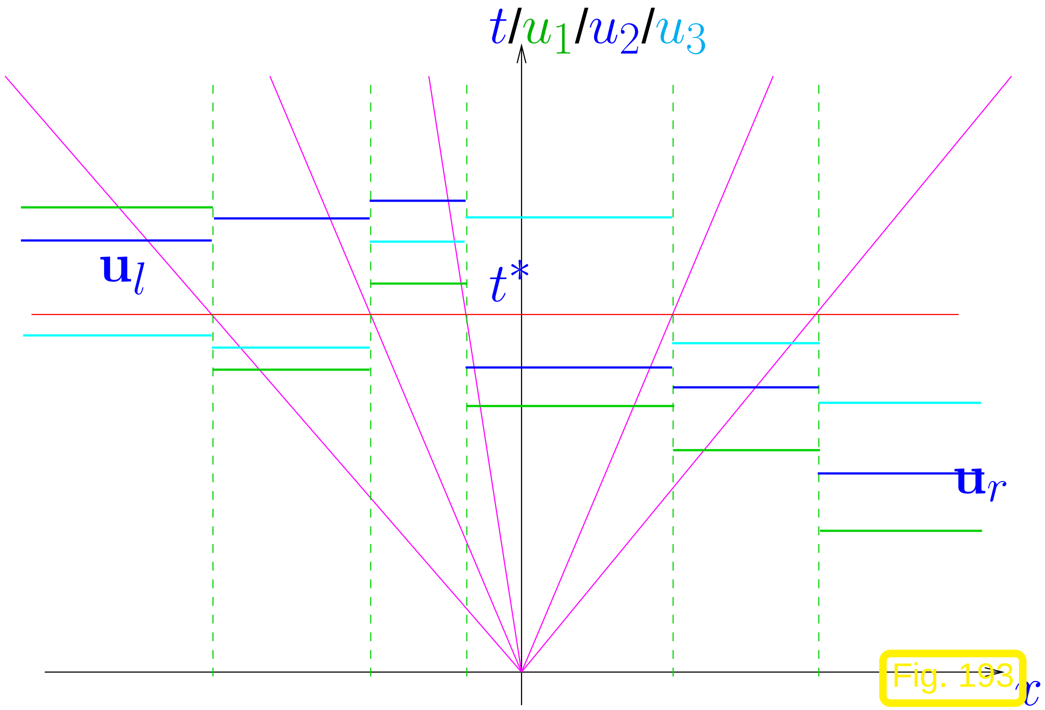
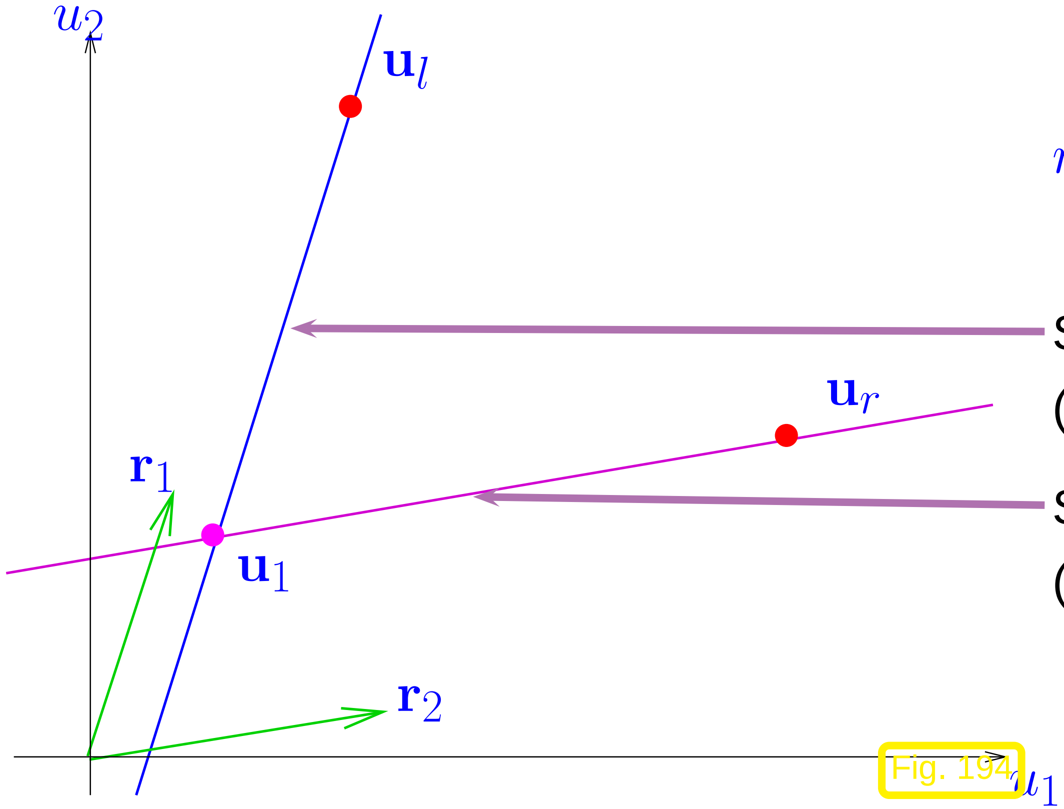


Fig. 193



$m = 2$: visualization of Riemann solution in $u_1 - u_2$-plane **=** phase plane:

States separated from $\mathbf{u}_l$ by a jump with speed λ_1 (slow discontinuity)

States separated from $\mathbf{u}_r$ by a jump with speed λ_2 (fast discontinuity)

5.3.2 Hugoniot loci and shocks

Setting: Cauchy problem for 1D non-linear system of conservation laws (5.0.1) + (5.0.2)

Analoguous to Thm. 2.3.2 (same proof, [29, Lemma 4.1.6]):

Theorem 5.3.1 (Rankine-Hugoniot jump conditions for systems)**.**
Let a C^1-curve $\Gamma := (\gamma(\tau), \tau)$, $0 \leq \tau \leq T$, separate

$$\widetilde{\Omega}_l := \{(x,t) \in \mathbb{R}\times]0,T[: x < \gamma(t)\} \quad , \quad \widetilde{\Omega}_r := \{(x,t) \in \mathbb{R}\times]0,T[: x > \gamma(t)\} \; .$$

$\mathbf{u} \in L^1_{\mathrm{loc}}(\mathbb{R}\times]0,T[)$ *and* $\mathbf{u}_{|\widetilde{\Omega}_l}$/$\mathbf{u}_{|\widetilde{\Omega}_r}$ *can be extended to* $\mathbf{u}_l \in C^1(\overline{\widetilde{\Omega}_l})$, $\mathbf{u}_r \in C^1(\overline{\widetilde{\Omega}_r})$, *which solve* $\frac{\partial \mathbf{u}}{\partial t} + \frac{\partial}{\partial x}\mathbf{F}(\mathbf{u}) = 0$ *in a classical sense (→ Def. 2.2.1) in* $\overline{\widetilde{\Omega}_l}$/$\overline{\widetilde{\Omega}_r}$*. Then* $\mathbf{u}$ *is a weak solution (→ Def. 5.0.1) of* (5.0.1)*, **if and only if***

$$\frac{d\gamma}{d\tau}(\tau)\left(\mathbf{u}_l(\gamma(\tau),\tau) - \mathbf{u}_r(\gamma(\tau),\tau)\right) = \mathbf{F}(\mathbf{u}_l(\gamma(\tau),\tau)) - \mathbf{F}(\mathbf{u}_r(\gamma(\tau),\tau)) \quad \forall 0 < \tau < T \; .$$

$$\Updownarrow$$

$$\boxed{\dot{s}(\mathbf{u}_l - \mathbf{u}_r) = \mathbf{F}_l - \mathbf{F}_r} \quad , \quad \dot{s} := \frac{d\gamma}{d\tau} \quad \text{“propagation speed of discontinuity”} \tag{5.3.3}$$

$m > 1$: Rankine-Hugoniot jump conditions (5.3.3) may not be possible for all $\mathbf{u}_l, \mathbf{u}_r \in \mathbb{R}^m$ **!**
(necessary $\mathbf{u}_l - \mathbf{u}_r \parallel \mathbf{F}_l - \mathbf{F}_r$)

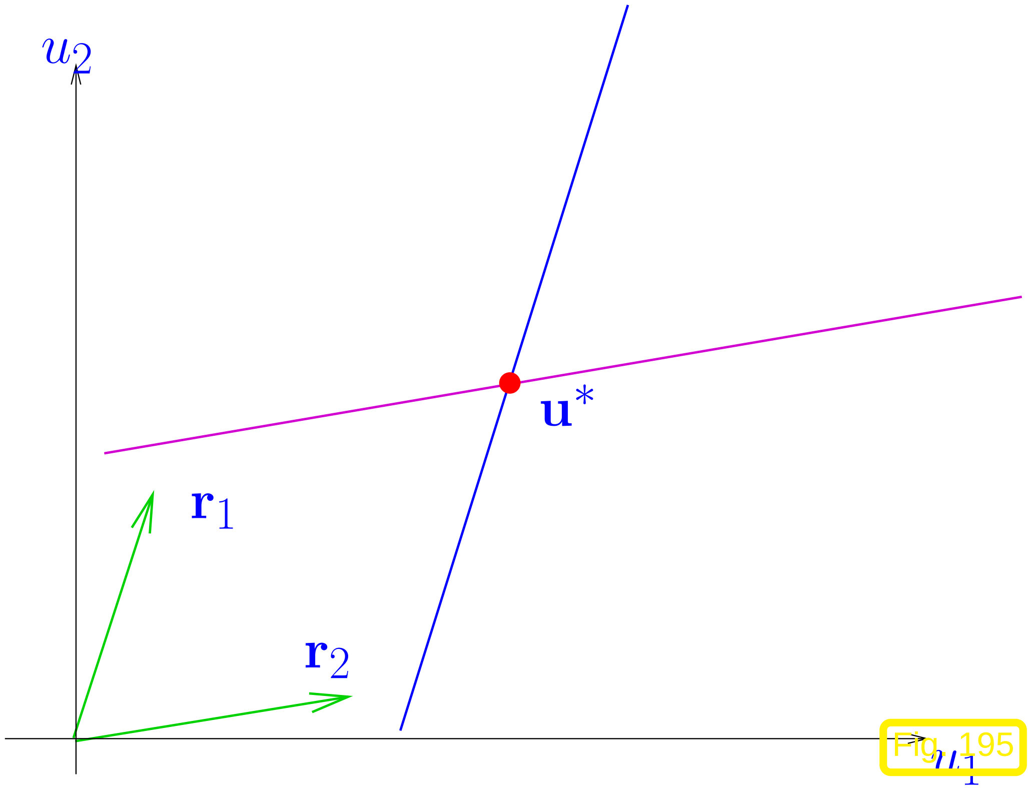
Definition 5.3.2 (Hugoniot locus)**.**
The Hugoniot locus for $\mathbf{u}^* \in U$ *(w.r.t.* (5.0.1)*) is the set*

$$\mathcal{HL}(\mathbf{u}^*) := \{\mathbf{u} \in U\colon \quad \exists \dot{s} \in \mathbb{R}\colon \quad \dot{s}(\mathbf{u}^* - \mathbf{u}) = \mathbf{F}(\mathbf{u}^*) - \mathbf{F}(\mathbf{u})\} \ .$$

► $\mathbf{u} \in \mathcal{HL}(\mathbf{u}^*)$ $\Leftrightarrow$ constant states $\mathbf{u}^*, \mathbf{u}$ separated by discontinuity (shock) provide weak solution of Riemann problem

What is the structure of Hugoniot loci **?**

❶ Special case: linear system of conservation laws $\longrightarrow$ Sect. 5.3.1



For $$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{A}\frac{\partial \mathbf{u}}{\partial x} = 0:$$

Hugoniot locus **=** union of straight lines parallel to eigenvectors of $\mathbf{A}$

$$\mathcal{HL}(\mathbf{u}^*) = \left\{ \mathbf{u} \in \mathbb{R}^m\text{:} \quad \mathbf{u} - \mathbf{u}^* \in \operatorname{Span}\{\mathbf{r}_j\} \text{ for some } j \in \{1, \dots, m\} \right\}$$

◁ situation for $m = 2$

❷ General non-linear case (5.0.1):

(5.3.3) $\longleftrightarrow$ m equations for $m+1$ unknowns $\dot{s}, \mathbf{u}$ ➤ expect 1-dimensional solution manifolds (**=** curves) $\mathbf{u} = \mathbf{u}(s)$, $s \in I \subset \mathbb{R}$

In general case **assume:** (5.0.1) strictly hyperbolic ($\rightarrow$ Def. 5.1.1), $\mathbf{F}$ smooth

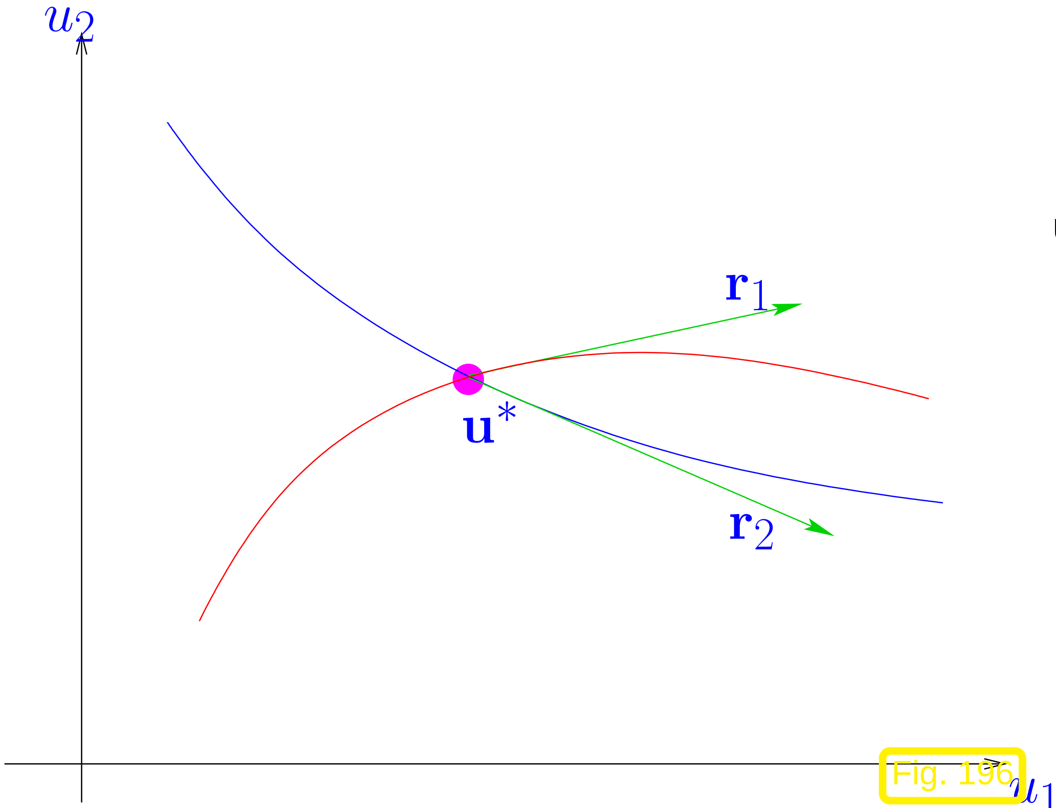


Fig. 196 ◁ local situation for $m = 2$

$\frac{d}{d\mathbf{u}}(\mathbf{F}(\mathbf{u}^*) - \mathbf{F}(\mathbf{u})) - \dot{s}(\mathbf{u}^* - \mathbf{u}))$ rank deficient in $(\mathbf{u}^*, \dot{s})$, if $\dot{s} \in \sigma(D\mathbf{F}(\mathbf{u}^*))$

➥ we find m smooth curves $\mathbf{u}_k = \mathbf{u}_k(s)$, $k = 1, \ldots, m$, $s \in$ *neighborhood* of $\lambda_k(\mathbf{u}^*)$:

- $\mathbf{u}_k(\lambda_k(\mathbf{u}^*)) = \mathbf{u}^*$,
- $s(\mathbf{u}^* - \mathbf{u}_k(s)) = \mathbf{F}(\mathbf{u}^*) - \mathbf{F}(\mathbf{u}_k(s))$,
- $\dfrac{d}{ds}\mathbf{u}_k(\lambda_k) = \mathbf{r}_k$.

Definition 5.3.3 (k-shock)**.**
A discontinuity separating the constant states $\mathbf{u}_l, \mathbf{u}_r \in U$ *with* $\mathbf{u}_r \in \mathcal{HL}(\mathbf{u}_l)$ *is a k-shock, if* $\mathcal{HL}(\mathbf{u}_l)$ *consists of smooth curves in phase space, and* $\mathbf{u}_r$ *is located on a curve with tangent vector* $\mathbf{r}_k$, $k = 1, \ldots, m$, *in* $\mathbf{u}_l$.

Example 103 (Hugoniot loci for shallow water equations). $\longrightarrow$ Ex. 99

Rankine-Hugoniot jump conditions (5.3.3) for shallow water equations (5.1.5):

$$\dot{s}(\mathbf{u}^* - \mathbf{u}) = \mathbf{F}(\mathbf{u}^*) - \mathbf{F}(\mathbf{u}) \quad \Leftrightarrow \quad \begin{aligned} \dot{s}(h^* - h) &= h^*v^* - hv \;, \\ \dot{s}(h^*v^* - hv) &= h^*(v^*)^2 - hv^2 + \tfrac{1}{2}g((h^*)^2 - h^2) \;. \end{aligned}$$

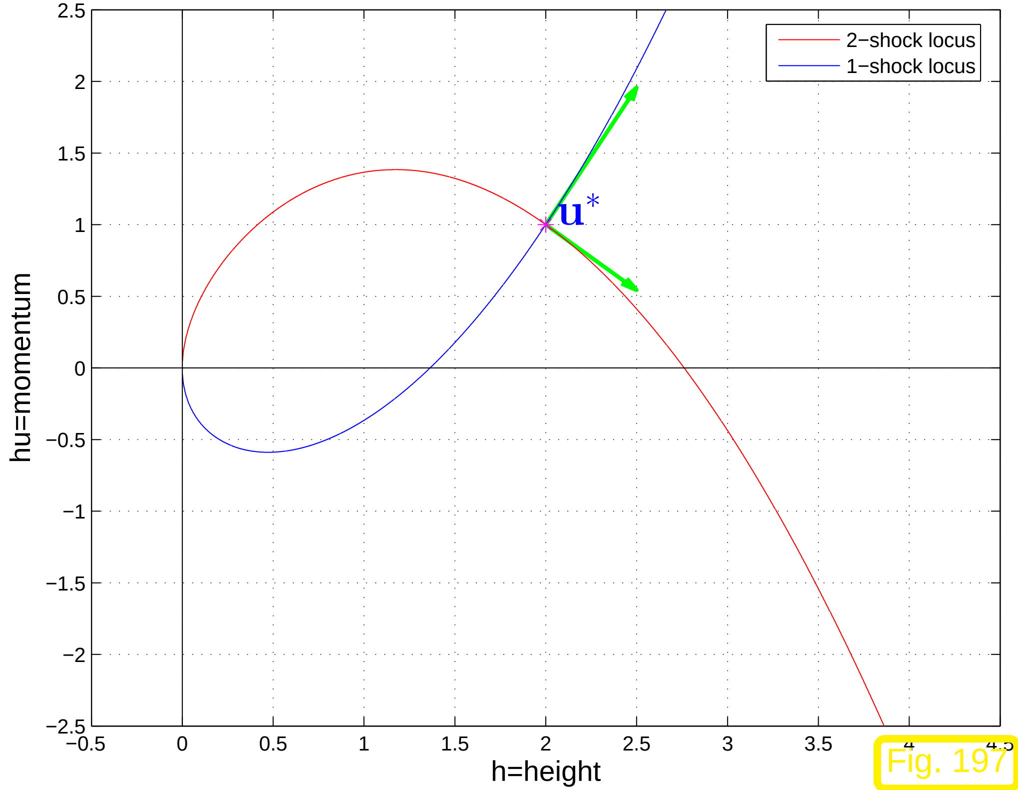


Fig. 197

(elimination of $\dot{s}$) $\longrightarrow$ [31, Sect. 13.7]

$$\blacktriangleright \quad v(h) = v^* \pm \sqrt{\frac{g}{2}\left(\frac{h^*}{h} - \frac{h}{h^*}\right)(h^* - h)} \;.$$

$\lhd$ curves of right states $\mathbf{u}$ satisfying (5.3.3) w.r.t. $\mathbf{u}^* = (2, 0.5)$ ($g = 1$)

$* \;\hat{=}\; (h^*, h^*v^*)$

$\nearrow \;\hat{=}\; \mathbf{r}_1/\mathbf{r}_2$

$$\mathbf{u} \in \mathcal{HL}(\mathbf{u}^*) \quad \Rightarrow \quad \mathbf{u}^* \in \mathcal{HL}(\mathbf{u}) \;,$$

but $\mathcal{HL}(\mathbf{u}^*)$ and $\mathcal{HL}(\mathbf{u})$ may only intersect in a few isolated points !

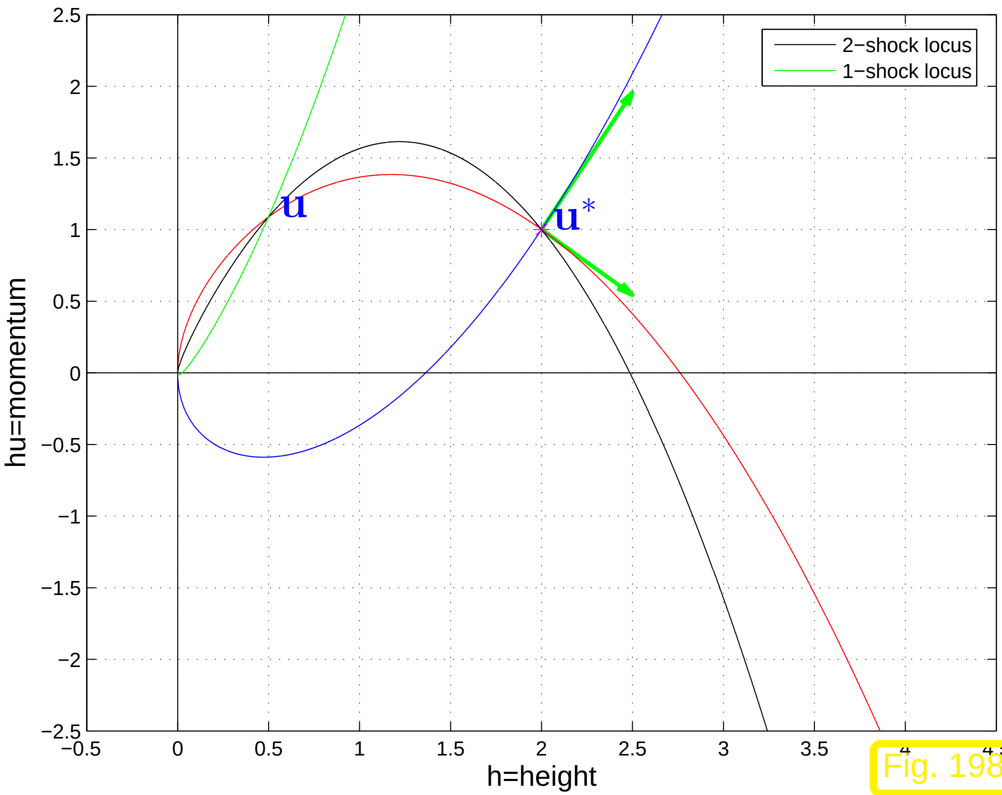


Fig. 198

◇

☞ Computation of all-shock solution of Riemann problem for (5.0.1) and states $\mathbf{u}_l, \mathbf{u}_r \in U$: determine $\mathbf{u}_k$, $k = 1, \dots, m-1$, such that $\quad$ ($\mathbf{u}_0 := \mathbf{u}_l$, $\mathbf{u}_m := \mathbf{u}_r$)

❶ $\quad \dot{s}_k(\mathbf{u}_k - \mathbf{u}_{k-1}) = \mathbf{F}(\mathbf{u}_k) - \mathbf{F}(\mathbf{u}_{k-1}) \;, \quad k = 1, \dots, m \;,$

❷ $\quad \dot{s}_k < \dot{s}_{k+1} \;, \quad k = 1, \dots, m-1 \;.$

Example 104 (All-shock solution of shallow water equations). $\quad \rightarrow$ Ex. 103

- $h_l = h_r = 1$, $v_l = 1/2$, $v_r = -1/2$ (colliding water fronts)

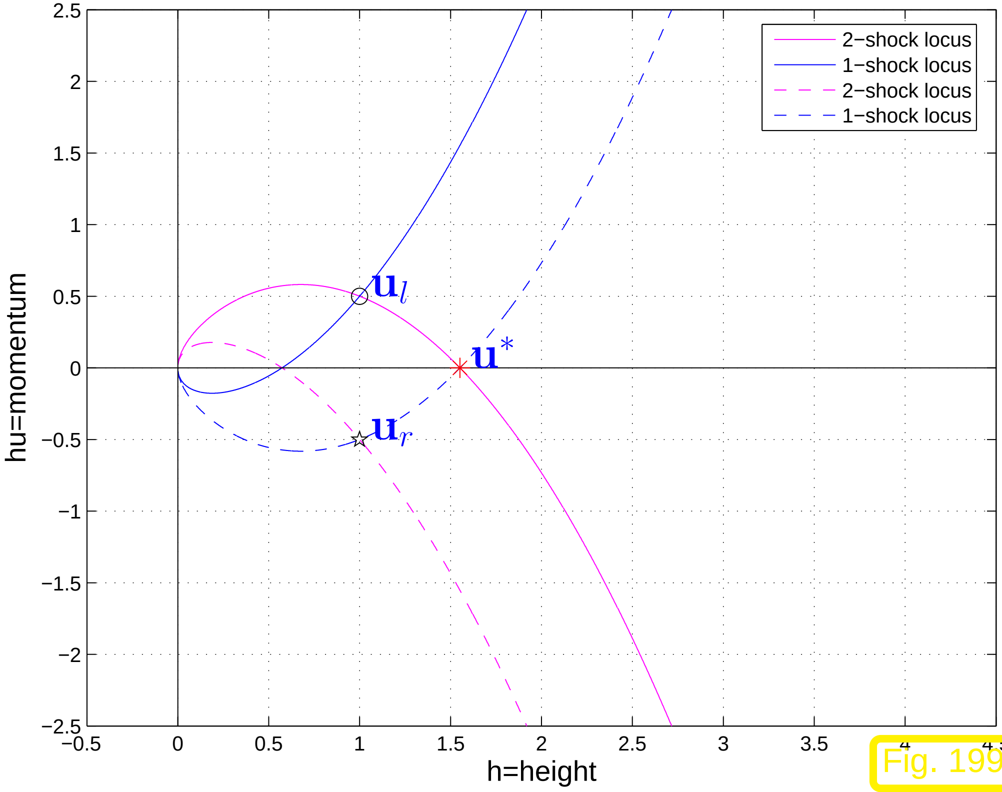


Fig. 199

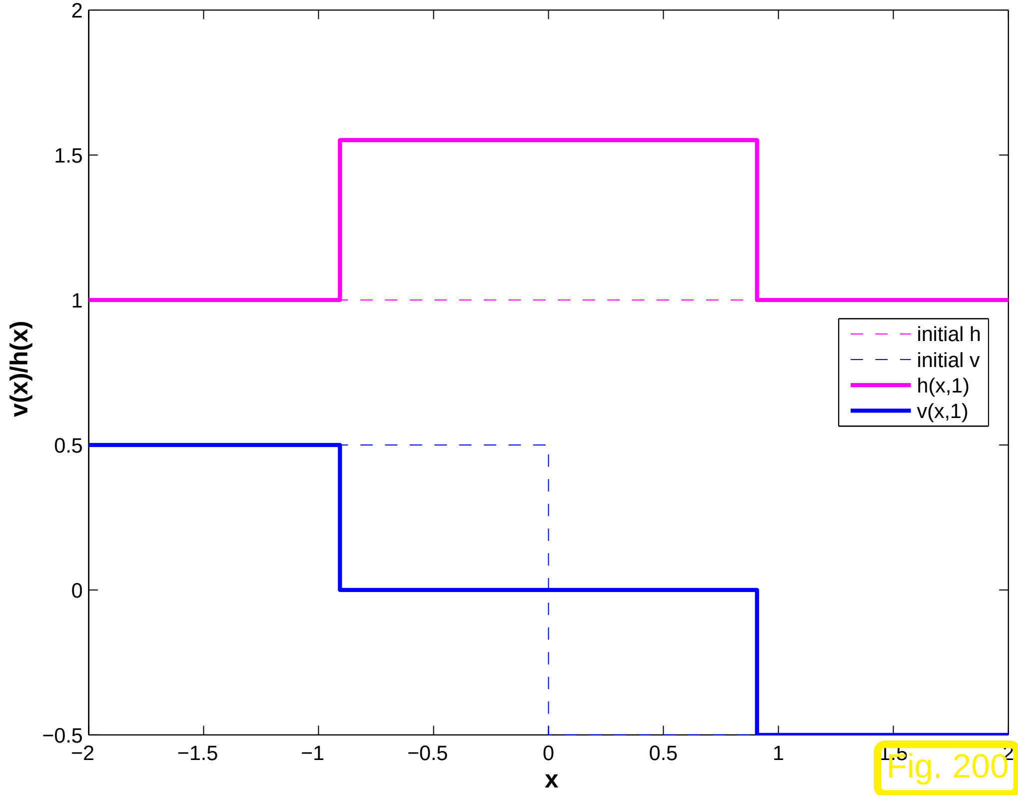


Fig. 200

- $h_l = 1$ $h_r = 3$, $v_l = v_r = 0$ (dam break problem)

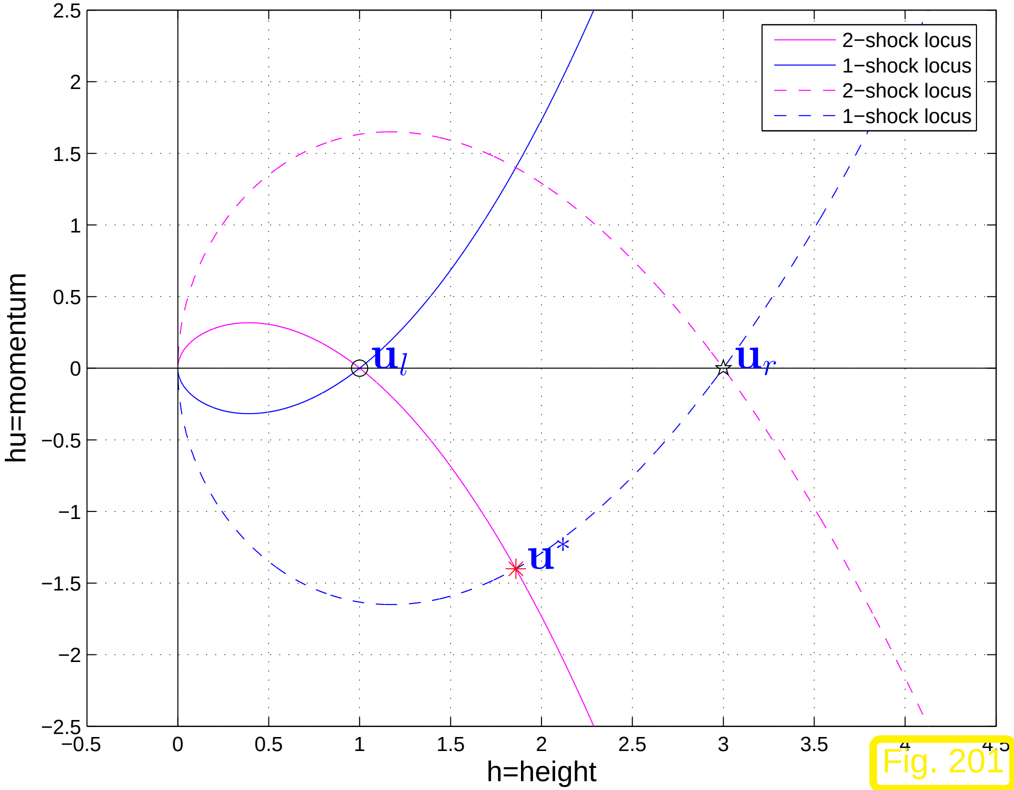


Fig. 201

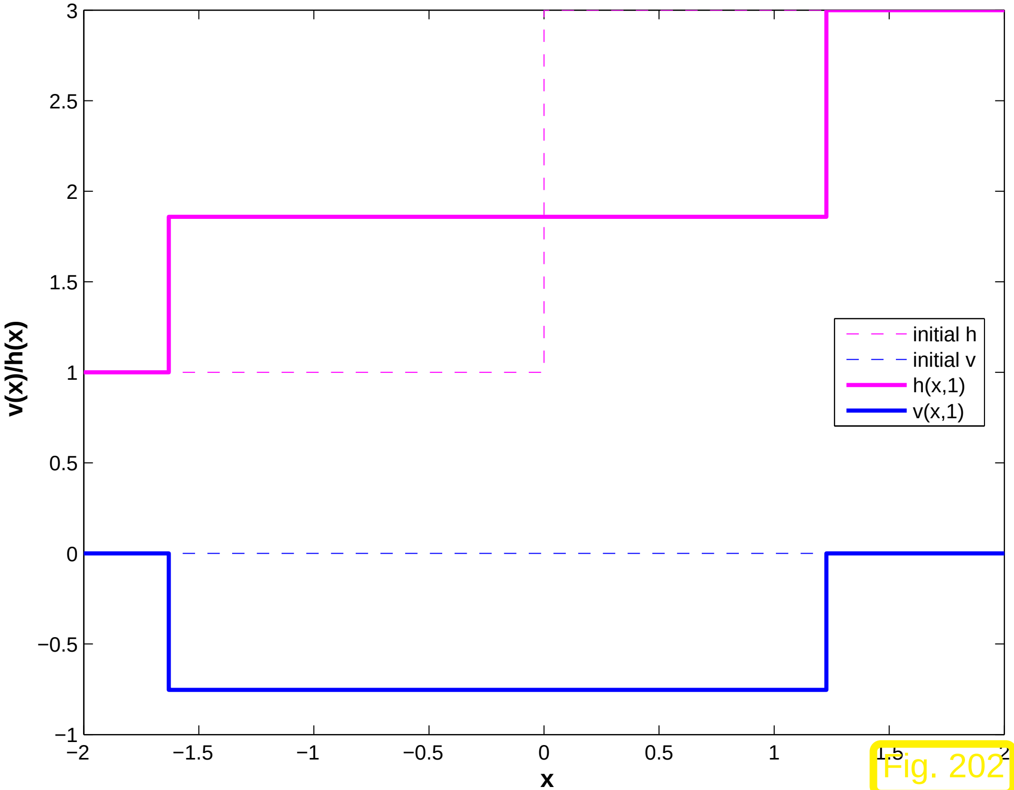


Fig. 202

◇

5.3.3 Simple waves and rarefaction

Setting: Cauchy problem for 1D non-linear system of conservation laws (5.0.1) + (5.0.2)

Recall Sect 2.4.2: construction of rarefaction waves as similarity solutions $\longrightarrow$ Lemma 2.4.4

Again for $m > 1$: only special pairs of states $\mathbf{u}_l$, $\mathbf{u}_r$ can be “connected” by similarity solution

Definition 5.3.4 (Integral curves)**.** *cf. calculus of ODEs*
A smooth curve $\boldsymbol{\kappa} : I \subset \mathbb{R} \mapsto U$, $\tau \in I \subset \mathbb{R}$, is an integral curve for the vectorfield $\mathbf{u} \mapsto \mathbf{r}_k(\mathbf{u})$, if $\mathbf{r}_k$ is tangent to $\boldsymbol{\kappa}$ at each point $\boldsymbol{\kappa}(\tau)$, $\tau \in I$.

$\boldsymbol{\kappa}$ integral curve $\Leftrightarrow$ $\exists \alpha : I \mapsto \mathbb{R} \setminus \{0\}$: $\dfrac{d}{d\tau}\boldsymbol{\kappa}(\tau) = \alpha(\tau)\mathbf{r}_k(\boldsymbol{\kappa}(\tau)) \quad \forall \tau \in I$. **(5.3.4)**

Note: Hugoniot loci ($\longrightarrow$ Def. 5.3.2) **not** composed of integral curves !

Example 105 (Integral curves for shallow water equations)*.* $\longrightarrow$ Ex. 99, [31, Sect. 13.8.1]

Integral curves $\boldsymbol{\kappa}_1$, $\boldsymbol{\kappa}_2$ for eigenvectorfields $\mathbf{r}_1(\mathbf{u})$, $\mathbf{r}_2(\mathbf{u})$ from (5.1.7) with $\kappa(h^*) = (h^*, h^*v^*)^T \in U$:

$$\frac{d}{d\tau}\boldsymbol{\kappa}_{1/2}(\tau) = \begin{pmatrix} 1 \\ \kappa_2/\kappa_1 \mp \sqrt{g\kappa_1} \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{\kappa}_{1/2}(\tau) = \begin{pmatrix} \tau \\ \tau v^* \pm 2\tau(\sqrt{gh^*} - \sqrt{g\tau}) \end{pmatrix} .$$

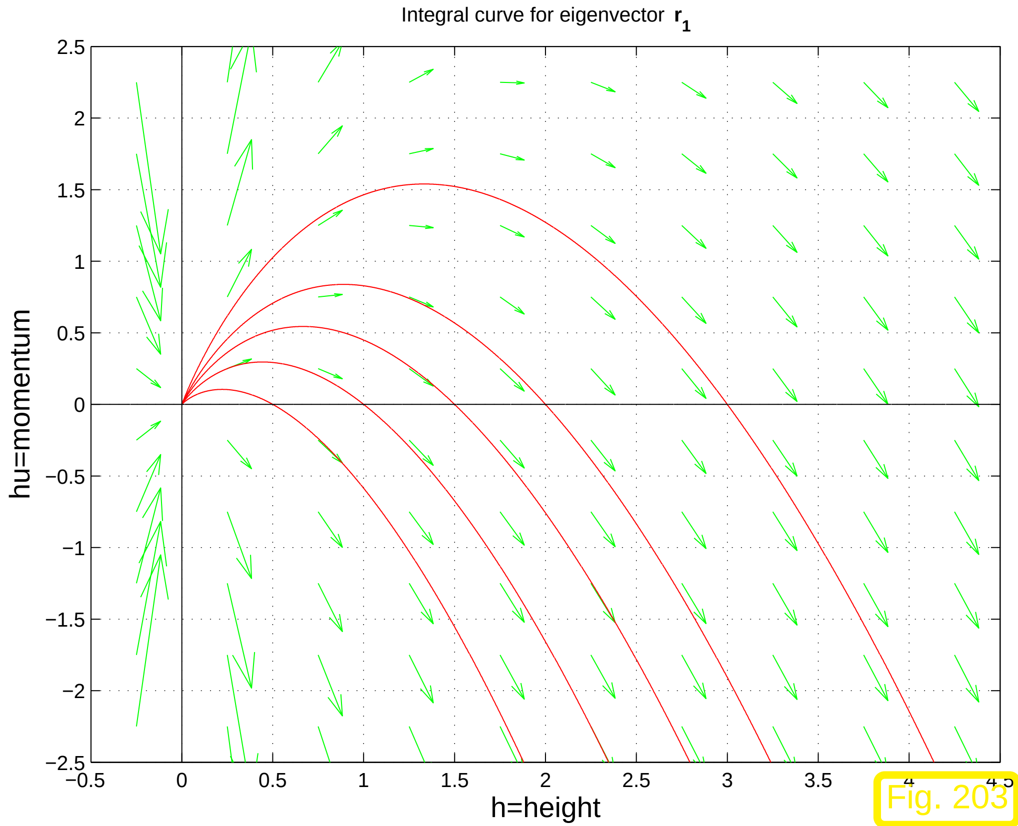


Fig. 203

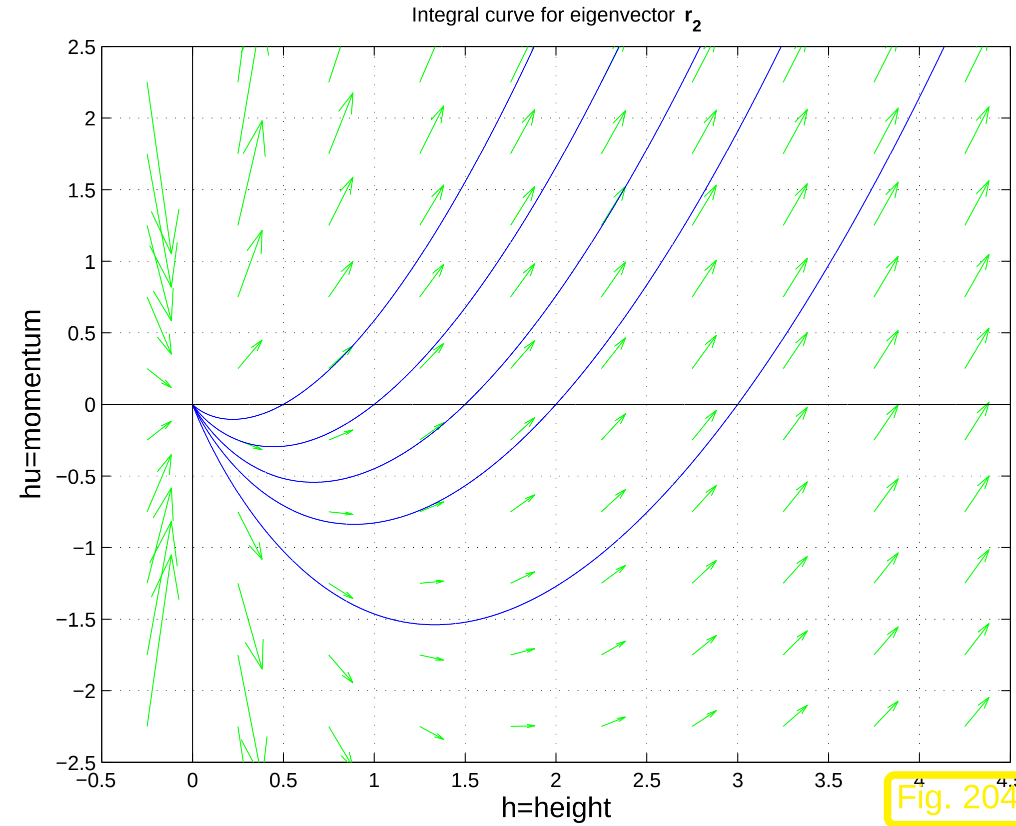


Fig. 204

◇

Definition 5.3.5 (Simple wave). *Let $\boldsymbol{\kappa} : I \subset \mathbb{R} \mapsto U$ be a an integral curve (→ Def. 5.3.4) for $\mathbf{u} \mapsto \mathbf{r}_k(\mathbf{u})$, $k \in \{1, \ldots, m\}$. A weak solution $\mathbf{u}$ of the Cauchy problem for* (5.0.1) *is a simple wave, if*

$$\mathbf{u}(x,t) = \boldsymbol{\kappa}(\xi(x,t)) \;, \quad \textit{a.e. in } \mathbb{R}\times]0,T[\;, \quad \textit{for some function} \quad \xi : \mathbb{R}\times]0,T[\mapsto I \;.$$

If $\mathbf{u} \stackrel{!}{=}$ classical solution of (5.0.1) ($\rightarrow$ Def. 2.2.1)

$$\blacktriangleright \quad \frac{\partial \xi}{\partial t}(x,t) \cdot \frac{d}{d\tau}\boldsymbol{\kappa}(\xi(x,t)) + \frac{\partial \xi}{\partial x}(x,t) D\mathbf{F}(\boldsymbol{\kappa}(\xi(x,t)))\frac{d}{d\tau}\boldsymbol{\kappa}(\xi(x,t)) = 0\ .$$

$$\blacktriangledown \leftarrow D\mathbf{F}(\kappa(\xi))\tfrac{d}{d\tau}\boldsymbol{\kappa}(\xi) = \lambda_k(\boldsymbol{\kappa}(\xi))\tfrac{d}{d\tau}\boldsymbol{\kappa}(\xi)$$

$$\left(\frac{\partial \xi}{\partial t} + \lambda_k(\boldsymbol{\kappa}(\xi))\frac{\partial \xi}{\partial x}\right)\underbrace{\frac{d}{d\tau}\boldsymbol{\kappa}(\xi)}_{\neq 0} = 0\ . \tag{5.3.5}$$

$\longleftrightarrow$ *scalar* hyperbolic evolution equation for ξ: $\frac{\partial \xi}{\partial t} + v(\xi)\frac{\partial \xi}{\partial x} = 0$, $v(\xi) := \lambda_k(\boldsymbol{\kappa}(\xi))$

$\Rightarrow$ ξ constant on characteristics $(\gamma(\tau),\tau)$ ($\rightarrow$ Def. 2.2.2) $\frac{d}{d\tau}\gamma(\tau) = v(\xi(\gamma(\tau),\tau))$, *cf.* Lemma 2.2.3 ➥ characteristics are straight lines !

In simple waves: non-linear system (5.0.1) $\rightarrow$ non-linear scalar hyperbolic equation (5.3.5)

Thm. 2.2.4 ➢ if $\mathbf{u}_0(x) = \boldsymbol{\kappa}(\xi_0(x))$, then for $0 \leq t \leq T_\infty \leq T$, $x \in \mathbb{R}$

$$\mathbf{u}(x,t) = \boldsymbol{\kappa}(\xi(x,t)) \quad \text{where} \quad \begin{aligned} &\frac{\partial \xi}{\partial t} + \lambda_k(\boldsymbol{\kappa}(\xi))\frac{\partial \xi}{\partial x} = 0 \quad \text{in } \mathbb{R}\times]0,T_\infty[\ , \\ &\xi(x,0) = \xi_0(x) \quad \text{in } \mathbb{R}\ . \end{aligned}$$

finite time breakdown of simple waves possible ! $\longrightarrow$ Sect. 2.2

Special situation: $x \mapsto \lambda_k(\boldsymbol{\kappa}(\xi_0(x)))$ increasing $\Rightarrow$ $T_\infty = T$ (simple wave solution exists $\forall t$)

Recall (Sect. 2.4, Lemma 2.4.4): Simple structure of Riemann solutions of 1D scalar conservation laws, if f strictly convex/concave

Generalization to systems (5.0.1):

Definition 5.3.6 (Genuine non-linearity)**.**
The k*-th field for* (5.0.1) *is genuinely non-linear, if*

$$\mathbf{grad_u}\,\lambda_k(\mathbf{u}) \cdot \mathbf{r}_k(\mathbf{u}) \neq 0 \quad \forall \mathbf{u} \in U \ .$$

genuine non-linearity $\Leftrightarrow$ $\tau \mapsto \lambda_k(\boldsymbol{\kappa}(\tau))$ strictly monotone

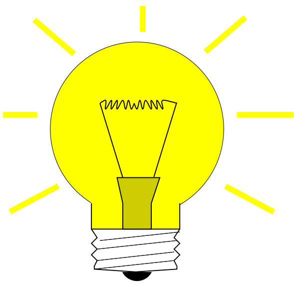
Example 106 (Genuine non-linearity for shallow water equations). $\longrightarrow$ Ex. 99

For (5.1.5): $\lambda_{1/2}(\mathbf{u}) = \frac{u_2}{u_1} \mp \sqrt{g u_1}$, $\mathbf{r}_{1/2}(\mathbf{u}) = \begin{pmatrix} 1 \\ \lambda_{1/2}(\mathbf{u}) \end{pmatrix}$

$$\blacktriangleright \qquad \mathbf{grad}_{\mathbf{u}}\, \lambda_{1/2} \cdot \mathbf{r}_{1/2}(\mathbf{u}) = \mp \frac{3}{2}\sqrt{\frac{g}{u_1}} \neq 0 \quad \forall \mathbf{u} \in \mathbb{R}^+ \times \mathbb{R} \,.$$

$\diamond$

Assume: genuine non-linearity of k-th field $\longrightarrow$ Def. 5.3.6



Idea: rarefaction waves for 1D systems $\longrightarrow$ Sect. 2.4.2

$\Updownarrow$

simple wave similarity solution of Riemann problem

$\leftrightarrow$ $\xi(x,t) = x/t$ in Def. 5.3.5

For given integral curve $\boldsymbol{\kappa} : I \mapsto U$ ($\leftrightarrow$ k-th eigenvector field $\mathbf{r}_k$ of $D\mathbf{F}(\mathbf{u})$, see Def. 5.3.4), and

$$\boxed{\mathbf{u}_l, \mathbf{u}_r \in \boldsymbol{\kappa}(I)} \quad , \quad \lambda_k(\mathbf{u}_l) < \lambda_k(\mathbf{u}_r) \,, \tag{5.3.6}$$

try *continuous* similarity solution

$$\mathbf{u}(x,t) = \begin{cases} \mathbf{u}_l & \text{for} \quad x < \dot{s}_l t \ , \\ \boldsymbol{\kappa}(x/t) & \text{for} \quad \dot{s}_l t < x < \dot{s}_r t \ , \\ \mathbf{u}_r & \text{for} \quad x > \dot{s}_r t \ , \end{cases} \tag{5.3.7}$$

$$\boldsymbol{\kappa}(\dot{s}_l) = \mathbf{u}_l \ , \quad \boldsymbol{\kappa}(\dot{s}_r) = \mathbf{u}_r \ , \qquad \dot{s}_l < \dot{s}_r \ . \tag{5.3.8}$$

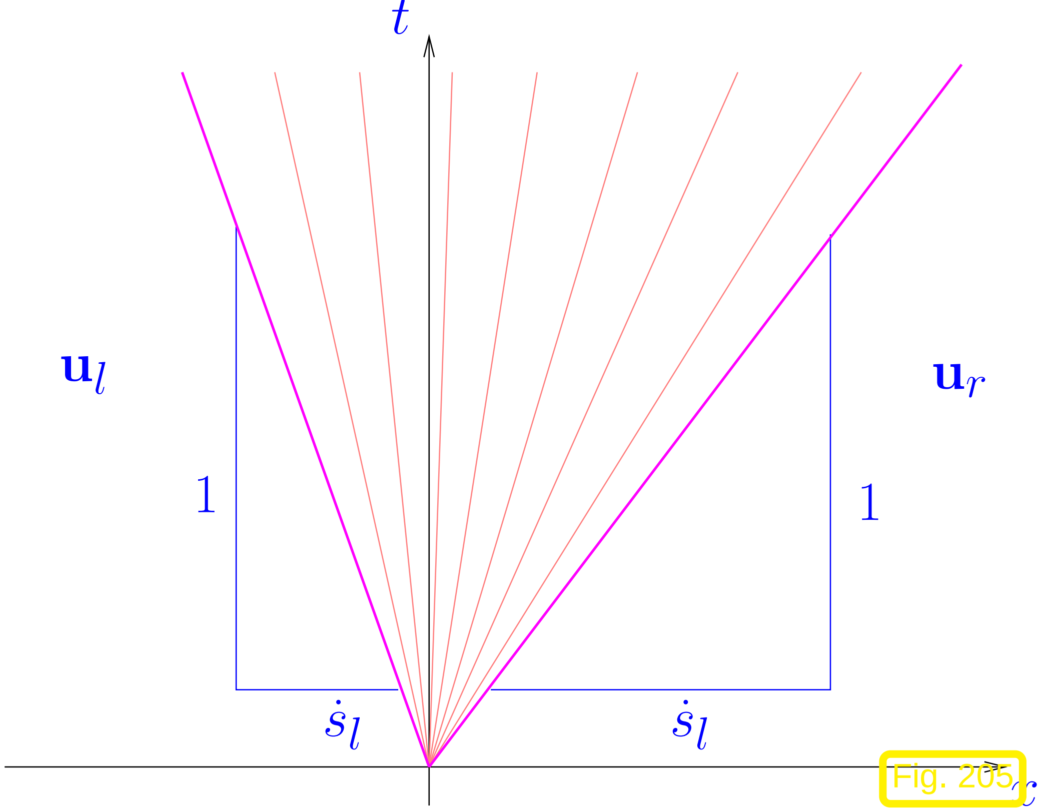


Fig. 205

$\mathbf{u}$ solves (5.0.1) $\Rightarrow \quad -\dfrac{x}{t^2} + \lambda_k(\boldsymbol{\kappa}(x/t))\dfrac{1}{t} = 0 \quad \Leftrightarrow \quad \lambda_k(\boldsymbol{\kappa}(x/t)) = x/t$.

$$\Rightarrow \quad \dot{s}_l = \lambda_k(\mathbf{u}_l) \ , \quad \dot{s}_r = \lambda_k(\mathbf{u}_r) \ , \quad \lambda_k(\boldsymbol{\kappa}(\tau)) = \tau \ . \tag{5.3.9}$$

(5.3.9) $\overset{(5.3.4)}{\Rightarrow} \quad \mathbf{grad}_{\mathbf{u}}\, \lambda_k(\boldsymbol{\kappa}(\tau)) \cdot \alpha(\tau)\mathbf{r}_k(\boldsymbol{\kappa}(\tau)) = 1$ with $\alpha : I \mapsto \mathbb{R}^+$.

$$\blacktriangleright \quad \frac{d}{d\tau}\boldsymbol{\kappa}(\tau) = \frac{1}{\underbrace{\mathbf{grad}_{\mathbf{u}}\, \lambda_k(\boldsymbol{\kappa}(\tau)) \cdot \mathbf{r}_k(\boldsymbol{\kappa}(\tau))}_{\text{well defined by genuine non-linearity}}}\ \mathbf{r}_k(\boldsymbol{\kappa}(\tau)) \ , \quad \tau \in I \ . \tag{5.3.10}$$

(5.3.7) **+** parameterization (5.3.10) **=** rarefaction wave solution of Riemann problem for 1D system of conservation laws

Example 107 (Rarefaction wave for shallow water equations). Ex. 99, Ex. 106, [31, Ex. 13.9]

Parameterization of integral curve for 1-rarefaction for $\mathbf{F}(\mathbf{u}) = \begin{pmatrix} u_2 \\ u_2^2 u_1^{-1} + \frac{1}{2} g u_1^2 \end{pmatrix}$

(5.3.10) ➡ $\dfrac{d}{d\tau}\boldsymbol{\kappa}(\tau) = -\dfrac{2}{3}\sqrt{\dfrac{\kappa_1}{g}} \begin{pmatrix} 1 \\ \kappa_2/\kappa_1 - \sqrt{g\kappa_1} \end{pmatrix} \Rightarrow \kappa_1(\tau) = \dfrac{1}{9g}(C-\tau)^2 , \quad C \in \mathbb{R}$.

C fixed by $\kappa_1(\lambda_1(\mathbf{u}_l)) = h_l$, $\kappa_1(\lambda_1(\mathbf{u}_r)) = h_r$ ➔ possible **?**

Note Riemann invariant: $w_1(\boldsymbol{\kappa}(\tau)) \equiv \text{const}$ for $w_1(\mathbf{u}) = u_2/u_1 + 2\sqrt{gu_1}$

$$\begin{aligned} \kappa_1(\tau) &= \frac{1}{9g}(v_l + 2\sqrt{gh_l} - \tau)^2 , \\ \kappa_2(\tau) &= \kappa_1(\tau)v_l + 2\kappa_1(\tau)(\sqrt{gh_l} - \sqrt{g\kappa_1(\tau)}) . \end{aligned} \tag{5.3.11}$$

☞ rarefaction solution from formula (5.3.7).

- rarefaction evolution for $h_l = 2$, $h_r = 0.5$, $v_l = 0$, $v_r = 1.414214$ ($g = 1$)

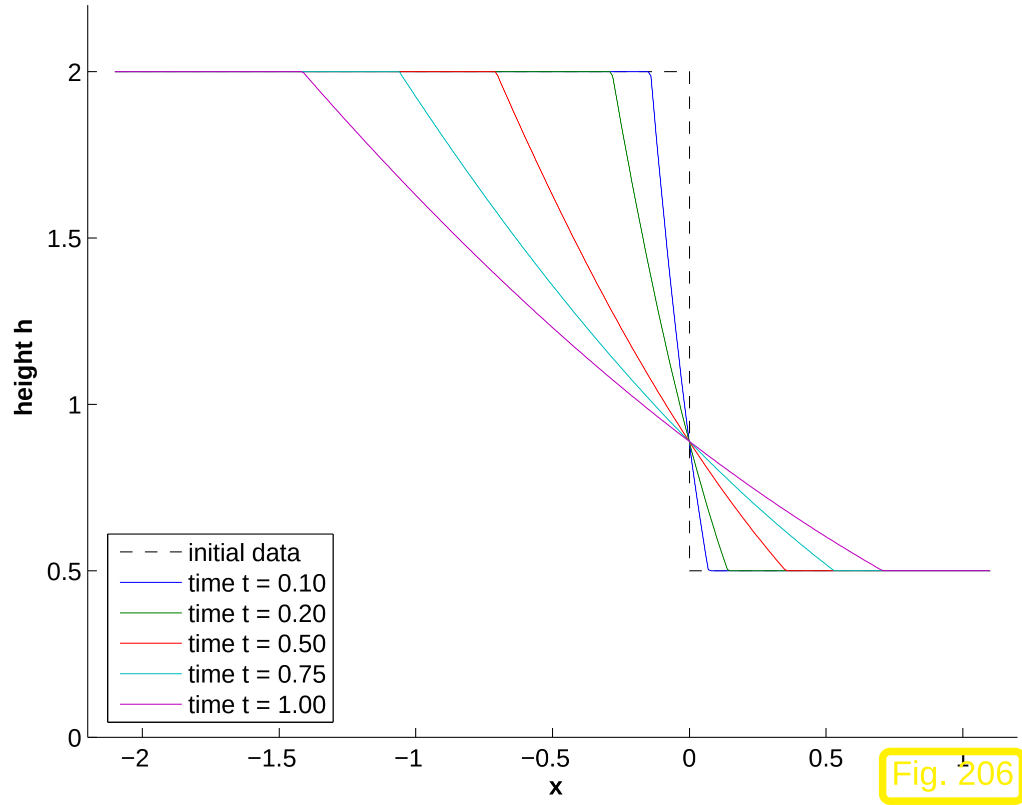


Fig. 206

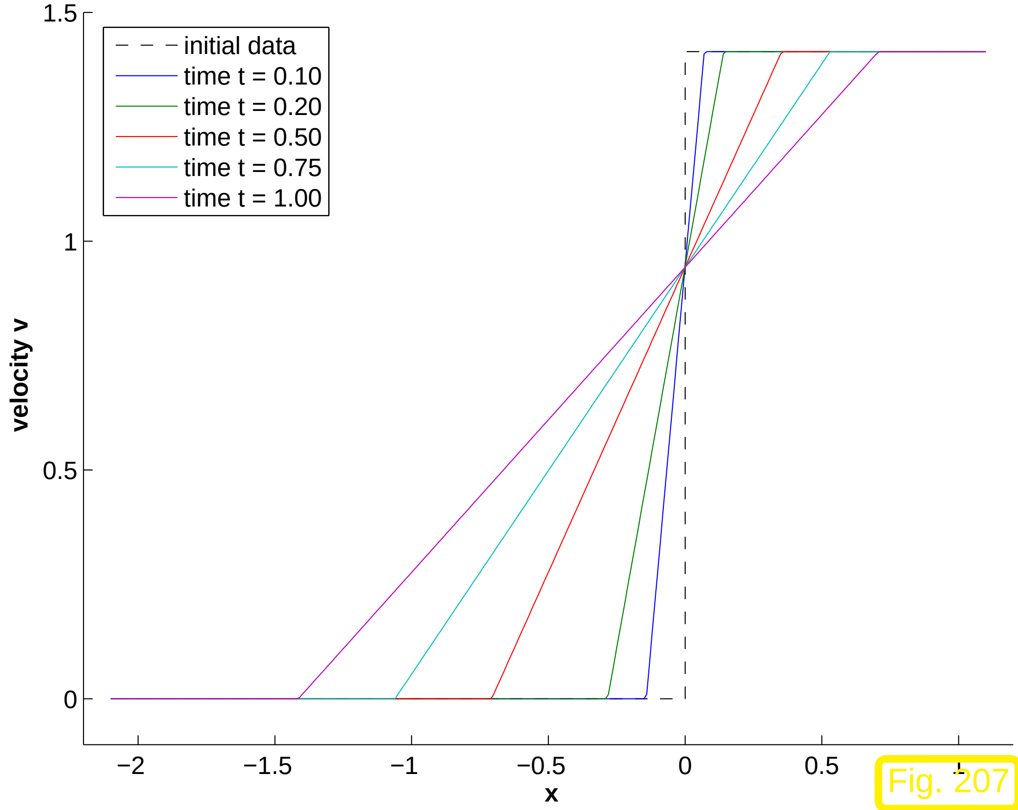


Fig. 207

◇

Example 108 (All-rarefaction solution for Riemann problem for shallow water equations).

Given $\mathbf{u}_l, \mathbf{u}_r \in U$ find two integral curves (→ Def. 5.3.4) $\boldsymbol{\kappa}_1$, $\boldsymbol{\kappa}_2$ and intermediate state $\mathbf{u}^*$, *cf.* Ex. 104, such that

① $\boldsymbol{\kappa}_1$ is associated with eigenvectorfield $\mathbf{r}_1(\mathbf{u})$ **&** connects $\mathbf{u}_l$ and $\mathbf{u}^*$

② $\boldsymbol{\kappa}_2$ is associated with eigenvectorfield $\mathbf{r}_2(\mathbf{u})$ **&** connects $\mathbf{u}^*$ and $\mathbf{u}_r$

③ $\lambda_1(\mathbf{u}_l) < \lambda_1(\mathbf{u}^*)$ and $\lambda_2(\mathbf{u}_r) > \lambda_2(\mathbf{u}^*)$

- Riemann problems as in Ex. 104: possible rarefaction solutions ?

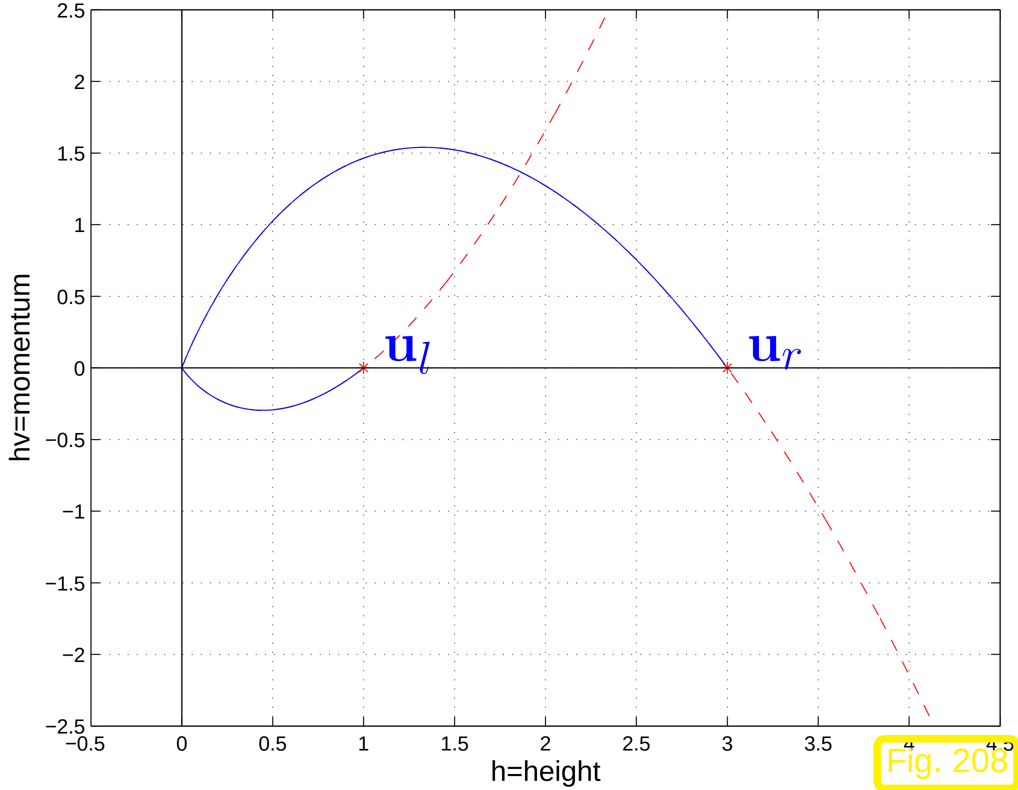


Fig. 208

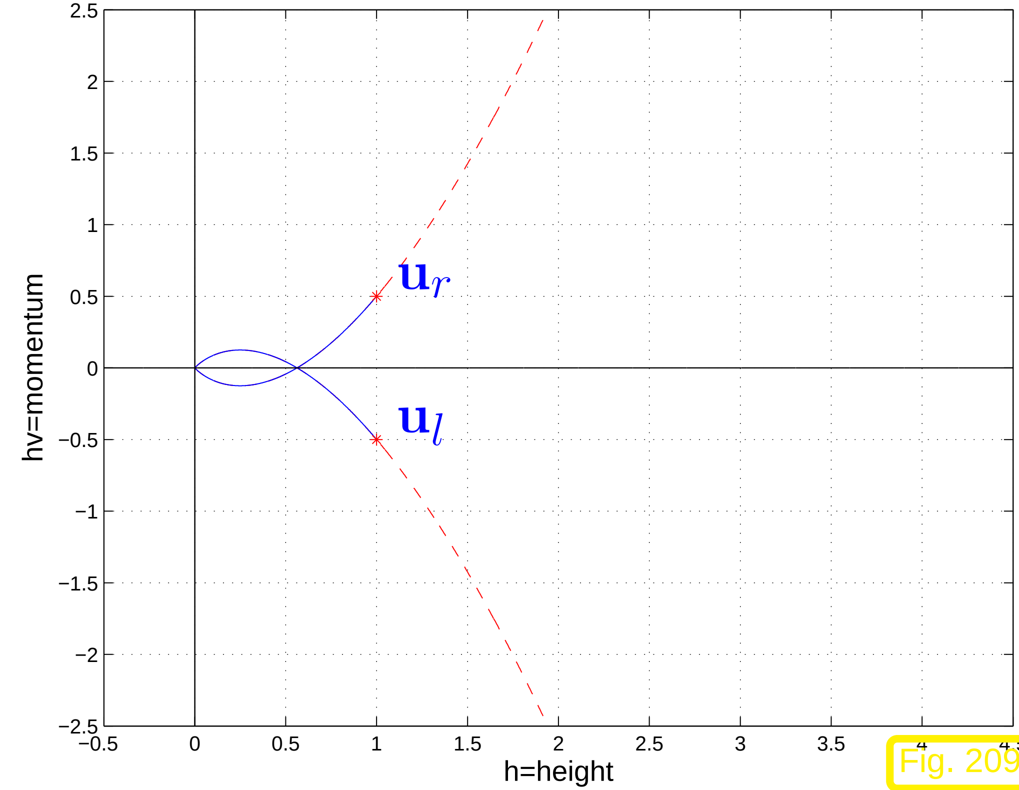


Fig. 209

◇

As for $m = 1$, Sect. 2.4: **Non-uniqueness of weak solutions !**

5.4 Entropy conditions

As in Sect. 2.5.1: vanishing viscosity limit selects "physically meaningful" solutions:

$$\mathbf{u} = \lim_{\epsilon \to 0} \mathbf{u}_\epsilon \quad \text{where} \quad \begin{aligned} \frac{\partial \mathbf{u}_\epsilon}{\partial t} + \frac{\partial}{\partial x}\mathbf{F}(\mathbf{u}_\epsilon) &= \epsilon \frac{\partial^2}{\partial x^2}\mathbf{u}_\epsilon \quad \text{in } \mathbb{R}\times]0,T[\;, \\ \mathbf{u}_\epsilon(x,0) &= \mathbf{u}_0(x) \quad \text{a.e. in } \mathbb{R}\;. \end{aligned}$$

▶ As in Sect. 2.5.2:

Definition 5.4.1 (Pair of entropy functions for systems). *cf. Def. 2.5.2*
$\eta, \psi \in C^2(U,\mathbb{R})$ *is a pair of entropy functions for* (5.0.1), *if*

$$\eta \text{ is strictly convex} \quad \text{and} \quad D\mathbf{F}(\mathbf{u})^T\,\mathbf{grad}\,\eta(\mathbf{u}) = \mathbf{grad}\,\psi(\mathbf{u}) \text{ for all } \mathbf{u} \in U.$$

➨ notations for derivatives:

$$\mathbf{F}(\mathbf{u}) = \begin{pmatrix} F_1(u_1,\dots,u_m) \\ \vdots \\ F_m(u_1,\dots,u_m) \end{pmatrix} \;:\; D\mathbf{F}(\mathbf{u}) = \begin{pmatrix} \frac{\partial F_1}{\partial u_1} & \cdots & \frac{\partial F_1}{\partial u_m} \\ \vdots & & \vdots \\ \frac{\partial F_m}{\partial u_1} & \cdots & \frac{\partial F_m}{\partial u_m} \end{pmatrix} .$$

$$\eta : U \subset \mathbb{R}^m \mapsto \mathbb{R} : \quad \mathbf{grad}\,\eta(\mathbf{u}) := \begin{pmatrix} \frac{\partial \eta}{\partial u_1} \\ \vdots \\ \frac{\partial \eta}{\partial u_m} \end{pmatrix} , \quad D\eta(\mathbf{u}) := \begin{pmatrix} \frac{\partial \eta}{\partial u_1} & \cdots & \frac{\partial \eta}{\partial u_m} \end{pmatrix} .$$

Definition 5.4.2 (Entropy consistency of weak solutions). *cf. Def. 2.5.3*

A weak solution $\mathbf{u}$ *(→ Def. 5.0.1) of a Cauchy problem for* (5.0.1) *is consistent with the entropy pair* (η, ψ) *(→ Def. 2.5.2), if*

$$\frac{\partial}{\partial t}\eta(\mathbf{u}(x,t)) + \frac{\partial}{\partial x}\psi(\mathbf{u}(x,t)) \leq 0 \quad \textit{in } \mathbb{R}\times]0,T[\tag{5.4.1}$$

in weak sense, see Def. 2.5.3.

☞ If $\mathbf{u}$ is classical solution (→ Def. 2.2.1), then (5.4.1) becomes pointwise equality, *cf.* (2.5.3).

How to find entropy pairs **?**

▷ $m = 1$: every smooth convex function belongs to an entropy pair, see Sect. 2.5.2

▷ $m = 2$: existence of entropy pairs for smooth flux functions

▷ $m \geq 3$: existence of entropy pairs **?**

entropy pairs available for “physically meaningful” systems of conservation laws

Example 109 (Entropy pair for shallow water equations). $\longrightarrow$ Ex. 99

(5.1.5): $\mathbf{F}(\mathbf{u}) = \begin{pmatrix} u_2 \\ u_2^2 u_1^{-1} + \frac{1}{2} g u_1^2 \end{pmatrix}$

➥ “energy as entropy”:

$$\eta(\mathbf{u}) = {}^1\!/\!_2 h v^2 + {}^1\!/\!_2 g h^2 = {}^1\!/\!_2 u_2^2 / u_1 + {}^1\!/\!_2 g u_1^2 \;,$$
$$\psi(\mathbf{u}) = {}^1\!/\!_2 h v^3 + g h^2 v = {}^1\!/\!_2 u_2^3 / u_1^2 = g u_2 u_1 \;.$$

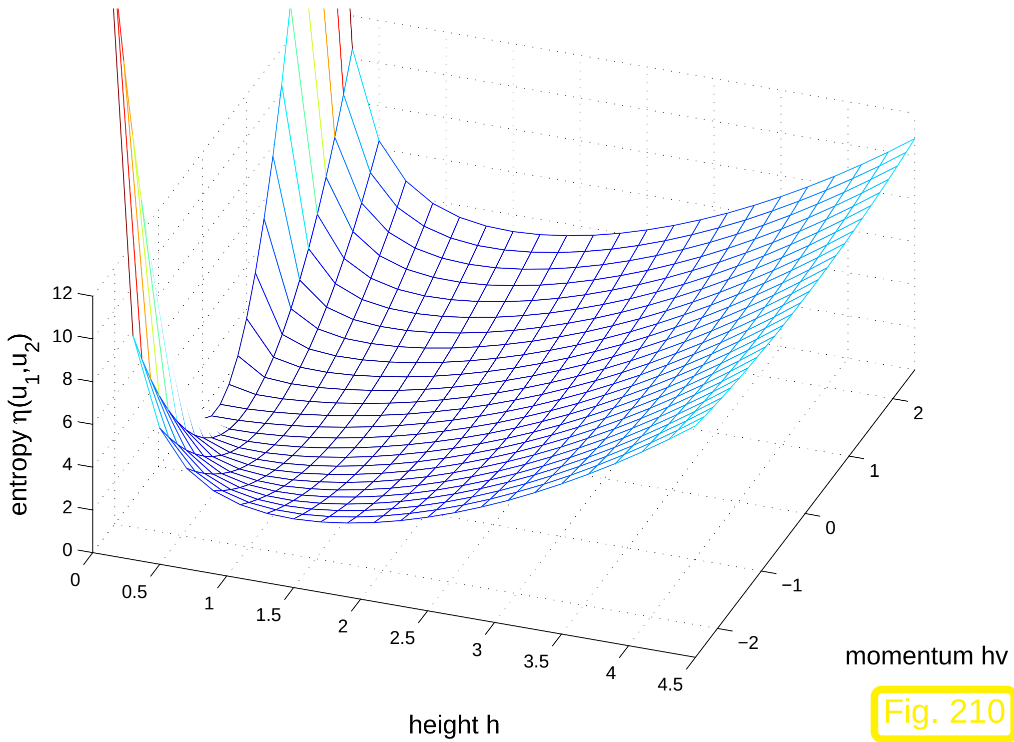


Fig. 210

◇

Example 110 (Entropy for symmetric hyperbolic systems). $\longrightarrow$ Def. 5.2.4, [15, Ex. 3.2]

$$\eta(\mathbf{u}) = \frac{1}{2}|\mathbf{u}|^2 \quad , \quad \psi(\mathbf{u}) = D\mathbf{F}(\mathbf{u})^T\mathbf{u} - \Psi(\mathbf{u}) \; , \tag{5.4.2}$$

where $\Psi : U \mapsto \mathbb{R}$ is scalar potential for $\mathbf{F}(\mathbf{u})$, see proof of Lemma 5.2.5.

$\Diamond$

Example 111 (Entropy consistent shocks for shallow water equation). $\longrightarrow$ Ex. 99, Ex. 104, Ex. 109

entropy inequality (5.4.1) applied to locally piecewise constant weak solution of (5.0.1), *cf.* (2.5.4),

$$\Rightarrow \qquad \dot{s}(\eta(\mathbf{u}_l) - \eta(\mathbf{u}_r)) \leq \psi(\mathbf{u}_l) - \psi(\mathbf{u}_r) \; , \tag{5.4.3}$$

notations from Thm. 5.3.1, $\dot{s} \hat{=}$ local speed of discontinuity (shock).

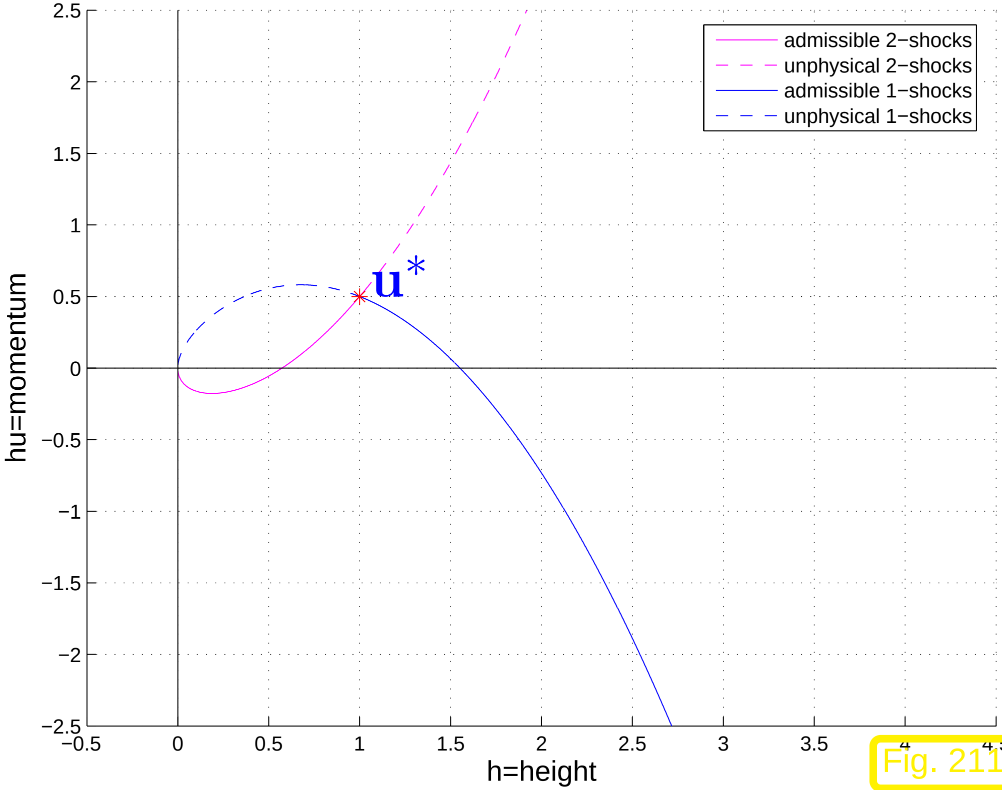


Fig. 211

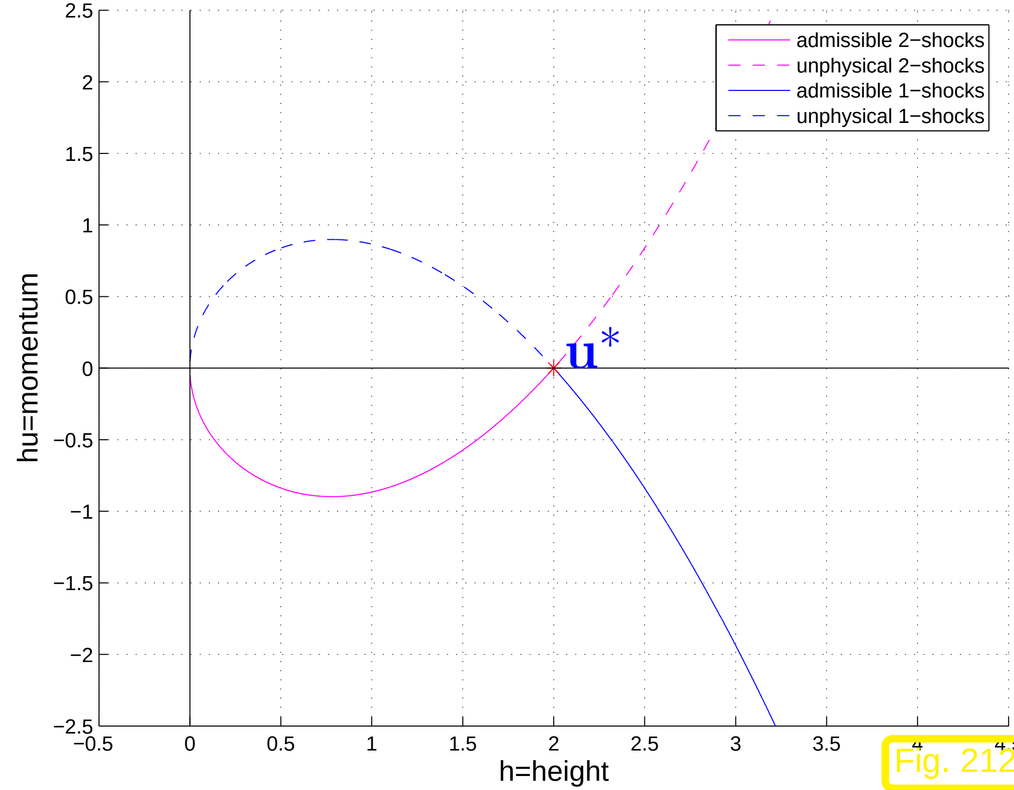


Fig. 212

dashed lines: parts of Hugoniot locus ($\longrightarrow$ Def. 5.3.2) corresponding to entropy violating shocks

➢ application to Riemann problems of Ex. 104

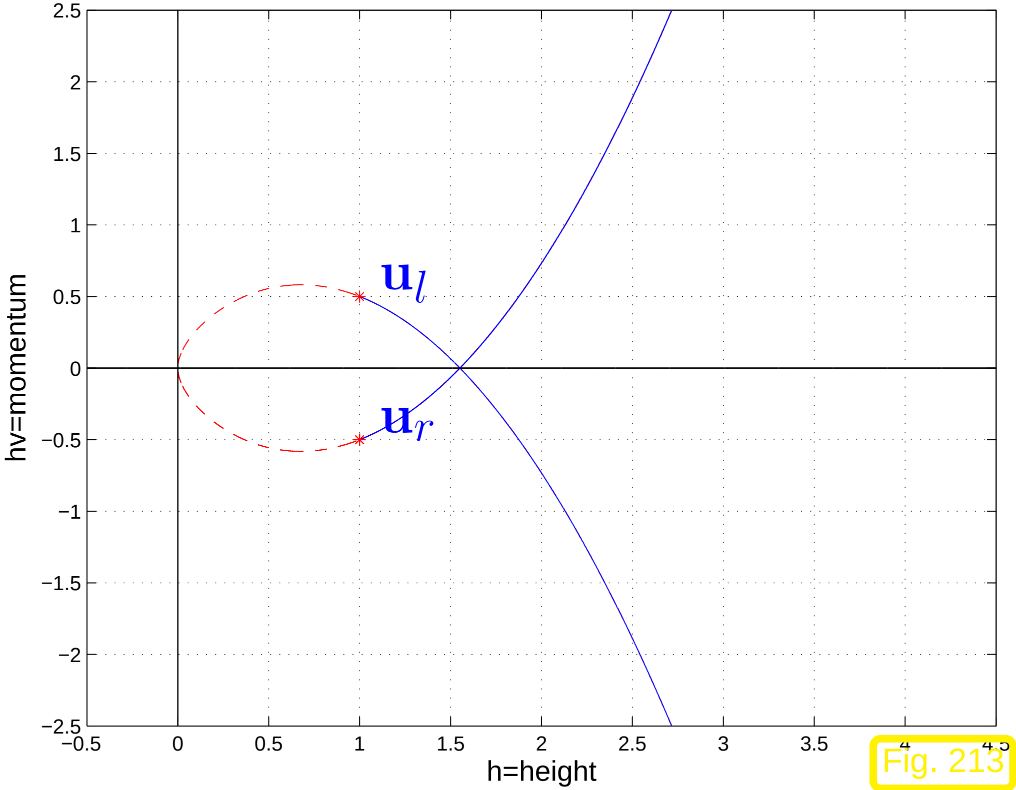


Fig. 213

admissible all-shock solution

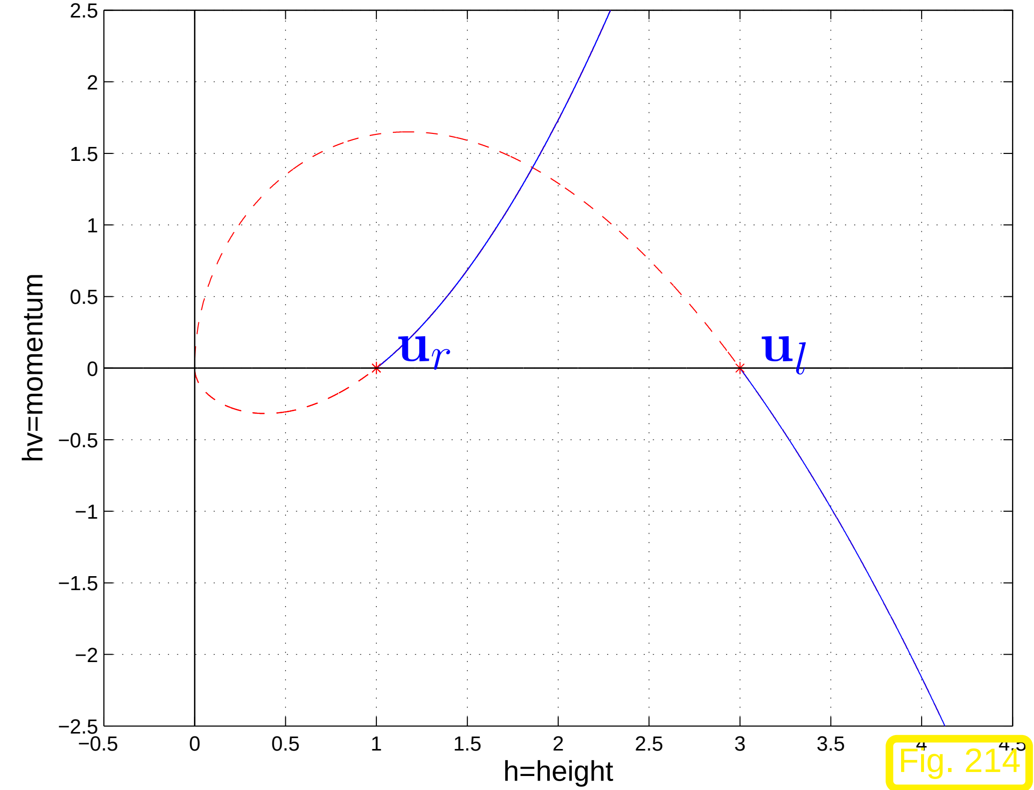


Fig. 214

all-shock solution not admissible

◇

Assume: all fields $k = 1, \ldots, m$ are genuine non-linear $\longrightarrow$ Def.5.3.6

➜ simpler criterion for entropy consistent shocks $\leftrightarrow$ analoguous to Lemma 2.5.6

Definition 5.4.3 (Lax entropy condition, *cf.* Def. 2.5.7, for systems). $\longrightarrow$ *[29, Def. 4.1.22]*

A discontinuity separating states $\mathbf{u}_l$ and $\mathbf{u}_r$ and propagating at speed $\dot{s}$ satisfies the Lax entropy condition, if

(i) $\exists k \in \{1,\ldots,m\}$: $\lambda_k(\mathbf{u}_l) > \dot{s} > \lambda_k(\mathbf{u}_r)$

(ii) $\forall j < k$: $\lambda_j(\mathbf{u}_l), \lambda_j(\mathbf{u}_r) < \dot{s}$

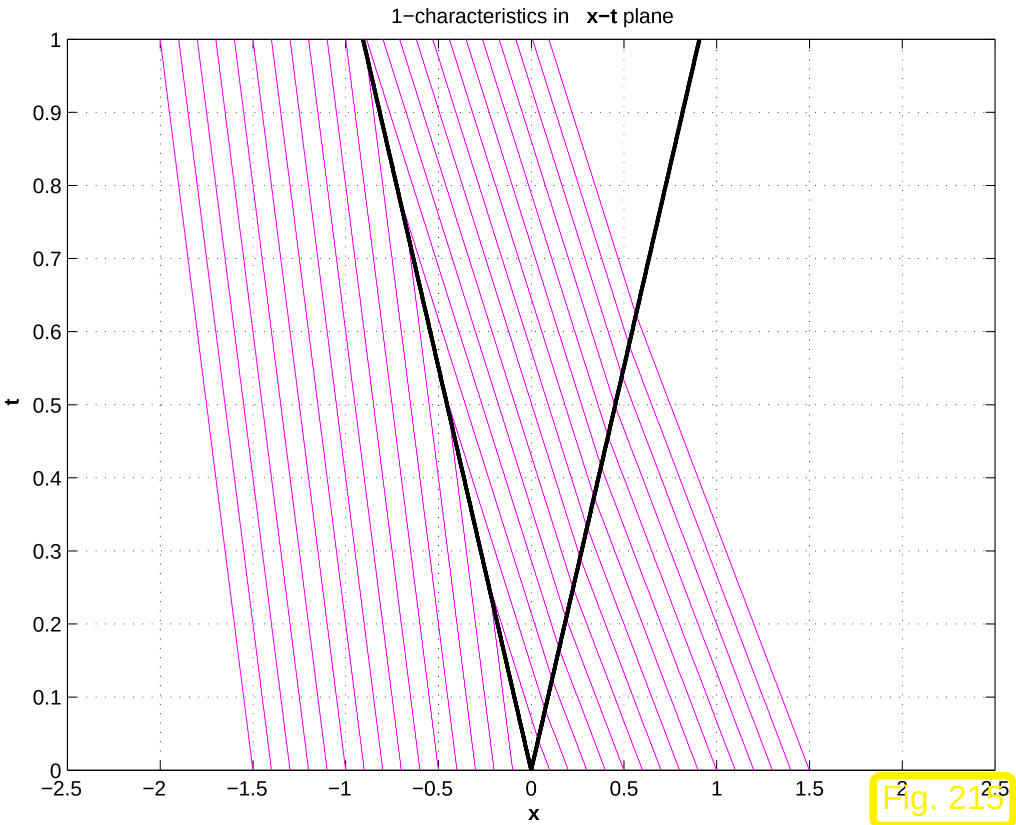
(iii) $\forall j > k$: $\lambda_j(\mathbf{u}_l), \lambda_j(\mathbf{u}_r) > \dot{s}$

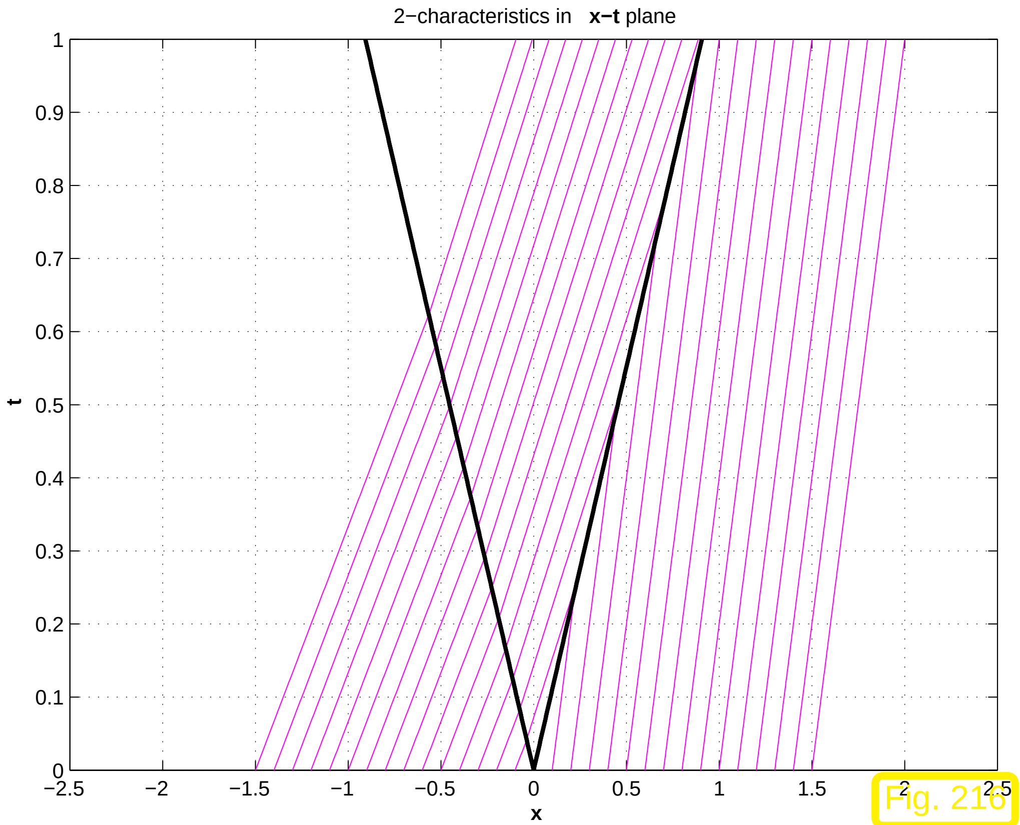
☞
- k-characteristics ($\longrightarrow$ Def. 5.2.1) impinge on shock (*cf.* discussion in Sect. 2.5.3)
- j-characteristics, $j < k$, cross shock from right to left
- j-characteristics, $j > k$, cross shock from left to right

Example 112 (Characteristics for all-shock solution of Riemann problem for shallow water equation).
$\longrightarrow$ Ex. 104

Plots of k-characteristics ($\longrightarrow$ Def. 5.2.1), $k = 1, 2$ for entropy consistent all-shock solution:

- Riemann problem for (5.1.5): $h_l = h_r = 0$, $v_l = 0.5$, $v_r = -0.5$, see Ex. 104, Figs. 199, 200

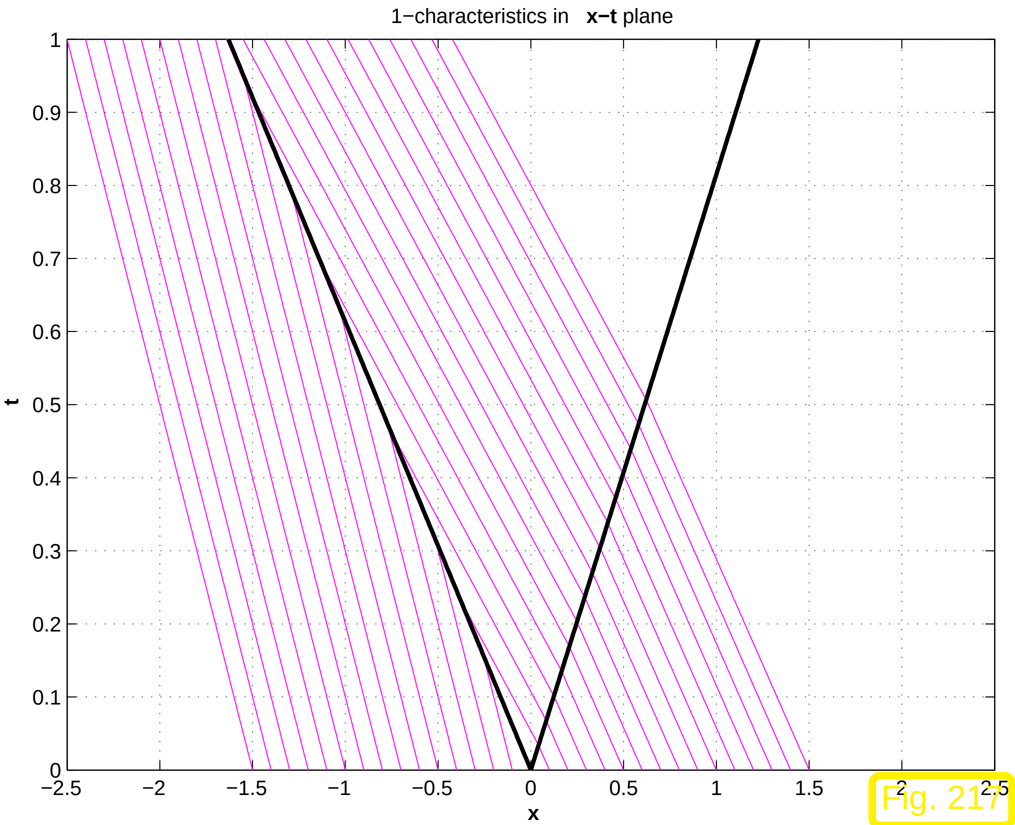




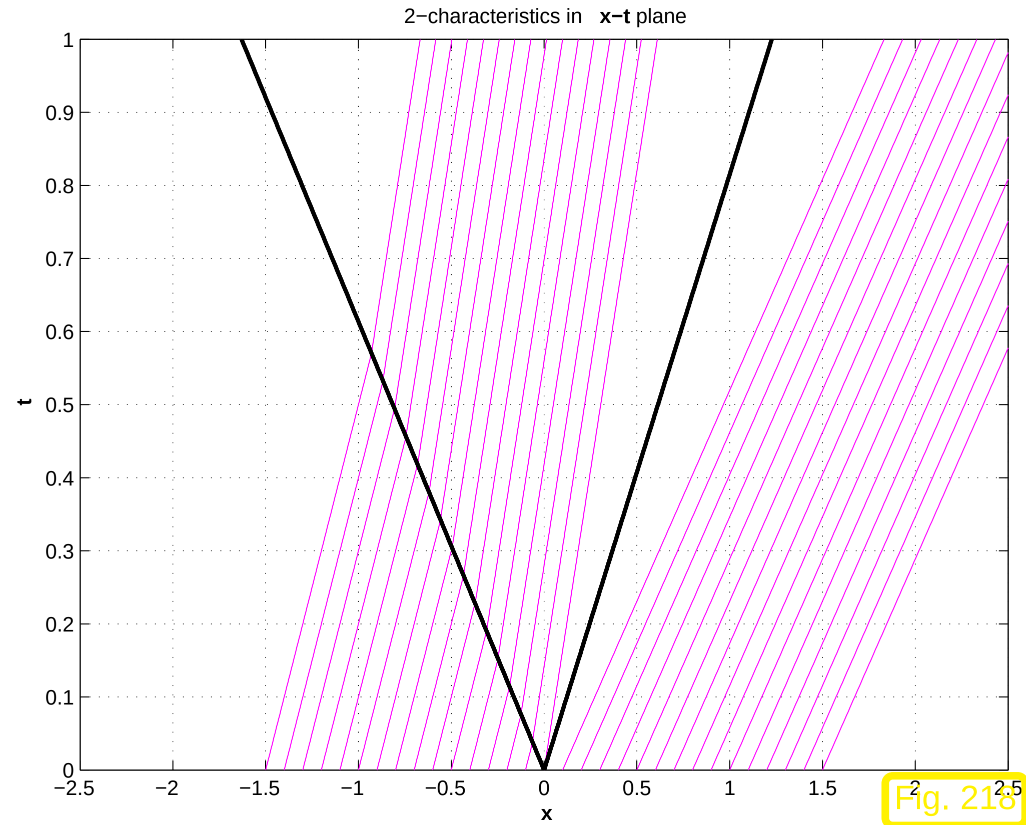
1-characteristics

2-characteristics

- Riemann problem for (5.1.5): $h_l = 1$, $h_r = 3$, $v_l = 0$, $v_r = 0$, see Ex. 104, Figs. 201, 202



1-characteristics



Lax condition violated !

◇

Example 113 (Lax entropy condition for shallow water equations). $\longrightarrow$ Ex. 111

Def. 5.4.3 applied to 1-shock ("slow shock") $\longrightarrow$ Ex. 103, Figs. 211, 212:

$$\lambda_1(\mathbf{u}_l) = v_l - \sqrt{gh_l} > \dot{s} := \frac{h_l v_l - h_r v_r}{h_l - h_r} > v_r - \sqrt{gh_r q} \,,$$

$$v_r - v_l = -(h_r - h_l)\sqrt{\frac{g}{2}\left(\frac{1}{h_r} + \frac{1}{h_l}\right)} \quad \Rightarrow \quad \boxed{h_l < h_r} \,.$$

Analoguously for 2-shock ("fast shock"): $\boxed{h_l > h_r}$

◇

Theorem 5.4.4 (Selection by Lax entropy condition from Def. 5.4.3)**.** $\longrightarrow$ *[29, Thm. 4.1.25]*
Assume that a 1D non-linear system of conservation laws (5.0.1) *possesses an entropy pair* (η, ψ) *and all fields are genuinely non-linear (*$\longrightarrow$ *Def. 5.3.6). Then, if* $\mathbf{u}$ *is a piecewise classical solution with a sufficiently small jump, the Lax entropy condition (*$\longrightarrow$ *Def. 5.4.3) is equivalent to inequality* (5.4.3)

► Lax entropy condition ensures uniqueness of solutions of Riemann problem

Example 114 (Riemann entropy solution for shallow water equations)*.* [31, Sect. 13.10]

Height for intermediate state that can be connected with left state (h_l, v_l):

$$G_l(h) = \begin{cases} v_l + 2\sqrt{g}(\sqrt{h_l} - \sqrt{h}) & \text{for } h < h_l \quad \rightarrow \text{1-rarefaction, Sect. 5.3.3 ,} \\ v_l - (h - h_l)\sqrt{\frac{g}{2}(1/h + 1/h_l)} & \text{for } h > h_l \quad \rightarrow \text{1-shock, Ex.103 .} \end{cases}$$

Height for intermediate state that can be connected with right (h_r, v_r):

$$G_r(h) = \begin{cases} v_r - 2\sqrt{g}(\sqrt{h_r} - \sqrt{h}) & \text{for } h < h_r \quad \rightarrow \text{2-rarefaction, Sect. 5.3.3 ,} \\ v_r + (h - h_r)\sqrt{\frac{g}{2}(1/h + 1/h_r)} & \text{for } h > h_r \quad \rightarrow \text{2-shock, Ex.103 .} \end{cases}$$

➥ intermediate state (h_m, v_m): $h_m > 0$: $G_l(h_m) = G_r(h_m) \Rightarrow v_m := G_l(h_m)$ **(5.4.4)**

- dam break problem $\longrightarrow$ Ex. 104: $h_l = 3$, $h_r = 1$, $v_l = v_r = 0$, $T = 2$

➤ movie: evolution of height $h(x,t)$

➤ movie: evolution of velocity $v(x,t)$

◇

Existence of "entropy solutions" for Riemann problem for (5.0.1) **?** (*cf.* Thm. 2.5.4)

☞ only guaranteed for $\mathbf{u}_r - \mathbf{u}_l$ "sufficiently small", [29, Thm. 4.1.33]

5.5 Multidimensional systems of conservation laws

Multidimensional system $\leftrightarrow$ conservation laws (2.1.3) for spatial dimension $d > 1$ phase space dimension $m > 1$

➥ Cauchy problem:

$$\frac{\partial}{\partial t}\mathbf{u} + \operatorname{div}_{\boldsymbol{x}} \mathbf{F}(\mathbf{u}) = 0 \quad \text{in } \mathbb{R}^d \times]0, T[\ , \qquad (5.5.1)$$

$$\mathbf{u}(\boldsymbol{x}, 0) = \mathbf{u}_0(\boldsymbol{x}) \quad \text{in } \mathbb{R}^d\ ,$$

with matrix valued flux function $\mathbf{F} : U \subset \mathbb{R}^m \mapsto \mathbb{R}^{m,d}$ ($\operatorname{div}_{\boldsymbol{x}}$ acts on rows!).

☛ Important examples: Euler equations (inviscid fluid flow)
magnetohydrodynamics (fluid **+** electromagnetic fields)

Projection of (5.5.1) onto direction $\boldsymbol{n} \in \mathbb{R}^d$, $|\boldsymbol{n}| = 1$, *cf.* (3.4.21), $\mathbf{u}(\xi, t) = \mathbf{u}(\xi\boldsymbol{n}, t)$,

$$\frac{\partial}{\partial t}\mathbf{u}(\xi, t) + \frac{\partial}{\partial \xi}(\mathbf{F}(\mathbf{u}) \cdot \boldsymbol{n}) = 0\ . \qquad (5.5.2)$$

Definition 5.5.1 (Hyperbolicity of multidimensional systems of conservation laws)**.**

(5.5.1) *(strictly) hyperbolic* $:\Longleftrightarrow$ (5.5.2) *(strictly) hyperbolic for any* $\boldsymbol{n} \in \mathbb{R}^d \setminus \{0\}$ *(→ Def. 5.1.1).*

Example 115 (2D shallow water equations). → Ex. 99

Inviscid incompressible fluid (→ water) in a shallow (infinite) basin:

Assume:
- vanishing vertical flow velocity component: $v_z = 0$
- no vertical variational of flow velocity

Physical quantities: $h(\boldsymbol{x}, t)$: height of fluid ($[h] = \mathrm{m}$), ➥ $h \geq 0$

$v_x(\boldsymbol{x}, t)$/$v_y(\boldsymbol{x}, t)$: fluid velocity (x/y-components) ($[\mathbf{v}] = \mathrm{ms}^{-1}$)

conservation of mass ► $$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(v_x h) + \frac{\partial}{\partial y}(v_y h) = 0 \;, \tag{5.5.3}$$

conservation of momentum ► $$\frac{\partial}{\partial t}(h v_x) + \frac{\partial}{\partial x}(h v_x^2 + \tfrac{1}{2} g h^2) + \frac{\partial}{\partial y}(h v_x v_y) = 0 \;, \tag{5.5.4}$$

conservation of momentum ► $$\frac{\partial}{\partial t}(h v_y) + \frac{\partial}{\partial x}(h v_x v_y) \frac{\partial}{\partial y}(h v_y^2 + \tfrac{1}{2} g h^2) + = 0 \;. \tag{5.5.5}$$

➢ conserved quantities mass $u_1 := h$, momenta $u_2 := hv_x, u_3 := hv_y$ ➞ $m = 3$

$$\mathbf{F}(\mathbf{u}) = \begin{pmatrix} u_2 & u_3 \\ u_2^2/u_1 + \frac{1}{2}gu_1^2 & u_2u_3/u_1 \\ u_3u_2/u_1 & u_3^2/u_1 + \frac{1}{2}gu_1^2 \end{pmatrix} .$$

◇

6 Finite Volume Methods for 1D Systems of Conservation Laws

Consider: Cauchy problem for 1D system of conservation laws:

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{\partial}{\partial x}\mathbf{F}(\mathbf{u}) = 0 \quad \text{in } \mathbb{R}\times]0,T[\quad , \quad u(\cdot,0) = u_0 \quad \text{in } \mathbb{R} \ , \tag{6.0.1}$$

unknown function $\mathbf{u} : \mathbb{R}\times]0,T[\mapsto U \subset \mathbb{R}^m$ with flux function $\mathbf{F} : U \mapsto \mathbb{R}^m$, $\mathbf{F} \in C^1(U,\mathbb{R}^m)$, see Ch. 5.

Model problems:

- Linear wave equation (5.2.4) ($\rightarrow$ Ex. 100): $m = 2$, $\mathbf{F}(\mathbf{u}) = \begin{pmatrix} 0 & -c \\ -c & 0 \end{pmatrix} \mathbf{u}$
- shallow water equations (5.1.5): $m = 2$, $\mathbf{F}(\mathbf{u}) = \begin{pmatrix} u_2 \\ u_2^2 u_1^{-1} + \frac{1}{2} g u_1^2 \end{pmatrix}$, $U := \mathbb{R}^+ \times \mathbb{R}$

Setting for discretization $\longrightarrow$ Ch. 3, Sect. 3.1:

☞ infinite equidistant space time tensor product grid $\mathcal{M}$ of $\mathbb{R}\times]0,T[$ $\longrightarrow$ (3.1.1), meshwidth Δx, timestep Δt, ratio $\gamma := \Delta t/\Delta x$

➨ vector space of vector valued spatial grid functions: $\mathbf{C}^0(\mathcal{G}_{\Delta x}) := \{\mathcal{G}_{\Delta x} \mapsto \mathbb{R}^m\}$
notation for grid functions $\in \mathbf{C}^0(\mathcal{G}_{\Delta x})$: $\vec{\boldsymbol{\mu}}, \vec{\boldsymbol{\eta}}$, etc.

Adopt interpretation ($\longrightarrow$ Sect. 3.2): $$\boldsymbol{\mu}_j^{(k)} \approx \frac{1}{\Delta x} \int\limits_{x_{j-1/2}}^{x_{j+1/2}} \mathbf{u}(x,t_k)\,\mathrm{d}x \quad \text{(cell average)}$$

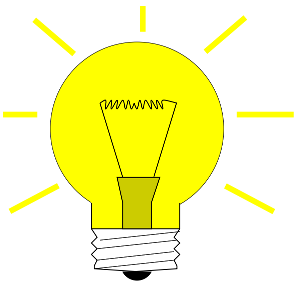
6.1 Linear systems of conservation laws

Special case: $\mathbf{F}(\mathbf{u}) = \mathbf{A}\mathbf{u}$, $\mathbf{A} \in \mathbb{R}^{m,m}$ $\longrightarrow$ Sect. 5.2

Recall: diagonalization approach of Sect. 5.2 ($\leftarrow$ notations): $(\mathbf{R}^{-1}\mathbf{A}\mathbf{R} = \mathbf{D})$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{A}\frac{\partial \mathbf{u}}{\partial x} = 0 \quad \xrightarrow{\mathbf{w}:=\mathbf{R}^{-1}\mathbf{u}} \quad \frac{\partial \mathbf{w}}{\partial t} + \mathbf{D}\frac{\partial \mathbf{w}}{\partial x} = 0 \,, \quad \mathbf{D} := \operatorname{diag}(\lambda_1, \dots, \lambda_m) \,. \tag{6.1.1}$$

decoupled advection equations, *cf.* (5.2.1)



Idea: ▷ pick FDM ($\longrightarrow$ Def. 3.1.1) for 1D scalar advection

▷ formulate FDM for diagonalized system $\frac{\partial \mathbf{w}}{\partial t} + \operatorname{diag}(\lambda_1, \dots, \lambda_m)\frac{\partial \mathbf{w}}{\partial x} = 0$

▷ undo transformation $\mathbf{w} \longrightarrow \mathbf{u} := \mathbf{R}\mathbf{w}$

① 1st-order upwind 3-point finte difference scheme (3.1.26) $\longrightarrow$ Ex. 53

$$\boldsymbol{\omega}_j^{(k)} = (1 - \gamma|\mathbf{D}|)\boldsymbol{\omega}_j^{(k-1)} + \gamma\mathbf{D}^+\boldsymbol{\omega}_{j-1}^{(k-1)} - \gamma\mathbf{D}^-\boldsymbol{\omega}_{j+1}^{(k-1)} \,. \tag{6.1.2}$$

➡ notations: $\boldsymbol{\omega}_j^{(k)} \approx$ cell averages for $\mathbf{w}(\cdot, t_k)$,
$|\mathbf{D}| := \operatorname{diag}(|\lambda_1|, \dots, |\lambda_m|)$,
$\mathbf{D}^{\pm} := \operatorname{diag}(\lambda_1^{\pm}, \dots, \lambda_m^{\pm})$, $\xi^+ := \max\{0, \xi\} \geq 0$, $\xi^- := \min\{0, \xi\} \leq 0$

▶

$$\boldsymbol{\mu}_j^{(k)} = (1 - \gamma|\mathbf{A}|)\boldsymbol{\mu}_j^{(k-1)} + \gamma\mathbf{A}^+\boldsymbol{\mu}_{j-1}^{(k-1)} - \gamma\mathbf{A}^-\boldsymbol{\mu}_{j+1}^{(k-1)} \,. \tag{6.1.3}$$

➡ notations:

$$\boxed{|\mathbf{A}| := \mathbf{R}|\mathbf{D}|\mathbf{R}^{-1}, \quad \mathbf{A}^+ := \mathbf{R}\mathbf{D}^+\mathbf{R}^{-1}, \quad \mathbf{A}^- := \mathbf{R}\mathbf{D}^-\mathbf{R}^{-1}}$$

rewriting (6.1.3) in conservation form (→ Def. 3.2.1):

$$\begin{aligned}\boldsymbol{\mu}_j^{(k)} &= \boldsymbol{\mu}_j^{(k-1)} - \gamma\mathbf{A}^+(\boldsymbol{\mu}_j^{(k-1)} - \boldsymbol{\mu}_{j-1}^{(k-1)}) - \gamma\mathbf{A}^-(\boldsymbol{\mu}_{j+1}^{(k-1)} - \boldsymbol{\mu}_j^{(k-1)}) \\ &= \boldsymbol{\mu}_j^{(k-1)} - \gamma(\mathbf{F}_{j+1/2} - \mathbf{F}_{j-1/2}) \,,\end{aligned} \tag{6.1.4}$$

with numerical flux $\mathbf{F}_{j+1/2} = F_{\mathrm{uw}}(\boldsymbol{\mu}_j^{(k-1)}, \boldsymbol{\mu}_{j+1}^{(k-1)})$, $\boxed{F_{\mathrm{uw}}(\mathbf{v},\mathbf{w}) = \mathbf{A}^+\mathbf{v} + \mathbf{A}^-\mathbf{w}}$

① 1st-order Lax-Friedrichs 3-point finite difference scheme (3.1.29)

$$\boldsymbol{\omega}_j^{(k)} = \tfrac{1}{2}(\boldsymbol{\omega}_{j+1}^{(k-1)} + \boldsymbol{\omega}_{j-1}^{(k-1)}) - \tfrac{1}{2}\gamma\mathbf{D}(\boldsymbol{\omega}_{j+1}^{(k-1)} - \boldsymbol{\omega}_{j-1}^{(k-1)}) \,. \tag{6.1.5}$$

► $$\boldsymbol{\mu}_j^{(k)} = \tfrac{1}{2}(\boldsymbol{\mu}_{j+1}^{(k-1)} + \boldsymbol{\mu}_{j-1}^{(k-1)}) - \tfrac{1}{2}\gamma\mathbf{A}(\boldsymbol{\mu}_{j+1}^{(k-1)} - \boldsymbol{\mu}_{j-1}^{(k-1)}) \,. \tag{6.1.6}$$

➢ Lax-Friedrichs numerical flux function, *cf.* (3.2.9),

$$F_{\mathrm{LF}}(\mathbf{v},\mathbf{w}) = \tfrac{1}{2}\mathbf{A}(\mathbf{v}+\mathbf{w}) - \tfrac{1}{2\gamma}(\mathbf{w}-\mathbf{v}) \,. \tag{6.1.7}$$

② *2nd-order* Lax-Wendroff 3-point finite difference scheme (3.1.12)

$$\boldsymbol{\omega}_j^{(k)} = (1-(\gamma\mathbf{D})^2)\boldsymbol{\omega}_j^{(k-1)} + \tfrac{1}{2}\gamma\mathbf{D}(\gamma\mathbf{D}+\mathbf{I})\boldsymbol{\omega}_{j-1}^{(k-1)} + \tfrac{1}{2}\gamma\mathbf{D}(\gamma\mathbf{D}-\mathbf{I})\boldsymbol{\omega}_{j+1}^{(k-1)} \,. \tag{6.1.8}$$

$$\boldsymbol{\mu}_j^{(k)} = (1-(\gamma\mathbf{A})^2)\boldsymbol{\mu}_j^{(k-1)} + \tfrac{1}{2}\gamma\mathbf{A}(\gamma\mathbf{A}+\mathbf{I})\boldsymbol{\mu}_{j-1}^{(k-1)} + \tfrac{1}{2}\gamma\mathbf{A}(\gamma\mathbf{A}-\mathbf{I})\boldsymbol{\mu}_{j+1}^{(k-1)} . \tag{6.1.9}$$

➢ Lax-Wendroff numerical flux function, *cf.* (3.2.25)

$$F_{\mathrm{LW}}(\mathbf{v},\mathbf{w}) = \tfrac{1}{2}\mathbf{A}(\mathbf{v}+\mathbf{w}) - \tfrac{1}{2}\gamma\mathbf{A}^2(\mathbf{w}-\mathbf{v}) \;,$$

$$\boxed{F_{\mathrm{LW}}(\mathbf{v},\mathbf{w}) = F_{\mathrm{uw}}(\mathbf{v},\mathbf{w}) + \tfrac{1}{2}|\mathbf{A}|(1-\gamma|\mathbf{A}|)(\mathbf{w}-\mathbf{v})} \;. \tag{6.1.10}$$

anti-diffusive flux, *cf.* (3.3.9)

For all these schemes: CFL-condition (→ Def. 3.1.4) ⇔ $\gamma \max\{|\lambda_1|, |\lambda_m|\} \leq 1$

Remark 116 (Lax-Friedrichs method for non-linear systems of conservation laws).

(3.2.9) **&** (6.1.7) ➥ Lax-Friedrichs nunerical flux for $\frac{\partial \mathbf{u}}{\partial t} + \frac{\partial}{\partial x}\mathbf{F}(\mathbf{u}) = 0$:

$$\boxed{F_{\mathrm{LF}}(\mathbf{v},\mathbf{w}) = \tfrac{1}{2}\left(\mathbf{F}(\mathbf{v})+\mathbf{F}(\mathbf{w})\right) - \tfrac{1}{2\gamma}(\mathbf{w}-\mathbf{v})} \;. \tag{6.1.11}$$

△

Example 117 (Lax-Friedrichs scheme for shallow water equations).

Numerical solution of dam break problem, see Ex. 123: convergence rates and movie

◇

Remark 118 (Implementation of boundary conditions for linear wave equation).

1D linear wave equation (1.10.1) in conservation form $\rightarrow$ Ex. 100:

$$\Rightarrow \quad \frac{\partial}{\partial t}\underbrace{\begin{pmatrix} v \\ w \end{pmatrix}}_{=:\mathbf{u}} + \underbrace{\begin{pmatrix} 0 & -c \\ -c & 0 \end{pmatrix}}_{=:\mathbf{A}} \frac{\partial}{\partial x}\begin{pmatrix} v \\ w \end{pmatrix} = 0 \quad \text{in }]0,\infty[\times]0,T[\,. \tag{6.1.12}$$

with reflecting boundary conditions at $x = 0$: $\quad v(0,t) = 0 \quad \forall 0 \leq t \leq T \quad \rightarrow$ Sect. 1.10.

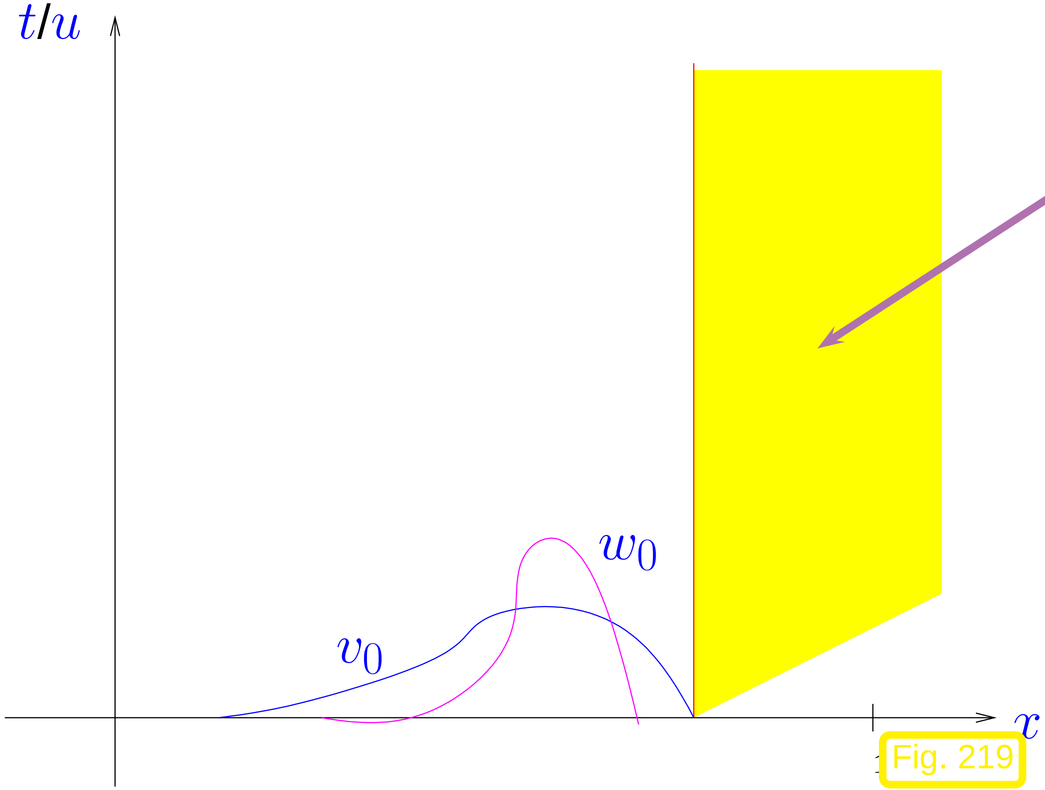
Truncated spatial computational domain $\quad D :=]0,1[$

➤ absorbing boundary conditions at $x = 1 \quad \rightarrow$ Sect. 1.12

Equidistant spatial mesh $\quad \mathcal{G}_{\Delta x} = \{(j - 1/2)\Delta x\colon j = 1,\ldots,N\}$, $\Delta x := N^{-1}$, $N \in \mathbb{N} \hat{=}$ no. of cells

Assume: initial data v_0, w_0 compactly supported in D

☛ **Absorbing boundary conditions**:

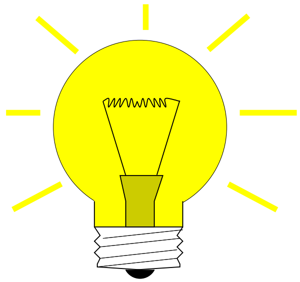


In this zone: $\mathbf{u}(x,t) \in \mathrm{Span}\,\{\mathbf{r}_1\}$ (only right propagating states)

➤ $\mathbf{A}^{-}\mathbf{u}(x,t) = 0$

▷ $\boldsymbol{\mu}_{j+1}^{(k-1)}$ irrelevant for upwind FDM (6.1.4) for $j = N$!

▷ Lax-Friedrichs scheme (6.1.6) and Lax-Wendroff scheme (6.1.9) need $\boldsymbol{\mu}_{N+1}^{(k-1)}$, *but* little impact, if $\boldsymbol{\mu}_{N+1}^{(k-1)} \in \mathrm{Span}\,\{\mathbf{r}_2\}$!



Idea: Ghost cell approach:

Set $\boldsymbol{\mu}_{N+1}^{(k)} = \boldsymbol{\mu}_{N}^{(k)}$ for all k

☛ **Reflecting boundary conditions**:

Recall Ex. 20: reflected solution = solution (on $\mathbb{R}^+$)of Cauchy problem with reflected initial data u odd ➤ $v := \frac{\partial}{\partial t} u$ odd

$w := c \frac{\partial}{\partial x} u$ even

$$w_0^{(k)} = w_1^{(k)} ,$$
$$v_0^{(k)} = -v_1^{(k)}$$

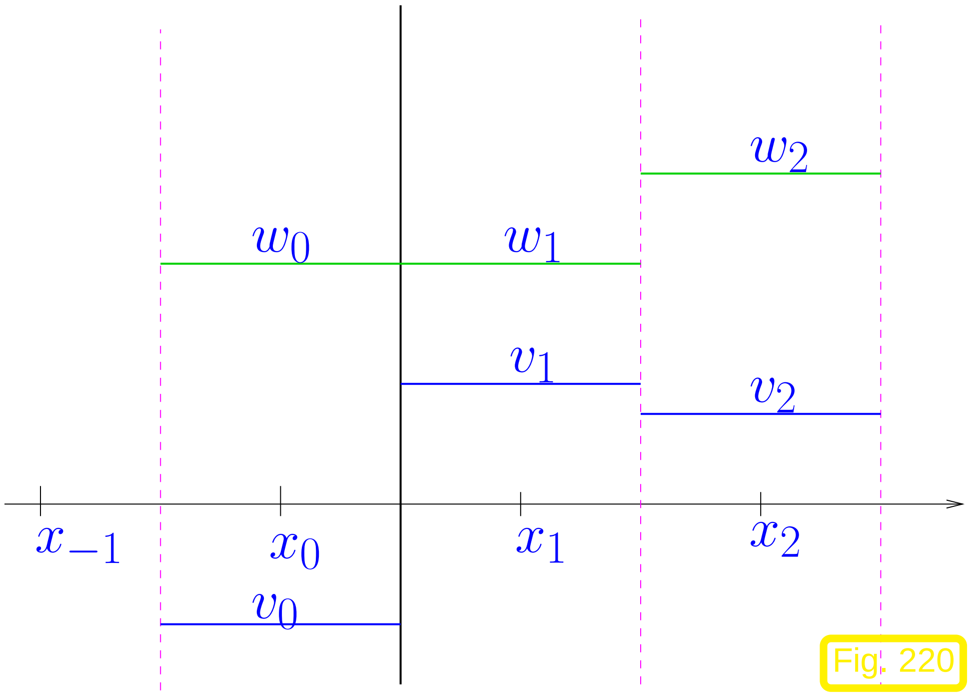


Fig. 220

△

Example 119 (FVM for linear wave equation). → Ex. 100

* initial boundary value problem from Rem. 118, $c=1$ ($\rightarrow$ Ex. 24):

  ☞ absorbing b.c. at $x=0$

  ✌ reflecting b.c. at $x=1$

* $u_0 = \chi_{]1/4,3/4[}\cos^2(2\pi(x-1/2))$ ▷
* $\frac{\partial u}{\partial t}(x,0) = 0$

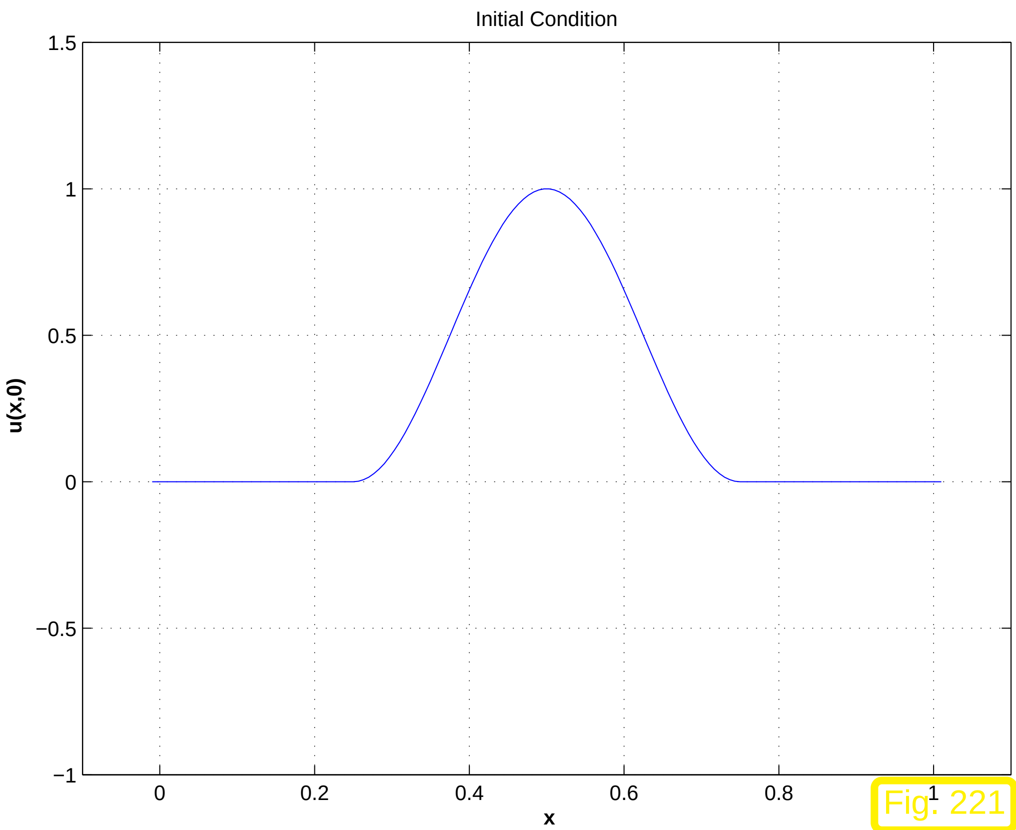


Fig. 221

☞ plots of (integrated) solutions for $u(x,t) = c^{-1}\int w(\xi,t)\,\mathrm{d}\xi$ for $N=150$ mesh cells, $t\in\{0,25,0.5,1.0\}$, $\gamma=0.9$

❶ upwind scheme (6.1.3):

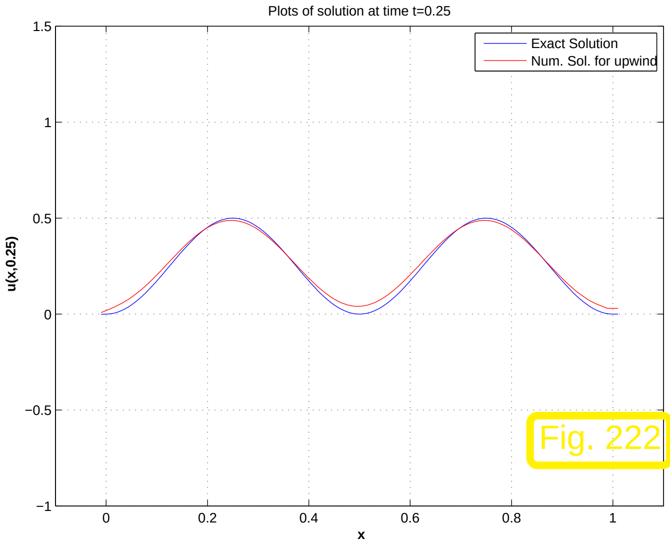


Fig. 222

$t = 0.25$

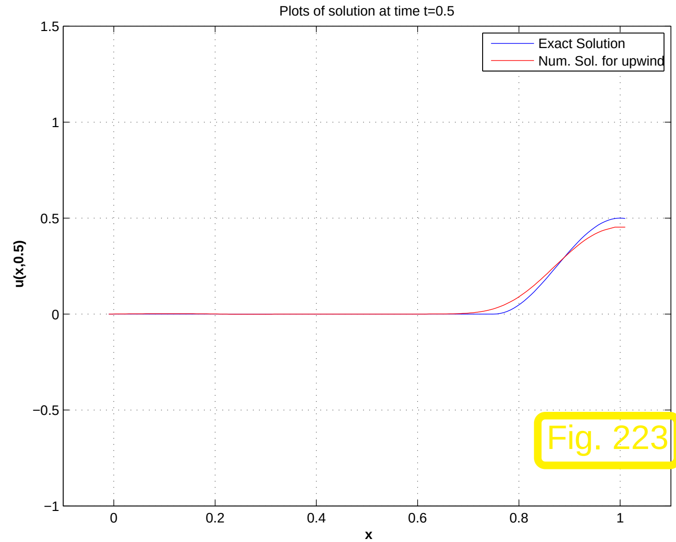


Fig. 223

$t = 0.5$

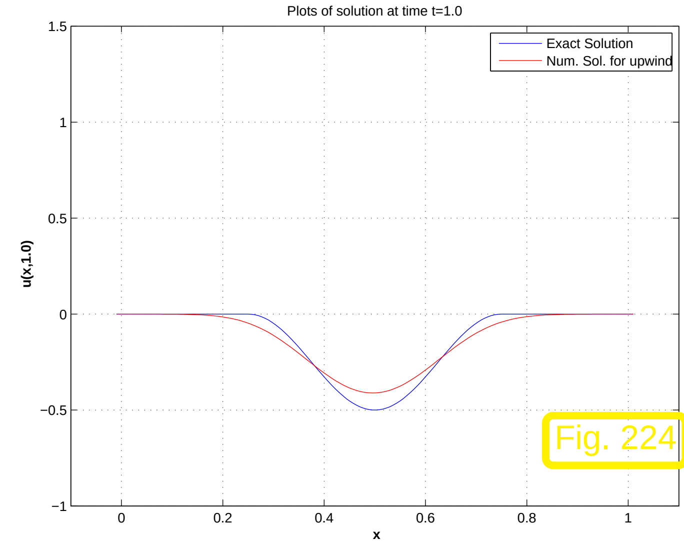


Fig. 224

$t = 1$

❷ Lax-Friedrichs scheme (6.1.6):

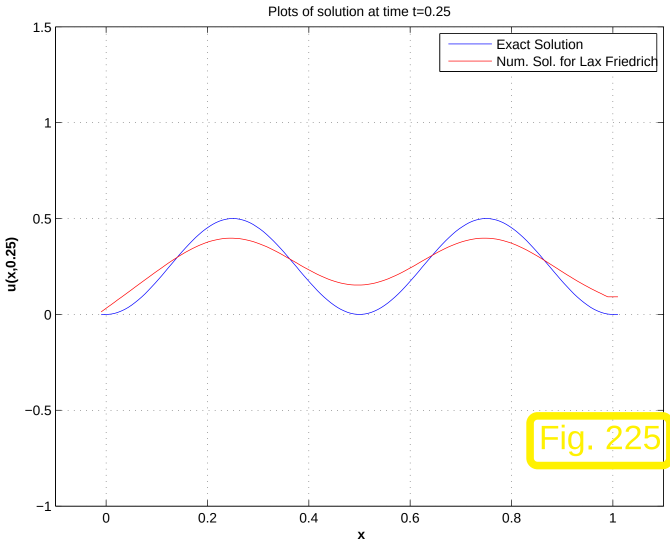


Fig. 225

$t = 0.25$

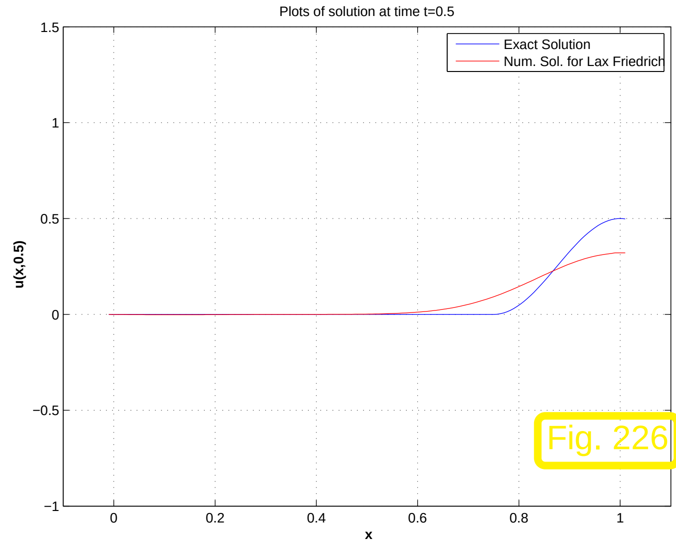


Fig. 226

$t = 0.5$

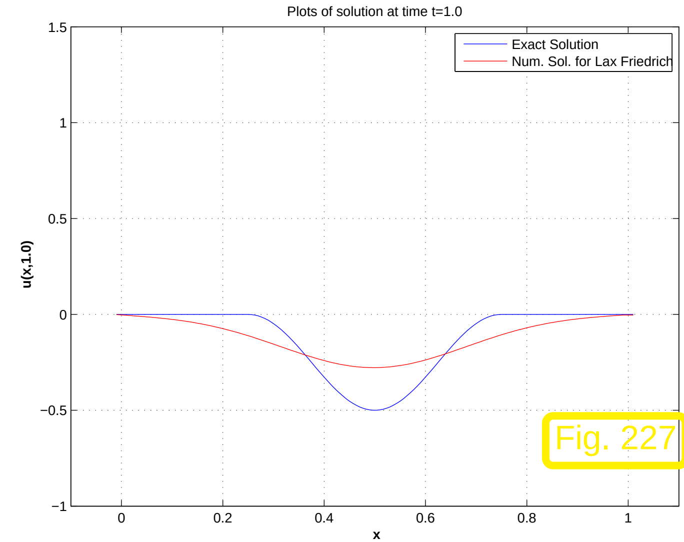
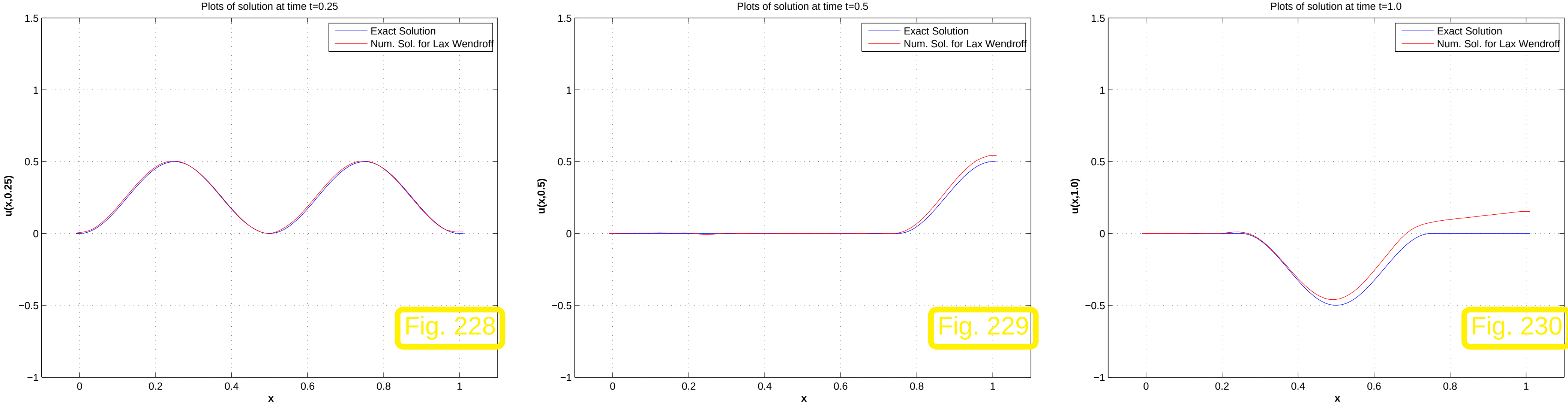


Fig. 227

$t = 1$

➡ excessive damping of waves in Lax-Friedrichs solution, *cf.* Ex. 64

❸ Lax-Wendroff scheme (6.1.9):



$t = 0.25$ $t = 0.5$ $t = 1$

➡ "overshoots" in Lax-Wendroff solution, *cf.* Ex. 74

- l^2/l^∞-norms of discretization error at $t = 1$ for w-component **+** approximate convergence rates, *cf.* Ex. 79.

❶ upwind scheme (6.1.3), Lax-Friedrichs scheme (6.1.6):

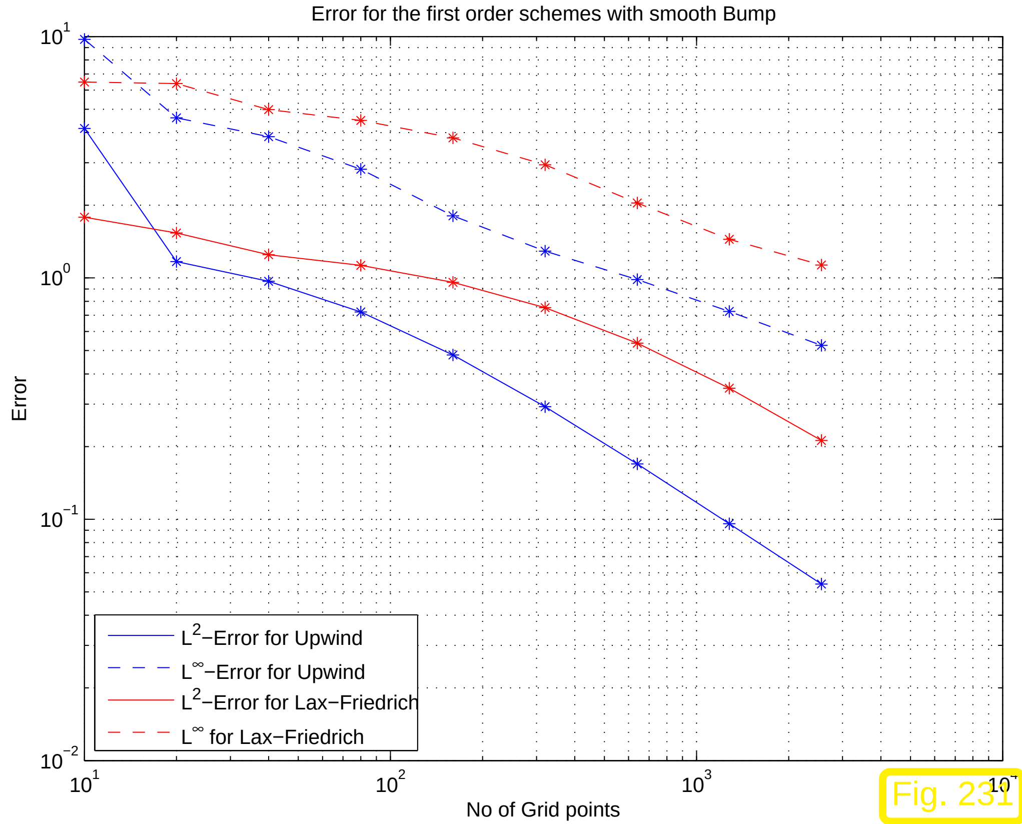


Fig. 231

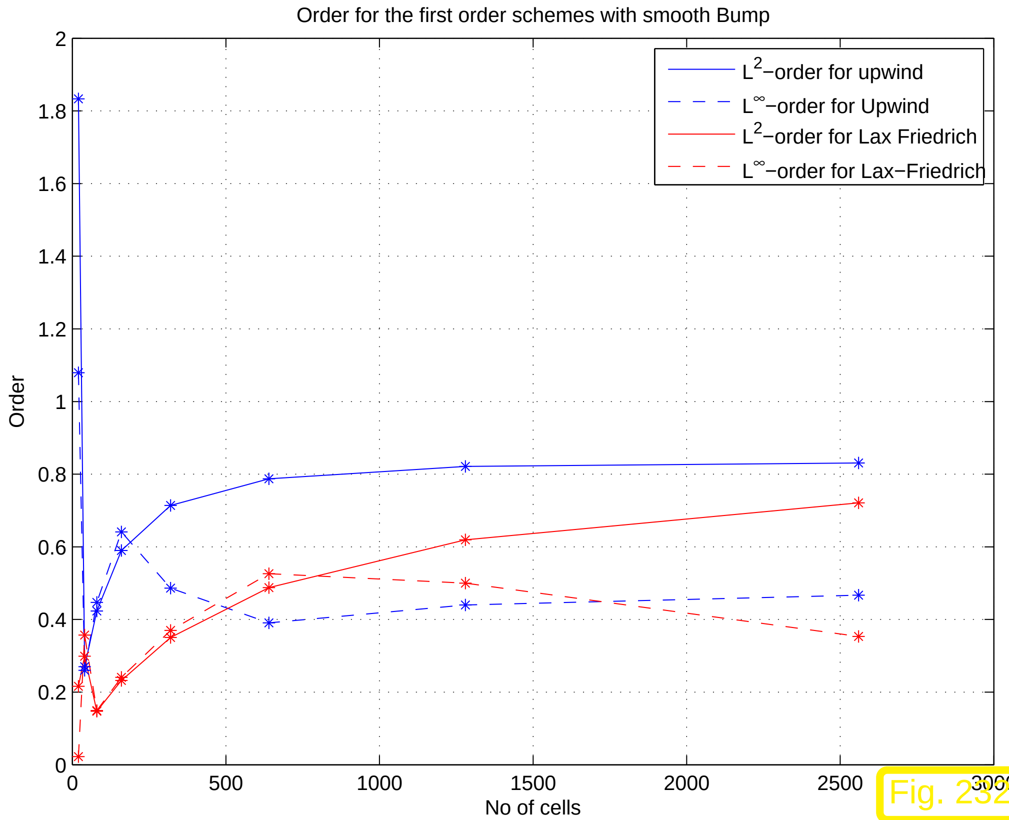


Fig. 232

Observation: algebraic convergence, slower than 1st-order

❸ Lax-Wendroff scheme (6.1.9) and wave limited FVM:

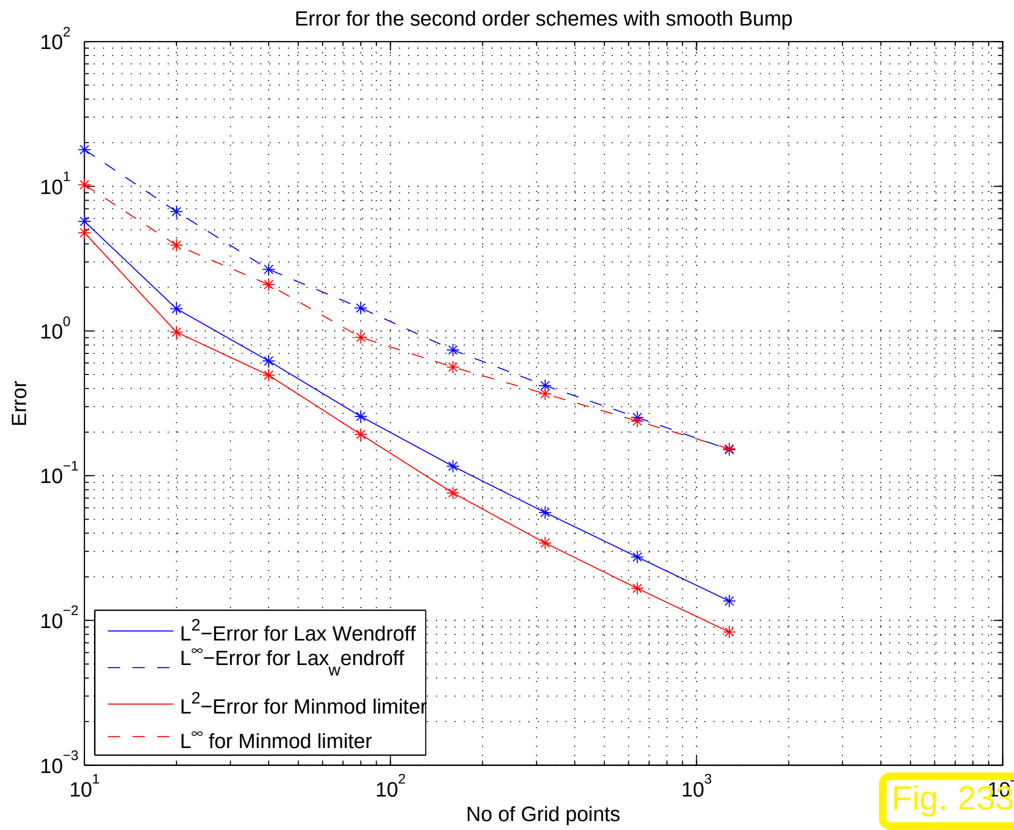


Fig. 233

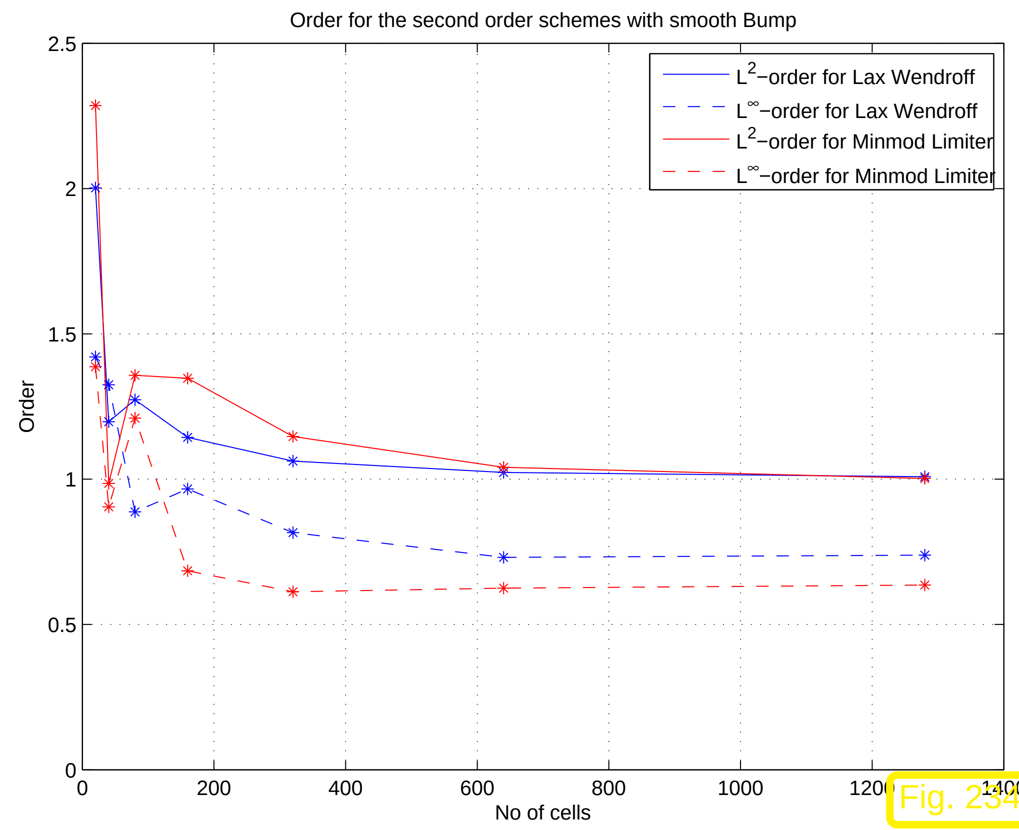


Fig. 234

Observation: only first-order algebraic convergence

➡ conjecture: merely C^0 initial data foil 2nd-order convergence

- Evolution of total energy during discrete evolutions ▷

➢ Numerical dissipation:

Lax-Friedrichs $>$ Upwind $>$ 2nd-order schemes

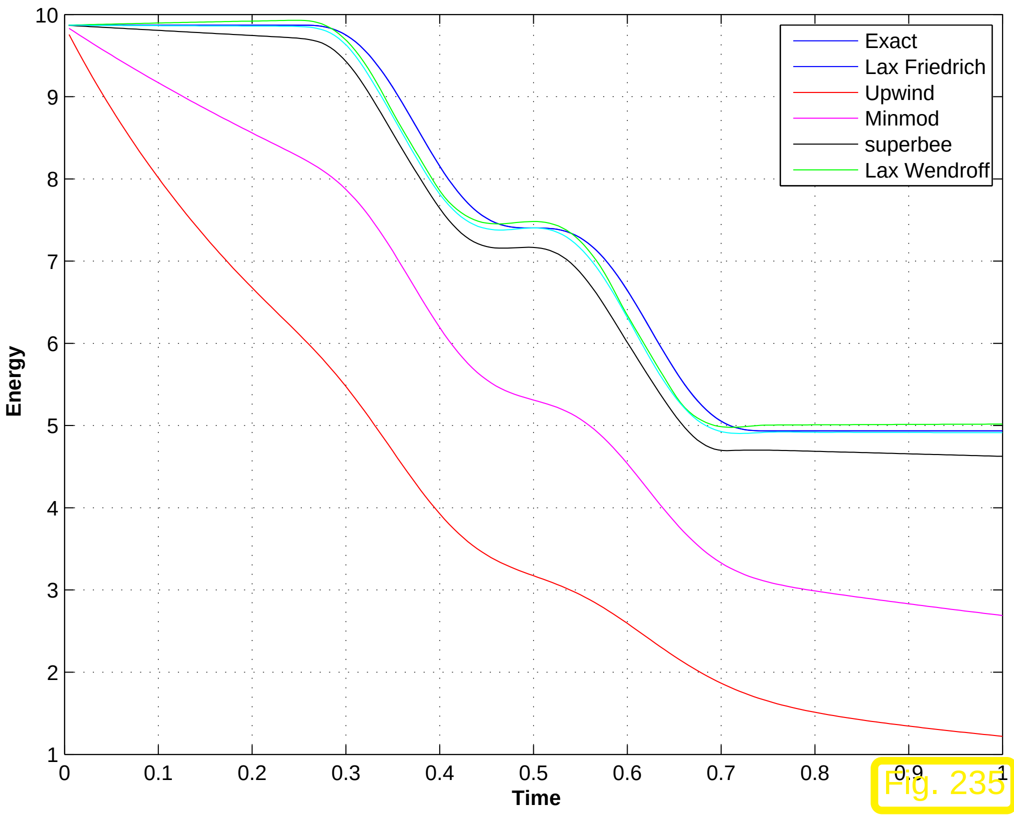


Fig. 235

◇

Remark 120 (Boundary conditions for shallow water equations).

Physically meaningful boundary conditions by ghost cell approach:

Absorbing boundary conditions: constant extrapolation $\boldsymbol{\mu}_{N+1}^{(k)} = \boldsymbol{\mu}_{N}^{(k)}$

Reflecting boundary conditions: constant extrapolation of u_1 (height): $\mu_{1,N+1}^{(k)} = \mu_{1,N+1}^{(k)}$

antisymmetric extrapolation of u_2 (momentum): $\mu_{2,N+1}^{(k)} = -\mu_{2,N+1}^{(k)}$

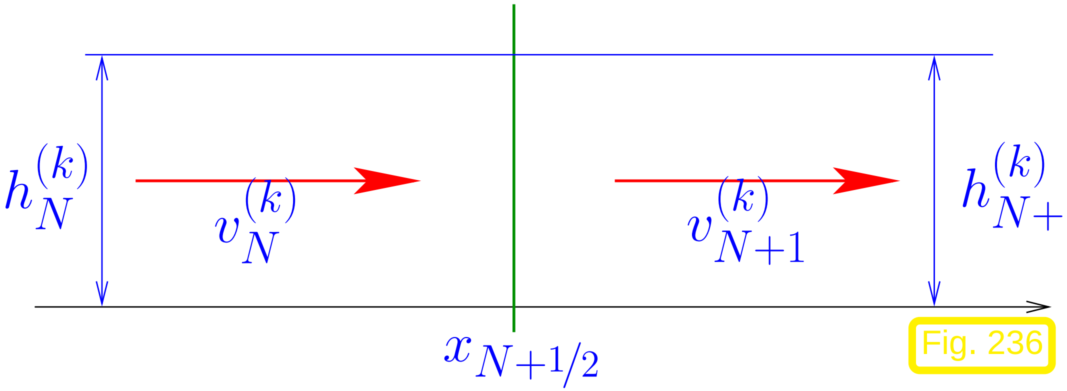


Fig. 236

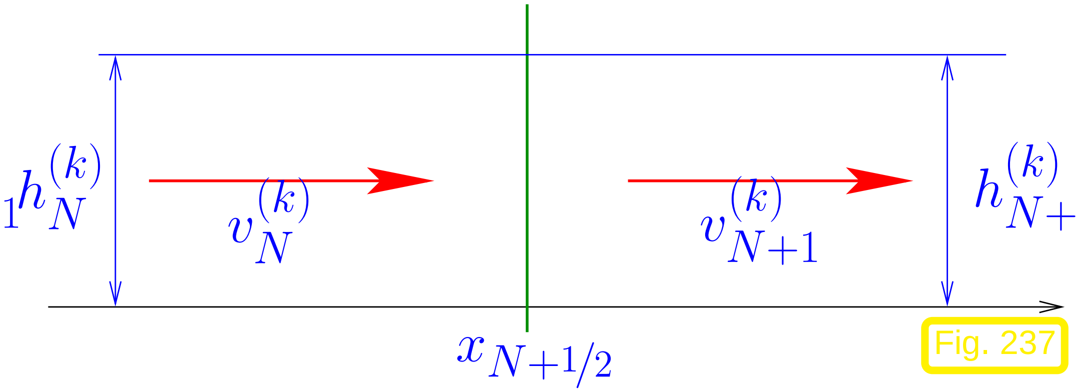


Fig. 237

High resolution methods

Recall: numerical flux for flux limited FVM with flux limiter function $\varphi : \mathbb{R} \mapsto \mathbb{R}$ for constant scalar linear advection $\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial x} = 0$ $\longrightarrow$ Sect. 3.3.1.3

$$f_{j+1/2} = v^+\mu_j^{(k-1)} + v^-\mu_j^{(k-1)} + \tfrac{1}{2}|v|(1-\gamma|v|)\varphi(\theta_{j+1/2}^{(k-1)})(\mu_{j+1}^{(k-1)} - \mu_j^{(k-1)}) \,, \tag{3.3.13}$$

$$\theta_{j+1/2}^{(k-1)} := \begin{cases} \Delta\mu_{j-1/2}^{(k-1)} : \Delta\mu_{j+1/2}^{(k-1)} & \text{, if } v > 0 \,, \\ \Delta\mu_{j+3/2}^{(k-1)} : \Delta\mu_{j+1/2}^{(k-1)} & \text{, if } v < 0 \,. \end{cases} \tag{3.3.11}$$

for diagonalized system (6.1.1) ▶ $\boldsymbol{\omega}_j^{(k)} = \boldsymbol{\omega}_j^{(k-1)} - \gamma(\mathbf{G}_{j+1/2} - \mathbf{G}_{j-1/2})$,

$$(\mathbf{G}_{j+1/2})_l = (\lambda_l^+ \boldsymbol{\omega}_j^{(k-1)} + \lambda_l^- \boldsymbol{\omega}_{j+1}^{(k-1)})_l + \tfrac{1}{2}|\lambda_k|(1-\gamma|\lambda_l|)\varphi(\theta_{j+1/2,l}^{(k-1)})(\boldsymbol{\omega}_{j+1}^{(k-1)} - \boldsymbol{\omega}_j^{(k-1)})_l \ ,$$

$$\theta_{j+1/2,l}^{(k-1)} := \begin{cases} (\Delta\boldsymbol{\omega}_{j-1/2}^{(k-1)})_l : (\Delta\boldsymbol{\omega}_{j+1/2}^{(k-1)})_l & \text{, if } \lambda_l > 0 \ , \\ (\Delta\boldsymbol{\omega}_{j+3/2}^{(k-1)})_l : (\Delta\boldsymbol{\omega}_{j+1/2}^{(k-1)})_l & \text{, if } \lambda_l < 0 \ , \end{cases} \quad l = 1,\dots,m \ . \tag{6.1.13}$$

Principle: flux limiter function applied to $\mathbf{w}$-components = wave limiting

▷ wave limited numerical flux

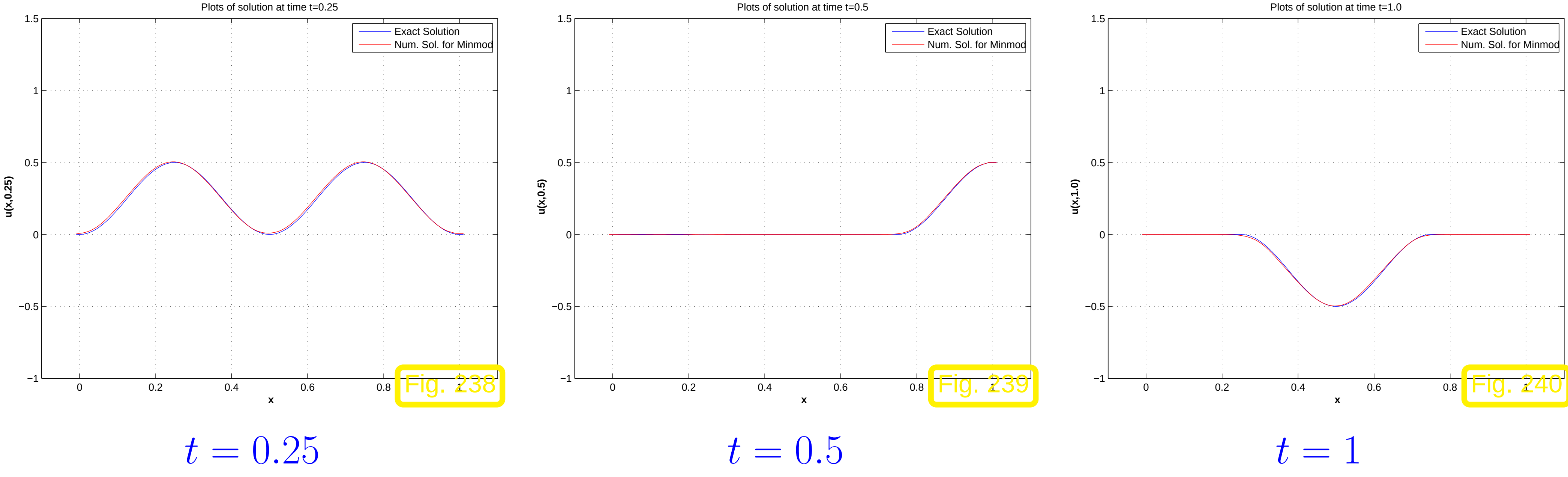
$$\mathbf{F}_{j+1/2} = F_{\mathrm{uw}}(\boldsymbol{\mu}_j, \boldsymbol{\mu}_{j+1}) + 1/2|\mathbf{A}|(1-\gamma|\mathbf{A}|)(\mathbf{R}\mathbf{D}\mathbf{R}^{-1})(\boldsymbol{\mu}_{j+1} - \boldsymbol{\mu}_j) \ , \tag{6.1.14}$$

$$\mathbf{D} := \mathrm{diag}(\varphi(\theta_{j+1/2,1}^{(k-1)}),\dots,\varphi(\theta_{j+1/2,m}^{(k-1)})) \ .$$

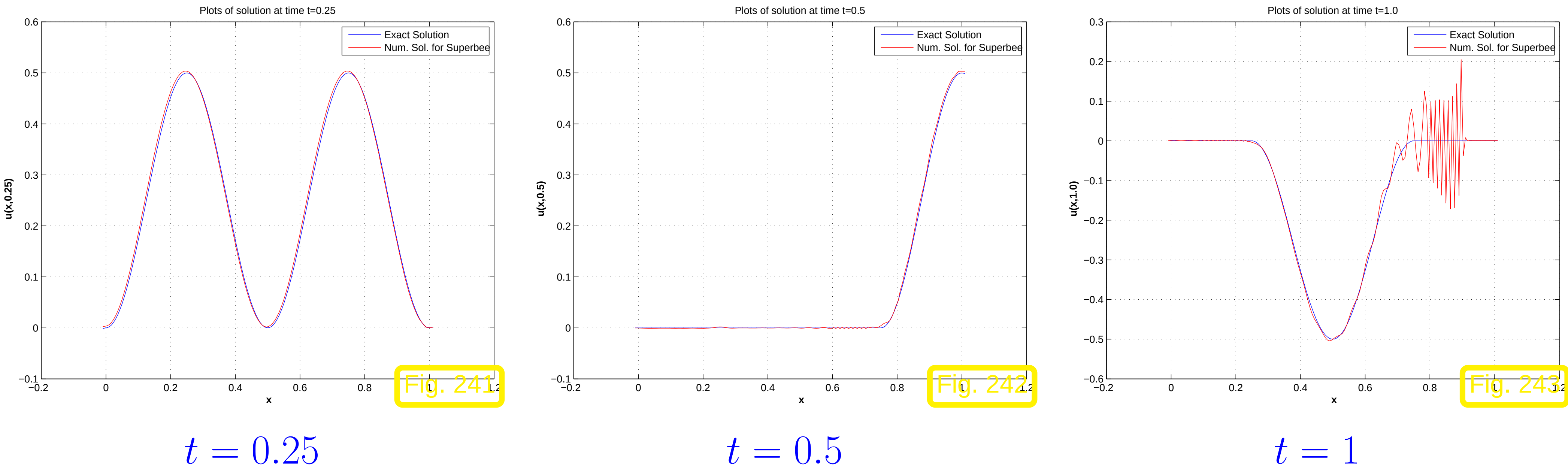
Example 121 (Flux limited FVM for linear wave equation). → Ex. 119

- initial boundary value problem from Ex. 119
- same evaluations as in Ex. 119 for wave limited FVM with
  - φ = minmod limiter (→ Def. 3.3.3): $\varphi(\theta) = \max\{0, \min\{\theta, 1\}\}$
  - φ = superbee limiter → (76): $\varphi(\theta) = \max\{0, \min\{2\theta, 1\}, \min\{\theta, 2\}\}$

❶ minmod wave limited FVM:



❷ superbee wave limited FVM:



Observation: spurious oscillations (instability of "overcompressive" superbee-limiter ?)

❶ Asymptotics of discretization error $\longrightarrow$ Fig. 233, 234

◇

6.2 Godunov's method

➡ extend time-local piecewise constant REA-algorithm of Sect. 3.2.2 ($m = 1$) to systems (5.0.1), case $m > 1$:

Assume:
- existence of (entropy) solutions for all Riemann problems for (5.0.1)
- all Riemann solutions $\mathbf{u}$ are similarity solutions: $\mathbf{u}(x,t) = \boldsymbol{\psi}(x/t)$ $\longrightarrow$ Sect. 5.3.3

$\longleftarrow$ CFL-condition $\boxed{\sup_{\mathbf{u}} \gamma \max\{|\lambda_1(\mathbf{u})|, |\lambda_m(\mathbf{u})|\} < 1}$

$$\boldsymbol{\mu}_j^{(k)} = \boldsymbol{\mu}_j^{(k-1)} - \gamma \left(F_{\mathsf{GD}}(\boldsymbol{\mu}_j^{(k-1)}, \boldsymbol{\mu}_{j+1}^{(k-1)}) - F_{\mathsf{GD}}(\boldsymbol{\mu}_{j-1}^{(k-1)}, \boldsymbol{\mu}_j^{(k-1)}) \right) , \tag{6.2.1}$$

where $F_{\mathsf{GD}}(\mathbf{v},\mathbf{w}) = \mathbf{F}(\mathbf{u}^{\downarrow}(\mathbf{v},\mathbf{w})) = \mathbf{F}(\boldsymbol{\psi}(0))$.

⇒ Notations: $\mathbf{u}(\mathbf{v},\mathbf{w})$ Riemann (entropy) solution for left state $\mathbf{u}_l = \mathbf{v}$, right state $\mathbf{u}_r = \mathbf{w}$

$\mathbf{u}^{\downarrow} = \mathbf{u}(0,t)$ = constant $= \boldsymbol{\psi}(0)$ for similarity solution $\mathbf{u}(x,t) = \boldsymbol{\psi}(x/t)$

☛ Lax-Wendroff theorem Thm. 3.2.6 holds for (6.2.1):

"convergence $\Rightarrow$ convergence to weak solution"

☛ As in Sect. 2.5: if (η, ψ) **=** entropy pair ($\rightarrow$ Def. 5.4.1) ➢ discrete entropy inequality for (6.2.1), *cf.* Def. 3.2.14

$$\eta(\boldsymbol{\mu}_j^{(k)}) \leq \eta(\boldsymbol{\mu}_j^{(k-1)}) - \gamma(\psi_{j+1/2}^{(k-1)} - \psi_{j-1/2}^{(k-1)}) \ ,$$

$\psi_{j+1/2}^{(k-1)} = \boldsymbol{\Psi}(\boldsymbol{\mu}_j^{(k-1)}, \boldsymbol{\mu}_{j+1}^{(k-1)})$, $\boldsymbol{\Psi}$ **=** ψ-consistent numerical entropy flux function.

Convergence ?

No general (L^1/L^∞/TV) stability results for Cauchy problem for system (5.0.1) !

no stability theory for discrete evolutions

no convergence theory

Feasibility/efficiency of Godunov's method (6.2.1) **?**

Recall: $m = 1$ ➢ simple formula (3.2.17) for Godunov flux F_{GD}

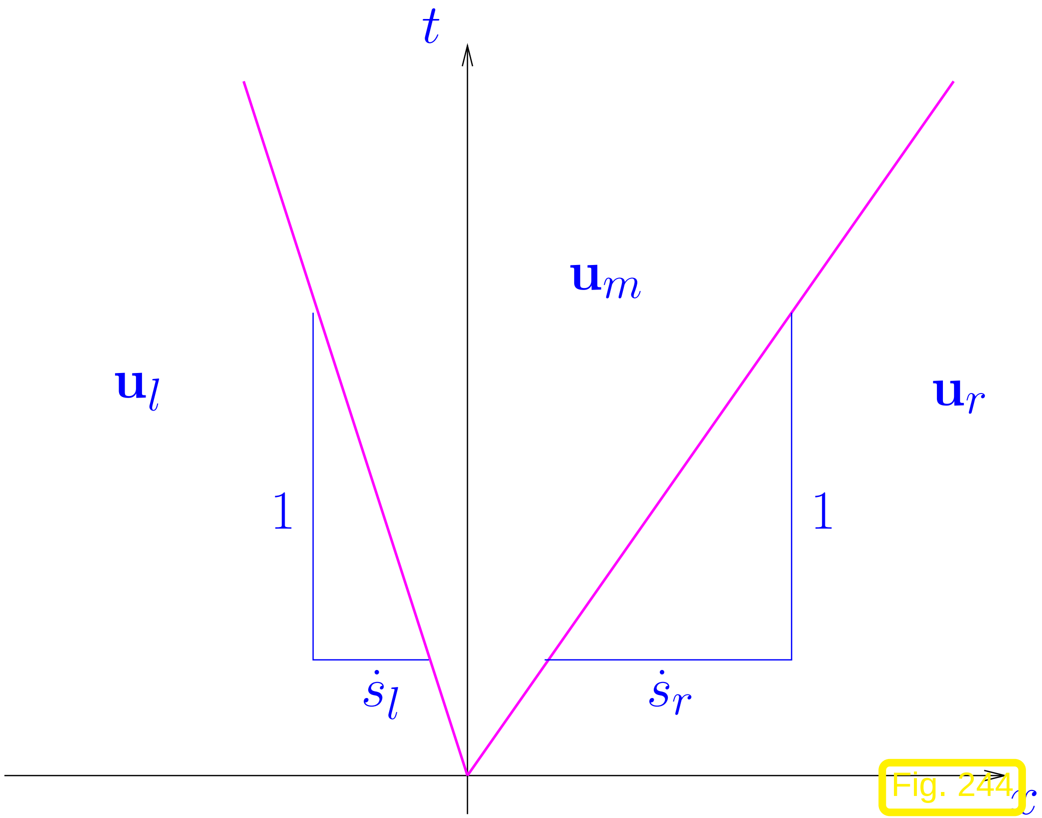
Example 122 (Computation of Godunov flux for shallow water equations). $\longrightarrow$ Ex. 99

Given: $\mathbf{v} \leftrightarrow$ left state $\mathbf{u}_l = (h_l, v_l h_l)$, $\mathbf{w} \leftrightarrow$ right state $\mathbf{u}_r = (h_r, v_r h_r)$

Use results of Ex. 113, Ex. 114 to compute Riemann solution:

① solve nonlinear equation (5.4.4) ➜ intermediate state $\mathbf{u}_m \leftrightarrow (h_m, v_m h_m)$

② Determine structure of Riemann solution:
(Rankine-Hugoniot speeds $\dot{s}_x = \frac{h_m v_m - h_x v_x}{h_m - h_x}$, $x \in \{l, r\}$)



- $h_l, h_r < h_m$: all-shock solution

$$\mathbf{u}^{\downarrow}(\mathbf{v}, \mathbf{w}) = \begin{cases} \mathbf{u}_l & \text{, if } \dot{s}_l > 0 \ , \\ \mathbf{u}_m & \text{, if } \dot{s}_l < 0 < \dot{s}_r \ , \ \triangleright \\ \mathbf{u}_r & \text{, if } \dot{s}_r < 0 \ . \end{cases}$$

Fig. 244

- $h_r < h_m < h_l$: 1-rarefaction & 2-shock

$$\mathbf{u}^{\downarrow}(\mathbf{v},\mathbf{w}) = \begin{cases} \mathbf{u}_l & \text{, if } \lambda_1(\mathbf{u}_l) > 0 \ , \\ \mathbf{u}_m & \text{, if } \lambda_1(\mathbf{u}_m) < 0 \ , \\ \mathbf{u}_r & \text{, if } \dot{s}_r < 0 \ , \\ \boldsymbol{\kappa}_1(0) & \text{, otherwise.} \end{cases} \quad \triangleright$$

transsonic rarefaction case

$\boldsymbol{\kappa}_1(\tau)$ = integral curve (for $\mathbf{r}_1$), parameterization (5.3.10) $\longrightarrow$ (5.3.11)

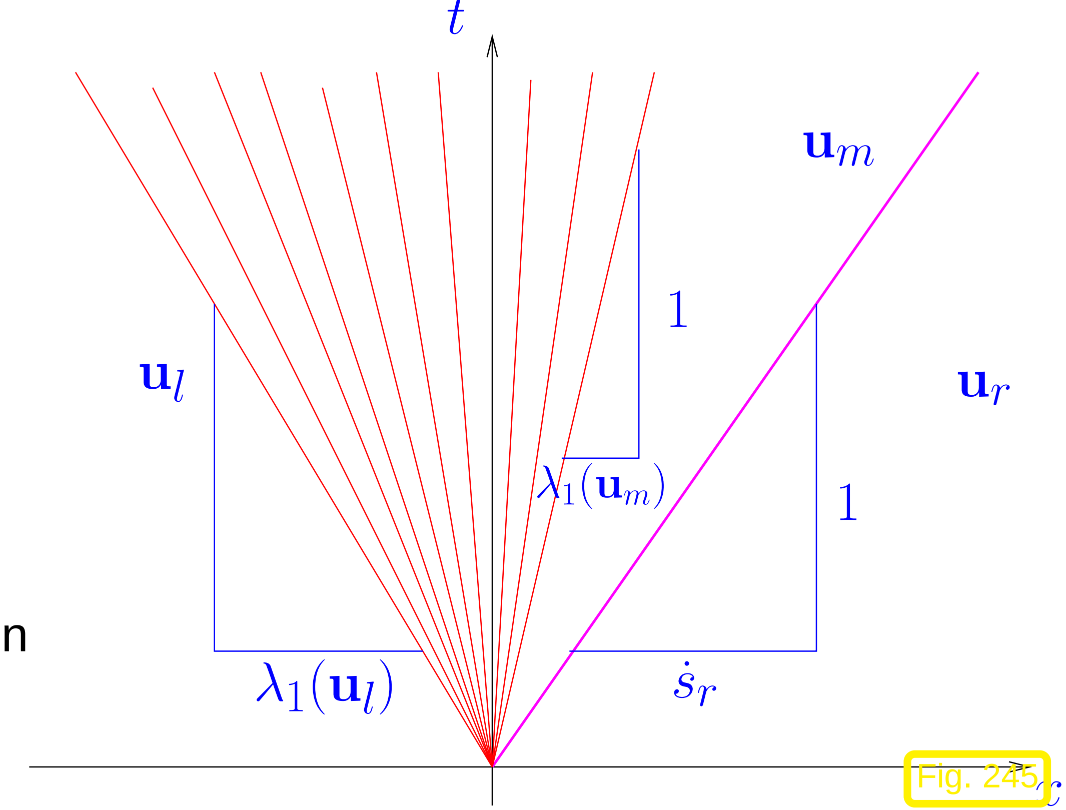


Fig. 245

- $h_l < h_m < h_r$: 1-shock & 2-rarefaction

$$\mathbf{u}^{\downarrow}(\mathbf{v},\mathbf{w}) = \begin{cases} \mathbf{u}_l & \text{, if } \dot{s}_l > 0 \text{ ,} \\ \mathbf{u}_m & \text{, if } \dot{s}_l < 0 < \lambda_2(\mathbf{u}_m) \text{ ,} \\ \mathbf{u}_r & \text{, if } \lambda_2(\mathbf{u}_r) < 0 \text{ ,} \\ \boldsymbol{\kappa}_2(0) & \text{, otherwise.} \end{cases} \quad \triangleright$$

transonic rarefaction case

$\boldsymbol{\kappa}_2(\tau)$ = integral curve (for $\mathbf{r}_2$), parameterization (5.3.10), *cf.* (5.3.11)

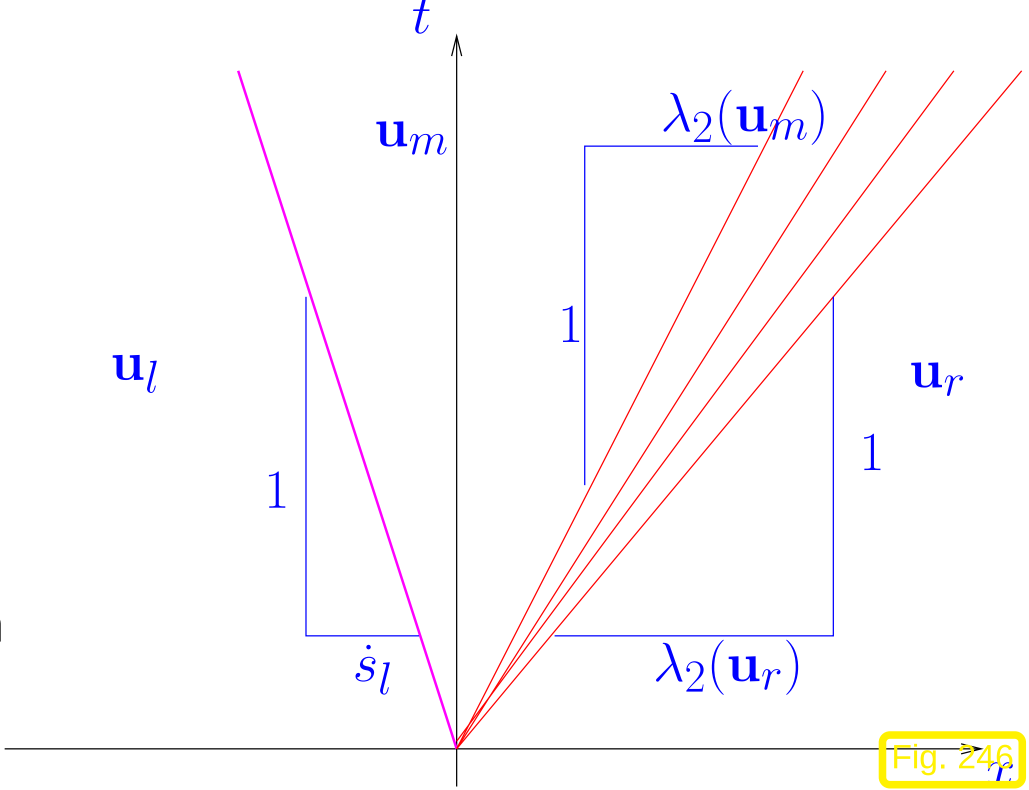


Fig. 246

- $h_m < h_l, h_r$: all-rarefaction solution

$$\mathbf{u}^{\downarrow}(\mathbf{v},\mathbf{w}) = \begin{cases} \mathbf{u}_l & \text{, if } \lambda_1(\mathbf{u}_l) > 0 \;, \\ \mathbf{u}_m & \text{, if } \lambda_1(\mathbf{u}_m) < 0 < \lambda_2(\mathbf{u}_m) \;, \\ \mathbf{u}_r & \text{, if } \lambda_2(\mathbf{u}_r) < 0 \;, \\ \boldsymbol{\kappa}_1(0) & \text{, if } \lambda_1(\mathbf{u}_l) < 0 < \lambda_1(\mathbf{u}_m) \;, \\ \boldsymbol{\kappa}_2(0) & \text{, if } \lambda_2(\mathbf{u}_m) < 0 < \lambda_2(\mathbf{u}_r) \;. \end{cases} \quad \triangleright$$

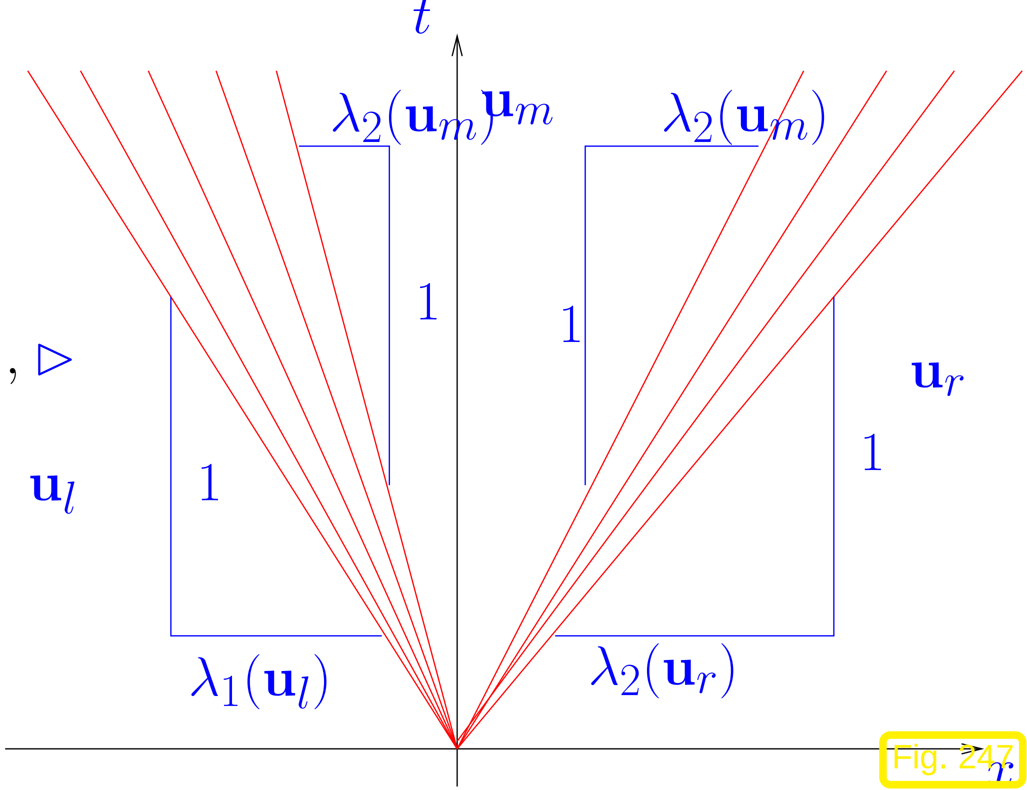


Fig. 247

◇

Example 123 (Godunov method for shallow water equations)*.*

- "dam break" Riemann problem ($h_l = 3$, $h_r = 1$, $v_l = v_r = 0$) for shallow water hyperbolic system of conservation laws (5.1.5), analytic solution from Ex. 114

- Godunov FVM on equidistant space time mesh, fixed ratio $\gamma = \Delta t/\Delta x$

Monitored:

- l^1-norm of discretization error for $t = 1$, $\Delta x \in \{\frac{1}{20}, \frac{1}{40}, \frac{1}{80}, \frac{1}{160}, \frac{1}{320}, \frac{1}{640}\}$ and approximate convergence rates
- evolution of entropy from Ex. 109
- movie: evolution of discrete solution for $\Delta x = \frac{1}{40}$

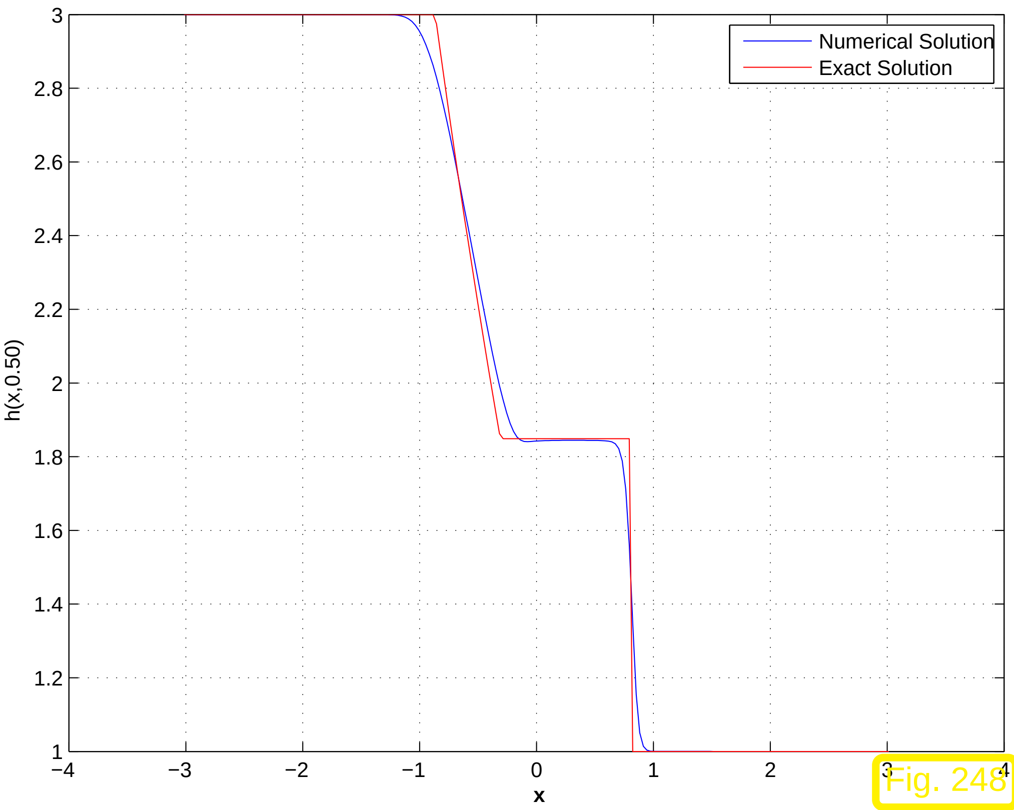


Fig. 248

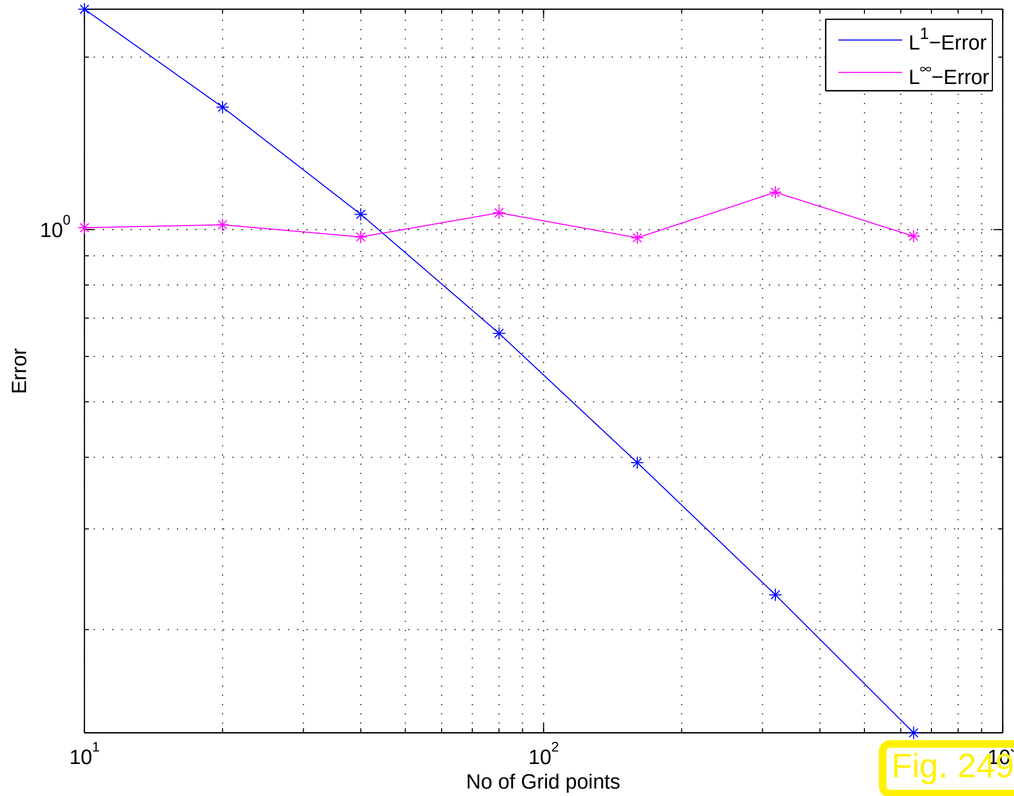


Fig. 249

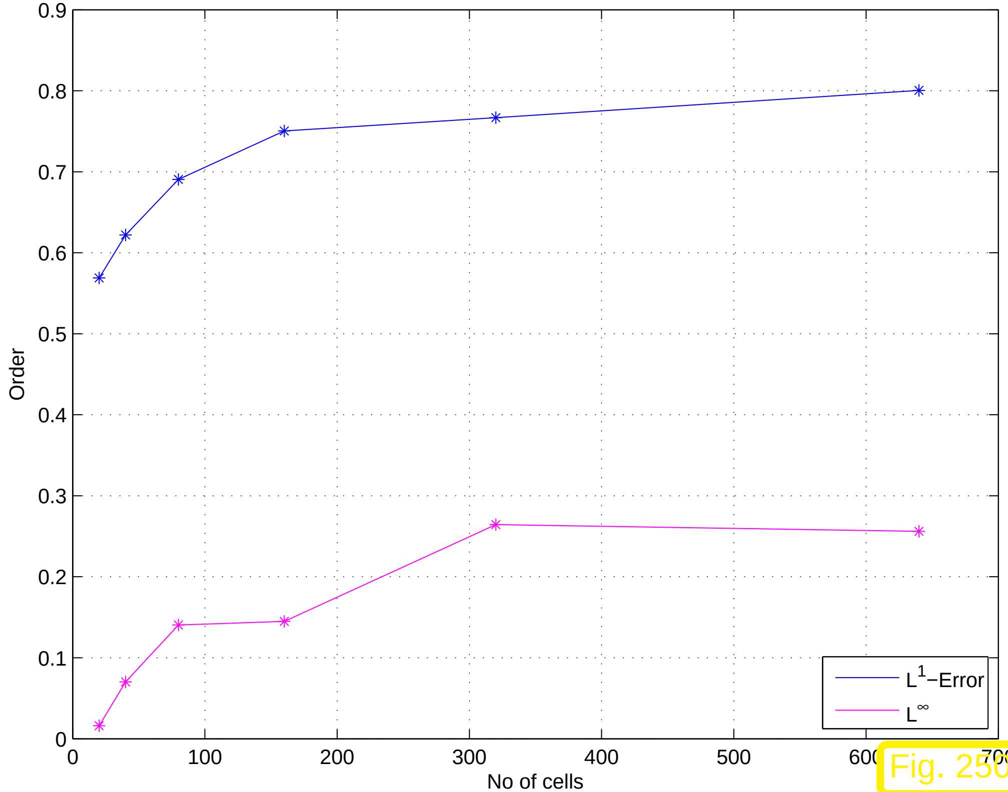


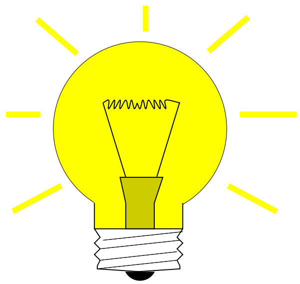
Fig. 250

evaluation of $F_{\mathsf{GD}}(\mathbf{v},\mathbf{w})$ expensive !

(non-linear equations and many ($\approx 2^m$) cases)

6.3 Approximate Riemann solvers

Task: for hyperbolic system (5.0.1) and $\vec{\mu}^{(k-1)}$, $k = 1, \dots .M$, compute numerical fluxes $\mathbf{F}_{j+1/2}$, aim at $\mathbf{F}_{j+1/2} \approx F_{\text{GD}}(\boldsymbol{\mu}_j^{(k-1)}, \boldsymbol{\mu}_{j+1}^{(k-1)})$, $j \in \mathbb{Z}$



Idea: Find similarity solution $\widetilde{\mathbf{u}} : \mathbb{R}\times]0,T[\mapsto \mathbb{R}^m$ of Riemann problem at $x = x_{j+1/2}$ for simplified flux function $\widetilde{\mathbf{F}} : U \mapsto \mathbb{R}^m$

$$\widetilde{\mathbf{u}}: \quad \frac{\partial \widetilde{\mathbf{u}}}{\partial t} + \frac{\partial}{\partial x}\widetilde{\mathbf{F}}(\widetilde{\mathbf{u}}) = 0 \;, \quad \widetilde{\mathbf{u}}(x,0) = \begin{cases} \boldsymbol{\mu}_j^{(k-1)} & \text{, if } x \leq 0 \;, \\ \boldsymbol{\mu}_{j+1}^{(k-1)} & \text{, if } x > 0 \;. \end{cases} \tag{6.3.1}$$

➢ approximate Godunov flux ($\rightarrow$ Sect. 6.2) at $x = x_{j+1/2}$

$$\mathbf{F}_{j+1/2} = \mathbf{F}_{j+1/2}(\boldsymbol{\mu}_j^{(k-1)}, \boldsymbol{\mu}_{j+1}^{(k-1)}) = \mathbf{F}(\widetilde{\mathbf{u}}^{\downarrow}) \;, \quad \widetilde{\mathbf{u}}^{\downarrow} := \widetilde{\mathbf{u}}(0,t) \;, \tag{6.3.2}$$

(More popular) alternative numerical fluxes/numerical flux functions:

$$\mathbf{F}_{j+1/2} = \widetilde{\mathbf{F}}(\widetilde{\mathbf{u}}^{\downarrow}) - \tfrac{1}{2}\big(\widetilde{\mathbf{F}}(\boldsymbol{\mu}_j^{(k-1)}) + \widetilde{\mathbf{F}}(\boldsymbol{\mu}_{j+1}^{(k-1)})\big) + \tfrac{1}{2}\big(\mathbf{F}(\boldsymbol{\mu}_j^{(k-1)}) + \mathbf{F}(\boldsymbol{\mu}_{j+1}^{(k-1)})\big) \;. \tag{6.3.3}$$

Both (6.3.2) & (6.3.3) ➡ consistent numerical flux functions $\rightarrow$ Def. 3.2.2

Observations (guiding choice of $\widetilde{\mathbf{F}} \leftrightarrow \widetilde{\mathbf{u}}$):

☞ Ex. 122 ➤ F_{GD} uses only one value (at $x/t = 0$) of the Riemann solution.

☞ Usually: solution $\mathbf{u}$ of Cauchy problem for (5.0.1) smooth almost everywhere

☞ Usually: discontinuities of $\mathbf{u} \leftrightarrow$ simple shocks $\rightarrow$ Thm. 5.3.1 (Riemann problem "artificial")

6.3.1 Local linearization

$\widetilde{\mathbf{u}}$ **=** Riemann solution for locally (at cell boundaries) linearized system of conservation laws:

$$\text{in (6.3.1):}\quad \frac{\partial\widetilde{\mathbf{u}}}{\partial t} + \mathbf{A}_{j+1/2}\frac{\partial\widetilde{\mathbf{u}}}{\partial x} = 0\,,\quad \widetilde{\mathbf{u}}(x,0) = \begin{cases} \boldsymbol{\mu}_j^{(k-1)} & \text{, if } x < 0\,, \\ \boldsymbol{\mu}_{j+1}^{(k-1)} & \text{, if } x \geq 0\,. \end{cases} \tag{6.3.4}$$

$\boxed{\mathbf{A}_{j+1/2} = \mathbf{A}(\boldsymbol{\mu}_j^{(k-1)}, \boldsymbol{\mu}_{j+1}^{(k-1)})}$ **=** approximation of $D\mathbf{F}(\mathbf{u}(x_{j+1/2}, t_k))$ based on data $\boldsymbol{\mu}_j^{(k-1)}, \boldsymbol{\mu}_{j+1}^{(k-1)}$

Requirements for matrix $\mathbf{A} = \mathbf{A}(\mathbf{v}, \mathbf{w})$:

- $\mathbf{A}$ similar to real diagonal matrix (→ hyperbolicity, Def. 5.1.1),
- $(\mathbf{v}, \mathbf{w}) \mapsto \mathbf{A}(\mathbf{v}, \mathbf{w})$ Lipschitz-continuous,
- $\mathbf{A}(\mathbf{v}, \mathbf{w}) \rightarrow D\mathbf{F}(\mathbf{u})$ as $\mathbf{w}, \mathbf{v} \rightarrow \mathbf{u}$ (→ consistency, *cf.* Def. 3.2.2).

Sect. 5.3.1, (5.3.1) ▶ approximate Riemann solution (wave fan)

$$\widehat{\mathbf{u}}(x,t) = \begin{cases} \boldsymbol{\mu}_j^{(k-1)} & \text{, if } x \leq \widehat{\lambda}_1 t \text{ ,} \\ \boldsymbol{\mu}_j^{(k-1)} + \sum_{i=1}^{l} \delta_i \widehat{\mathbf{r}}_i & \text{, if } \widehat{\lambda}_l t < x \leq \widehat{\lambda}_{l+1} t \text{ ,} \\ \boldsymbol{\mu}_{j+1}^{(k-1)} & \text{, if } x \geq \widehat{\lambda}_m t \text{ ,} \end{cases} \quad \text{with} \quad \Delta\boldsymbol{\mu}_{j+1/2}^{(k-1)} = \sum_{i=1}^{m} \delta_i \widehat{\mathbf{r}}_i \ . \tag{6.3.5}$$

☛ notations: $\sigma(\mathbf{A}_{j+1/2}) = \{\widehat{\lambda}_1 < \widehat{\lambda}_2 < \cdots < \widehat{\lambda}_m\}$, eigenvectors $\widehat{\mathbf{r}}_i$, $i = 1, \ldots, m$

$$\blacktriangleright \quad \widetilde{\mathbf{u}}^{\downarrow} = \boldsymbol{\mu}_j^{(k-1)} + \sum_{\widehat{\lambda}_i < 0} \delta_i \widehat{\mathbf{r}}_i \ .$$

(6.3.2) $\Rightarrow$ $$F(\mathbf{v},\mathbf{w}) = \mathbf{F}(\mathbf{v} + \sum_{\widehat{\lambda}_i<0} \delta_i \widehat{\mathbf{r}}_i) \;, \quad \mathbf{w}-\mathbf{v} = \sum_{i=1}^{m} \delta_i \widehat{\mathbf{r}}_i \;, \tag{6.3.6}$$

(6.3.3) $\Rightarrow$ $$\begin{aligned} F(\mathbf{v},\mathbf{w}) &= \mathbf{A}^+\mathbf{v} + \mathbf{A}^-\mathbf{w} - \tfrac{1}{2}\mathbf{A}(\mathbf{v}+\mathbf{w}) + \tfrac{1}{2}(\mathbf{F}(\mathbf{v}) + \mathbf{F}(\mathbf{w})) \\ &= \tfrac{1}{2}(\mathbf{F}(\mathbf{v}) + \mathbf{F}(\mathbf{w})) - \tfrac{1}{2}|\mathbf{A}|(\mathbf{w}-\mathbf{v}) \;. \end{aligned} \tag{6.3.7}$$

centered flux

viscous modification $\longrightarrow$ Sect. 3.2.9, Rem. 49,
compare Lax-Friedrichs numerical flux (6.1.7)

Simplest choice: state average $\mathbf{A}(\mathbf{v},\mathbf{w}) = D\mathbf{F}(\tfrac{1}{2}(\mathbf{v}+\mathbf{w}))$

Example 124 (State average based linearization for shallow water equations). $\longrightarrow$ Ex. 99

- Riemann problem for (5.1.5) with $h_l = 3$, $v_l = 0.25$, $h_r = 0.5$ $v_r = -2.450309$, $g = 1$

➤ $\mathbf{u}_r \in \mathcal{HL}(\mathbf{u}_l)$, see Ex. 103

➤ $\mathbf{u}_l$, $\mathbf{u}_r$ connected by admissible 1-shock, see Ex. 111

BUT two shocks in approximate Riemann solution based on $\mathbf{A} := D\mathbf{F}(\frac{1}{2}(\mathbf{u}_l + \mathbf{u}_r))$ ▷

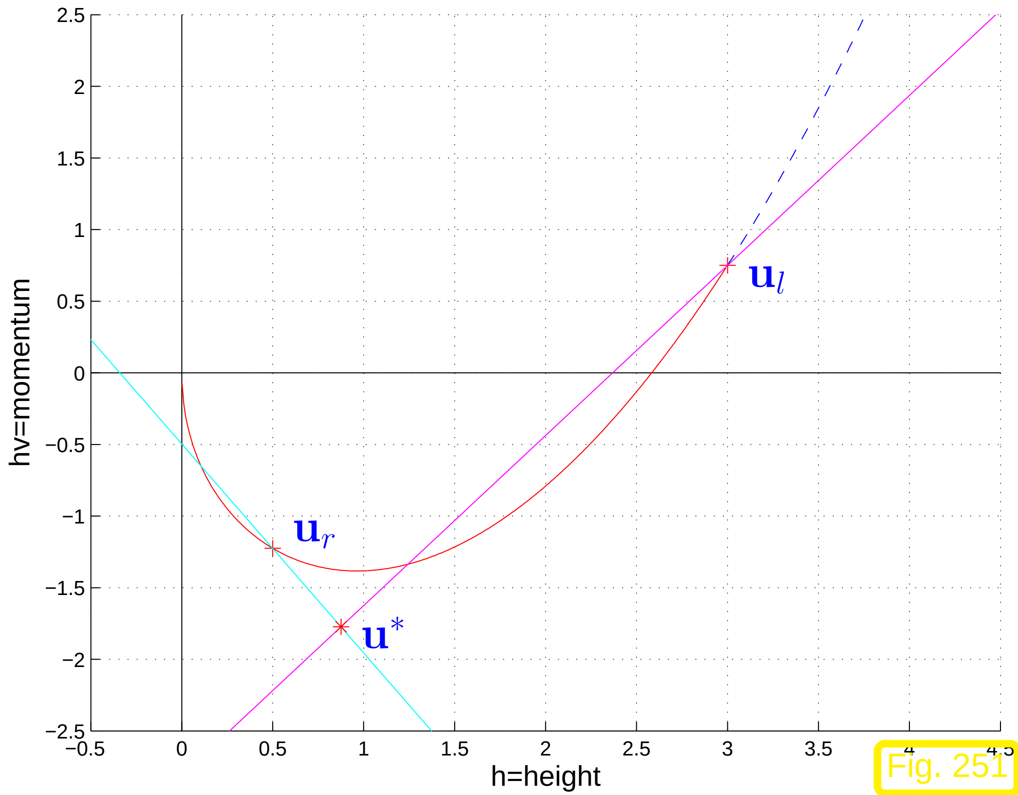


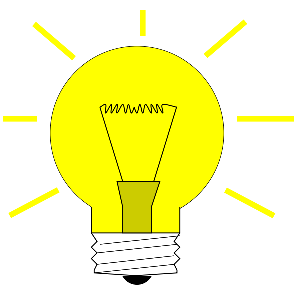
Fig. 251

Numerical simulation of simple shock shallow water Riemann solution based on local linearization at the simple state average. Does this approach lead to increased shock smearing

◇

6.3.2 Roe linearization

If $\boldsymbol{\mu}_j^{(k-1)}$, $\boldsymbol{\mu}_{j+1}^{(k-1)}$ vastly different	$\Rightarrow$	hints at simple shock at $x_{j+1/2}$
	$\Rightarrow$	$\boldsymbol{\mu}_j^{(k-1)} \in \mathcal{HL}(\boldsymbol{\mu}_{j+1}^{(k-1)})$
	$\Rightarrow$	$\boxed{\exists \dot{s} \in \mathbb{R}\text{:}\quad \dot{s}(\boldsymbol{\mu}_{j+1}^{(k-1)} - \boldsymbol{\mu}_j^{(k-1)}) = \mathbf{F}(\boldsymbol{\mu}_{j+1}^{(k-1)}) - \mathbf{F}(\boldsymbol{\mu}_j^{(k-1)})}$



Idea: enforce correct simple shock representation for linearized problem!

$$\dot{s}(\mathbf{w} - \mathbf{v}) = \mathbf{F}(\mathbf{w}) - \mathbf{F}(\mathbf{v}) \;, \quad \mathbf{v}, \mathbf{w} \in U$$

require: $\Downarrow$ (6.3.8)

$$\mathbf{A}(\mathbf{v}, \mathbf{w})(\mathbf{w} - \mathbf{v}) = \dot{s}(\mathbf{w} - \mathbf{v}) = \mathbf{F}(\mathbf{w}) - \mathbf{F}(\mathbf{v})$$

$\Rightarrow$ $\mathbf{w} - \mathbf{v}$ eigenvector of $\mathbf{A}(\mathbf{v}, \mathbf{w})$!

Remark 125 (Linearization and conservation).

$\mathbf{u}$: solution of Riemann problem for (5.0.1) with $\mathbf{u}_l = \mathbf{v}$, $\mathbf{u}_r = \mathbf{w}$

$\widetilde{\mathbf{u}}$: solution of same Riemann problem for $\frac{\partial \widetilde{\mathbf{u}}}{\partial t} + \mathbf{A}\frac{\partial \widetilde{\mathbf{u}}}{\partial x} = 0$

► $$\frac{d}{dt}\int_{\mathbb{R}} \mathbf{u}(x,t)\,\mathrm{d}x = \mathbf{F}(\mathbf{v}) - \mathbf{F}(\mathbf{w}) \quad\longleftrightarrow\quad \frac{d}{dt}\int_{\mathbb{R}} \widetilde{\mathbf{u}}(x,t)\,\mathrm{d}x = \mathbf{A}(\mathbf{v} - \mathbf{w}) \;.$$

(6.3.8) $\Rightarrow$ global conservation (↔ accurate for simple shocks, Ex. 67) (6.3.8) $\Rightarrow$ correct spee

$\triangle$

How to find suitable $\mathbf{A}$ **?**

! mean value theorem: $\displaystyle\int_0^1 D\mathbf{F}(\mathbf{v}+\tau(\mathbf{w}-\mathbf{v}))\,\mathrm{d}\tau\cdot(\mathbf{w}-\mathbf{v}) = \mathbf{F}(\mathbf{w})-\mathbf{F}(\mathbf{v}) \quad \forall \mathbf{v},\mathbf{w}\in U$

Candidate for $\mathbf{A}(\mathbf{v},\mathbf{w})$ **?** ☹ not necessarily similar to real diagonal matrix !

Theorem 6.3.1 (Existence of Roe matrix)**.** → *[25, Thm. 2.1]*
If (5.0.1) *is hyperbolic with convex phase space* U, $\mathbf{F}\in C^1$, *and there is an entropy pair (→ Def. 5.4.1), then we can find* $\mathbf{A}: U\times U\mapsto\mathbb{R}^{m,m}$ *such that*

(i) $\mathbf{A}(\mathbf{u},\mathbf{u}) = D\mathbf{F}(\mathbf{u})$ *for all* $\mathbf{u}\in U$,

(ii) $\mathbf{A}(\mathbf{v},\mathbf{w})(\mathbf{w}-\mathbf{v}) = \mathbf{F}(\mathbf{w})-\mathbf{F}(\mathbf{v})$ *for all* $\mathbf{v},\mathbf{w}\in U$,

(iii) $\mathbf{A}(\mathbf{v},\mathbf{w})$ *is similar to a real diagonal matrix.*

Terminology: $\mathbf{A}(\mathbf{v},\mathbf{w})$ as in Thm. 6.3.1 **=** Roe matrix

Tool for proof: entropy variables ($\rightarrow$ [44]) for entropy pair (η,ψ)

$$\mathbf{q} := \mathbf{grad}\,\eta(\mathbf{u}): \quad \mathbf{q} \leftrightarrow \mathbf{u} \quad \text{is one-to-one (conjugate variables).} \tag{6.3.9}$$

Use idea of the proof for *construction* of $\mathbf{A}(\mathbf{v},\mathbf{w})$ (not necessarily based on entropy variables):

Example 126 (Roe matrix for shallow water equations). $\rightarrow$ [31, Sect. 15.3.3]

$$\text{(5.1.5):}\quad \mathbf{F}(\mathbf{u}) = \begin{pmatrix} u_2 \\ \frac{u_2^2}{u_1} + \frac{1}{2}gu_1^2 \end{pmatrix} \quad , \quad D\mathbf{F}(\mathbf{u}) = \begin{pmatrix} 0 & 1 \\ -(u_2/u_1)^2 + gu_1 & 2u_2/u_1 \end{pmatrix} .$$

$$\text{new variables:}\quad \mathbf{q}(\mathbf{u}) = \frac{1}{\sqrt{u_1}}\mathbf{u} \quad \Leftrightarrow \quad \mathbf{u}(\mathbf{q}) = \begin{pmatrix} q_1^2 \\ q_1 q_2 \end{pmatrix} \quad \Rightarrow \quad \frac{d\mathbf{u}}{d\mathbf{q}} = \begin{pmatrix} 2q_1 & 0 \\ q_2 & q_1 \end{pmatrix}$$

$$\blacktriangleright \quad \widehat{\mathbf{F}}(\mathbf{q}) = \begin{pmatrix} q_1 q_2 \\ q_2^2 + \frac{1}{2}gq_1^4 \end{pmatrix} \quad \Rightarrow \quad D_{\mathbf{q}}\widehat{\mathbf{F}} = \begin{pmatrix} q_2 & q_1 \\ 2gq_1^3 & 2q_2 \end{pmatrix} \tag{6.3.13}$$

➜ in (6.3.13): matrix entries polynomial in $\mathbf{q}$!

Generalization of technique of proof of Thm. 6.3.1:

$$\mathbf{F}(\mathbf{w})-\mathbf{F}(\mathbf{v})=\underbrace{\int_0^1 D\widehat{\mathbf{F}}(\mathbf{q}(\mathbf{v})+\tau(\mathbf{q}(\mathbf{w})-\mathbf{q}(\mathbf{v})))\,\mathrm{d}\tau}_{=:\mathbf{C}}\,(\mathbf{q}(\mathbf{w})-\mathbf{q}(\mathbf{v}))\;,$$

$$\mathbf{w}-\mathbf{v}=\underbrace{\int_0^1 \frac{d\mathbf{u}}{d\mathbf{q}}(\mathbf{q}(\mathbf{v})+\tau(\mathbf{q}(\mathbf{w})-\mathbf{q}(\mathbf{v})))\,\mathrm{d}\tau}_{=:\mathbf{B}}\,(\mathbf{q}(\mathbf{w})-\mathbf{q}(\mathbf{v}))\;.$$

➤ $\mathbf{A}(\mathbf{v},\mathbf{w}):=\mathbf{C}\mathbf{B}^{-1}$.

(6.3.13) $\Rightarrow$ $\mathbf{B}=\begin{pmatrix}2\bar{q}_1 & 0\\ \bar{q}_2 & \bar{q}_1\end{pmatrix}$, $\mathbf{C}=\begin{pmatrix}\bar{q}_2 & \bar{q}_1\\ 2g\bar{q}_1\bar{u}_1 & 2\bar{q}_2\end{pmatrix}$, $\bar{\mathbf{q}}:=\frac{1}{2}(\mathbf{q}(\mathbf{w})+\mathbf{q}(\mathbf{v}))$, $\bar{\mathbf{u}}:=\frac{1}{2}(\mathbf{w}+\mathbf{v})$.

➤ $$\mathbf{A}(\mathbf{v},\mathbf{w})=\mathbf{C}\mathbf{B}^{-1}=\begin{pmatrix}0 & 1\\ -\bar{q}_2^2\bar{q}_1^{-2}+g\bar{u}_1 & 2\bar{q}^2\bar{q}_1^{-1}\end{pmatrix}=\begin{pmatrix}0 & 1\\ -\widehat{v}+g\bar{h} & 2\widehat{v}\end{pmatrix}\;, \qquad (6.3.14)$$

with Roe average $$\widehat{v}:=\frac{\bar{q}_2}{\bar{q}_1}=\frac{w_2w_1^{-1/2}+v_2v_1^{-1/2}}{w_1^{1/2}+v_1^{1/2}}=\frac{\sqrt{h_l}v_l+\sqrt{h_r}v_r}{\sqrt{h_l}+\sqrt{h_r}}\;,$$

with non-conservative state variables $(h_l,v_l)\leftrightarrow\mathbf{v}$, $(h_r,v_r)\leftrightarrow\mathbf{w}$.

Note: $\boxed{\mathbf{A}(\mathbf{v},\mathbf{w})=D\mathbf{F}(\begin{pmatrix}\bar{h}\\ \bar{h}\widehat{v}\end{pmatrix})}$ ➤ similar to real diagonal matrix

Riemann problem of Ex. 124

one-shock solution
of Roe linearized
Riemann problem

(by construction of $\mathbf{A}(\mathbf{v},\mathbf{w})$!)

▷

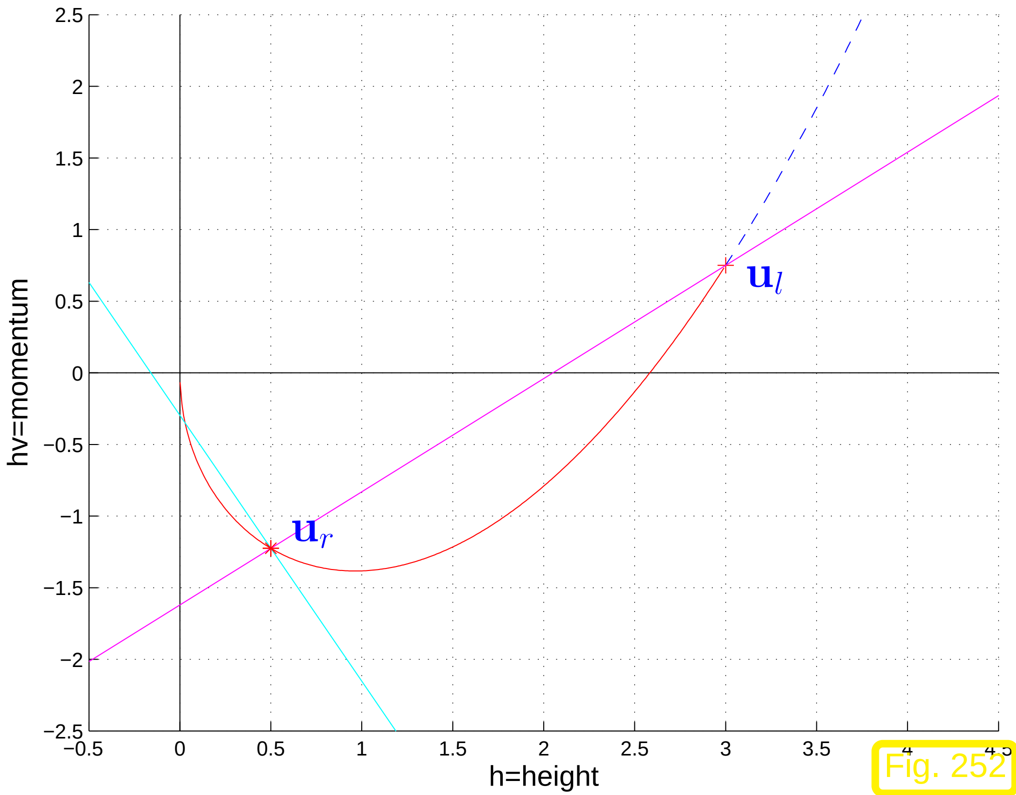


Fig. 252

◇

Example 127 (Breakdown of Roe linearization).

Roe linearization:

➤ approximate Riemann solution
= all-shock solution

Problems in all-rarefaction case **?**

- shallow water equations (5.1.5), $h_l = h_r = 1$, $-v_l = v_r = 2$

➢ non-physical ($h^* < 0$) state in Riemann solution of linearized problem !

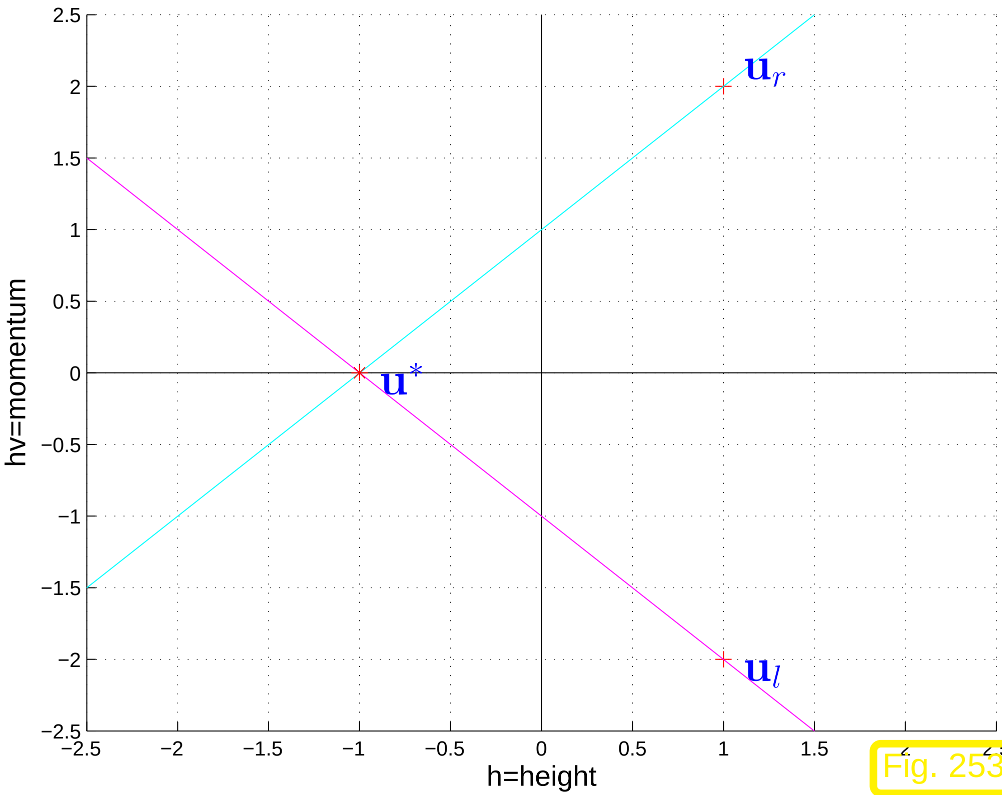


Fig. 253

➡ Must use better (exact) Riemann solution! (*positively conservative* methods [13]) ◇

Example 128 (Roe scheme for shallow water equations). $\longrightarrow$ Ex. 126

* “dam break” Riemann problem of Ex. 123
* Godunov-type FVM with Roe linearization according to Ex. 126 on equidistant space-time mesh.
* same evaluations as in Ex. 123

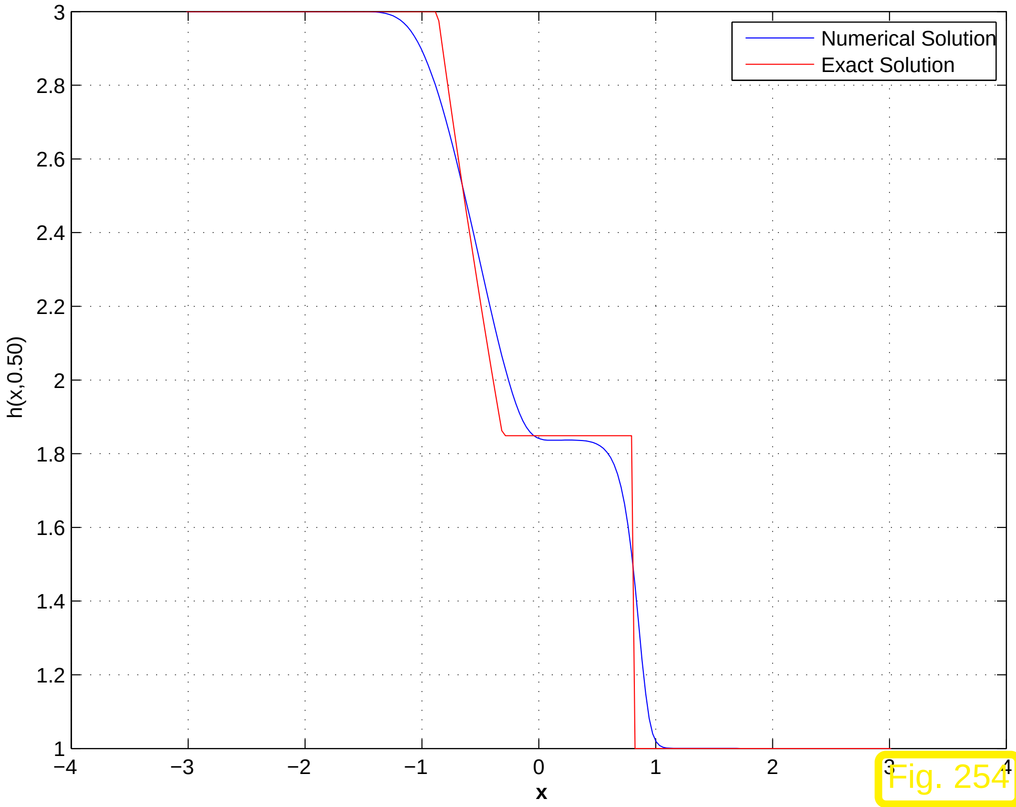


Fig. 254

➤ movie: $h(x, t)$ for Roe scheme

◇

6.3.3 Entropy fixes

$m = 1$: approximate Godunov method & Roe linearization for (2.2.1) **=** simple upwinding (3.2.6)

!

Ex. 70 ➤ convergence to non-physical shock possible !
(failure to capture transsonic rarefaction)

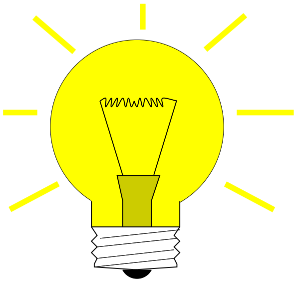
Necessary: entropy fix, see Sect. 3.2.9

☛ notations: $\mathbf{A} = \mathbf{A}(\mathbf{v}, \mathbf{w})$ = Roe matrix for states $\mathbf{v}, \mathbf{w} \in U$,
$\widehat{\lambda}_i$/$\widehat{\mathbf{r}}_i$ = sorted eigenvalues/eigenvectors of $\mathbf{A}$, $\widetilde{\lambda}_0 := -\infty$, $\widetilde{\lambda}_{m+1} = +\infty$,
$\widetilde{\mathbf{u}}$ = approximate Riemann solution used in Godunov-type method $\longrightarrow$ (6.3.1)

6.3.3.1 Harten-Hyman entropy fix

Approximate Riemann solution from (6.3.5):

$$\widetilde{\mathbf{u}}(x,t) = \mathbf{u}^{(l)} \quad \text{for} \quad \widehat{\lambda}_l t < x \leq \widehat{\lambda}_{l+1} t \ , \quad l = 0, \dots, m \ , \quad \mathbf{u}^{(l)} := \mathbf{v} + \sum\nolimits_{i=1}^{l} \delta_i \widehat{\mathbf{r}}_i \ , \quad \mathbf{u}^{(m)} = \mathbf{w} \ .$$



Idea: ➤ detect discontinuities of $\widetilde{\mathbf{u}}$ that should be transsonic rarefactions

(violation of Lax entropy condition Thm. 5.4.4)

$$\Updownarrow$$

for some $l \in \{1,\dots,m-1\}$: $\lambda_l(\mathbf{u}^{(l-1)}) < 0 < \lambda_l(\mathbf{u}^{(l)})$ (6.3.15)

Assume (6.3.15) for single $l \in \{1,\dots,m\}$ ➢ split l-th shock ! [23]

wave decomposition: $$\widetilde{\mathbf{u}}(x,t) = \mathbf{v} + \sum_{i=1}^{m} \mathbf{q}_i(x,t) \;, \quad \mathbf{q}_i(x,t) := \begin{cases} 0 & \text{, if } x \leq \widehat{\lambda}_i t \;, \\ \delta_i \widehat{\mathbf{r}}_i & \text{, if } x > \widehat{\lambda}_i t \;. \end{cases} \qquad (6.3.16)$$

Modified approximate Riemann solution: with $0 < \beta < 1$

$$\widetilde{\mathbf{u}} \rightarrow \check{\mathbf{u}}(x,t) = \mathbf{v} + \sum_{i \neq l} \mathbf{q}_i(x,t) + \check{\mathbf{q}}(x,t) \;, \quad \check{\mathbf{q}}(x,t) = \begin{cases} 0 & \text{, if } x \leq \lambda_l(\mathbf{u}^{(l-1)})t \;, \\ \beta \mathbf{q}_l(x,t) & \text{, if } \lambda_l(\mathbf{u}^{(l-1)})t < x \leq \lambda_l(\mathbf{u}^{(l)})t \;, \\ \mathbf{q}_l(x,t) & \text{, if } x > \lambda_l(\mathbf{u}^{(l)})t \;. \end{cases}$$

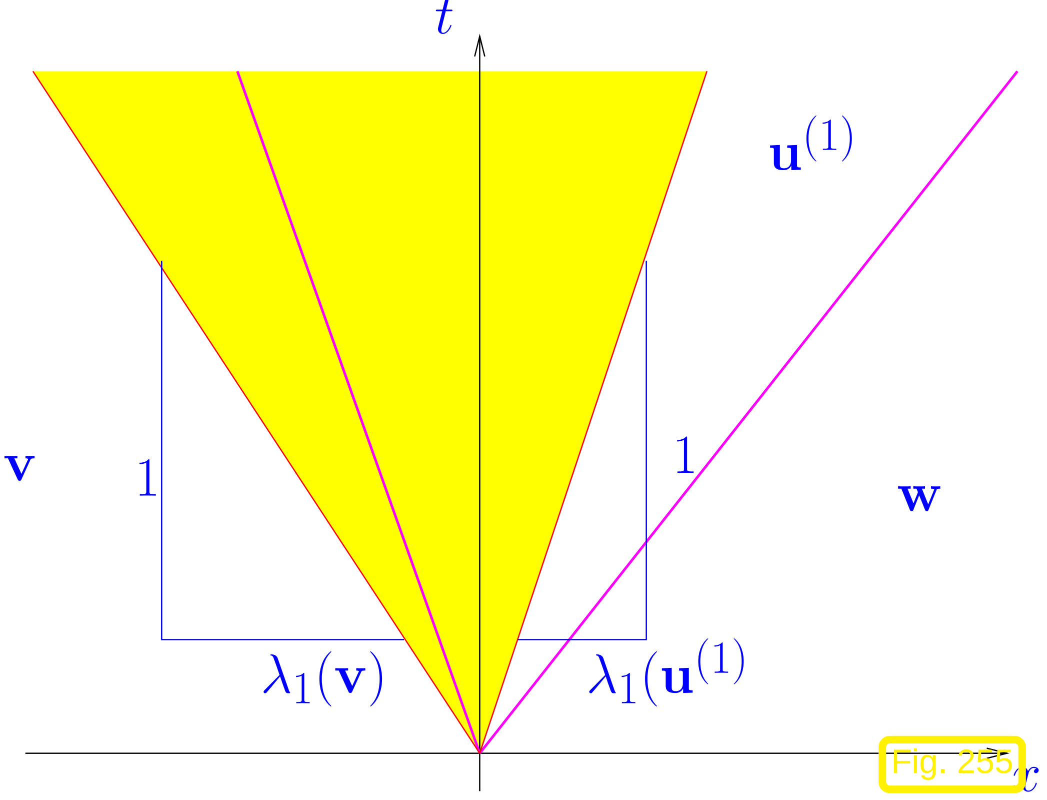


Fig. 255

$m = 2$:

— = shocks of $\widetilde{\mathbf{u}}$

— = 1-characteristics

$$\lambda_1(\mathbf{v}) < 0 < \lambda_1(\mathbf{u}^{(1)})$$

➢ characteristics emanate from 1-shock

■ = area, in which new intermediate state is introduced

How to chose β **?**

Consider $\widetilde{\mathbf{u}} : \mathbb{R}\times]0,T[\mapsto \mathbb{R}^m$ **=** "all-shock" self-similar function, *cf.* (6.3.5): $\mathbf{v}, \widetilde{\mathbf{d}}_i \in U$

$$\widetilde{\mathbf{u}}(x,t) = \mathbf{v} + \sum_{i=1}^{l} \widetilde{\mathbf{d}}_i \quad \text{for} \quad \dot{s}_l t \leq x < \dot{s}_{l+1} t \;, \quad -\infty = \dot{s}_0 < \dot{s}_1 < \cdots < \dot{s}_m < \dot{s}_{m+1} := \infty \;.$$

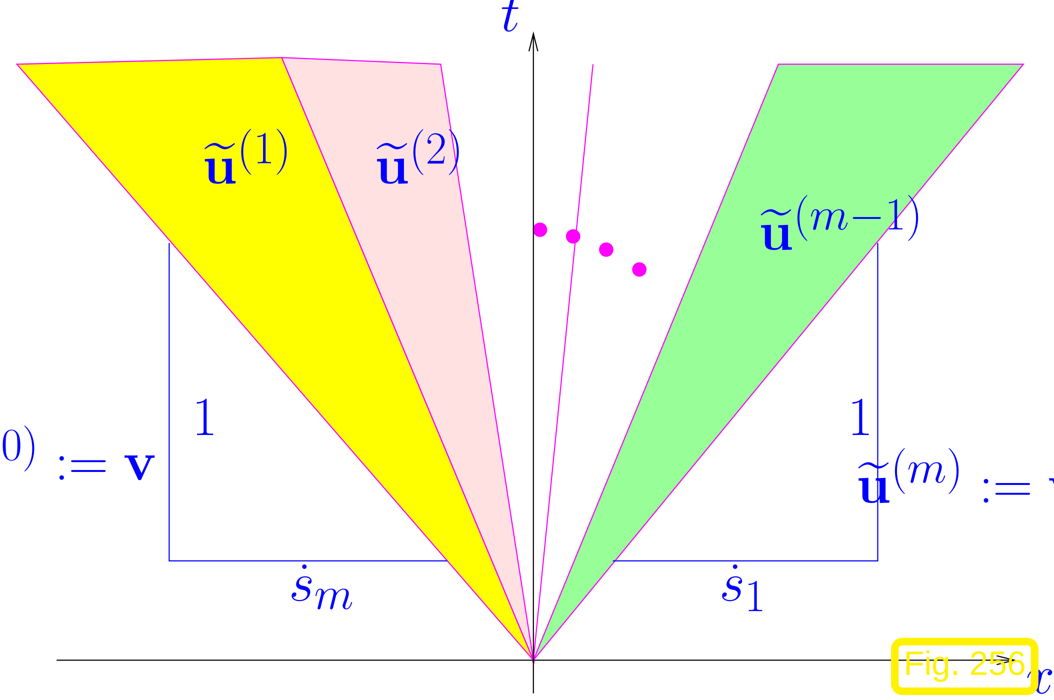


Fig. 256

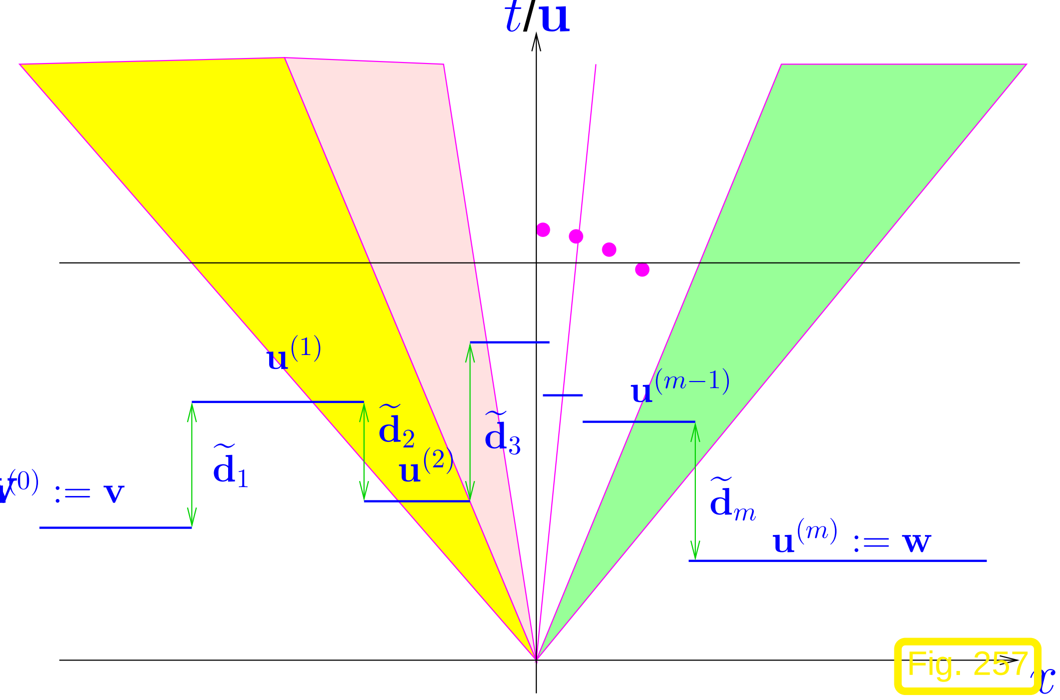


Fig. 257

▶ $$-\frac{d}{dt}\int_{\mathbb{R}} \widetilde{\mathbf{u}}(x,t)\,\mathrm{d}x = \sum_{i=1}^{m} \dot{s}_i \widetilde{\mathbf{d}}_i \overset{!}{=} \mathbf{F}(\mathbf{w}) - \mathbf{F}(\mathbf{v})$$, if $\widetilde{\mathbf{u}}$ ($\approx$) Riemann solution . (6.3.17)

We demand: global conservation property for $\check{\mathbf{u}}$, *cf.* Rem. 125: (6.3.17) $\Rightarrow$

$$\sum_{i<l} \widehat{\lambda}_i \delta_i \widehat{\mathbf{r}}_i + \lambda_l(\mathbf{u}^{(l-1)})\beta\delta_l\widehat{\mathbf{r}}_l + \lambda_l(\mathbf{u}^{(l)})(1-\beta)\delta_l\widehat{\mathbf{r}}_l + \sum_{i>l} \widehat{\lambda}_i \delta_i \widehat{\mathbf{r}}_i \overset{!}{=} \mathbf{A}(\mathbf{w}-\mathbf{v}) = \mathbf{F}(\mathbf{w}) - \mathbf{F}(\mathbf{v}) \,.$$

$\mathbf{A}$ Roe matrix $\Rightarrow$ $\mathbf{A}(\mathbf{w}-\mathbf{v}) = \sum_{i=1}^{m} \widehat{\lambda}_i \delta_i \widehat{\mathbf{r}}_i$ $\Rightarrow$ $\beta = \dfrac{\lambda_l(\mathbf{u}^{(l)}) - \widehat{\lambda}_l}{\lambda_l(\mathbf{u}^{(l)}) - \lambda_l(\mathbf{u}^{(l-1)})}$.

in (6.3.2) $\breve{\mathbf{u}}^{\downarrow} = \mathbf{v} + \sum_{\widehat{\lambda}_i<0, i\neq l} \delta_i \widehat{\mathbf{r}}_i + \beta\delta_l \widehat{\mathbf{r}}_l$

Elaborate Harten-Hyman entropy fix for scalar conservation law with convex flux function and demonstrate viability for Burger's equation with transsonic rarefaction.

6.3.3.2 Enhanced viscosity

For (6.3.7): "entropy fix" in the spirit of Sect. 3.2.9:

$$F(\mathbf{v},\mathbf{w}) = \tfrac{1}{2}(\mathbf{F}(\mathbf{v}) + \mathbf{F}(\mathbf{w})) - \tfrac{1}{2} m_\epsilon(\mathbf{A})(\mathbf{w}-\mathbf{v}) ,$$

$$m_\epsilon(\mathbf{A}) = \mathbf{R}\,\mathrm{diag}(m_\epsilon(\widehat{\lambda}_1),\dots,m_\epsilon(\widehat{\lambda}_m))\mathbf{R}^{-1} , \quad m_\epsilon(\xi) = \begin{cases} \frac{\xi^2}{4\epsilon} + \epsilon & \text{, if } |\xi| < 2\epsilon , \\ |\xi| & \text{, if } |\xi| > 2\epsilon . \end{cases}$$

Choice of "regularization parameter" ϵ **?** $\epsilon \sim \Delta x$ $\longrightarrow$ Ex. 73

6.3.4 Two wave approximations

Sect. 6.3.2 ☞ all-shock approximate Riemann solutions

Now, *cf.* Sect. 6.3.3.1: piecewise constant approximate Riemann solution for (5.0.1) (left state $\mathbf{v} \in U$, right state $\mathbf{w}$) $\longrightarrow$ [25] of "rarefaction type":

$$\widetilde{\mathbf{u}}(x,t) = \begin{cases} \mathbf{v} & \text{, if } x < \dot{s}^- t \ , \\ \mathbf{u}^* & \text{, if } \dot{s}^- t \leq x < \dot{s}^+ t \ , \\ \mathbf{w} & \text{, if } \dot{s}^+ t \leq x \ . \end{cases} \qquad (6.3.18)$$

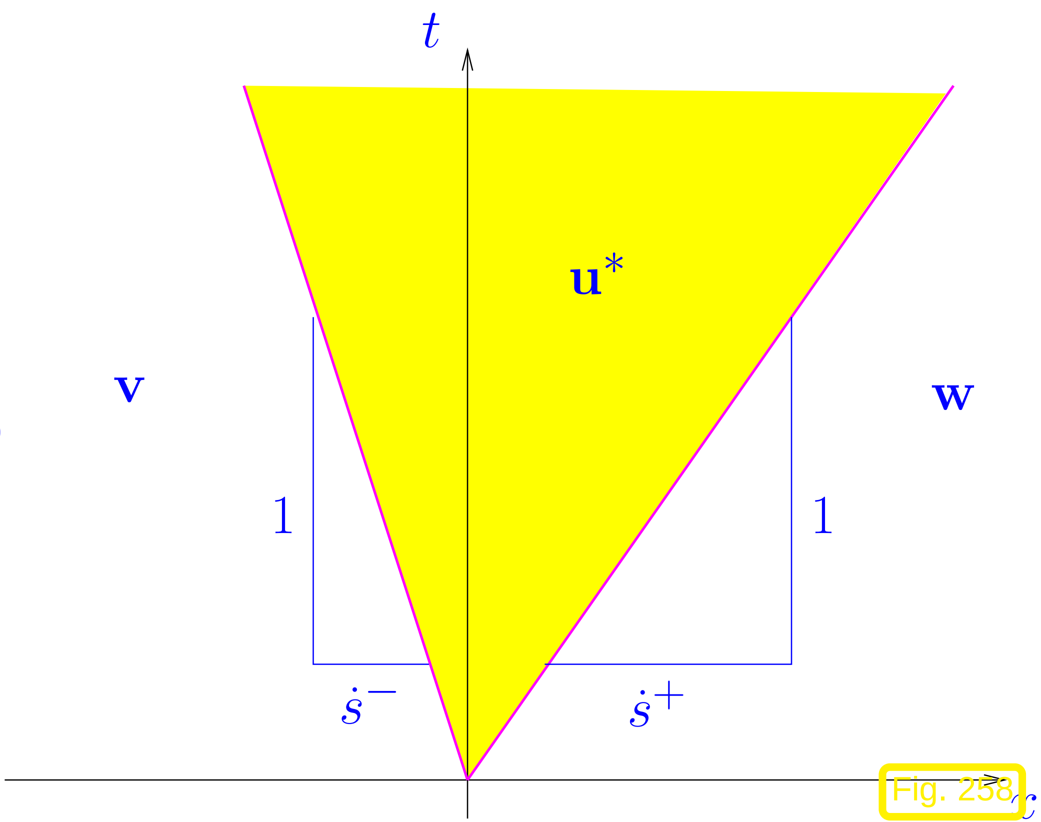


Fig. 258

We demand: global conservation (6.3.17)

$$\mathbf{F}(\mathbf{w}) - \mathbf{F}(\mathbf{v}) = \dot{s}^-(\mathbf{u}^* - \mathbf{v}) + \dot{s}^+(\mathbf{w} - \mathbf{u}^*) \quad \Rightarrow \quad \mathbf{u}^* = \frac{\mathbf{F}(\mathbf{w}) - \mathbf{F}(\mathbf{v}) - \dot{s}^+\mathbf{w} + \dot{s}^-\mathbf{v}}{\dot{s}^- - \dot{s}^+}$$

Choice of "fan edge speeds" $\dot{s}^-$, $\dot{s}^+$ **?**

➡ approximate extremal local signal speeds $\longrightarrow$ [25, 12]: HLLE-FVM

$$\dot{s}^{-} = \min_{1\leq i\leq m} \min\{\widehat{\lambda}_i, \lambda_i(\mathbf{v})\} \quad , \quad \dot{s}^{+} = \max_{1\leq i\leq m} \max\{\widehat{\lambda}_i, \lambda_i(\mathbf{w})\} \;, \tag{6.3.19}$$

$\widehat{\lambda}_i$ **=** eigenvalues of a Roe matrix.

► numerical flux: $$F_{\mathrm{HLLE}}(\mathbf{v},\mathbf{w}) = \begin{cases} \mathbf{F}(\mathbf{v}) & \text{, if } \dot{s}^{-} > 0 \;, \\ \mathbf{F}(\mathbf{w}) & \text{, if } \dot{s}^{+} < 0 \;, \\ \mathbf{F}(\mathbf{u}^*) & \text{, if } \dot{s}^{-} < 0 < \dot{s}^{+} \;. \end{cases} \tag{6.3.20}$$

Special case: $m = 1$ $\leftrightarrow$ scalar 1D conservation law $\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u) = 0 \longrightarrow$ Ch. 3

Assume: f strictly convex ➢ $u^{\downarrow}$ for exact solution of R.P. from (3.2.16)

HLLE-approximation of Riemann solution (left state $v \in \mathbb{R}$, right state $w \in \mathbb{R}$):

$$v > w \quad \text{(shock):} \qquad \dot{s}^{-} = \dot{s}^{+} = \dot{s} \;,$$
$$v < w \quad \text{(rarefaction):} \qquad \dot{s}^{-} = f'(v) \quad , \quad \dot{s}^{+} = f'(w) \;,$$

shock speed $\dot{s} := \frac{f(w)-f(v)}{w-v}$ $\hat{=}$ “Roe matrix” for $m = 1$.

$$\blacktriangleright \quad \widetilde{u}^{\downarrow} = \begin{cases} v & \text{, if } \dot{s}^{-} > 0 \,, \\ w & \text{, if } \dot{s}^{+} < 0 \,, \\ u^* := \frac{f(w)-f(v)-f'(w)w+f'(v)v}{f'(v)-f'(w)} & \text{, if } f'(v) < 0 < f'(w) \,. \end{cases}$$

transsonic rarefaction case

➩ another entropy fix, *cf.* Sect. 3.2.9

Example 129 (HLLE-solver for Burgers equation)*.*

Burgers equation (2.1.7): $f(u) = \frac{1}{2}u^2$ convex, $f'(u) = u$

$$\blacktriangleright \quad F_{\mathrm{HLLE}}(v,w) = \begin{cases} \frac{1}{2}v^2 & \text{, if } \left(v > w \wedge \frac{1}{2}(v+w) > 0\right) \text{ or } 0 < v < w \,, \\ \frac{1}{2}w^2 & \text{, if } \left(v > w \wedge \frac{1}{2}(v+w) < 0\right) \text{ or } v < w < 0 \,, \\ \frac{1}{8}(v+w)^2 & \text{, if } v < 0 < w \,. \end{cases} \tag{6.3.21}$$

➢ F_{HLLE} discontinuous **!**

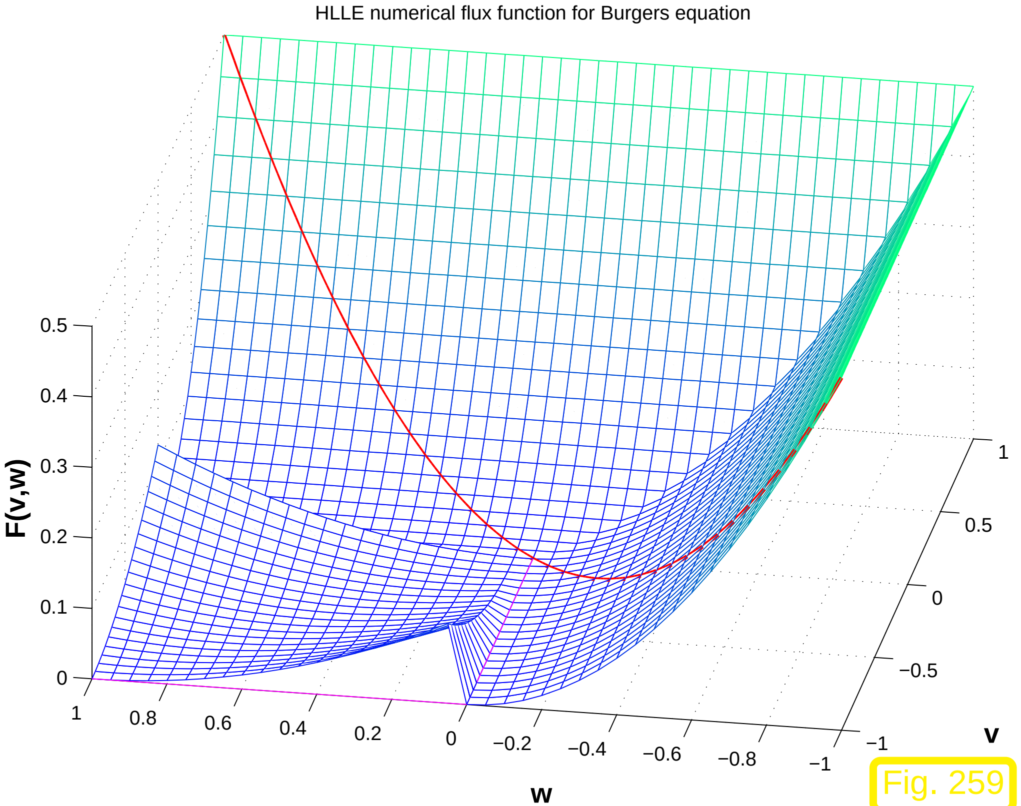


Fig. 259

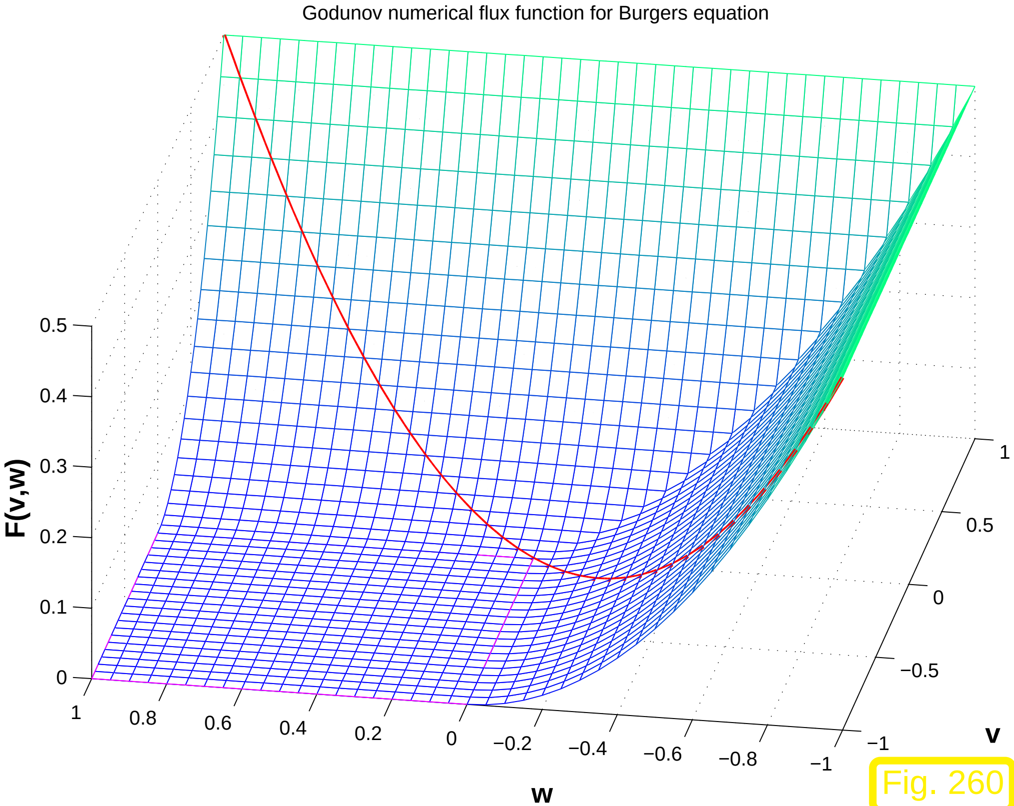


Fig. 260

* Cauchy problem of Ex. 70 (solution is transsonic rarefaction wave)
* equidistant space-time mesh, $\Delta x = 0.06$, $\gamma = 1$
* FVM with HLLE numerical flux

solution for $T = 1$, *cf.* Ex. 73 ▷

➡ movie: HLLE discrete evolution

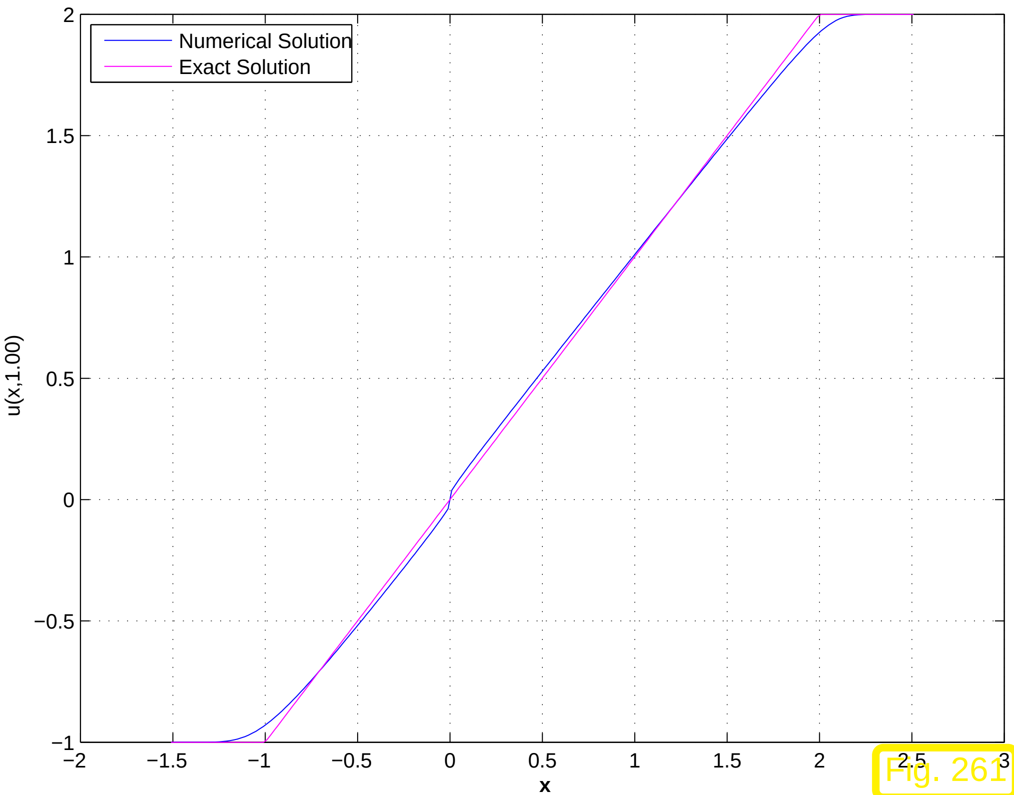


Fig. 261

◇

Example 130 (HLLE-FVM solver for shallow water equations).

* “dam break” Riemann problem of Ex. 123
* HLLE FVM (6.3.20) based on Roe linearization according to Ex. 126 on equidistant space-time mesh.
* same evaluations as in Ex. 123

movie: $h(x,t)$ for HLLE scheme

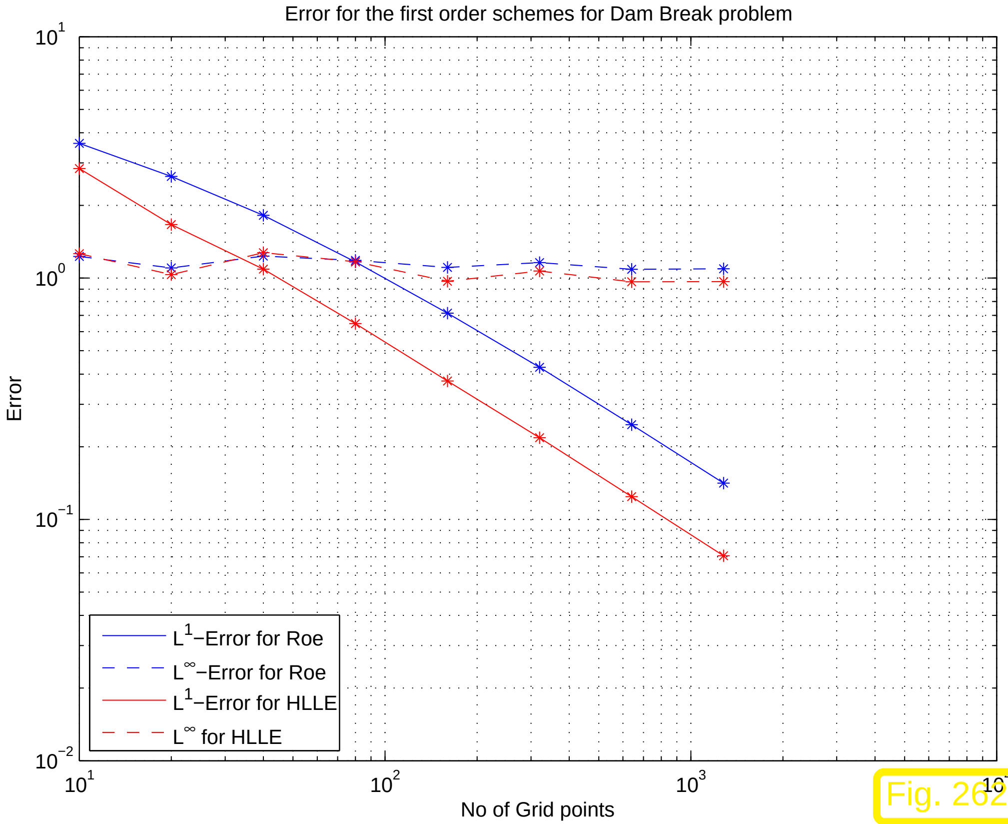


Fig. 262

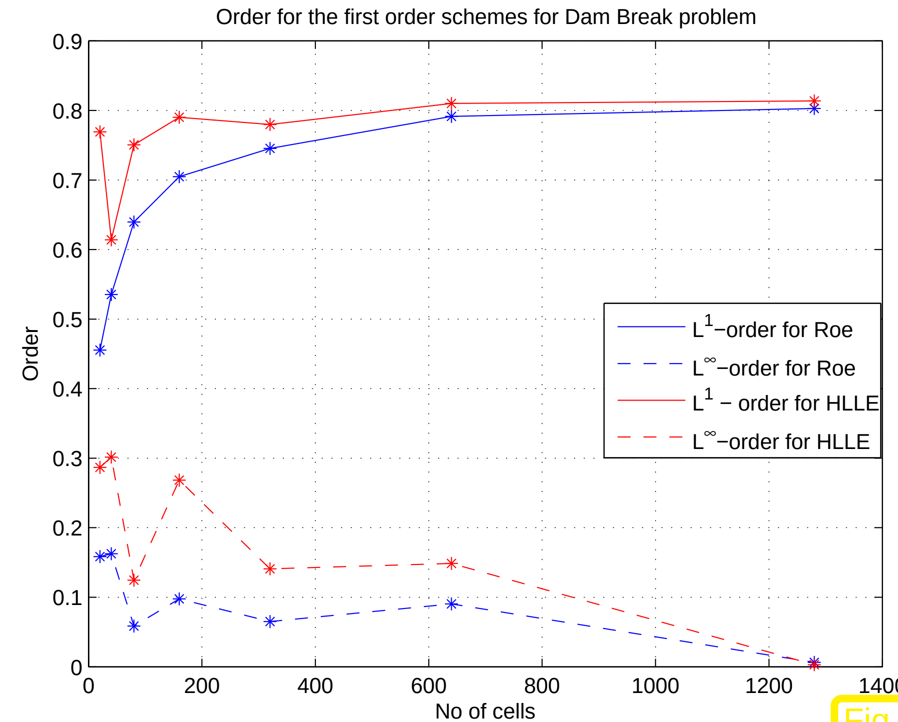


Fig. 263

▷ algebraic convergence rate < 1 due to discontinuous/non-smooth solution

◇

6.4 High resolution FVM

Numerical flux for wave limited (flux limiter function $\varphi : \mathbb{R} \mapsto \mathbb{R}$) high resolution method (for linear systems) from Sect. 6.1:

$$\mathbf{F}_{j+1/2} = F_{\mathrm{uw}}(\boldsymbol{\mu}_j^{(k-1)}, \boldsymbol{\mu}_{j+1}^{(k-1)}) + \tfrac{1}{2}|\mathbf{A}|(1-\gamma|\mathbf{A}|)\boldsymbol{\Phi}(\boldsymbol{\theta}_{j+1/2}^{(k-1)})\Delta\boldsymbol{\mu}_{j+1/2}^{(k-1)} , \tag{6.4.1}$$

$$\boldsymbol{\Phi}(\boldsymbol{\theta}_{j+1/2}^{(k-1)}) := \widehat{\mathbf{R}}\,\mathrm{diag}\big(\varphi(\theta_{j+1/2,1}^{(k-1)}), \dots, \varphi(\theta_{j+1/2,m}^{(k-1)})\big)\widehat{\mathbf{R}}^{-1} . \tag{6.4.2}$$

slope ratios from (6.1.13)

➜ Adapt (6.4.1), (6.4.2) to non-linear system (5.0.1) ! $\longrightarrow$ (3.3.15)

replace $\mathbf{A}$ $\leftarrow$ Roe matrix w.r.t $\boldsymbol{\mu}_j^{(k-1)}, \boldsymbol{\mu}_{j+1}^{(k-1)}$ **or** e.g., $\mathbf{A} = D\mathbf{F}(\tfrac{1}{2}(\boldsymbol{\mu}_j^{(k-1)} + \boldsymbol{\mu}_{j+1}^{(k-1)}))$

F_{uw} $\leftarrow$ Godunov-type numerical flux function (6.3.2), (6.3.3)

How to obtain slope ratios between different cell boundaries **?**

For cell boundary $x_{j+1/2}$ ➢ Roe matrix $\mathbf{A}_{j+1/2} = \mathbf{A}(\boldsymbol{\mu}_j^{(k-1)}, \boldsymbol{\mu}_{j+1}^{(k-1)}) = \widehat{\mathbf{R}}\,\mathrm{diag}(\widehat{\lambda}_1, \dots, \widehat{\lambda}_m)\widehat{\mathbf{R}}^{-1}$

① $\Delta\boldsymbol{\omega}_{j-i1/2} := \widehat{\mathbf{R}}^{-1}\Delta\boldsymbol{\mu}_{j-i1/2}\ , \quad i \in \{-1,1,3\}\ ,$

② $\quad \theta^{(k-1)}_{j+1/2,l} = \begin{cases} (\Delta\boldsymbol{\omega}_{j-1/2})_l : (\Delta\boldsymbol{\omega}_{j+1/2})_l & \text{, if } \widehat{\lambda}_l > 0\ , \\ (\Delta\boldsymbol{\omega}_{j+3/2})_l : (\Delta\boldsymbol{\omega}_{j+1/2})_l & \text{, if } \widehat{\lambda}_l < 0\ , \end{cases} \quad l = 1,\dots,m\ .$ (6.4.3)

Example 131 (Lax-Wendroff and flux limited FVM for shallow water equations).

- "dam break" Riemann problem of Ex. 123

- "Lax-Wendroff": unlimited scheme (6.4.1), (6.4.2), (6.4.3), $\varphi(\theta) \equiv 1$, based on Roe linearization according to Ex. 126 on equidistant space-time mesh $\longrightarrow$ (3.2.27)

- Flux limited FVM (6.4.1), (6.4.2), (6.4.3), based on Roe linearization according to Ex. 126 on equidistant space-time mesh, using
  - φ = minmod limiter ($\longrightarrow$ Def. 3.3.3): $\quad \varphi(\theta) = \max\{0, \min\{\theta, 1\}\}$
  - φ = superbee limiter $\longrightarrow$ (76): $\quad \varphi(\theta) = \max\{0, \min\{2\theta, 1\}, \min\{\theta, 2\}\}$

- same evaluations as in Ex. 123

➨ movie: Lax-Wendroff evolution of $h(x,t)$

➤ movie: $h(x,t)$ for minmod flux limited FVM

➤ movie: $h(x,t)$ for superbee flux limited FVM

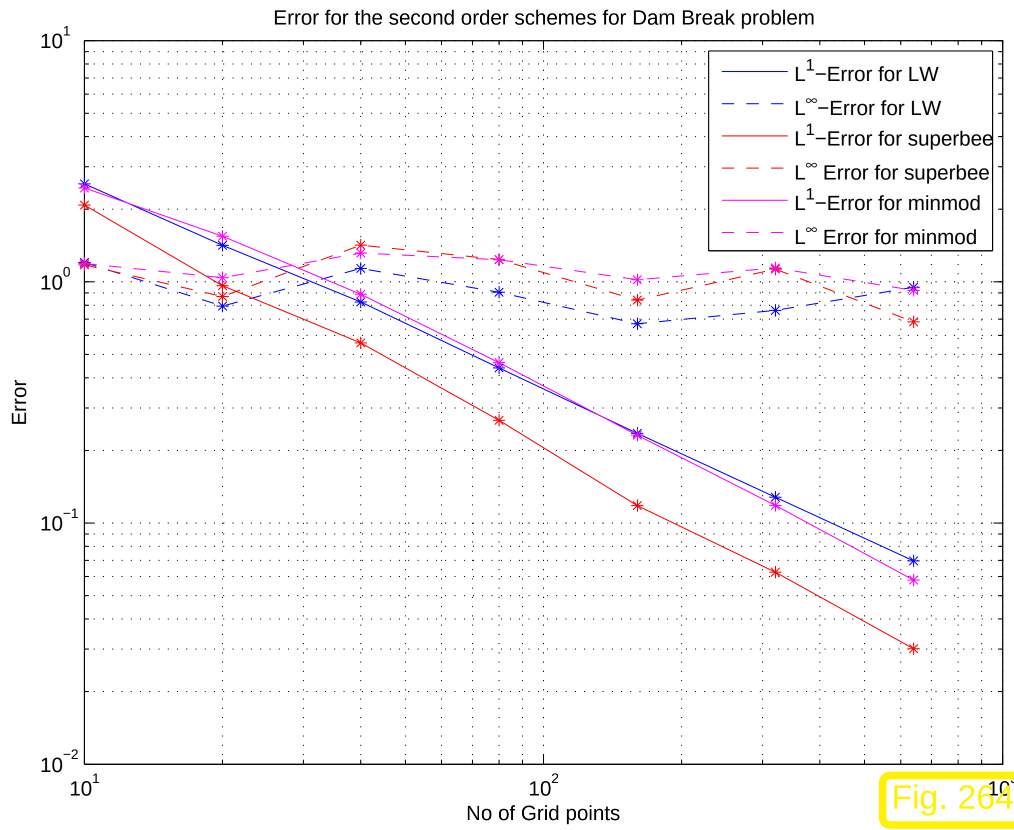


Fig. 264

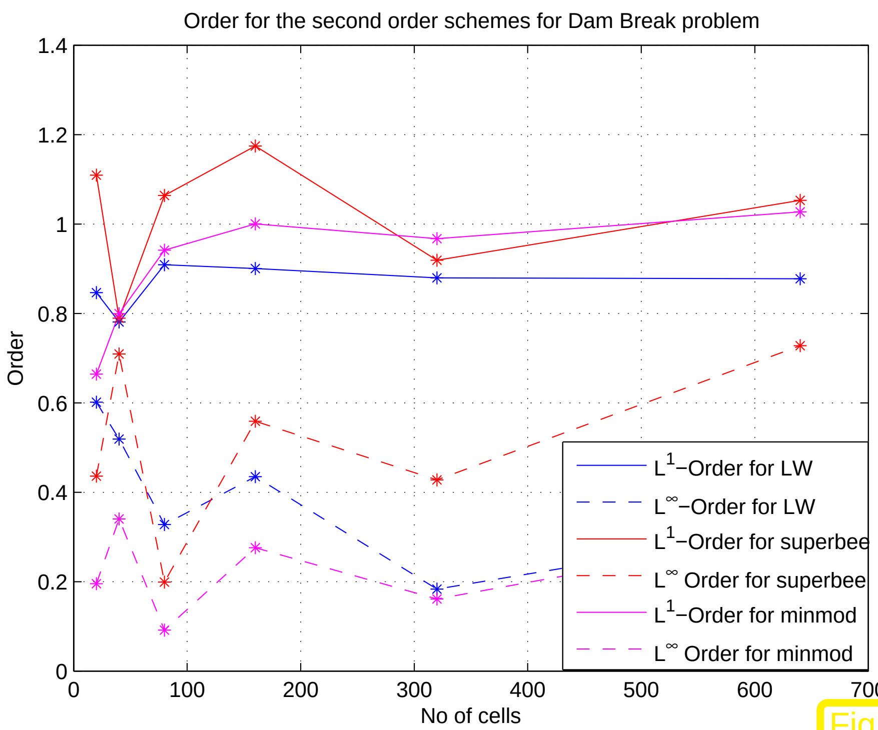


Fig. 265

▷ algebraic convergence rate $\lesssim 1$ due to discontinuous/non-smooth solution

◇

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Bibliography

[1] M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions*, Dover Publications, New York, 1970.

[2] B. Alpert, L. Greengard, and T. Hagstrom, *Rapid evaluation of non-reflecting boundary kernels for time-domain wave propagation*, SIAM J. Num. Anal., 37 (2000), pp. 1138–1164.

[3] ——, *Nonreflecting boundary conditions for the time-dependent wave equation*, J. Comp. Phys., 180 (2002), pp. 270–296.

[4] R. Artebrant and H. Schroll, *Conservative logarithmic reconstructions and finite volume methods*, SIAM J. Sci. Comp., 27 (2005), pp. 294–314.

[5] T. Bridges and S. Reich, *Numerical methods for hamiltonian pdes*, J. Phys. A: Math. Gen., 39 (2005), pp. 5287–5320.

[6] M. Crandall and A. Majda, *The method of fractional steps for conservation laws*, Numerische Mathematik, 34 (1980), pp. 285–314.

[7] M. Crandall and A. Majda, *Monotone difference approximations for scalar conservation laws*, Math. Comp., 34 (1980), pp. 1–21.

[8] C. Dafermos, *Hyperboloic conservation laws in continuum physics*, vol. 325 of Grundlehren der mathematischen Wissenschaften, Springer, Berlin, 2000.

[9] P. Deuflhard and F. Bornemann, *Numerische Mathematik II*, DeGruyter, Berlin, 2 ed., 2002.

[10] P. Deuflhard and A. Hohmann, *Numerische Mathematik I*, DeGruyter, Berlin, 3 ed., 2002.

[11] R. DiPerna, *Convergence of approximate solutions to conservation laws*, Archive for Rational Mechanics and Analysis, 82 (1983), pp. 27–70.

[12] B. Einfeldt, *On Godunov type methods for gas dynamics*, SIAM J. Numer. Anal., 25 (1988), pp. 294–318.

[13] B. Einfeldt, C. Munz, P. Roe, and B. Sjogreen, *On Godunov type methods near low densities*, J. Comp. Phys., 92 (1991), pp. 273–295.

[14] L. Evans, *Partial differential equations*, vol. 19 of Graduate Studies in Mathematics, American Mathematical Society, Providence, RI, 1998.

[15] E. Godlewski and P.-A. Raviart, *Hyperbolic systems of conservation laws*, no. 3/4 in Matematiques & Applications, Ellipsis, Paris, France, 1991.

[16] S. GOTTLIEB, C.-W. SHU, AND E. TADMOR, *Strong stability-preserving high-order time discretization methods*, SIAM Review, 43 (2001), pp. 89–112.

[17] W. HACKBUSCH, *Elliptic Differential Equations. Theory and Numerical Treatment*, vol. 18 of Springer Series in Computational Mathematics, Springer, Berlin, 1992.

[18] ——, *Iterative Solution of Large Sparse Systems of Equations*, vol. 95 of Applied Mathematical Sciences, Springer-Verlag, New York, 1993.

[19] E. HAIRER, C. LUBICH, AND G. WANNER, *Geometric numerical integration*, vol. 31 of Springer Series in Computational Mathematics, Springer, Heidelberg, 2002.

[20] E. HAIRER, S. NORSETT, AND G. WANNER, *Solving Ordinary Differential Equations I. Nonstiff Problems*, Springer-Verlag, Berlin, Heidelberg, New York, 2 ed., 1993.

[21] M. HANKE-BOURGEOIS, *Grundlagen der Numerischen Mathematik und des Wissenschaftlichen Rechnens*, Mathematische Leitfäden, B.G. Teubner, Stuttgart, 2002.

[22] A. HARTEN, *High resolution schemes for hyperbolic conservation laws*, J. Comp. Phys., 49 (1983), pp. 357–393.

[23] A. HARTEN AND J. HYMAN, *Self-adjusting grid methods for one-dimensional hyperbolic conservation laws*, J. Comp. Phys., 50 (1983), pp. 235–269.

[24] A. HARTEN, K. HYMAN, AND P. LAX, *On finite-difference approximations and entropy conditions for shocks*, Commun. Pure Appl. Math., 29 (1976), pp. 297–322.

[25] A. HARTEN, P. LAX, AND B. VAN LEER, *On upstream differencing and godunov-type schemes for hyperbolic conservation laws*, SIAM Review, 25 (1983), pp. 35–61.

[26] H. HEUSER, *Lehrbuch der Analysis 2*, Teubner–Verlag, Stuttgart, 2 ed., 1986.

[27] R. HIPTMAIR AND C. SCHWAB, *Numerics of elliptic and parabolic boundary value problems*. Lecture slides. Available at http://www.sam.math.ethz.ch/~hiptmair/NAPDE_06.pdf, March 2006.

[28] C. KANE, J. MARSDEN, M. ORTIZ, AND M. WEST, *Variational integrators and the Newmark algorithm for conservative and dissipative mechanical systems*, Int. J. Numer. Meth. Engr., 49 (2000), pp. 479–504.

[29] D. KRÖNER, *Numerical Schemes for Conservation Laws*, Wiley-Teubner, Chichester, 1997.

[30] R. LEVEQUE, *Numerical Methods for Conservation Laws*, Birkhäuser Verlag, Basel, Boston, Berlin, 1992.

[31] R. LEVEQUE, *Finite Volume Methods for Hyperbolic Problems*, Cambridge Texts in Applied Mathematics, Cambridge University Press, Cambridge, UK, 2002.

[32] A. MARQUINA, *Local piecewise hyperbolic reconstruction of numerical fluxes for nonlinear scalar conservation laws*, SIAM J. Sci. Comp., 15 (1994), pp. 892–915.

[33] R. MCLACHLAN AND G. QUISPEL, *Splitting methods*, Acta Nmerica, 11 (2002).

[34] K. MORTON, *Discretization of unsteady hyperbolic conservation laws*, SIAM J. Numer. Anal., 39 (2001), pp. 1556–1597.

[35] H. NESSAHU AND E. TADMOR, *Non-oscillatory central differencing for hyperbolic conservation laws*, J. Comp. Phys., 87 (1990), pp. 408–463.

[36] S. OSHER, *Riemann solvers, the entropy condition, and difference approximations*, SIAM J. Numer. Anal., 21 (1984), pp. 217–235.

[37] S. OSHER AND E. TADMOR, *On the convergence of difference approximations to scalar conservation laws*, Math. Comp., 50 (1988), pp. 19–51.

[38] B. PERTHAME AND M. WESTDICKENBERG, *Total oscillation diminishing property for scalar conservation laws*, Numer. Math., 100 (2005), pp. 331–349.

[39] R. REMMERT, *Funktionentheorie I*, no. 5 in Grundwissen Mathematik, Springer, Berlin, 1984.

[40] M. RENARDY AND R. ROGERS, *An Introduction to Partial Differential Equations*, Springer–Verlag, New York, 1993.

[41] C.-W. SHU, *Essentially non-oscillatory and weighted essentially non-oscillatory schemes for hyperbolic conservation laws*, in Advanced Numerical Approximation of Non-linear Hyperbolic Equations, A. Quarteroni, ed., vol. 1697 of Lecture Notes in Mathematics, Springer, Berlin, 1998, pp. 325–432.

[42] C.-W. SHU AND S. OSHER, *Efficient implementation of essentially non-oscillatory shock-capturing schemes*, J. Comp. Phys., 77 (1988), pp. 439–471.

[43] E. TADMOR, *Viscosity and the entropy condition for conservative finite difference schemes*, Math. Comp., 43 (1984), pp. 369–381.

[44] ——, *Entropy stability theory for difference approximations of nonlinear conservation laws and related time dependent problems*, Acta Numerica, (2003), pp. 451–512.

[45] Z.-H. TENG, *On the accuracy of fractional step methods for conservation laws in two dimensions*, SIAM J. Numer. Anal., 31 (1994), pp. 43–63.

[46] E. ZEIDLER, *Nonlinear Functional Analysis and its Applications. II/B: Non-Linear Monotone Operators*, Springer–Verlag, New York, Berlin, Heidelberg, 1990.