Functional Analysis I

Exercise classes

Hjalti Isleifsson

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These are notes for my exercises classes in Functional Analysis I in fall semester 2023 at ETH Zürich. If you find mistakes in the notes, please let me know by sending me an email at hjalti.isleifsson@math.ethz.ch.

Exercise class 1

Exercise 1. Let $(X, \|\cdot\|)$ be a normed space. Show that X is a Banach space if and only if every absolutely converging serie in X converges (a serie $\sum_{i=1}^{\infty} x_i$ is said to be *absolutely convergent* if $\sum_{i=1}^{\infty} \|x_i\|$ converges).

Solution. Assume that X is a Banach space and let $\sum_{i=0}^{\infty} x_i$ be an absolutely converging serie. For every $n \geq 0$, we let $s_n := \sum_{i=1}^{n} x_i$ denote the n-th partial sum. We will show that (s_n) converges by showing that it is a Cauchy sequence. So let $\varepsilon > 0$. As $\sum_{i=1}^{\infty} \|x_i\|$ converges, there is $N \geq 0$ such that if $N \leq n < m$ then $\sum_{i=0}^{m} \|x_i\| < \sum_{i=0}^{n} \|x_i\| + \varepsilon$ so

$$\|s_m - s_n\| = \left\| \sum_{i=n+1}^{m} x_i \right\| \leq \sum_{i=n+1}^{m} \|x_i\| < \varepsilon.$$

This shows that (s_n) is Cauchy and hence it converges as X is complete.

Conversely, assume that every abolutely convergent serie converges. Let $(x_i)_{i=0}^{\infty}$ be a Cauchy sequence. It suffices to show that there is a converging subseqeunce (it is a general fact that any Cauchy sequence in a metric space which has a convergent subsequence is convergent; if you have not seen this before you should convince yourself that this is correct). As (x_i) is Cauchy, there is a sequence $i_0 < i_1 < \cdots$ such that $\|x_{i_{j+1}} - x_{i_j}\| < 2^{-j}$ for every j. Now, $x_{i_k} = x_{i_0} + \sum_{j=1}^{k} (x_{i_j} - x_{i_{j-1}})$ and the serie $\sum_{j=1}^{\infty} (x_{i_j} - x_{i_{j-1}})$ is absolutely convergent and hence convergent, so (x_{i_k}) is convergent. This finishes the proof. ■

Example 2. (i) Let c_{00} denote the space of sequences $(x_i)_{i=0}^{\infty}$ such that $x_i = 0$ for all but finitely many i, endowed with the supremum norm $\|(x_i)\|_\infty = \sup_{i \in \mathbb{N}} |x_i|$. This space is not complete as can be seen as follows: Let (x_i) be any sequence of positive numbers such that $x_i \to 0$ as $i \to \infty$ and consider the sequence $x^j = (x_0, x_1, \ldots, x_j, 0, 0, \ldots) \in c_{00}$. Now, let $\varepsilon > 0$. As $x_i \to \infty$, there exists an N such that if $i \geq N$, then $x_i < \varepsilon$. Now, if $N \leq j < k$ then $\|x^j - x^k\|_\infty = \max_{i=k+1}^{j} |x_i| < \varepsilon$ so (x^j) is Cauchy. However, it does not have a limit in c_{00} which can be seen as follows: Suppose for a contradiction that $\bar{x} = \lim_{j\to\infty} x^j$. Then $|\bar{x}_i - (x^j)_i| \leq \|x - x^j\|_\infty \to 0$ as $j \to \infty$ so $\bar{x}_i = \lim_{j\to\infty}(x^j)_i = x_i$ i.e. $\bar{x} = x$ and as we assumed that all the entries of x are positive, $x \notin c_{00}$.

(ii) Let c_0 denote the space of sequences $(x_i)_{i=0}^\infty$ such that $x_i \to 0$ as $i \to \infty$, endowed with the supremum norm. We begin by showing that c_0 is complete: So let (x^j) be a Cauchy sequence in c_0. For each i it holds that $|(x^j)_i - (x^k)_i| \leq \|x^j - x^k\|_\infty$ so $((x^j)_i)_{i\in\mathbb{N}}$ is also Cauchy and hence has a limit x_i; let $x := (x_i)$. We now show that $\|x - x^j\|_\infty \to 0$ as $j \to \infty$: Let $\varepsilon > 0$. As (x^j) is Cauchy, there is an N such that if $j, k \geq N$ then $\|x^j - x^k\|_\infty < \varepsilon$ and then $|(x^k)_i - (x^j)_i| \leq \|x^k - x^j\|_\infty < \varepsilon$ for every $i \in \mathbb{N}$ so

$$|x_i - (x^j)_i| \leq \limsup_{k\to\infty} |(x^k)_i - (x^j)_i| \leq \varepsilon$$

for every $i \in \mathbb{N}$ and hence $\|x - x^j\|_\infty \leq \varepsilon$. This shows that $\|x - x^j\|_\infty \to 0$ as $j \to \infty$.

It remains to show that $x_i \to \infty$ as $i \to \infty$. So let $\varepsilon > 0$ and let N be such that if $j \geq N$ then $\|x - x^j\|_\infty < \varepsilon$. As $(x^N)_j \to 0$ as $j \to \infty$, there is an M such that if $i \geq M$ then $|(x^N)_i| < \varepsilon$. Now, for $i \geq M$, it holds that

$$|x_i| \leq |x_i - (x^N)_i| + |(x^N)_i| < \|x - x^N\|_\infty + \varepsilon < 2\varepsilon$$

which shows that $x_i \to 0$ as $i \to \infty$.

(iii) Now it is easy to see that c_0 is the completion of c_{00}: Let $x \in c_0$, $\varepsilon > 0$ and N be such that if $i \geq N$ then $|x_i| < \varepsilon$. Then $(x' = (x_0, \ldots, x_N, 0, 0, \ldots) \in c_{00}$ and $\|x - x'\|_\infty < \varepsilon$. This shows that c_{00} is dense in c_0 and as c_0 is complete, we conclude that c_0 is the completion of c_{00}.

We will now cover the following classical theorem.

Theorem 3. *Let $(X, \|\cdot\|)$ be a normed space. Then X is finite dimensional if and only if its closed unit ball $B(0,1)$ is compact.*

A standard way to prove this is to use the following lemma due to F. Riesz.

Lemma 4. *(Riesz' lemma) Let $(Y, \|\cdot\|)$ be a normed space and $X \subseteq Y$ a subspace which is not dense in Y. Then for every $0 < \alpha < 1$ there is $y \in Y$ with $\|y\| = 1$ and $d(y, X) > \alpha$.*

Proof. Let $y_0 \in Y$ be a vector which is not in the closure of X. Then $R := \inf_{x\in X} \|y - x\| > 0$. Let $x_0 \in X$ be such that $\|y_0 - x_0\| < R/\alpha$ and $y := (y_0 - x_0)/\|y_0 - x_0\|$. Then

$$\begin{aligned} d(y, X) &= \inf_{x\in X} \|y - x\| = \inf_{x\in X} \left\| \frac{y_0 - x_0}{\|y_0 - x_0\|} - x \right\| \\ &= \inf_{x\in X} \left\| \frac{y_0 - x_0}{\|y_0 - x_0\|} - \frac{x}{\|y_0 - x_0\|} \right\| = \inf_{x\in X} \frac{\|y_0 - x\|}{\|y_0 - x_0\|} \\ &> \alpha \end{aligned}$$

which finishes the proof. ■

Proof of Theorem 3. Assume that X is infinite dimensional. We now define inductively a sequence of unit vectors $x_1, x_2, \ldots$ such that $\|x_i - x_j\| > 1/2$ for every $i \neq j$: Let $x_1 \in X$

be any unit vector. Having defined $x_1, \dots, x_n$ we let $V_n := \langle x_1, \dots, x_n \rangle$ and use Riesz' lemma to find a unit vector $x_{n+1} \in X \setminus V_n$ with $d(x_{n+1}, V_n) > 1/2$. It is clear that the sequence (x_n) satisfies $\|x_i - x_j\| > 1/2$ and hence does not have a convergent subsequence so $B(0,1)$ can not be compact.

Now we assume that X is finite dimensional and show that $B(0,1)$ is compact. The trick here is to use that we know this property for $\mathbb{R}^n$. So let n denote the dimension of X. Fix a basis $b_1, \dots, b_n$ of X, consisting of unit vectors. Now, define a map $T : \mathbb{R}^n \to X$, $Tx = \sum_{i=1}^n x_i b_i$. This is a linear bijection. It is continuous since

$$\|Tx\| = \left\| \sum_{i=1}^n x_i b_i \right\| \leq \sum_{i=1}^n |x_i| \leq n \cdot \|x\|.$$

Further, $\|T(x)\| > 0$ for every $x \in \mathbb{R}^n$ with $\|x\| = 1$ and as the unit sphere in $\mathbb{R}^n$ is compact, there is $C > 0$ such that $\|T(x)\| \geq 1/C$ for every $x \in \mathbb{R}^n$ with $\|x\| = 1$ and hence $\|T(x)\| \geq \frac{1}{C} \cdot \|x\|$ for every $x \in \mathbb{R}^n$. Without loss of generality, we assume that $C \geq n$. We have shown that

$$\frac{1}{C} \cdot \|x\| \leq \|Tx\| \leq C \cdot \|x\|$$

for every $x \in \mathbb{R}^n$. Thus, if $x \in B_X(0,1)$, then $\|T^{-1}x\| \leq C$ so $B_X(0,1) \subseteq T(B_{\mathbb{R}^n}(0,C))$. As T is continuous and $B_{\mathbb{R}^n}(0,C)$ is compact, $T(B_{\mathbb{R}^n}(0,C))$ is compact and hence is $B_X(0,1)$ compact as it is a closed subset of a compact set. ■

Exercise class 2

On last exercise sheet, you were supposed to show that Hilbert spaces are *uniformly convex* i.e. for any $0 < \varepsilon \leq 2$ there is $\delta > 0$ such that if x, y are unit vectors with $\|x - y\| \geq \varepsilon$ then $\|\frac{1}{2}(x + y)\| \leq 1 - \delta$. The same holds true for the L^p spaces when $1 < p < \infty$, as follows from *Clarkson's inequalities.*

Lemma 5 (Clarkson's inequalites)**.** *Let (X, μ) be a measure space. For every $f, g \in L^p(X)$ it holds that*

$$\left\| \frac{f+g}{2} \right\|_p^q + \left\| \frac{f-g}{2} \right\|_p^q \leq \left(\frac{1}{2}\|f\|_p^p + \frac{1}{2}\|g\|_p^p \right)^{\frac{q}{p}}$$

when $1 < p < 2$ and

$$\left\| \frac{f+g}{2} \right\|_p^p + \left\| \frac{f-g}{2} \right\|_p^p \leq \frac{1}{2} \left(\|f\|_p^p + \|g\|_p^p \right)$$

when $2 \leq p < \infty$. Here q is the unique number such that $\frac{1}{p} + \frac{1}{q} = 1$.

For the proof, see e.g. Brezis.

Exercise 6. Use Clarkson's inequalities to show that L^p, $1 < p < \infty$, is uniformly convex.

Exercise 7. Let $\mathcal{H}$ be a Hilbert space an $U : \mathcal{H} \to \mathcal{H}$ a linear operator. Show that the following is equivalent:

(i) U is bounded and $U^*U = UU^* = \mathrm{id}_{\mathcal{H}}$.

(ii) U is surjective and $\langle Ux, Uy\rangle = \langle x, u\rangle$ for all $x, y \in \mathcal{H}$.

Such an operator is said to be *unitary*.

Solution. Assume that (i) holds. Given $y \in \mathcal{H}$,

$$y = \mathrm{id}_{\mathcal{H}}\, y = (UU^*)y = U(U^*y)$$

so U is surjective. Given $x, y \in \mathcal{H}$,

$$\langle Ux, Uy\rangle = \langle x, U^*Uy\rangle = \langle x, y\rangle.$$

This shows that (ii) holds. Assuming (ii), we let $x, y \in \mathcal{H}$ and write as before

$$\langle x, y\rangle = \langle Ux, Uy\rangle = \langle U^*Ux, y\rangle.$$

As this holds for all $y \in \mathcal{H}$ we conclude that $x = U^*Ux$ and hence that $U^*U = \mathrm{id}_{\mathcal{H}}$. Let $y \in \mathcal{H}$. As U is surjective, there is an $x \in \mathcal{H}$ such that $Ux = y$. Now,

$$(UU^*)y = (UU^*)(Ux) = U(U^*U)x = U(\mathrm{id}_{\mathcal{H}})x = Ux = y$$

so $UU^* = \mathrm{id}_{\mathcal{H}}$ as well. ∎

Exercise 8. Let $U : \mathcal{H} \to \mathcal{H}$ be a unitary operator and assume that λ is an eigenvalue of U. Show that $|\lambda| = 1$.

Solution. Let λ be an eigenvalue and $x \in \mathcal{H}$ a corresponding unit eigenvector. Then

$$1 = \|x\|^2 = \|Ux\|^2 = |\lambda x|^2 = |\lambda|^2$$

which finishes the proof. ∎

Exercise 9. Let Γ be a countable group, endowed with the counting measure. In last lecture, the left action λ by Γ on $\ell^2(\Gamma)$, which is given by $(\lambda(\gamma)f)(\eta) = f(\gamma^{-1}\eta)$, was introduced. For each $\gamma \in \Gamma$, the operator $\lambda(\gamma) : \ell^2(\Gamma) \to \ell^2(\Gamma)$ is a unitary operator. Show that $\lambda(\gamma)$ has an eigenvector if and only if γ is of finite order.

Solution. For convenience, we sometimes write $f \in \ell^2(\Gamma)$ as a formal sum $f = \sum_{\gamma\in\Gamma} f(\gamma)\gamma$.

Assume that $\gamma \in \Gamma$ is of finite order n. Then $1 + \gamma + \cdots + \gamma^{n-1}$ is an eigenvector of $\lambda(\gamma)$, corresponding to the eigenvalue 1. For the other direction, assume that $\lambda(\gamma)$ has eigenvector f corresponding to the eigenvalue μ. As $f \neq 0$, there exists $\eta \in \Gamma$ such that $f(\eta) \neq 0$. Then

$$x^n f(\gamma^n\eta) = (\lambda(\gamma)f)^n(\gamma^n\eta) = (\lambda(\gamma^n)f)(\gamma^n\eta) = f(\eta)$$

so $f(\gamma^n\eta) = x^{-n}f(\eta)$ for every n and hence $|f(\gamma^n\eta)| = |f(\eta)|$ as $|x| = 1$ since $\lambda(\gamma)$ is unitary. Now, let n denote the order of γ. Then

$$n \cdot |f(\eta)|^2 = \sum_{k=0}^{n-1} |f(\gamma^k\eta)|^2 = \|f\|_2^2 < \infty$$

so the order is finite. ∎

Example 10 (Brezis Exercise 5.8). Let (X,μ) be a measure space, $h : X \to [0,\infty)$ a measurable function and

$$C := \{g \in L^2(X) \mid |g(x)| \leq h(x) \text{ for almost every } x\}.$$

Then C is a closed and convex subset and the projection $P_C : L^2(X) \to C$ onto C is given by

$$P_C f(x) := \begin{cases} 0 & \text{if } f(x) = 0 \\ \frac{f(x)}{|f(x)|} \cdot \min\{|f(x)|,\, h(x)\} & \text{else} \end{cases}.$$

Example 11 (Brezis Exercise 5.7). Let $\mathcal{H}$ be a Hilbert space and $C \subseteq \mathcal{H}$. One says that C is a cone with apex 0 if for every $u, v \in C$ and every $\mu, \nu \geq 0$ it holds that $\mu u + \nu v \in C$. Assume that K is a closed cone and let $P_C : \mathcal{H} \to C$ denote the projection. Let us show that $P_C u$ is the unique vector $v \in K$ such that

$$\langle u - v, v\rangle = 0 \quad \text{and} \quad \langle u - v, w\rangle \leq 0 \quad \text{for every} \quad w \in C. \tag{1}$$

As C is a cone, we know that for every $t \geq 0$ and every $w \in C$ it holds that $\|u - (P_C u + tv)\| \geq \|u - P_C u\|$ which gives that $\langle u - P_C u, v\rangle \leq \frac{t}{2}\|v\|^2$. As this holds for every $t \geq 0$, we conclude that $\langle u - P_C u, w\rangle \leq 0$ for every $w \in K$. The property $\langle u - P_C u, P_C u\rangle = 0$ follows from the fact that $t \mapsto \|u - tP_C u\|^2$ attains a minima at $t = 1$.

Now assume that there are two vectors $v, v' \in K$ which satisfy (1). Consider the function $f(t) := \frac{1}{2}\|u - (1-t)v - tv'\|^2$. It holds that

$$f'(t) = \langle u - (1-t)v - tv', v - v'\rangle$$

so from (1), it follows that

$$f'(0) = \langle u - v, v - v'\rangle \geq 0 \quad \text{and} \quad f'(1) = \langle u - v', v - v'\rangle \leq 0.$$

However, if $v \neq v'$, then f is a strictly convex function which contradicts that $f'(0) \geq 0$ and $f'(1) \leq 0$. Hence, we conclude that $v = v'$.

Exercise class 3

Example 12 (Brezis Exercise 1.3). Consider the vector space

$$X = \{f \in C([0,1], \mathbb{R}) \mid f(0) = 0\}$$

endowed with the supremum norm. Let $\lambda : X \to \mathbb{R}$ be given by $\lambda(f) := \int_0^1 f(x)\,dx$. Then $|\lambda(f)| \leq \int_0^1 |f(x)|\,dx \leq \|f\|_\infty$ so $\|\lambda\|_{X^*} \leq 1$. Let us show that $\|\lambda\|_{X^*} = 1$: For $\varepsilon > 0$, let

$$f_\varepsilon(x) := \begin{cases} \frac{x}{\varepsilon} & \text{if } 0 \leq x \leq \varepsilon \\ 1 & \text{if } \varepsilon < x \leq 1 \end{cases}.$$

Then $\|f\|_\infty = 1$ and $|\lambda(f)| = 1 - \varepsilon/2$ so $\|\lambda\|_{X^*} \geq 1 - \varepsilon/2$. As this holds for every $\varepsilon > 0$, $\|\lambda\|_{X^*} = 1$.

Note however, that there does not exist $f \in X$ with $\|f\|_\infty = 1$ such that $|\lambda(f)| = \|\lambda\|_{X^*} = 1$ because such an f would have to satisfy $f(x) = 1$ for almost every x and as f is assumed to be continuous, $f(x) = 1$ for every $x \in [0,1]$ which is impossible as $f(0) = 0$.

Recall that any vector space X has a *Hamel basis* i.e. there exists a family $(e_i)_{i\in I}$ of vectors in X which are linearly independent and such that any vector $x \in X$ can be written as

$$x = \sum_{i\in J} x_i e_i$$

where $J \subseteq I$ is finite and $(x_i)_{i\in J}$ are numbers in the underlying field.

Example 13. Let X be an infinite dimensional normed space and let $(e_i)_{i\in I}$ be a Hamel basis for X. Without loss of generality, we may assume that $\|e_i\| = 1$ for every $i \in I$. Now, let $(x_i)_{i\in I}$ be an unbounded family of numbers and define a functional $\lambda : X \to \mathbb{R}$ such that $\lambda(e_i) = x_i$ for every $i \in I$ and extend by linearity. Then λ is unbounded and hence not in X^*.

Exercise 14. Let X be an infinite dimensional Banach space. Show that the cardinality of any Hamel basis of X is uncountable.

Solution. Let $(e_n)_{n\in\mathbb{N}}$ be a countable family of vectors in X and let us show that they can not span X. For $n \in \mathbb{N}$ let X_n denote the span of $(e_k)_{k\leq n}$. Then each of the sets X_n is closed (finite dimensional vector spaces are always closed as they are complete) and has empty interior (else, there would exist an open ball $U(x, \varepsilon) \subseteq X_n$. As X_n is a subspace, we could then deduce that $U(0, 1) \subseteq X$ and hence that $X_n = X$). Thus, as X is complete, $\bigcup_{n\in\mathbb{N}} X_n$ has empty interior by the Baire category theorem so $\bigcup_{n\in\mathbb{N}} X_n$ must be a proper subspace of X and therefore do the vector $(e_n)_{n\in\mathbb{N}}$ not span X. ■

Recall that a *hyperplane* in a normed space X is a subspace of the form

$$\{x \in X \mid \lambda(x) = 0\}$$

where X is a linear functional on X. Let $H = \{x \in X \mid \lambda(x) = 0\}$ be a hyperplane. For every $x \notin H$ and every $y \in X$ it holds that

$$y = \left(y - \frac{\lambda(y)}{\lambda(x)} \cdot x\right) + \frac{\lambda(y)}{\lambda(x)} \cdot x$$

and as $\lambda(y - \frac{\lambda(y)}{\lambda(x)} \cdot x) = 0$, this shows that x and H span X.

Example 15. Let X be a normed space, λ a linear functional on X and $H = \{x \in X \mid \lambda(x) = 0\}$. Then $\bar{H}$ is still a subspace of X. If $H \neq \bar{H}$ then there exists $x \in \bar{H} \setminus H$ and by the remark above, x and H span X so $\bar{H} = H$. This shows that hyperplanes are either closed or dense.

Exercise 16 (Brezis Proposition 1.5). Let X be a normed space, λ a linear functional on X and $H = \{x \in X \mid \lambda(x) = 0\}$. Show that H is closed if and only if λ is continuous.

Solution. It is clear that H is closed if λ is continuous so assume that H is closed. Let $x_0 \notin H$ and take $r > 0$ such that $B(x_0, r) \cap H = \emptyset$. Then λ has fixed sign on $B(x_0, r)$, say that $\lambda(x) < 0$ for every $x \in B(x_0, r)$. Now, let $x \in X$. Then

$$\frac{r}{2\|x - x_0\|}(x - x_0) + x_0 \in B(x_0, r)$$

so

$$\lambda(x) < \left(1 - \frac{2}{r}\|x - x_0\|\right) \lambda(x_0)$$

which implies that λ is bounded. ■

Exercise class 4

Exercise 17. Let $(\lambda_i)_{i\in\mathbb{N}}$ be a sequence and define a linear operator $T : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ by $T((x_i)_{i\in\mathbb{N}}) = (\lambda_i x_i)_{i\in\mathbb{N}}$. Show that T is compact if and only if $\lambda_i \to 0$ as $i \to \infty$.

Solution. Assume first that $\lambda_i \to 0$ as $i \to \infty$. Let $(x^j)_{j\in\mathbb{N}}$ be a bounded sequence in $\ell^2(\mathbb{N})$. Fix $\varepsilon > 0$ and let $N \in \mathbb{N}$ be such that if $j \geq N$ then $|\lambda_i| < \varepsilon$. Let $(x^{j_k})_{k\in\mathbb{N}}$ be a subsequence such that $((x^{j_k})_i)_{k\in\mathbb{N}}$ converges for each $0 \leq i < N$ as $k \to \infty$. Let $M \in \mathbb{N}$ be such that if $k, l \geq M$ then $\sum_{i=0}^{N-1} |\lambda_j (x^{j_k})_i - \lambda_j (x^{j_l})_i|^2 < \varepsilon^2$. Then for $k, l \geq M$ it holds that

$$\begin{aligned}\|Tx^{i_k} - Tx^{i_l}\|_2^2 &= \sum_{i=0}^{N-1} |\lambda_i (x^{j_k})_i - \lambda_i (x^{j_l})_i|^2 + \sum_{i=N}^{\infty} |\lambda_i (x^{j_k})_i - \lambda_i (x^{j_l})_i|^2 \\ &< (1 + 4C^2) \cdot \varepsilon^2.\end{aligned}$$

This shows that $(Tx^j)_{j\in\mathbb{N}}$ is Cauchy and hence convergent.

For the other direction, assume that $(\lambda_i)_{i\in\mathbb{N}}$ does not converge to 0 as $i \to \infty$. Then we may find $\varepsilon > 0$ and subsequence $(i_j)_{j\in\mathbb{N}}$ such that $|\lambda_{i_j}| \geq \varepsilon$ for every $j \in \mathbb{N}$. Now, consider the sequence $(x^j)_{j\in\mathbb{N}}$ given by $(x^j)_k = 0$ for $k \neq i_j$ and $(x^j)_{i_j} = 1$. This is a bounded sequence but $\|Tx^j - Tx^k\| \geq \sqrt{2} \cdot \varepsilon$ for every $j \neq k$ so it does not have a convergenct subsequence. This proves that T is not compact. ■

Example 18 (Brezis Exercise 6.2.3). Consider the operator $T : C([0,1]) \to C([0,1])$ given by $(Tf)(t) \coloneqq \int_0^t f(\tau)\, d\tau$. Note that for $0 \leq s \leq t \leq 1$,

$$|(Tf)(t) - (Tf)(s)| \leq \int_s^t |f(\tau)|\, d\tau \leq (t - s) \cdot \|f\|_\infty. \tag{2}$$

Now, let $(f_i)_{i\in\mathbb{N}}$ be a bounded sequence in $C([0,1])$. Then by (2), $(Tf_i)_{i\in\mathbb{N}}$ is uniformly Lipschitz and hence in particular equicontinuous. The sequence is also bounded, so by Arzela-Ascoli, it has a convergent subsequence. Hence is the operator T compact.

Now, note that $T(B(0,1))$ consists of all continuously differentiable functions g on $[0,1]$ which satisfy $g(0) = 0$ and $\|g'\|_\infty \leq 1$. This is not a closed set as for example $g_0(t) = \frac{1}{2}(1 - |t - \frac{1}{2}|)$ is in the closure of $T(B(0,1))$ but not in $T(B(0,1))$.

Exercise 19 (Stein-Shakarchi Exercise III.4.7.31). Let K be the function which is defined on $[-\pi, \pi)$ by $K(x) \coloneqq i(\operatorname{sgn}(x)\pi - x)$ and then extended 2π-periodically to $\mathbb{R}$. Given $f \in L^1([0,1])$, let

$$Tf(x) \coloneqq \frac{1}{2\pi} \int_{-\pi}^{\pi} K(x - \xi) f(\xi)\, d\xi.$$

(a) Show that $F(x) = Tf(x)$ is absolutely continuous and if $\int_{-\pi}^{\pi} f(y)\, dy = 0$ then $F'(x) = if(x)$ for a.e. x.

(b) Show that the mapping $f \mapsto Tf$ is compact and symmetric on $L^2([-\pi, \pi])$.

(c) Prove that the eigenfunctions of T are $\varphi_n(x) = ce^{inx}$ where $c \neq 0$ is a constant and $n \in \mathbb{Z}$ and that the eigenvalue corresponding to φ_n is $1/n$ if $n \neq 0$ and 0 if $n = 0$.

(d) Conclude that $(e^{inx})_{n\in\mathbb{N}}$ is an orthonormal basis of $L^2([-\pi,\pi])$.

Solution. (a) Let $-\pi \le x < y < \pi$. Note that

$$K(y) - K(x) = \begin{cases} 2\pi - (y-x) & \text{if } x < 0 < y \\ x - y & \text{else} \end{cases}$$

so

$$\begin{aligned} |Tf(y) - Tf(x)| &\le \frac{1}{2\pi}\int_{-\pi}^{\pi} |K(y-\xi) - K(x-\xi)| \cdot |f(\xi)|\, d\xi \\ &\le \int_x^y |f(\xi)|\, d\xi + |y-x| \cdot \|f\|_1 \end{aligned}$$

from which it is clear that Tf is absolutely continuous.

Now, let f be a function with $\int_{-\pi}^{\pi} f(y)\, dy = 0$, let x be a Lebesgue point of f and $0 < h < \pi$. After extending f to a 2π-periodic function on $\mathbb{R}$ we can compute as follows

$$\begin{aligned} \frac{Tf(x+h) - Tf(x)}{h} &= \frac{1}{2\pi}\int_{x-\pi}^{x+\pi} \frac{K(x-\xi+h) - K(x-\xi)}{h} \cdot f(\xi)\, d\xi \\ &= \frac{1}{2\pi}\left(\int_x^{x+h} i \cdot \frac{2\pi - h}{h} \cdot f(\xi)\, d\xi - \int_{[x-\pi,x+\pi]\setminus[x,x+h]} i \cdot f(\xi)\, d\xi\right) \\ &= \frac{1}{h}\int_x^{x+h} f(\xi)\, d\xi \\ &\to i \cdot f(\xi) \end{aligned}$$

as $h \searrow 0$. The computations are similar for $h < 0$.

(b) Let $f, g \in L^2([-\pi,\pi])$. Then

$$\begin{aligned} \langle Tf, g\rangle &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} K(x-\xi) f(\xi)\, d\xi\right) \overline{g(x)}\, dx \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} K(\xi - x)\overline{g(\xi)}\, d\xi\right) dx \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) \overline{\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} K(x-\xi) g(\xi)\, d\xi\right)} dx \\ &= \langle f, Tg\rangle \end{aligned}$$

where we used that $\overline{K(-x)} = K(x)$. As K is bounded, $\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} |K(x-y)|^2\, dx\, dy < \infty$ so T is a Hilbert-Schmidt operator and hence compact.

(c) Let $\varphi \in L^2([-\pi,\pi])$ be an eigenfunction corresponding to the eigenvalue λ. Assume first that $\lambda = 0$. Let $c := \frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi(y)\, dy$. Note that $T(\varphi - c) = 0$ as T sends constants functions to 0, so by (a), $\varphi - c = 0$ which gives that $\varphi = c$ i.e. φ is constant. This shows that the eigenfunctions corresponding to the eigenvalue 0 are exactly the constant functions.

Now assume that $\lambda \neq 0$ and $T\varphi = \lambda\varphi$. Note that $\int_{-\pi}^{\pi} T\varphi(y)\,dy = 0$ so $\int_{-\pi}^{\pi} \varphi(y)\,dy = 0$ so we may apply (a) to conclude that $\lambda\varphi' = i\varphi$ which gives that $\varphi(x) = c \cdot e^{i\frac{1}{\lambda}x}$ where $c \neq 0$ is a constant. As $\int_{-\pi}^{\pi} \varphi(y)\,dy = 0$, $\lambda = 1/n$ for some integer n. Conversely, one checks that all functions of the form $\varphi_n(x) = c \cdot e^{inx}$, where $c \neq 0$ is a constant and n is an integer, are eigenfunctions, so we are done.

(d) By the spectral theorem for compact self-adjoint operators, we know that $(e^{inx})_{n\in\mathbb{Z}}$ form an orthogonal basis for $L^2([-\pi,\pi])$. As $\langle e^{inx}, e^{inx}\rangle = 1$, it is an orthonormal basis. ■

Exercise class 5

Exercise 20 (Stein-Shakarchi III.4.7.29). Let $\mathcal{H}$ be a Hilbert space, $T : \mathcal{H} \to \mathcal{H}$ a compact symmetric operator and $\lambda \neq 0$.

(a) Show that the range of $\lambda - T$ is closed.

(b) Show that the conclusion of (a) may fail if $\lambda = 0$.

(c) Show that $\lambda - T$ is surjective if and only if $\bar{\lambda} - T^*$ is injective.

Solution. (a) Let $g \in \mathcal{H}$ and assume that there is a sequence $g_j := (\lambda - T)f_j$, $j \in \mathbb{N}$, such that $g_j \to g$ as $j \to \infty$. As each $f \in \mathcal{H}$ can be written as $f = f^\perp + f^\parallel$ where $f^\perp$ is orthogonal to the eigenspace of λ and $f^\parallel$ is contained in it, and $(\lambda - T)f^\parallel = 0$, we may assume that each of the f_j are orthogonal to the eigenspace of λ. Let $0 < \lambda_1 \leq \lambda_2 \leq \cdots \to \infty$ be an enumeration of the eigenvalues of T with multiplicity, and $(e_i)_{i=1}^\infty$ an orthonormal basis of eigenvectors such that $Te_i = \lambda_i e_i$. Given $f \in \mathcal{H}$, let $f_0 := f - \sum_{i=1}^\infty \langle f, e_i\rangle e_i$. Now

$$\|g_j\|^2 = \|(\lambda - T)f_j\|^2 = \|(f_j)_0\|^2 + \sum_{i=1}^{\infty} |\langle f_j, e_i\rangle|^2 \cdot |\lambda - \lambda_i|^2.$$

As the sequence (g_j) is bounded, $\langle f_j, e_i\rangle = 0$ for every i such that $\lambda_i = \lambda$ and $\lambda_i \to \infty$, we can conclude that (f_j) is a bounded sequence. But then, after possibly passing to a subsequence, we may assume that Tf_j converges and then $f_j = \frac{1}{\lambda}(g_j + Tf_j)$ converges to say f and passing to the limit, $f = \frac{1}{\lambda}(g + Tf)$ i.e. $g = (\lambda - T)f$.

(b) We saw this in last class.

(c) Let us show that $R(\lambda - T)^\perp = N(\bar{\lambda} - T^*)$. So let y be orthogonal to $R(\lambda - T)$. Then

$$\|(\bar{\lambda} - T^*)y\|^2 = \langle(\bar{\lambda} - T^*)y, (\bar{\lambda} - T^*)y\rangle = \langle y, (\lambda - T)(\bar{\lambda} - T^*)y\rangle = 0$$

so $y \in N(\bar{\lambda} - T^*)$. If $y \in N(\lambda - T)$ then for every $x \in \mathcal{H}$ it holds that

$$0 = \langle x, (\bar{\lambda} - T^*)y\rangle = \langle(\lambda - T)x, y\rangle$$

so y is orthogonal to $R(\lambda - T)$, as $x \in \mathcal{H}$ was arbitrary.

Now the claim follows trivally, using that $R(\lambda - T)$ is closed. ■

Example 21. As $\mathbb{R}$ is complete, by Baire's category theorem, it can not be written as a countable union of closed sets with empty interior. Now, singletons are closed and have empty interior so as a consequence, $\mathbb{R}$ is uncountable.

Exercise 22 (Stein-Shakarchi III.4.7.35). Let $\mathcal{H}$ be a Hilbert space.

(a) Let S and T be two linear symmetric and compact operators on $\mathcal{H}$ that commute. Show that there exists an orthonormal basis $(e_i)_{i\in\mathbb{N}}$ of $\mathcal{H}$ such that for every $i \in \mathbb{N}$, e_i is an eigenvector of S and T.

(b) A linear operator $\mathcal{H}$ is said to be normal if $TT^* = T^*T$. Show that if T is normal and compact then T can be diagonalized.

(c) Let U be a unitary operator of the form $U = \lambda - T$ where T is compact. Show that U can be diagonalized.

Solution. (a) Let x be an eigenvector of T corresponding to the eigenvalue λ. Then $S(Tx) = TSx = T\lambda x = \lambda Tx$ so Tx is also an eigenvector of S with eigenvalue λ. From here, the claim is obvious.

(b) Note that $T = T_1 + iT_2$ where $T_1 = \frac{1}{2}(T+T^*)$ and $T_2 = \frac{1}{2}(T-T^*)$. As $TT^* = T^*T$, it follows that T_1 and T_2 commute and as they are compact, they can be diagonalized simultaneously by (a) and therefore can T be diagonalized.

(c) As U is unitary,

$$(\lambda - T)(\bar{\lambda} - T^*) = I = (\bar{\lambda} - T^*)(\lambda - T)$$

which gives that $TT^* = T^*T$. By (b), T can thus be diagonalized so $U = \lambda - T$ can also be diagonalized. ∎

Exercise class 6

In last week's lectures the closed graph theorem was proven:

Theorem 23 (Closed graph theorem)**.** *Let X, Y be Banach spaces and $T : X \to Y$ a linear map. Then T is continuous if and only if the graph of T is closed.*

In general when one has a map $f : X \to Y$ between metric spaces, one has to show that given any point $x \in X$ and any convergent sequence (x_n) in X with $x = \lim x_n$ it holds that $f(x) = \lim f(x_n)$. Note that this includes two steps: i) Showing that the sequence $(f(x_n))$ is convergent i.e. that there exists $y \in Y$ such that $y = \lim f(x_n)$. ii) Showing that $y = f(x)$. The great thing about the closed graph theorem is that it allows us to skip the first step, i.e. it suffices to show that for every sequence $(x_n, Tx_n) \in X \times Y$ which converges to some $(x, y) \in X \times Y$ it holds that $y = Tx$.

Exercise 24 (Stein-Shakarchi Exercise IV.4.12). Let X, Y, Z be Banach spaces and $T : X \times Y \to Z$ a linear map such that

(i) for every $x \in X$, the map $Y \to Z$, $y \mapsto T(x, y)$ is continuous.

(ii) for every $y \in Y$, the map $X \to Z$, $x \mapsto T(x, y)$ is continuous.

Solution. By (i), we have a linear map $X \to \mathcal{B}(Y,Z)$, $x \mapsto T_x$ where $T_x : Y \to Z$, $T_x y = T(x,y)$. Let $(x_n, T_{x_n}) \to (x, \bar{T})$ where $\bar{T} \in \mathcal{B}(Y,Z)$. As $T_{x_n} \to \bar{T}$ in the strong operator topology, it holds in particular for every $y \in Y$ that

$$\bar{T}y = \lim_{n\to\infty} T_{x_n}(y) = \lim_{n\to\infty} T(x_n, y) = T(x,y)$$

where we used (ii) in the last step. Hence is $\bar{T} = T_x$. Now, $\mathcal{B}(Y,Z)$ is a Banach space as Z is a Banach space so the closed graph theorem implies that $x \mapsto T_x$ is a continuous map. Hence, there exists a constant $C \geq 0$ such that $\|T_x\| \leq C\|x\|$ for every $x \in X$ and thus

$$\|T(x,y)\| = \|T_x y\| \leq \|T_x\| \cdot \|y\| \leq C \cdot \|x\| \cdot \|y\|.$$

This finishes the solution. ■

Exercise 25 (Stein-Shakarchi Exercise IV.4.14). Let X be a complete metric space and $T : X \to X$ a continuous map. An element $x \in X$ is said to be *universal* for T if the orbit $(T^n(x))_{n\in\mathbb{N}}$ is dense in X. Show that the set of universal elements for T is either empty or generic.

Solution. Assume that there exists a universal element x. For $j, k, N \geq 1$, let

$$F_{j,k,N} \coloneqq \{y \in X \mid d(T^n(y), T^j(x) < \frac{1}{k} \text{ for some } n \geq N\}.$$

As T is continuous, the sets $F_{j,k,N}$ are open. Further, for each $m \geq 0$, $T^m(x) \in F_{j,k,N}$ as $(T^n(x))_n$ is dense in X. Therefore is $F_{j,k,N}$ open and dense so $F \coloneqq \bigcap_{j,k,N} F_{j,k,N}$ is generic. Now, if $y \in F$ then for every $j \in \mathbb{N}$, there exists a sequence $n_i \to \infty$ such that $T^{n_i}(y) \to T^j(x)$ as $i \to \infty$. As $(T^j(x))_j$ is dense in X, it follows that $(T^n(y))_n$ is dense in X and hence that y is universal. ■

In last week's lectures, we also saw the uniform boundedness principle.

Theorem 26 (Uniform boundedness principle). *Let X be a Banach space, Y a normed space and $(T_\lambda)_{\lambda\in\Lambda}$ a family in $\mathcal{B}(X,Y)$. If for every $x \in X$ it holds that $\sup_{\lambda\in\Lambda} \|T_\lambda x\| < \infty$ then $\sup_{\lambda\in\Lambda} \|T\| < \infty$.*

Exercise 27 (Corollary IV.18). Let X be a Banach space and $B^* \subseteq X^*$ a subset such that $\{f(x) \mid f \in B^*\}$ is bounded for every $x \in X$. Show that B^* is bounded in X^*.

Solution. This follows directly from the uniform boundedness principle: For every $x \in X$, $\sup_{f\in B^*} |f(x)| < \infty$ so by the uniform boundedness principle, $\sup_{f\in B^*} \|f\| < \infty$ i.e. B^* is bounded. ■

Example 28. Let $X \subseteq \ell^2(\mathbb{N})$ be the set of those $x = \sum_{j=0}^{\infty} a_j e_j \in \ell^2(X)$ such that $a_j = 0$ for all but finitely many j. Then X is a normed space but it is not complete. For $n \in \mathbb{N}$, let $T_n : X \to X$ be given by $T_n e_j = j e_j$ if $j \leq n$ but $T_n e_j = 0$ for $j > n$. Then $\sup_{n\in\mathbb{N}} \|T_n x\| < \infty$ for every $x \in X$ but $\sup_{n\in\mathbb{N}} \|T_n\| = \infty$ as $\|T_n\| = n$.

Exercise class 7

Today, we will prove that for any point $x_0 \in [-\pi, \pi]$ there exists a continuous function on $[-\pi, \pi]$ whose Fourier serie diverges at x_0. We follow Section IV.4.2.1 in Stein and Shakarchi.

Recall that, given a complex valued function $f \in L^1([-\pi, \pi])$, its Fourier coefficients $a_n(f)$, $n \in \mathbb{Z}$, are defined by

$$a_n(f) := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx}\, dx$$

and its Fourier serie $\mathcal{F}(f)$ is given by

$$\mathcal{F}(f)(x) = \sum_{n=-\infty}^{\infty} a_n(f) e^{inx}.$$

For $N \in \mathbb{N}$, we let

$$S_N(f)(x) := \sum_{n=-N}^{N} a_n(f) e^{inx}$$

denote the N-th partial sum of f. Note that

$$\begin{aligned} S_N(f)(x) &= \sum_{n=-N}^{N} a_n(f) e^{inx} = \sum_{n=-N}^{N} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) e^{-iny}\, dy \cdot e^{inx} \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\sum_{n=-N}^{N} e^{in(x-y)} \right) f(y)\, dy \\ &= (D_N * f)(x) \end{aligned}$$

where

$$\begin{aligned} D_N(x) &:= \sum_{n=-N}^{N} e^{inx} = e^{-iNx} \cdot \frac{e^{i(2N+1)x} - 1}{e^{ix} - 1} = \frac{e^{i(N+1/2)x} - e^{-i(N+1/2)x}}{e^{ix/2} - e^{-ix/2}} \\ &= \frac{\sin((N+1/2)x)}{\sin(x/2)} \end{aligned}$$

and

$$(f * g)(x) := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-y) g(y)\, dy.$$

Now, we assume without loss of generality that $x_0 = 0$. In order to show the existence of a continuous function f on $C([-\pi, \pi])$ we use the uniform boundedness principle in the following way: For every $N \in \mathbb{N}$, let

$$\ell_N : C([-\pi, \pi]) \longrightarrow \mathbb{C}, \qquad f \longmapsto S_N(f)(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(-y) D_N(y)\, dy.$$

We will show that each of the linear functionals ℓ_N is bounded but that $\|\ell_N\| \to \infty$ as $N \to \infty$. Hence, by the uniform boundedness principle, there has to be a function $f \in C([-\pi, \pi])$ such that $\sup_{N \in \mathbb{N}} |\ell_N(f)| = \infty$ which means that the Fourier serie of f diverges at x.

Lemma 29. *For every $N \in \mathbb{N}$ it holds that $\|\ell_N\| = L_N$ where $L_N := \frac{1}{2\pi}\int_{-\pi}^{\pi} |D_N(y)|\, dy$.*

Proof. First,

$$|\ell_N(f)| \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(-y)||D_n(y)|\, dy \leq L_N \|f\|_\infty$$

so $\|\ell_N\| \leq L_N$. For the other inequality, let $g(x) := \operatorname{sgn}(D_N(x))$. Then

$$L_N = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(-y) D_N(y)\, dy.$$

Now, by standard approximation results, there exists a seqeunce $(f_k)_{k\in\mathbb{N}}$ of functions in $C([-\pi,\pi])$ with $\|f_k\| \leq 1$ and such that $\|g - f_k\|_1 \to 0$ as $k \to \infty$. Then $\ell_N(f_k) \to L_N$ as $k \to \infty$ so $\|\ell_N\| \geq L_N$. ■

Lemma 30. *There exists a constant $c > 0$ such that $L_N \geq c \cdot \ln N$ for every $N \geq 1$.*

Proof. Making use of the fact that $|\sin(x)/x| \leq 1$, we obtain

$$\begin{aligned}
L_N &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |D_N(x)|\, dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|\frac{\sin((N+1/2)x)}{\sin(x/2)}\right| dx \\
&\geq \frac{2}{\pi}\int_0^{\pi} \left|\frac{\sin((N+1/2)x)}{x}\right| dx = \frac{2}{\pi}\int_0^{(N+1/2)\pi} \left|\frac{\sin(x)}{x}\right| dx \\
&\geq \frac{2}{\pi}\sum_{k=0}^{N-1}\int_{k\pi}^{(k+1)\pi} \left|\frac{\sin(x)}{x}\right| dx \geq \frac{2}{\pi}\sum_{k=1}^{N-1}\frac{1}{(k+1)\pi}\int_{k\pi}^{(k+1)\pi} |\sin(x)|\, dx \\
&\geq \frac{2}{\pi^2}\int_0^{\pi} |\sin(x)|\, dx \cdot \ln N
\end{aligned}$$

which finishes the proof. ■

Exercise class 8

Exercise 31 (Brezis Exercise 3.1). Let X be Banach space and let $A \subseteq X$ be a subset which is compact in the weak topology. Show that A is bounded.

Solution. We use the Banach-Steinhaus theorem: As A is compact, it holds for every $f \in X^*$ that $f(A)$ is bounded and hence is A bounded by Banach-Steinhaus. ■

Exercise 32 (Brezis Exercise 3.3). Let X be a Banach space and $A \subseteq X$ a convex set. Show that the closure of A in the weak topology and in the strong topology agree.

Solution. We only have to show that the closure of A in the strong topology, $\bar{A}$, is closed in the weak topology. So let $x \notin \bar{A}$. Then by Hahn-Banach, there exists $f \in X^*$ such that $f(x) > 0$ and $f(y) \leq 0$ for every $y \in \bar{A}$. Then $V := f^{-1}((0,\infty)$ is a neighborhood of x in the weak topology which does not intersect $\bar{A}$. This shows that $\bar{A}$ is closed in the weak topology. ■

Exercise 33 (Brezis Exercise 3.4). Let X be a Banach space and (x_n) a sequence in X such that $x_n \rightharpoonup x$ in the weak topology as $n \to \infty$.

(i) Prove that there exists a sequence (y_n) in X such that

$$y_n \in \operatorname{conv}(\{x_i \mid i = n, n+1, \ldots\})$$

for every n and $y_n \to x$ in the strong topology.

(ii) Prove that there exists a seqeunce (z_n) in X such that

$$z_n \in \operatorname{conv}(\{x_i \mid i = 1, \ldots, n\})$$

for every n and $z_n \to x$ in the strong topology.

Solution. (i) For every n it holds that y is contained in the weak closure of the convex set $\operatorname{conv}(\{x_i \mid i = n, n+1, \ldots\})$ which is also its strong closure by the previous exercise. Hence the result.

(ii) Note that

$$\operatorname{conv}(\{x_i \mid i \in \mathbb{N}\}) = \bigcup_{n \in \mathbb{N}} \operatorname{conv}(\{x_i \mid i = 1, \ldots, n\})$$

and as y is in the closure of this set (by the previous exercise), the result follows. ■

Exercise 34 (Brezis Exercise 3.5). Let X be a Banach space $K \subseteq X$ a set which is compact in the strong topology and let $x_n, x \in K$, $n \in \mathbb{N}$, be such that $x_n \rightharpoonup x$ in the weak topology as $n \to \infty$. Show that $x_n \to x$ in the strong topology.

Solution. Assume for a contradiction that (x_n) does not converge to x in the strong topology as $n \to \infty$. By using that K is strongly compact, we may then find $x' \in K$ such that $x' \neq x$ and such that after possibly passing to a subsequence, $x_n \to x'$ in the strong topology. But then $x_n \rightharpoonup x'$ in the weak topology and as the weak topology is Hausdorff, $x = x'$, which is a contradiction. ■

Exercise 35 (Brezis Exercise 3.8). Let X be an infinite dimensional Banach space. Show that the weak topology on X is not metrizable.

Solution. Suppose for a contradiction that there exists a metric d on X which induced the weak topology. For each intger $k \geq 1$, there are $f_{k,1}, \ldots, f_{k,n_k} \in X^*$ and an $\varepsilon_k > 0$ such that

$$V_k := \{x \in X \mid |f_{k,i}(x)| < \varepsilon_k \text{ for every } i = 1, \ldots, n_k\} \subseteq \left\{x \in X \mid d(x, 0) < \frac{1}{k}\right\}.$$

Now, let $g \in X^*$. Recall that all elements of X^* are continuous with respect to the weak topology (the weak topology is defined as the coarsest topology with respect to which all elements of X^* are continuous). Hence there exists $k \geq 1$ such that if $d(x, 0) < \frac{1}{k}$ then $|g(x)| < 1$ and thus, that if $x \in V_k$, then $|g(x)| < 1$. Note that if $f_{k,i}(x) = 0$ for $i = 1, \ldots, n_k$ then the same holds for rx, for every $r \in \mathbb{R}$ so $rx \in V_k$ for every $r \in \mathbb{R}$ and hence it follows that $g(x) = 0$ because else we can let $r \in \mathbb{R}$ be such that $|g(rx)| \geq 1$ and then get a contradiction. It follows that g is linear comination of $f_{k,1}, \ldots, f_{k,n_k}$.

We have shown that the countable family $f_{k,i}$, $k \geq 1$, $1 \leq i \leq n_k$ forms a Hamel basis for X^*. But then X^* is finite dimensional by the Baire category theorem and hence is X finite dimensional. ■

Exercise class 9

Let us begin with a short recap of the weak topology and the weak* topology.

Let X be any set and $((f_i, Y_i))_{i\in I}$ a family of pairs such that for each $i \in I$, Y_i is a topological space and $f_i : X \to Y_i$ is a map. There exists a unique coarsest topology τ on X which makes all the maps f_i, $i \in I$, continuous; the topology τ is called the topology which the family $((f_i, Y_i))_{i\in I}$ induces. A fundamental property of the topology τ is the following: Let Y be a topological space and $f : Y \to X$ a map. Then f is continuous when X is endowed with the topology τ if and only if for each $i \in I$, the map $f_i \circ f : Y \to Y_i$ is continuous.

Example 36. Let $(X_i)_{i\in I}$ be a family of topological spaces. The product topology on $\prod_{i\in I} X_i$ is the topology which the projection maps

$$\pi_i : \prod_{j\in I} X_j \longrightarrow X_i, \quad (x_j)_{j\in I} \longmapsto x_i$$

induce. Hence, given a topological space X and a map $f : X \to \prod_{j\in I} X_j$, the map f is continuous if and only if $\pi_i \circ f : X \to X_i$ is continuous for each $i \in I$.

Let now X be a Banach space. Then the weak topology on X is defined as the topology which the family $((f, \mathbb{R}))_{f\in X^*}$ induces. Given $x \in X$, we let $\hat{x} : X^* \to \mathbb{R}$, $\bar{x}(f) \coloneqq f(x)$. The topology on X^* which the family $((\hat{x}, \mathbb{R}))_{x\in X}$ induces, is called the weak* topology on X^*.

In last week's lectures, we learned about the Banach-Alaoglu theorem which says that given a Banach space X, the closed unit ball $B_{X^*}(0, 1)$ in the dual space X^* is compact in the weak*-topology. The proof actually goes by viewing $B_{X^*}(0, 1)$ as a suitable subset of $\prod_{x\in X} \mathbb{R}$ which is compact by Tychonoff. Then one notices that on this subset, the weak* topology and the product topology agree and hence the result follows.

Example 37. Let X be a reflexive Banach space, that is the embedding $\iota : X \to X^{**}$, $x \mapsto \hat{x}$, is surjective. Then ι is a homomorphism when X is endowed with the weak topology and X^{**} is endowed with the weak*-topology: To show that, we let Y be a topological space, $F : Y \to X$ and show that F is continuous if and only if $\iota \circ F : Y \to X^{**}$ is continous when X is endowed with the weak topology and X^{**} with the weak* topology. Now, F is continuous if and only if for each $f \in X^*$, $y \mapsto f(F(y))$ is continuous and $\iota \circ F$ is continuous if and only if for each $f \in X^*$, $y \mapsto \hat{f}(\iota \circ F(y)) = (\iota \circ F(y))(f) = f(F(y))$ is continuous. Hence it is equivalent that F and $\iota \circ F$ are continuous.

From this one may conclude that $B_X(0, 1)$ endowed with the weak topology is compact as it is homeomorphic to $B_{X^{**}}(0, 1)$ endowed with the weak*-topology, and the latter is compact by Banach-Alaoglu.

Example 38 (Brezis Exercise 3.10). Let X, Y be Banach spaces and $T : X \to Y$ a bounded linear map. The dual map $T^* : Y^* \to X^*$ is defined by $T^*g = g \circ T$. Let us show that T^* is continuous when X^*, Y^* are endowed with the weak*-topologies. Recall that the weak topology on X^* is the coarsest topology such that for each $x \in X$, the map $\hat{x} : X^* \to \mathbb{R}$, $\hat{x}(f) = f(x)$, is continuous. Hence it suffices to show that for each $x \in X$, the map $Y^* \to \mathbb{R}$, $g \mapsto \hat{x}(T^*g) = g(Tx) = \widehat{Tx}(f)$ is continuous when Y^* is endowed with the weak*-topology. But that simply holds since the weak* topology on Y^* is the coarses topology such that for each $y \in Y$, the map $\hat{y} : Y^* \to \mathbb{R}$, $\hat{y}(g) = g(y)$, is continuous and hence in particular, are all the maps $\widehat{Tx}$ weak*-continuous.

Exercise 39 (Brezis Exercise 3.20). Let X be a Banach space. Show that there exists a compact topological space K and an isometric embedding $\iota : X \to C(K)$.

Proof. Take $K := B_{X^*}(0,1)$ endowed with the weak* topology and let $\iota : X \to C(K)$ be given by $\iota(x)(f) := f(x)$ for every $f \in K$. It is clear that ι is linear. Furthermore,

$$\|\iota(x)\|_\infty = \sup_{f \in K} |\iota(x)(f)| = \sup_{\substack{f \in X^* \\ \|f\| \le 1}} |f(x)| = \|x\|$$

so ι is an isometric embedding. ■

Exercise class 10

Today, we will prove the following theorem.

Theorem 40 (Brezis Theorem 3.28 and Theorem 3.29). *Let X be a Banach space. Then*

(i) $B_{X^*}(0,1)$ is metrizable in the weak-topology if and only if X is separable.*

(ii) $B_X(0,1)$ is metrizable in the weak topology if and only if X^* is separable.

Proof. (i) Assume that X is separable. Let $(x_n)_{n\ge1}$ be a dense sequence in $B_X(0,1)$ and define a metric d on $B_{X^*}(0,1)$ by

$$d(f,g) := \sum_{n=0}^{\infty} \frac{1}{2^n} |(f-g)(x_n)|.$$

Let us show that d induces the weak*-topology on $B_{X^*}(0,1)$. So let $f_0 \in B_{X^*}(0,1)$, $\varepsilon > 0$, $y_1, \dots, y_k \in X$ with $\|y_i\| \le 1$ for $i = 1, \dots, k$ and

$$V := \{f \in B_{X^*}(0,1) \mid |(f - f_0)(y_i)| < \varepsilon \text{ for all } i = 1, \dots, k\}.$$

As $(x_n)_{n\ge1}$ is dense in $B_X(0,1)$, for every $i = 1, \dots, k$, there is n_i such that $\|y_i - x_{n_i}\| < \varepsilon/4$. Let $r > 0$ be small enough so that $2^{n_i} r < \varepsilon/2$ for every $i = 1, \dots, k$. Then if $d(f, f_0) < r$ it holds for every $i = 1, \dots, k$ that

$$\frac{1}{2^{n_i}} |(f - f_0)(x_{n_i})| < r$$

so

$$|(f - f_0)(y_i)| \le |(f - f_0)(y_i - x_{n_i})| + |(f - f_0, x_{n_i})| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

and hence is $f \in V$. This shows that the topology induced by d is finer than the weak*-topology.

Let now $f_0 \in B_{X^*}(0,1)$ and $r > 0$. For $\varepsilon := r/2$ and $k \ge 1$ such that $1/2^{k-1} < r/2$ it holds that if $|(f - f_0)(x_i)| < \varepsilon$ for $i = 1, \dots, k$, then

$$d(f, f_0) = \sum_{n=1}^{k} \frac{1}{2^n} |(f - f_0)(x_n)| + \sum_{n=k+1}^{\infty} \frac{1}{2^n} |(f - f_0)(x_n)| < \varepsilon + 2 \sum_{n=k+1}^{\infty} \frac{1}{2^n}$$
$$< r.$$

This shows that the weak* topology is finer than the topology which d induces.

(ii) The proof is exactly the same as in (i). ■

As a corollary, one gets the following.

Corollary 41 (Brezis Corollary 3.30). *Let X be a separable Banach space and $(f_n)_{n\in\mathbb{N}}$ a bounded sequence in X^*. Then there exists a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ which converges in the weak* topology.*

The preceding corollary allows one to prove the following theorem.

Theorem 42 (Brezis Theorem 3.18). *Let X be a reflexive Banach space and $(x_n)_{n\in\mathbb{N}}$ a bounded seqeunce in X. Then there exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ that converges in the weak topology.*

Proof. Let M_0 be the vector space generated by $(x_n)_{n\in\mathbb{N}}$ and $M := \bar{M}_0$. Then M is separable and reflexive as a closed subspace of a reflexive space is reflexive (see Proposition 3.20 in Brezis). By (ii) in the previous theorem, $B_M(0,1)$ is compact and metrizable in the weak topology since M^* is separable (here we must use that a Banach space is reflexive and seperable if and only if the same holds for its dual space (see Corollary 3.27 in Brezis)). Now the result follows. ■

Exercise class 11

In last lecture we saw the following theorem.

Theorem 43 (Mercer's theorem). *Let (X,d) be a compact metric space and μ a Borel regular probability measure on X such that $\mu(U) > 0$ for every open set $U \subseteq X$. Let $K \in C(X\times X)$ be a continuous positive semi-definite kernel on X (that is, $K(x,y) = K(y,x)$ for every $x, y \in X$ and for every $x_1,\dots,x_n \in X$, the matrix $(K(x_i,x_j))_{i,j=1}^n$ is positive semi definite) and let $T_K : L^2(X,\mu) \longrightarrow L^2(X,\mu)$ be the operator given by*

$$T_K f(x) := \int_X K(x,y)f(y)\, d\mu(y).$$

Then there exists a sequence $(\varphi_i)_{i=1}^\infty$ of continuous eigenfunctions of T_K which form an orthonormal basis of $\ker(T_K)^\perp$ and for each i, the eigenvalue λ_i corresponding to φ_i satisfies $\lambda_i > 0$. Furthermore,

$$K(x,y) = \sum_{i,j=1}^\infty \lambda_i \varphi_i(x)\varphi_j(y)$$

for all $x, y \in X$ and the sum is absolutely and uniformly convergent.

Let us now consider an application of Mercer's theorem to stochastic processes, namely the Karhunen-Loeve theorem. We follow the Wikipedia page on that theorem.

Theorem 44 (Karhunen-Loeve). *Let $(\Omega,\mathbb{P})$ be a probability space and $(X_t)_{t\in[0,1]}$ a stochastic process on Ω such that*

(a) *The function*

$$[0,1]\times\Omega \to \mathbb{R}, \qquad (t,\omega) \longmapsto X_t(\omega)$$

is in $L^2([0,1]\times\Omega)$.

(b) *For every* $t \in [0,1]$, $\mathbb{E}[X_t] = 0$ *(i.e.* X_t *has zero-mean).*

(c) *For every* $t \in [0,1]$, $\mathbb{E}[X_t^2] < \infty$ *(i.e.* X_t *has bounded variance).*

(d) *The covariance function* $K_X(s,t) \coloneqq \mathbb{E}[X_s X_t]$, $s,t \in [0,1]$, *is continuous.*

Then there exists a sequence $(e_i)_{i=1}^{\infty}$ *of continuous functions in which are eigenfunctions of* T_{K_X} *and form an orthonormal basis of* $L^2([0,1])$ *such that for the random variables*

$$Z_i : \Omega \to \mathbb{R}, \qquad Z_i(\omega) \coloneqq \int_0^1 X_t(\omega) e_i(t)\, dt$$

it holds that

(i) *As* $N \to \infty$,

$$\sup_{t\in[0,1]} \left\| X_t - \sum_{i=1}^{N} Z_i e_i(t) \right\|_{L^2(\Omega,\mathbb{P})} \to 0.$$

(ii) *For every* i, $\mathbb{E}[Z_i] = 0$.

(iii) *For every* i,j, $\mathbb{E}[Z_i Z_j] = 0$ *if* $i \neq j$ *and* $\mathbb{E}[Z_i^2] = \lambda_i$ *where* λ_i *is the eigenvalue of* T_{K_X} *corresponding to* e_i.

Proof. Note first that K_X is positive semi-definite kernel: It is clear that K_X is symmetric and for every $t_1, \dots, t_n \in [0,1]$ and all real numbers $c_1, \dots, c_n$ it holds that

$$\sum_{i,j=1}^{n} c_i c_j K_X(t_i, t_j) = \sum_{i,j=1}^{n} c_i c_j \mathbb{E}[X_{t_i} X_{t_j}] = \mathbb{E}\left[c_1 X_{t_1} + \cdots + c_n X_{t_n}\right] \geq 0$$

so K_X is positive semi-definite. Now, Mercer's theorem gives the existence of an orthonormal basis $(e_i)_{i=1}^{\infty}$ of $L^2([0,1])$, consisting of eigenfunctions T_{K_X}. As $t \mapsto X_t(\omega)$ is in $L^2([0,1])$ for almost every $\omega \in \Omega$ by (a), we can for every $i \geq 1$ define $Z_i \in L^2(\Omega, \mathbb{P})$ by

$$Z_i(\omega) \coloneqq \int_0^1 X_t(\omega) e_i(t)\, dt.$$

For every i it holds that

$$\mathbb{E}[Z_i] = \mathbb{E}\left[\int_0^1 X_t e_i(t)\, dt\right] = \int_0^1 \mathbb{E}[X_t] e_i(t)\, dt = 0$$

and for every i,j it holds that

$$\begin{aligned}
\mathbb{E}[Z_i Z_j] &= \mathbb{E}\left[\int_0^1 \int_0^1 X_s X_t e_i(s) e_j(t)\, ds\, dt\right] = \int_0^1 \int_0^1 \mathbb{E}[X_s X_t] e_i(s) e_j(t)\, ds\, dt \\
&= \int_0^1 \int_0^1 K_X(s,t) e_i(s) e_j(t)\, ds\, dt = \int_0^1 \left(\int_0^1 K_X(s,t) e_i(s) ds\right) e_j(t)\, dt \\
&= \langle T_{K_X} e_i, e_j \rangle \\
&= \lambda_i \cdot \delta_{ij}
\end{aligned}$$

so we have shown (ii) and (iii). For (i), we let $S_N := \int_{i=1}^N Z_i e_i(t)$. Then

$$\begin{aligned}
\mathbb{E}\left[|X_t - S_N|^2\right] &= \mathbb{E}[X_t^2] + \mathbb{E}[S_N^2] - 2\mathbb{E}[X_t S_N] \\
&= K_X(t,t) + \mathbb{E}\left[\sum_{i,j=1}^N Z_i Z_j e_i(t) e_j(t)\right] - 2\mathbb{E}\left[X_t \sum_{i=1}^N Z_i e_i(t)\right] \\
&= K_X(t,t) + \sum_{i=1}^N \lambda_i e_i(t)^2 - 2\sum_{i=1}^N \int_0^1 \mathbb{E}[X_s X_t] e_i(s) e_i(t)\, ds \\
&= K_X(t,t) + \sum_{i=1}^N \lambda_i e_i(t)^2 - 2\sum_{i=1}^N \int_0^1 K_X(s,t) e_i(s) e_i(t)\, ds \\
&= K_X(t,t) + \sum_{i=1}^N \lambda_i e_i(t)^2 - 2\sum_{i=1}^N \lambda_i e_i(t)^2 \\
&= K_X(t,t) - \sum_{i=1}^N \lambda_i e_i(t)^2
\end{aligned}$$

and by Mercer, this goes uniformly to zero as $N \to \infty$. ■

Example 45. A Brownian motion is a stochastic process $(B_t)_{t\in\mathbb{R}_+}$ on probability space $(\Omega, \mathbb{P})$ such that

(i) $B_0 = 0$ almost surely.

(ii) For every $t, h \geq 0$ it holds that $B_{t+h} - B_t \sim \mathcal{N}(0, h)$ i.e. $B_{t+h} - B_t$ is a normal variable with zero mean and variance h.

(iii) For every $0 \leq t_1 < \cdots < t_n$ it holds that $B_{t_n} - B_{t_{n-1}}, \ldots, B_{t_1} - B_0$ are independent random variables.

(iv) For almost every $\omega \in \Omega$ it holds that $t \mapsto B_t(\omega)$ is continuous.

Let now $(B_t)_{t\in[0,1]}$. For every $t \in [0,1]$ it holds that $\mathbb{E}[B_t] = 0$ and $\mathbb{E}[B_t] = t$ and for every $0 \leq s \leq t \leq 1$ it holds that

$$K(s,t) := K_B(s,t) = \mathbb{E}[B_s B_t] = E[B_s^2] + E[B_s(B_t - B_s)] = s$$

since $\mathbb{E}[B_s(B_t - B_s)] = 0$ as B_s and $B_t - B_s$ are independent. Hence $K(s,t) = \min(s,t)$ for every $s, t \in [0,1]$ so the criterias of the Karhunan Loeve theorem are satisfied. Let us determine the eigenfunctions of T_K. For that we must solve the eigenvalue problem $T_K e = \lambda e$ i.e.

$$\lambda e(t) = \int_0^1 K(s,t) e(s)\, ds = \int_0^1 \min(s,t) \cdot e(s)\, ds = \int_0^t s e(s)\, ds + t \int_t^1 e(s)\, ds.$$

Now note that as $e \in L^2$, the right hand side is differentiable in t by the Lebesgue differentiation theorem. Assume first that $\lambda = 0$. Then we get by differentiating with respect to t that

$$te(t) - te(t) + \int_t^1 e(s)\, ds = 0$$

i.e. $\int_t^1 e(s) = 0$. As this holds for every $t \in [0,1]$ we conclude that $e = 0$. Now assume that $\lambda \neq 0$. Then e is differentiable as the right hand side is differentiable and by differentiating, we get

$$\lambda e'(t) = \int_t^1 e(s)\, ds.$$

The right hand side is differentiable so we can differentiate again to obtain

$$\lambda e''(t) + e(t) = 0.$$

We know by Karhunan-Loeve that $\lambda > 0$ so

$$e(t) = a \cdot \cos\left(\frac{1}{\sqrt{\lambda}} t\right) + b \cdot \sin\left(\frac{1}{\sqrt{\lambda}} t\right)$$

where a, b are constants. From $\lambda e(t) = \int_0^1 K(s,t)e(s)\, ds$ it follows that $e(0) = 0$ so $a = 0$. From $\lambda e'(t) = \int_t^1 e(s)\, ds$ it follows then that $\cos(1/\sqrt{\lambda}) = 0$ which gives that $1/\sqrt{\lambda} = (k+1/2) \cdot \pi$ for some $k \in \mathbb{N}$ i.e. $\lambda = 1/((k+1/2)^2\pi^2)$, $k \in \mathbb{N}$. Let $\lambda_k := 1/((k+1/2)^2\pi^2)$, $e_k(t) := b_k \cdot \sin((k+1/2) \cdot t)$. From $\int_0^1 e_k(t)\, dt = 1$ it follows that

$$1 = \int_0^1 b_k^2 \cdot \sin^2((k+1/2) \cdot \pi \cdot t)\, dt = \frac{b_k^2}{2} \int_0^1 (1 - \cos((2k+1) \cdot \pi \cdot t))\, dt = \frac{b_k^2}{2}$$

so $b_k = \sqrt{2}$ for every k so $e_k(t) = \sqrt{2} \cdot \sin((k+1/2) \cdot \pi \cdot t)$.