Functional Analysis II

Exercise classes

Hjalti Isleifsson

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These are notes for my exercises classes in Functional Analysis II in spring semester 2023 at ETH Zürich. If you find mistakes in the notes, please let me know by sending me an email at hjalti.isleifsson@math.ethz.ch.

Exercise class 1

Last lecture ended with the following theorem:

Theorem 1. *Let X and Y be Banach spaces, $A : D(A) \subseteq X \to Y$ a densily defined closed operator. Then* $\operatorname{ran}(A)$ *is closed if and only if* $\operatorname{ran} A = \operatorname{ann}\ker A^*$

Remark. The proof showed that $\operatorname{ran} A \subseteq \operatorname{ann}\ker A^*$ indeed always holds: If $y = Ax \in \operatorname{ran} A$ and $\lambda \in \ker A^*$ then $\lambda y = \lambda Ax = (A^*\lambda)x = 0$.

Let as now prove results of similar flavour.

Exercise 2 (Ex. 2.18 Brezis). Let X, Y be Banach spaces and $A : D(A) \subseteq X \to Y$ a densily defined operator. Then

(i) $\ker A^* = \operatorname{ann}\operatorname{ran} A$ and $\ker A \subseteq \operatorname{ann}\operatorname{ran} A^*$.

(ii) If A is closed, then $\ker A = \operatorname{ann}\operatorname{ran} A^*$.

Solution. (i) We have:

$$\begin{aligned}\lambda \in \ker A^* &\Leftrightarrow \lambda(Ax) = 0, \text{ for every } x \in D(A)\\ &\Leftrightarrow \lambda \in \operatorname{ann}\operatorname{ran} A.\end{aligned}$$

Let $x \in \ker A$ and $\lambda \in D(A^*)$. Then

$$(A^*\lambda)x = \lambda(Ax) = \lambda(0) = 0$$

i.e. $x \in \operatorname{ann}\operatorname{ran} A^*$.

(ii) Assume for a contradiction that there exists $x_0 \in \operatorname{ann}\operatorname{ran} A^* \setminus \ker A$. Then $(x_0, 0) \notin \Gamma_A$ and since $\Gamma_A \subseteq X \times Y$ is closed, Hahn-Banach gives the existence of a $\lambda \in (X \times Y)^*$ such that $\lambda|_{\Gamma_A} = 0$ and $\lambda(x_0, 0) \neq 0$. Now,

$$D(A) \ni x \longmapsto \lambda(0, Ax) = -\lambda(x, 0) + \lambda(x, Ax) = -\lambda(x, 0)$$

extends to a continuous functional on X so $\mu \in Y^*$ defined by $\mu(y) := \lambda(0, y)$ is in $D(A^*)$. However,

$$(A^*\mu)x_0 = \mu(Ax_0) = \lambda(0, Ax_0) = -\lambda(x_0, 0) \neq 0,$$

contradicting that $x_0 \in \operatorname{ann}\operatorname{ran} A^*$. ■

Example 3 (Ex. 2.22 Brezis). Let $X = \ell^1(\mathbb{N})$ so $X^* = \ell^\infty(\mathbb{N})$. Consider

$$A : D(A) \subseteq \ell^1(\mathbb{N}) \longrightarrow \ell^1(\mathbb{N})$$

where

$$D(A) := \{a = (a_n) \in \ell^1(\mathbb{N}) \mid \sum n a_n < \infty\}$$

and $A(a_n) = (na_n)$. Then $D(A) \subseteq \ell^1(\mathbb{N})$ is dense. Let us show that A is closed: Let $(a^{(k)}, Aa^{(k)}) \to (a, b)$. Then $b \in \ell^1(\mathbb{N})$ since it is a limit of a sequence in $\ell^1(\mathbb{N})$ and $na^{(k)} \to b_n$ and $a_n^{(k)} \to a_n$ for every n, which gives that $b_n = na_n$ so $b = Aa$, i.e. A is closed.

Now let $u = (u_n) \in \ell^\infty(\mathbb{N}) = (\ell^1(\mathbb{N}))^*$. Then

$$D(A) \ni a \longmapsto \langle u, Aa \rangle = \sum n u_n a_n$$

is continuous if and only if (nu_n) is bounded. Thus,

$$D(A^*) = \{u \in \ell^1(\mathbb{N}) \mid \sup n|u_n| < \infty\}.$$

One checks that $\overline{D(A^*)} = c_0(\mathbb{N}) = \{a = (a_n) \in \ell^\infty(\mathbb{N}) \mid \lim a_n = 0\}$ so A^* is not densily defined. Thus A is an example of a closed and densily defined operator whose adjoint is not densily defined.

Example 4 (Ex. 2.17.3 Brezis). Let $A : D(A) \subseteq C[0,1] \to C[0,1]$ where $D(A) = C^2[0,1]$ and $Au = u'$. Let $v \in C \setminus C^1$ and let (v_n) be a sequence in $C^1[0,1]$ (e.g. polynomials) such that $v_n \to v$ in the supremum norm. Then let $u_n(t) := \int_0^t v_n(s)ds$ and $u(t) := \int_0^t v(s)ds$. Then $\Gamma_A \ni (u_n, Au_n) \to (u, v)$ but $u \notin D(A)$ so A is not closed. However, if we had taken $D(A) = C^1[0,1]$, then A would have been closed.

Example 5. Let $[a, b] \subseteq \mathbb{R}$ and consider the operator

$$L : D(L) \subseteq L^2[a,b] \longrightarrow L^2[a,b],$$

$$D(L) := \{u \in C^2[a,b] \mid u(a) = u(b) = 0\},$$
$$Lu = a_2(x)u'' + a_1(x)u' + a_0(x)u$$

where $a_0, a_1, a_2 \in C[a,b]$, $a_2 < 0$. Be letting

$$p(x) := \exp \int_a^x \frac{a_1(\xi)}{a_2(\xi)} d\xi, \quad q(x) := -\frac{a_0(x)p(x)}{a_2(x)}, \quad \varrho(x) := -\frac{p(x)}{a_2(x)}$$

we can write L on the following form:

$$Lu = \frac{1}{\varrho}\left(-(pu')' + qu\right).$$

This is called the *Sturm-Liouville form* of the operator. Now, define the following inner product on $L^2[a, b]$:

$$\langle f, g\rangle := \int_a^b f(x)g(x)\varrho(x)dx.$$

Note that this inner product induces a norm which is equivalent to the standard one. One checks (by integrating by parts), that for $u, v \in D(L)$ it holds that $\langle Lu, v\rangle = \langle u, Lv\rangle$. Thus it is easy to verify that L is closable: Let (u_n) be a sequence in $D(A)$ such that $u_n \to 0$ and $Lu_k \to f$. Then for every $v \in D(A)$, we have

$$\langle f, v\rangle = \lim\langle Lu_k, v\rangle = \lim\langle u_k, Lv\rangle = 0.$$

As $D(A) \subseteq L^2[a, b]$ is dense, we conclude that $f = 0$.

Exercise class 2

We begin with the some exercises from the lectures.

Exercise 6. Let $A : H^2(\mathbb{S}^1) \subseteq L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$, $f \mapsto -f''$. Show that A is self-adjoint and

$$\sigma(A) = \sigma_p(A) = \{n^2 \mid n \in \mathbb{Z}\}.$$

Solution. Let $\tilde{A} : h^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$, $\tilde{A}(a_n) = (n^2 a_n)$. Then $A = \mathcal{F}^{-1} \circ \tilde{A} \circ \mathcal{F}$ where $\mathcal{F} : L^2(\mathbb{S}^1) \to \ell^2(\mathbb{Z})$ is the Fourier transform, so to solve the exercise for $\tilde{A}$. Now, let $b = (b_n) \in \ell^2(\mathbb{Z})$. Then

$$D(\tilde{A}) = h^2(\mathbb{Z}) \ni a = (a_n) \longmapsto \langle \tilde{A}a, b\rangle = \sum n^2 a_n \overline{b_n}$$

extends to a continuous linear functional on $\ell^2(\mathbb{Z})$ if and only if $b \in h^2(\mathbb{Z})$: It is clear that if $b \in h^2(\mathbb{Z})$, then we get a continuous linear functional. To see that it is necessary, take $a = b$. Further, it is clear that $\langle b, \tilde{A}a\rangle = \langle \tilde{A}b, a\rangle$ so $\tilde{A}$ is self-adjoint.

For the spectrum, let $\lambda \in \mathbb{C}$. Then $(A-\lambda)a = 0$ i.e. $((n^2 - \lambda)a_n) = 0$ has a non-trivial solution if and only if $\lambda = n^2$ for some $n \in \mathbb{Z}$. Therefore is $\sigma_p(\tilde{A}) = \mathbb{Z}$. Now, if $\lambda \neq n^2$ for every $n \in \mathbb{Z}$ then $\tilde{A} - \lambda$ is invertible with $(\tilde{A} - \lambda)^{-1}a = (a_n/(n^2 - \lambda))$ so $\lambda \notin \sigma(\tilde{A})$. ∎

Exercise 7. Let $A : D(A) \subseteq \mathcal{H} \to \mathcal{H}$ be self adjoint. Then A is closed.

Solution. Let $\Gamma_A \ni (x_n, Ax_n) \to (0, y)$. Then for every $z \in D(A)$,

$$\langle y, z\rangle = \lim\langle Ax_n, z\rangle = \lim\langle x_n, Az\rangle = 0$$

and as $D(A)$ is dense, $y = 0$. ∎

Exercise 8. Let $A : \mathcal{H} \to \mathcal{H}$ be linear and symmetric. Then A bounded.

Solution. One uses the same argument as in the previous exercise to see that A is closed and hence bounded by the closed graph theorem. ∎

Example 9. Recall that $U \in \mathcal{L}(\mathcal{H})$ is called *unitary* if it preserves the inner product, i.e. $\langle Ux, Uy\rangle = \langle x, y\rangle$ for every $x, y \in \mathcal{H}$ and it is surjective. A unitary operator is clearly bounded, and it is injective since $\|Ux\| = \|x\|$. Hence, it is bijective and thus invertible with a bounded inverse. To determine the inverse, one notices that

$$\langle U^*Ux, y\rangle = \langle Ux, Uy\rangle = \langle x, y\rangle$$

for every $x, y \in \mathcal{H}$ so $U^*U = \mathrm{id}_{\mathcal{H}}$ and as we know that U is invertible, we can conclude that $U^{-1} = U^*$. It is easy to see that this characterizes unitary operators, i.e. $U \in \mathcal{L}(\mathcal{H})$ is unitary if and only if it is invertible with $U^{-1} = U^*$.

Let now $A \in \mathcal{L}(\mathcal{H})$ be a self adjoint operator and $f(\lambda) := e^{i\lambda}$. Then f is a continuous function on the spectrum of A. By the continuous functional calculus, we have a C^*-algebra homomorphism $\varphi : C(\sigma(A)) \to \mathcal{L}(\mathcal{H})$. Now note that, since $\sigma(A) \subseteq \mathbb{R}$, we have

$$\mathrm{id}_{\mathcal{H}} = \varphi(1) = \varphi(f\bar{f}) = \varphi(f) \circ \varphi(f)^*$$

and similary $\mathrm{id}_{\mathcal{H}} = \varphi(f)^* \circ \varphi(f)$ so we conclude that $\varphi(f) = e^{iA}$ is a unitary operator.

Example 10. Let $U \in \mathcal{L}(\mathcal{H})$ be unitary. For $\lambda \in \mathbb{C}$, $|\lambda| \neq 1$, one can check that if $|\lambda| < 1$ then $\lambda - U$ is invertible with inverse

$$(\lambda - U)^{-1} = -U^* \sum_{n=0}^{\infty} (\lambda U^*)^n$$

and if $|\lambda| > 1$ then it is invertible with inverse

$$(\lambda - U)^{-1} = \frac{1}{\lambda} \sum_{n=0}^{\infty} \frac{1}{\lambda^n} U^n.$$

Hence is $\sigma(U) \subseteq \mathbb{U} = \{z \in \mathbb{C} \mid |z| = 1\}$.

Exercise 11. Let X, Y be Banach spaces, $A : D(A) \subseteq X \to Y$ a closed densily defined operator and assume that Y is reflexive. Then $D(A^*) \subseteq Y^*$ is dense.

Solution. Assume that $D(A^*) \subseteq Y^*$ is not dense. Then, since Y is reflexive, there exists $y_0 \in Y$, $y_0 \neq 0$, such that $\mu(y_0) = 0$ for every $\mu \in D(A^*)$. Now, Γ_A is closed and $(0, y_0) \notin \Gamma_A$ so there exists $\lambda \in (X \times Y)^*$ such that $\lambda|_{\Gamma_A} = 0$ and $\lambda(0, y_0) \neq 0$. Similarly as in Exercise 2, one verifies that $\mu \in Y^*$, $\mu(y) = \lambda(0, y)$, is in $D(A^*)$. But $\mu(y_0) = -\lambda(0, y_0) \neq 0$, a contradiction. ■

Exercise class 3

The following was proven in last week's lectures.

Theorem 12. *Let $\mathcal{H}$ be a separable Hilbert space and $T \in \mathcal{L}(\mathcal{H})$ be a normal operator. Then there exists a unique C^*-algeba homomorphism $\Phi : \mathcal{B}^\infty(\sigma(T)) \to \mathcal{L}(\mathcal{H})$ such that $\Phi(\lambda) = T$. Moreover, for any $v \in \mathcal{H}$ there exists a unique Radon measure μ_v on $\sigma(T)$ such that*

$$\langle f(T)v, v\rangle = \int_{\sigma(T)} f(\lambda) d\mu_v(\lambda)$$

for any $f \in \mathcal{B}^\infty(\sigma(T))$.

Remark. Recall that for $\Omega \subseteq \sigma(T)$ we defined $P(\Omega) := \mathbf{1}_\Omega(T)$ which is a self adjoint projection operator. Note that $\langle P(\Omega)v, v\rangle = \mu_v(\Omega)$ for any $v \in \mathcal{H}$.

Exercise 13. Let $T \in \mathcal{L}(\mathcal{H})$ be a normal operator and $f \in \mathcal{B}^\infty(\sigma(T))$ be real valued. Then $f(T)$ is self adjoint. If $f \geq 0$ then $f(T)$ is positive definite.

Solution. We have

$$\langle f(T)v, v\rangle = \int_{\sigma(T)} f d\mu_v = \int_{\sigma(T)} \bar{f} d\mu_v = \langle f(T)^* v, v\rangle$$

so $f(T) = f(T)^*$. ■

The remaining exercises are from the exercise sheets of Functional Analysis II in 2022.

Exercise 14. Let $T \in \mathcal{L}(\mathcal{H})$ be a nonzero normal operator. Then for any non-empty open set $\Omega \subseteq \sigma(T)$, $P(\Omega) \neq 0$.

Solution. Note first that if f, g are two functions such that $f \leq$ and $v \in \mathcal{H}$ is a vector then $\langle f(T)v, v\rangle \leq \langle g(T)v, v\rangle$. Indeed:

$$\langle g(T)v, v\rangle - \langle f(T)v, v\rangle = \langle (g-f)(T)v, v\rangle = \int_{\sigma(T)} (g-f) d\mu_v \geq 0.$$

As Ω is non-empty and open, there exists a non-zero function f such that $0 \leq f \leq \mathbf{1}_\Omega$. Now, $f(T)$ is non-trivial by the continuous functional calculus and it is positive definite, so there exists v such that $\langle f(T)v, v\rangle > 0$. ■

Exercise 15. Let $T \in \mathcal{L}(\mathcal{H})$ be a normal operator and $f \in C(\sigma(T))$. Then $\ker f(T) = \operatorname{im} P(f^{-1}(0))$.

Solution. For $v \in \mathcal{H}$ we have

$$\|f(T)v\|^2 = \langle f(T)^* f(T)v, v\rangle = \int_{\sigma(T)} |f|^2 d\mu_v$$

so $f(T)v = 0$ if and only if $\mu_v(\sigma(T) \smallsetminus f^{-1}(0)) = 0$ which is if and only if $\mu_v(f^{-1}(0)) = \|v\|^2$ as $\mu_v(\sigma(T)) = \|v\|^2$. Now, $\mu_v(f^{-1}(0)) = \langle P(f^{-1}(0))v, v\rangle$ so $\mu_v(\sigma(T) = \|v\|^2$ if and only if $\langle P(f^{-1}(0)v, v\rangle = \|v\|^2$ which is if and only if $v \in \operatorname{im} P(f^{-1}(0))$. ■

Exercise 16. Let $\lambda_0 \in \sigma(T)$ be an isolated point. Then λ_0 is an eigenvalue of T.

Proof. As $\lambda_0 \in \sigma(T)$ is isolated, $\{\lambda_0\}$ is open, so $P(\{\lambda_0\}) \neq 0$. Hence it follows from the last exercise, by taking $f(\lambda) := \lambda - \lambda_0$, that λ is indeed an eigenvalue of T. ■

Now we will prove von Neumann's ergodic theorem, a classical application of the spectral theorem.

Theorem 17. *Let $U \in \mathcal{L}(\mathcal{H})$ be a unitary operator and π be the projection onto the 1-eigenspace of U. Then for any $v \in \mathcal{H}$,*

$$\lim_{n\to\infty} \frac{1}{n}\left(v + Uv + \cdots + U^{n-1}v\right) = \pi(v).$$

Proof. The result is clear if v is in the 1-eigenspace so by linearity, it suffices to show that the left hand side is equal to 0 when v is othogonal to the 1-eigenspace i.e. when $\pi(v) = 0$. Now, let

$$f_n(\lambda) \coloneqq \frac{1}{n}\left(1 + \lambda + \cdots + \lambda^{n-1}\right) = \frac{1}{n}\frac{1-\lambda^n}{1-\lambda}.$$

Then $|f_n| \leq 1$ and $f_n \to \mathbf{1}_{\{1\}}$ pointwise so the dominated convergence theorem gives that

$$\|f_n(U)v\|^2 = \int_{\sigma(U)} |f_n|^2 d\mu_v \longrightarrow \int_{\sigma(U)} \mathbf{1}_{\{1\}} = \langle P(\{1\})v, v\rangle = \langle \pi(v), v\rangle = 0$$

as $P(\{1\}) = \pi$ since $\operatorname{im} P(\{1\}) = \ker(U - \mathrm{id})$. ■

Remark. If (X, μ) is a probability space and $\psi : X \to X$ is a measure preserving map, then $U : L^2(X, \mu) \to L^2(X, \mu)$, $U(f) \coloneqq f \circ \psi$ is a unitary transformation.

Exercise 18. Let $T \in \mathcal{L}(\mathcal{H})$ be a normal operator. Then $\|T\| = \sup\{|\langle Tv, v\rangle| \mid \|v\| \leq 1$.

Solution. It is clear that $\sup\{|\langle Tv, v\rangle| \mid \|v\| \leq 1\} \leq \|T\|$ by Cauchy-Schwarz. Now, if $T = 0$, then the result is clear, so we assume that $T \neq 0$ and hence $\|T\| > 0$. Since $\sup_{\sigma(T)} |\lambda| = \|T\|$ and $\sigma(T)$ is compact, $\lambda_0 \in \sigma(T)$ with $|\lambda_0| = \|T\|$ and $0 < \varepsilon < |\lambda_0|$. Then $P(B(\lambda_0, \varepsilon)) \neq 0$ so we can find a unit vector v in its image. Note that then μ_v is concentrated on $B(\lambda_0, \varepsilon)$ so

$$|\langle Tv, v\rangle| = \left|\int_{\sigma(T)} \lambda d\mu_v(\lambda)\right| \geq \int_{\sigma(T)\cap B(\lambda_0,\varepsilon)} |\lambda| d\mu_v(\lambda) \geq |\lambda_0| - \varepsilon = \|T\| - \varepsilon.$$

Since ε was arbitrary, the result follows. ■

Exercise class 4

Recall the following definition from class:

Definition 19. A map $U : \mathbb{R} \to \mathcal{L}(\mathcal{H})$ is called a strongly continuous one-parameter unitary group on $\mathcal{H}$ if the following holds

(i) $U(t)$ is unitary for every $t \in \mathbb{R}$.

(ii) $U(s + t) = U(s)U(t)$ for every $s, t \in \mathbb{R}$.

(iii) $\lim_{s\to t} U(t)v = U(t)v$ for every $t \in \mathbb{R}$.

Remark. Note that $U(0) = \mathrm{id}_{\mathcal{H}}$ since $U(0) = U(0+0) = U(0)U(0)$ and $U(0)$ is invertible. Thus, $\mathrm{id}_{\mathcal{H}} = U(0) = U(t - t) = U(t)U(-t)$ so $U(-t) = U(t)^{-1} = U(t)^*$.

The following theorem was stated in class but not proven. We will now go through the proof from the book *Quantum theory for mathematicians* by Brian Hall (see Theorem 10.15).

Theorem 20 (Stone's theorem)**.** *Let $\mathcal{H}$ be a separable Hilbert space and $t \mapsto U(t)$ a strongly continuous one-parameter unitary group on $\mathcal{H}$. Then there exists a unique self-adjoint operator $A : D(A) \subseteq \mathcal{H} \to \mathcal{H}$ such that $U(t) = e^{itA}$ for every $t \in \mathbb{R}$.*

Lemma 21. *Let $A : D(A) \subseteq \mathcal{H} \to \mathcal{H}$ be symmetric on $D(A)$. Then $\bar{A}$ is self adjoint if and only if* $\ker(A^* \pm i) = \{0\}$.

The proof of this lemma is left as an exercise.

Proof of Stone's theorem. Let $D(A)$ be the set of those vectors $v \in \mathcal{H}$ such that the limit

$$\lim_{t\to 0} \frac{U(t)v - v}{t}$$

exists and for vectors $v \in D(A)$ we let

$$Av := \lim_{t\to 0} \frac{U(t)v - v}{it}.$$

Let us now show that A is self-adjoint. Note that $D(A)$ is a subspace of $\mathcal{H}$. Let us show that it is dense. For $f \in C_c^\infty(\mathbb{R})$ we define $B_f \in \mathcal{L}(\mathcal{H})$ by

$$B_f v = \int_{-\infty}^{\infty} f(\tau)U(\tau)v \, d\tau.$$

For any $v \in \mathcal{H}$ it holds that

$$\begin{aligned} U(t)B_f v - B_f u &= \int_{-\infty}^{\infty} (f(\tau)U(\tau + t)v - f(\tau)U(\tau)v) dt \\ &= \int_{-\infty}^{\infty} (f(\tau - t) - f(\tau))U(\tau)v \, d\tau \end{aligned}$$

so $B_f v \in D(A)$ since

$$\lim_{t\to 0} \frac{U(t)B_f v - B_f v}{t} = -\int_{-\infty}^{\infty} f'(\tau)U(\tau)v \, d\tau.$$

Now let (f_n) be a sequence in $C_c^\infty(\mathbb{R})$ such that $f_n \geq 1$, $\operatorname{spt}(f_n) \subseteq [-1/n, 1/n]$ and $\int_{-\infty}^{\infty} f_n(\tau)d\tau = 1$. Then

$$\|B_{f_n} v - v\| \leq \int_{-\infty}^{\infty} f_n(\tau)\|U(\tau)v - v\| d\tau \leq \sup_{-1/n \leq \tau \leq 1/n} \|U(\tau)v - v\| \to 0$$

as $n \to \infty$. This shows that $D(A)$ is dense.

Now let $u, v \in D(A)$. Note that $U(t)^* = U(-t)$ so

$$\langle Au, v\rangle = \frac{1}{i}\frac{d}{dt}\bigg|_{t=0} \langle U(t)u, v\rangle = \frac{1}{i}\frac{d}{dt}\bigg|_{t=0} \langle u, U(t)^* v\rangle = \frac{1}{i}\frac{d}{dt}\bigg|_{t=0} \langle u, U(-t)v\rangle = \langle u, Av\rangle,$$

i.e. A is symmetric and hence closable. Let us show that $\bar{A}$ is self adjoint. For that, it suffices to show that $\ker(A^* \pm i) = \{0\}$. So assume that $v \in D(A^*)$ and $A^* v = iv$ and consider $f(t) := \langle U(t)u, v\rangle$. Now,

$$\frac{d}{dt}\langle U(t)u, v\rangle = \langle iAU(t)u, v\rangle = \langle iU(t)u, A^* v\rangle = \langle iU(t)u, iv\rangle = \langle U(t)u, v\rangle$$

so $f'(t) = f(t)$ and hence $f(t) = f(0)e^t$. As f is bounded, we must have $f(t) = 0$, and hence in particular $f(0) = 0$ i.e. $\langle u, v\rangle = 0$. As this holds for every $u \in D(A)$ and $D(A) \subseteq \mathcal{H}$ is dense, $v = 0$. Similarly, $\ker(A^* + i) = \{0\}$.

Let us now define $V(t) = e^{it\bar{A}}$. Take $v \in D(A)$ and set $\tilde{v}(t) = U(t)v - V(t)v$. Then

$$\frac{d}{dt}\tilde{v}(t) = iAU(t)v - iAV(t)v = iA\tilde{v}(t)$$

so

$$\frac{d}{dt}\|\tilde{v}(t)\|^2 = \langle \tilde{v}(t), \tilde{v}'(t)\rangle + \langle \tilde{v}'(t), \tilde{v}(t)\rangle = \langle \tilde{v}(t), iA\tilde{v}(t)\rangle + \langle iA\tilde{v}(t), \tilde{v}(t)\rangle = 0.$$

As $v(0) = 0$, it follows that $v(t) = 0$ for every t and hence is $U = V$ as for every $t \in \mathbb{R}$, $U(t)$ and $V(t)$ are bounded operators which agree on a dense set. This implies that $A = \bar{A}$ because by a theorem from class, the derivative of V is defined exactly on the domain of $\bar{A}$. Now, if B is another self adjoint operator such that $U(t) = e^{itB}$ then again by the theorem from class, $A = B$. ∎

Exercise class 5

Let $p \in C^1[0,1]$ with $p > 0$ and $q \in C[0,1]$. In this class, we will consider the following boundary problem of Sturm-Liouville type

$$\begin{cases} -(pu')' + qu = f \\ u(0) = u(1) = 0 \end{cases} \tag{1}$$

where $f \in L^2(a,b)$. The exercises and their solutions are based on the discussion about Sturm-Liouville type problems in chapter 8 in Brezis; see the book for more.

Exercise 22. Show that for each $f \in L^2[0,1]$ there exists a unique $u \in H^2(0,1)$ which solves (1). Moreover, show that if $f \in C[0,1]$, then $u \in C^2[0,1]$.

Solution. We consider the weak formulation of the problem, i.e. we look for $u \in H_0^1(0,1)$ such that

$$\int_0^1 pu'v' + quv = \int_0^1 v$$

for every $v \in H_0^1(0,1)$. Define the following bilinear form on α on $H_0^1(0,1)$:

$$\alpha(u,v) := \int_0^1 pu'v' + quv.$$

This form is symmetric, $\alpha(u,u) \geq 0$ for every $u \in H_0^1[0,1]$ and $\alpha(u,u) = 0$ if and only if $u = 0$, by the Poincaré inequality. Hence it defines an inner product on $H_0^1(0,1)$. Now, the functional

$$H_0^1(0,1) \ni u \longmapsto \int_0^1 fv$$

is continuous, so by Riesz, there exists a unique $u \in H_0^1(0,1)$ such that $\alpha(u,v) = \int_0^1 fv$ for every $v \in H_0^1(0,1)$ i.e.

$$\int_0^1 pu'v' + quv = \int_0^1 fv \tag{2}$$

for every $v \in H_0^1(0,1)$.

Now, from (2) we see that pu' has weak derivative $qu - f \in L^2(a,b)$ so $pu' \in H^1(0,1)$ and hence is $u' \in H^2(0,1)$ as $p \in C^1[a,b]$ and $p > 0$. Assuming that f is continuous, we use that

$$pu'(x) = \int_0^x q - fu$$

as $pu' \in H_0^1[0,1]$. Since $u \in H_0^1(0,1)$ it has a continuous representative so the integrand on the right hand side is continuous and thus is the right hand side continuously differentiable in x, by the fundamental theorem of calculus. But then, as $p > 0$ is a C^1 function, $u' \in C^1$ and hence $u \in C^2$. ■

Exercise 23 (Theorem 8.22 in Brezis). Show that there exists a sequence (λ_n) of real numbers together with a sequence (e_n) of functions in $C^2[0,1]$ such that for each n

$$\begin{cases} -(pe_n')' + qe_n = \lambda_n e_n \\ e_0(0) = e_n(1) = 0. \end{cases}$$

The functions (e_n) form a Hilbert basis for $L^2[0,1]$ and furthermore, $\lambda_n \to \infty$ as $n \to \infty$.

Solution. By the previous exercise, for each $f \in L^2(0,1)$, there exists a unique solution $u \in (H^2 \cap H_0^1)(0,1) \subseteq L^2(0,1)$ to (1); let T be the map which takes f to u. We will show that T considered as an operator from $L^2(0,1)$ to $L^2(0,1)$, is compact an self-adjoint.

For the compactness, let $f \in L^2$ and $u := Tf$ and put u into (2) to get

$$a\|u'\|_{L^2}^2 = \int_0^1 p(u')^2 + \int_0^1 qu^2 = \int_0^1 fu \leq \|f\|_{L^2}\|u\|_{L^2}$$

where $a := \inf_{x\in[0,1]} p(x) > 0$. By Poincaré, $\|u\|_{L^2} \leq C\|u'\|_{L^2}$ so $\|Tf\|_{H_0^1} \leq C\|f\|_{L^2}$. Now, the injection $H^1(0,1) \hookrightarrow L^2(0,1)$ is compact, so T is compact.

For the self-adjointness, let $f, g \in L^2(0,1)$. Put $u := Tf$, $v := Tg$ and then we get by partial integration that

$$\int_0^1 (Tf)g = \int_0^1 ug = \int_0^1 -u(pv')' + uqv = \int_0^1 -(pu')'v + quv = \int_0^1 fv = \int_0^1 f(Tg)$$

so T is self adjoint. It is moreover strictly positive definite.

Now the spectral theorem for compact self-adjoint operators gives a Hilbert basis (e_n) of $L^2(0,1)$ together with a bounded sequence (μ_n) of real numbers, for which 0 is the only accumulation point and such that for each n, $Te_n = \mu_n e_n$. Further, as T is strictly positive definite, we know that $\mu_n > 0$ for every n. Finally, $Te_n = \mu_n e_n$ gives that $-(pe_n')' + qe_n = \lambda_n e_n$ where $\lambda_n := 1/\mu_n$. ■

Exercise class 6

Let $\Omega \subseteq \mathbb{R}^n$ be an open set. Recall that the space of functions on Ω which are Hölder continuous with exponent $\alpha \geq 0$ is defined as

$$C^{0,\alpha}(\Omega) := \{u \in C(\Omega) \mid \|u\|_\infty + [u]_{C^{0,\alpha}} < \infty\}$$

where

$$[u]_{C^{0,\alpha}} := \sup_{x,y\in\Omega x\neq 0} \frac{|(x) - u(y)|}{\|x-y\|^\alpha}.$$

This space is endowed with the norm

$$\|u\|_{C^{0,\alpha}(\Omega)} := \|u\|_\infty + [u]_{C^{0,\alpha}(\Omega)}$$

and is a Banach space (this you can either prove as an exercise or see Satz 8.6.1 in Struwe).

Theorem 24 (Satz 8.6.2 in Struwe)**.** *Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded. Then for every $0 \leq \beta \leq \alpha \leq 1$, the embedding*

$$C^{0,\alpha}(\Omega) \hookrightarrow C^{0,\beta}(\Omega)$$

is compact.

Proof. We note first that for $u \in C^{0,\alpha}(\Omega)$, $x, y \in \Omega$ with $x \neq y$, it holds that

$$\frac{|u(x) - u(y)|}{\|x-y\|^\beta} \leq 2\|u\|_\infty^{1-\frac{\beta}{\alpha}} \left(\frac{|u(x) - u(y)|}{\|x-y\|^\alpha}\right)^{\beta/\alpha} \leq 2\|u\|_\infty^{1-\frac{\beta}{\alpha}} [u]_{C^{0,\alpha}}^{\beta/\alpha}$$

so

$$[u]_{C^{0,\beta}} \leq 2\|u\|_\infty^{1-\frac{\beta}{\alpha}} [u]_{C^{0,\alpha}}^{\beta/\alpha} \tag{3}$$

Now, let (u_k) be a bounded sequence in $C^{0,\alpha}(\Omega)$. Then for every $x, y \in \Omega$,

$$|u(x) - u(y)| \leq [u]_{C^{0,\alpha}} |x-y|^\alpha \leq \operatorname{diam}(\Omega)^\alpha [u]_{C^{0,\alpha}}$$

so (u_k) is equicontinuous. It is also bounded in $\|\cdot\|_\infty$ so by Arzela-Ascoli, we can assume that (u_k) is convergent in $\|\cdot\|_\infty$. From (3), it now follows that (u_k) is Cauchy in $C^{0,\beta}$ so it is convergent. ∎

Now, in last lecture, it was shown that for every $1 \leq p \leq \infty$ and every $I \subseteq \mathbb{R}$, $W^{1,p}(\Omega)$ embeds into $C^{0,\alpha}(\Omega)$ where $\alpha := 1 - 1/p$ (Theorem 21 in the lecture notes). Combining this with Theorem 24, one concludes the following theorem.

Theorem 25. *Let $I \subseteq \mathbb{R}$ be a bounded interval and $1 < p \leq \infty$. Then for every $\beta < 1 - 1/p$, the embedding*

$$W^{1,p}(I) \hookrightarrow C^{0,\beta}(I)$$

is compact.

Remark. In particular is $W^{1,p}(I) \hookrightarrow C(I)$ compact and thus $W^{1,p}(I) \hookrightarrow L^p(I)$ as well.

Remark. The theorem does not hold if I is unbounded. To see that, take a compactly supported smooth function and construct a sequence by shifting it.

Example 26 (Ex. 8.2.2 in Brezis). By Theorem 21 from the lectures, $W^{1,1}(I)$ embeds into $C^{0,0}(I) = C_0(I)$. Let us show that this embedding is not compact. Take $I = (0,1)$ and consider the sequence

$$u_n(x) := \begin{cases} 0 & \text{if } x \in [0, \frac{1}{2}], \\ n(x - 1/2) & \text{if } x \in (\frac{1}{2}, \frac{1}{2} + \frac{1}{n}), \\ 1 & \text{if } x \in [\frac{1}{2} + \frac{1}{n}, 1], \end{cases}$$

$n \geq 2$. It holds that

$$\|u_n\|_{W^{1,1}} = \int_0^1 u_n + \int_0^1 u_n' = \frac{3}{2} - \frac{1}{n}$$

so (u_n) is bounded in $W^{1,1}$. However, the sequence does not admit a convergent subsequence in $C(I)$.

Exercise 27 (Ex. 8.38 in Brezis). Let $T : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be the operator defined by $Tf := u$, where u is the unique weak solution to

$$-u'' + u = f.$$

(a) Show that T is symmetric, positive definite and $\|T\| \leq 1$.

(b) Show that T has no eigenvalues. Conclude that it is not compact.

(c) Show that $T - \lambda I$ is surjective for any $\lambda \in [0,1]$. Conclude that $\sigma(T) = [0,1]$.

Solution. (a) Let $f, g \in L^2$, $u := Tf$, $v := Tg$. Then

$$\langle Tf, g \rangle = \int_{\mathbb{R}} ug = \int_{\mathbb{R}} u'v' + uv = \int_{\mathbb{R}} fv = \langle f, Tg \rangle$$

so T is symmetric. Further,

$$\langle Tf, f \rangle = \int_{\mathbb{R}} uf = \int_{\mathbb{R}} (u')^2 + u^2 \geq 0$$

so it is positive definite. Finally,

$$\|Tf\|^2 = \langle Tf, Tf \rangle = \int_{\mathbb{R}} u^2 \leq \int_{\mathbb{R}} (u')^2 + u^2 = \langle Tf, f \rangle \leq \|Tf\| \cdot \|f\|$$

so $\|Tf\| \leq \|f\|$ and hence $\|T\| \leq 1$.

(b) Assume that $Tf = \lambda f$. Writing $u = Tf$, we have $u = \lambda(-u'' + u)$. The solutions to this equation will not be in L^2. By the spectral theorem for compact self adjoint operators, T can not be compact.

(c) First, T is not surjective (its image is H^2) so $T - \lambda I$ is not surjective for $\lambda = 0$. For $\lambda = 1$, we have $(T - I)f = g$ if and only if $u'' = g$, where $u = Tf$. Recall that $u \in H^2$ so $u'(y) - u'(x) = \int_x^y g$ so if we take for g a function with support contained in $[0,1]$, then u' is constant on $(-\infty, 0]$ and $[1, \infty)$ and as $u' \in L^2$, it has to be equal to 0 on these intervals. But then $0 = u'(1) - u'(0) = \int_0^1 g$ so if $\int_0^1 g \neq 0$, $(T - I)f = g$ has no solution.

Finally, if $\lambda \in (0,1)$, then $(T-\lambda I)f = g$ if and only if $\lambda u'' + (1-\lambda)u = g$ where $u = Tf$. Again, by picking g with support contained in $[0,1]$, we have that $\lambda u'' + (1-\lambda)u = 0$ outside of $[0,1]$. Solving this equation with standard methods and then imposing the restriction that $u \in L^2$, yields that $u = 0$ outside $(0,1)$. Now, let v be a solution to $\lambda v'' + (1-\lambda)v = 0$ on $(0,1)$. Then $\int_0^1 \lambda v'\varphi' = -\int_0^1 (1-\lambda)v\varphi$ for every $\varphi \in H^1(0,1)$ with $\varphi(0) = \varphi'(0) = \varphi(1) = \varphi'(1) = 0$. Hence, this holds in particular for $\varphi = u$. Using that together with $\lambda u'' + (1-\lambda)u = g$ gives

$$\int_0^1 gv = -\int_0^1 \lambda u'v' + \int_0^1 (1-\lambda)uv = -\int_0^1 (1-\lambda)uv + \int_0^1 (1-\lambda)uv = 0$$

i.e. $\int_0^1 gv = 0$ for any solution v to $\lambda v'' + (1-\lambda v) = 0$ on $(0,1)$. Picking a function g which does not satisfy this, shows that $T - \lambda I$ is not surjective.

Now, we know that $\sigma(T) \subseteq [0,1]$ since it is positive definite and $\|T\| \leq 1$. We have further shown that $[0,1] \subseteq \sigma(T)$ so we conclude that $\sigma(T) = [0,1]$. ■

Exercise class 7

Exercise 28 (Integral form of Minkowski's inequality). Let (X,μ) and (Y,ν) be σ-finite measure spaces and $f : X \times Y \to \mathbb{R}$ a measurable function. Show that for every $1 \leq p \leq \infty$ it holds that

$$\left(\int_Y \left|\int_X f(x,y)d\mu(x)\right|^p d\nu(y)\right)^{\frac{1}{p}} \leq \int_X \left(\int_Y |f(x,y)|^p d\nu(y)\right)^{\frac{1}{p}} d\mu(x)$$

i.e.

$$\left\|\int_X f(x,\cdot)d\mu(x)\right\|_{L^p(\nu)} \leq \int_X \|f(x,\cdot)\|_{L^p(\nu)} d\mu(x).$$

Solution. (From Stackexchange) The inequality follows from the triangle inequality and Fubini if $p = 1$. Hence we assume that $p > 1$. Let $1 \leq q < \infty$ be the conjugate number of p, i.e. the number such that $1/p + 1/q = 1$ and consider the map

$$\lambda : L^p(\nu) \longrightarrow L^q(\nu)^*, \quad g \mapsto \left(h \mapsto \int_Y g(y)h(y)d\nu(y)\right).$$

Recall that this is an isometry. Now, for every $g \in L^q(\nu)$, it holds that

$$\begin{aligned}
\left|\lambda\left(\int_X f(x,\cdot)d\mu(x)\right)(g)\right| &= \int_X \int_Y f(x,y)g(y)d\nu(y)d\mu(x) \\
&\leq \int_X \|f(x,\cdot)\|_{L^p(\nu)}\|g\|_{L^q(\nu)} d\mu(x) \\
&= \left(\int_X \|f(x,\cdot)\|_{L^p(\nu)} d\mu(x)\right)\|g\|_{L^q(\nu)}
\end{aligned}$$

so as λ is an isometry,

$$\left\|\int_X f(x,\cdot)d\mu(x)\right\|_{L^p(\nu)} = \left\|\lambda\left(\int_X f(x,\cdot)d\mu(x)\right)\right\|_{L^q(\nu)^*} \leq \int_X \|f(x,\cdot)\|_{L^p(\nu)} d\mu(x).$$

■

Exercise 29. Let $f \in L^p(\mathbb{R}^n)$ and $g \in L^1(\mathbb{R}^n)$. Then

$$\|f * g\|_p \leq \|f\|_p \|g\|_1.$$

Solution. We have

$$\begin{aligned}
\|f * g\|_p &= \left(\int_{\mathbb{R}^n} \left|\int_{\mathbb{R}^n} f(x-y)g(y)dy\right|^p dx\right)^{\frac{1}{p}} \leq \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |f(x-y)g(y)|^p dx\right)^{\frac{1}{p}} dy \\
&= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |f(x-y)|^p dx\right)^{\frac{1}{p}} |g(y)|dy = \|f\|_p \|g\|_1.
\end{aligned}$$

An alternative solution is to use Jensen's inequality: If (X, μ) is a probability space and φ a convex function, then

$$\varphi\left(\int_X f d\mu\right) \leq \int_X \varphi \circ f d\mu$$

for every measurable function f. Now, if $\|g\|_1 = 0$, then $f * g = 0$ so the result is clear. Assume that $\|g\|_1 \neq 0$. Then

$$\frac{\|f * g\|_p^p}{\|g\|_1^p} = \int_{\mathbb{R}^n} \left|\int_{\mathbb{R}^n} f(x-y)\frac{g(y)}{\|g\|_1}dy\right|^p dx \leq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |f(x-y)|^p \frac{g(y)}{\|g\|_1} dydx = \|f\|_p^p.$$

This finishes the proof. ∎

Exercise 30 (Hardy's inequality). Let $1 < p < \infty$. For $u \in W_0^{1,p}(0, \infty)$, let $v(x) := u(x)/x$. Show that $v \in L^p(0, \infty)$ and

$$\|v\|_p \leq C\|u'\|_p$$

where C is a constant depending only on p.

Proof. We have that $u(x) = \int_0^x u' = x\int_0^1 u'(xy)dy$ so $v(x) = \int_0^1 u'(xy)dy$ and hence

$$\begin{aligned}
\|v\|_p &= \left(\int_0^\infty \left|\int_0^1 u'(xy)dy\right|^p dx\right)^{\frac{1}{p}} \leq \int_0^1 \left(\int_0^\infty |u'(xy)|^p dx\right)^{\frac{1}{p}} dy \\
&= \int_0^1 \left(\int_0^\infty |u'(x)|^p dx\right)^{\frac{1}{p}} \frac{dy}{y^{\frac{1}{p}}} = \left(\int_0^1 \frac{dy}{y^{\frac{1}{p}}}\right) \|u'\|_p \\
&= \frac{p}{p-1}\|u'\|_p
\end{aligned}$$

where we used Minkowski in the second step. ∎

Theorem 31 (Satz 8.4.4 in Struwe)**.** *Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded with C^1-boundary. Then*

$$H^1(\Omega) = H_0^1(\Omega) \oplus \{u_0 \in H^1(\Omega) \mid \Delta u_0 = 0\}$$

and this decomposition is orthogonal with respect to the inner product $\langle u, v\rangle_{H_0^1} = \int_\Omega \langle \nabla u, \nabla v\rangle$ so

$$\|\nabla u\|_2^2 = \|\nabla u_0\|_2^2 + \|\nabla u_1\|_2^2$$

if $u = u_0 + u_1$ where $\nabla u_0 = 0$ and $u_1 \in H_0^1(\Omega)$. As a consequence, for every $u \in H^1(\Omega)$, there exists a unique $u_0 \in H^1(\Omega)$ such that $u|_{\partial\Omega} = u_0|_{\partial\Omega}$ and $\nabla u_0 = 0$, and that u_0 is characterized by the fact that it minimizes the Dirichlet energy

$$E(v) = \int_\Omega \|\nabla v\|^2$$

among all v satisfying $v|_{\partial\Omega} = v|_{\partial\Omega}$.

Proof. Let $u \in H^1(\Omega)$. the functional $H^1_0(\Omega) \ni v \mapsto \int_\Omega \langle \nabla v, \nabla u \rangle$ is continuous so by Riesz, there exists a unique $u_1 \in H^1_0(\Omega)$ such that

$$\int_\Omega \langle \nabla u_1, \nabla v \rangle = \int_\Omega \langle \nabla u, \nabla v \rangle$$

for every $v \in H^1_0(\Omega)$. Let $u_0 \coloneqq u - u_1$. Then for every $v \in C^\infty_c(\Omega)$, it holds that

$$\int_\Omega (\Delta u_0) v = -\int_\Omega \langle \nabla u_0, \nabla v \rangle = \int_\Omega \langle \nabla u_1, \nabla v \rangle - \int_\Omega \langle \nabla u, \nabla v \rangle = 0.$$

Hence is $\Delta u_0 = 0$, by the fundamental lemma of the calculus of variations.

Now let $u_0 \in H^1(\Omega)$ with $\Delta u_0 = 0$ and let $v \in C^\infty_c(\Omega)$. Then

$$\int_\Omega \langle \nabla u_0, \nabla v \rangle = -\int_\Omega (\Delta u_0) v = 0.$$

As $C^\infty_c(\Omega) \subset H^1_0(\Omega)$ is dense, we may conclude that the decomposition is orthogonal. ■

Exercise class 8

Recall that in last lecture, we saw the following theorem:

Theorem 32 (Gagliardo-Nirenberg)**.** *Let $1 \leq p < n$ and $p^* \geq p$ be the unique number such that $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$. Then for every $u \in W^{1,p}(\mathbb{R}^n)$ it holds that*

$$\|u\|_{p^*} \leq C \|\nabla u\|_p$$

where $C = \frac{(n-1)p}{n-p}$.

Interestingly, this inequality implies an isoperimetric inequality for domains in $\mathbb{R}^n$.

Example 33. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain with C^1 boundary. For $k \geq 1$, let $\varphi_k : [0, \infty) \to [0, 1]$ be a smooth nonincreasing function such that $\varphi_k(0) = 1$ and $\varphi_k(1/k) = 0$ and let u_k be defined by

$$u_k(x) \coloneqq \begin{cases} 1 & \text{if } x \in \Omega \\ \varphi_k(d(x, \partial\Omega)) & \text{if } x \notin \Omega. \end{cases}$$

Now,

$$\mathcal{L}^n(\Omega)^{\frac{n-1}{n}} = \lim_{k\to\infty} \left(\int_{\mathbb{R}^n} |u_k|^{\frac{n}{n-1}} \right)^{\frac{n-1}{n}} \leq \lim_{k\to\infty} \int_\Omega \|\nabla u_k\| \leq \mathcal{H}^{n-1}(\partial\Omega).$$

To prove the last step rigorously, one can for example use the coarea formula. It should be noted that this constant is not optimal.

Exercise 34 (Ex. 5.10.11 in Evans). Assume that Ω is connected and $u \in W^{1,p}(\Omega)$ satisfies $\nabla u = 0$ a.e. in U. Show that u is constant a.e. in Ω.

Solution. Let $x \in \Omega$, $r > 0$ be such that $r < d(x, \partial\Omega)$ and ϱ_ε be mollifiers. Now, $v * \varrho_\varepsilon \to W^{1,p}(B(x, r))$ and $\nabla(v * \varrho_\varepsilon) = (\nabla v) * \varrho_\varepsilon = 0$ on $B(x, r)$ for small enough ε so $v * \varrho_\varepsilon$ is constant on $B(x, r)$ for each ε so v is constant on $B(x, r)$ as it is a limit of constant functions. Connectedness of Ω implies that v is constant. ■

For a function u defined on a bounded domain Ω, we write $u_\Omega := \frac{1}{|\Omega|}\int_\Omega u$ for the average of u over Ω.

Exercise 35 (Poincaré's inequality, Thm. 5.8.1 in Evans). Let $\Omega \subseteq \mathbb{R}^n$ be a bounded and connected domain with C^1 boundary and $1 \leq p \leq \infty$. Then there exists a constant $C = C(n, p, \Omega)$ such that

$$\|u - u_\Omega\|_p \leq C\|\nabla u\|_p$$

for every $u \in W^{1,p}(\Omega)$.

The proof applies the Rellich-Konrachov compactness theorem that we saw in last week's lectures: If $\Omega \subseteq \mathbb{R}^n$ is a bounded domain with C^1 boundary, then the embedding $W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ is compact.

Solution. Assume for a contradiction that there exists a sequence (u_k) of functions in $W^{1,p}(\Omega)$ with

$$\|u_k - (u_k)_\Omega\|_p > k\|\nabla u_k\|p \tag{4}$$

for every k. Let

$$v_k := \frac{u_k - (u_k)_\Omega}{\|u_k - (u_k)_\Omega\|_p}.$$

Then $(v_k)_\Omega = 0$ and $\|v_k\|_p = 1$ so $\|\nabla v_k\|_p < \frac{1}{k}$ by (4). Now, by the Rellich-Kondrachov compactness theorem, there exists $v \in L^p(\Omega)$ such that after possibly passing to a subseqeunce, $v_k \to v$. Further, for every $\varphi \in C_c^\infty(\Omega)$,

$$\int_\Omega v\frac{\partial \varphi}{\partial x_i}dx = \lim_{k\to\infty}\int_\Omega v_k\frac{\partial \varphi}{\partial x_i} = -\lim_{k\to\infty}\int_\Omega \frac{\partial v_k}{\partial x_i} = 0$$

since $\|\nabla v_k\| \to 0$, so $\nabla v = 0$. Now, by the previous exercise, v is a constant and thus $v = 0$ as its average is 0. But this is a contradiction to $\|v_k\|_p = 1$. ■