# Functional Analysis II

# Solutions to exercises

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These are solutions to some exercises from the course *Functional analysis II* taught by Prof. Dr. Marc Burger in the spring semester of 2022 at ETH Zürich. If you find any mistakes, feel free to send an email at *hjalti.isleifsson@math.ethz.ch*.

**Exercise 1.1.** If $\Gamma$ is a countable group, then $\ell^1(\Gamma)$ with the product defined via convolution and involution defined by $f^*(\gamma) \coloneqq \overline{f(\gamma^{-1})}$ is an involutive Banach–algebra. Show that $\ell^1(\Gamma)$ is not a $C^*$-algebra unless $\Gamma = \{e\}$.

*Solution.* We need to show that

$$\|f * f^*\|_1 = \|f\|_1 \|f^*\|_1 \quad \text{for every} \quad f \in \ell^1(\Gamma) \tag{1}$$

if and only if $\Gamma = \{e\}$. We know that

$$f * f^*(\gamma) = \sum_{\eta \in \Gamma} f(\gamma\eta)\overline{f(\eta)}.$$

It is clear that if $\Gamma = \{e\}$, then (1) holds. Now, suppose that $\Gamma$ is nontrivial. Let $\gamma_0 \in \Gamma$ be a nontrivial element. Now consider the following two cases:
(i) Assume that $\gamma_0 = \gamma_0^{-1}$. Consider $f \coloneqq \delta_e + i\delta_{\gamma_0}$. We know that $\|f\|_1 = 2$ but

$$(f * f^*)(\gamma) = \begin{cases} |f(e)|^2 + |f(\gamma_0)|^2 = 2 & \text{if } \gamma = e \\ 2\,\mathrm{re}(f(\gamma_0)\overline{f(e)}) = 0 & \text{if } \gamma = \gamma_0 \\ 0 & \text{else} \end{cases}$$

so $\|f * f^*\|_1 = 2 < 4 = \|f\|_1 \|f^*\|_1$.
(ii) Assume that $\gamma_0 \neq \gamma_0^{-1}$. Then we consider $f \coloneqq \delta_e + i\delta_{\gamma_0} + i\delta_{\gamma_0^{-1}}$ and we get similarly that $\|f * f^*\|_1 = 3 < 9 = \|f\|_1 \|f^*\|_1$. ■

**Exercise 1.3.** Suppose $A$ is a unital $\mathbb{C}$–algebra with a Banach space norm $\|\cdot\|$ for which the multiplication is continuous in each factor separately. Then there exists an equivalent norm $\|\cdot\|'$ such that $(A, \|\cdot\|')$ is a unital Banach algebra verifying $\|e\|' = 1$ and $\|xy\|' \leq \|x\|'\|y\|'$ for all $x, y \in A$.

*Solution.* For $x \in A$ let $\|x\|' := \|L_x\|$ where $L_x$ denotes the left multiplication by $x$. It is clear that $\|e\|' = 1$ and $\|xy\|' \leq \|x\|'\|y\|'$. Now, $\|x\| = \|L_x e\| \leq \|L_x\| \cdot \|e\|$. Let $R_y$ denote the right multiplication by $y$. Note that $\|L_x\| = \sup_{\|y\|\leq 1} \|L_x y\| = \sup_{\|y\|=1} \|R_y x\|$ so $(R_y)_{\|y\|=1}$ is a family of pointwise bounded operators and hence uniformly bounded by the uniform boundedness principle so there is a constant $C > 0$ such that

$$\sup_{\|x\|=1} \|L_x\| = \sup_{\|y\|=1} \|R_y x\| = \sup_{\|y\|=1} \|R_y\| \leq C.$$

Thus $\|x\|' = \|L_x\| \leq C\|x\|$ for every $x \in A$. This shows that the two norms are equivalent. ∎

**Exercise 2.1.** If $X$ is a locally compact Hausdorff space, then for every $f \in C_0(X)$ the equality $\mathrm{Sp}_{C_0(X)}(f) = \overline{f(X)}$ holds.

*Solution.* Assume first that $X$ is compact. Then $C_0(X) = C(X)$ is unital. Let $f \in C_0(X)$. For $\lambda \in \mathbb{C}$, $f - \lambda$ is invertible if and only if there exists a $g \in C_0(X)$ such that $(f(x) - \lambda)g(x) = 1$ for every $x \in X$. As $\overline{f(X)} = f(X)$ since $X$ is compact, it is clear that $f - \lambda$ is invertible if and only if $\lambda \notin \overline{f(X)}$ so $\mathrm{Sp}_{C_0(X)}(f) = \overline{f(X)}$.

Now assume that $X$ is not compact. Then $C_0(X)$ is not unital so by definition

$$\mathrm{Sp}_{C_0(X)}(f) = \mathrm{Sp}_{C_0(X)_I}((f, 0)) = \{\lambda \in \mathbb{C} \mid (f, -\lambda) \text{ is not invertible } \}.$$

Let $\lambda \in \mathbb{C}$. By solving the equation $(f, -\lambda)(g, \mu) = (0, 1)$ we see that $(f, -\lambda)$ is invertible if and only if $\lambda \neq 0$ and $g(x) := \frac{1}{\lambda}\frac{f(x)}{f(x)-\lambda}$ is in $C_0(X)$. Now, assume that $\lambda \neq 0$ and $g \in C_0(X)$. Then as $f$ vanishes at infinity, there exists a compact $K \subseteq X$ such that $|f(x)| \leq |\lambda|/2$ for $x \notin K$. As $g$ is continuous and hence bounded on the compact set $K$, there is a constant $C > 0$ such that $C|f(x) - \lambda| \geq |f(x)|$ for all $x \in K$ so for $x \in K$, either $|f(x)| \leq |\lambda|/2$ and hence $|f(x) - \lambda| \geq |\lambda|/2 > 0$ or $C|f(x) - \lambda| > |\lambda|/2$. This shows that $\lambda \notin \overline{f(X)}$. Conversely, assume that $\lambda \notin \overline{f(X)}$. Then $\lambda \neq 0$ since $0 \in \overline{f(X)}$ (this follows from that $X$ is not compact and $f$ vanishes at infinity). Let $\varepsilon > 0$ be such that disc of radius $\varepsilon$ centered at $\lambda$ does not intersect $f(X)$. Then $|g(x)| \leq \frac{1}{|\varepsilon\lambda|}|f(x)|$ so $g$ vanishes at infinity as $f$ does. That $g$ is continuous is clear. ∎

**Exercise 2.2.**

(a) Show that for any $f, g \in L^1([0, 1])$,

$$f * g(t) := \int_0^t f(t - x)g(x)dx$$

exists for almost every $t \in [0, 1]$ and that $f * g \in L^1([0, 1])$.

(b) Show that $(f, g) \mapsto f * g$ gives a Banach algebra structure on $L^1([0, 1])$.

(c) Show that for every $f \in L^1([0, 1])$ it holds that $\mathrm{Sp}(f) = \{0\}$.

*Solution.* (a) We have that

$$\int_0^1 \left| \int_0^t f(t - x)g(x)dx \right| dt \leq \int_0^1 |g(x)| \int_0^t |f(t - x)|dxdt = \int_0^1 \int_x^1 |f(t - x)||g(x)|dtdx$$
$$\leq \|f\|_1 \int_0^1 |g(x)|dx = \|f\|_1\|g\|_1.$$

It follows that $f * g(t) := \int_0^t f(t-x)g(x)dx$ is finite for almost every $t$ and defines an element of $L^1([0,1])$.
(b) Trivial.
(c) Let $f(t) := 1$ for all $t \in [0,1]$. Now, $f * f(t) = t$, $f * f * f(t) = \frac{1}{2}t^2$ and generally

$$\underbrace{f * \cdots * f}_{n-\text{times}}(t) = \frac{1}{(n-1)!}t^{n-1}.$$

By Gelfand's formula, the spectral radius of $f$ is

$$\|f\|_{\mathrm{sp}} = \lim_{n\to\infty} \|\underbrace{f * \cdots * f}_{n-\text{times}}\|_1^{1/n} = \lim_{n\to\infty}\left(\frac{1}{(n-1)!}\int_0^1 t^{n-1}dt\right)^{1/n} = \lim_{n\to\infty}\frac{1}{(n!)^{1/n}} = 0$$

where the last step followed from the fact that

$$\ln\left((n!)^{1/n}\right) = \frac{1}{n}\sum_{k=1}^{n}\ln k \geq \frac{1}{n}\int_1^n \ln x dx = \ln n - \frac{n-1}{n} \longrightarrow \infty$$

as $n \to \infty$. Thus, as the spectrum is nonempty, $\mathrm{Sp}_{L^1([0,1])}(f) = \{0\}$. Every polynomial on $[0,1]$ can be written as $p(f)$ where $p$ is a polynomial which takes 0 to 0. Hence, by the spectral mapping theorem, the spectrum of any polynomial is $p(\mathrm{Sp}_{L^1([0,1])}(f)) = p(\{0\}) = \{0\}$ so its spectral radius is zero. Now, the polynomials are dense in $L^1([0,1])$ as they are dense in the supremum norm in $C([0,1])$ by Weierstrass, the supremum norm is finer than the $L^1$-norm on $L^1([0,1])$ and $C([0,1])$ is dense in $L^1([0,1])$. It follows that spectral radius vanishes identically as it vanishes on a dense set. ■

**Exercise 3.1.** Verify that $C^n([0,1])$ with pointwise multiplication and norm

$$\|f\| := \sum_{k=0}^{n}\frac{1}{k!}\|f^{(k)}\|_\infty$$

is a commutative unital Banach algebra for every integer $n$.

*Solution.* We begin by verifying that $C^n([0,1])$ is complete. So let $(f_j)$ be a Cauchy sequence in $C^n([0,1])$. Then for each $k = 0, 1, \ldots, n$, the sequence $(f_j^{(k)})$ of continuous functions is Cauchy in the supremum norm and hence converges uniformly to a continuous function which we denote by $g_k$. We show that $g_0 \in C^n([0,1])$ by showing that $g_{k+1} = g_k'$ for $k = 0, 1 \ldots, n-1$ and then we are done. Now, for $k = 0, 1, \ldots, n-1$ and $x \in [0,1]$ we have that

$$g_k(x) = \lim_{j\to\infty} f_j^{(k)}(x) = \lim_{j\to\infty}\left(f_j^{(k)}(0) + \int_0^x f_j^{(k+1)}(t)dt\right) = g_k(0) + \int_0^x g_{k+1}(t)dt$$

where the interchange of limit and integral is justified by the fact that we have uniform convergence on a compact set. Now the fundamental theorem of calculus gives that $g_k' = g_{k+1}$.

Let us also verify that $\|fg\| \leq \|f\|\|g\|$. So let $f, g \in C^n([0,1])$. We apply the Leibniz rule

$$(fg)^{(k)} = \sum_{j=0}^{k}\binom{k}{j} f^{(j)}g^{(k-j)}$$

to get

$$\begin{aligned}\|fg\| &= \sum_{k=0}^{n}\frac{1}{k!}\|(fg)^{(k)}\|_\infty \le \sum_{k=0}^{n}\frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}\|f^{(j)}\|_\infty\|g^{(k-j)}\|_\infty \\ &= \sum_{j=0}^{n}\sum_{k=j}^{n}\frac{1}{j!(k-j)!}\|f^{(j)}\|_\infty\|g^{(k-j)}\|_\infty = \sum_{j=0}^{n}\frac{1}{j!}\|f^{(j)}\|_\infty\sum_{k=0}^{n-j}\frac{1}{k!}\|g^{(k)}\|_\infty \\ &\le \sum_{j=0}^{n}\frac{1}{j!}\|f^{(j)}\|_\infty\|g\| = \|f\|\|g\|.\end{aligned}$$

■

**Exercise 3.2.** Show that $C^\infty([0,1])$ does not admit any Banach algebra norm.

*Solution.* We assume that there exists a Banach algebra norm $\|\cdot\|$ on $C^\infty([0,1])$ and derive a contraction. First, we show that the Banach algebra $C([0,1])$ is a semisimple. That follows from the fact that each maximal ideal has the form $\{f \in C([0,1]) \mid f(x) = 0\}$ for some fixed $x \in [0,1]$, and hence their intersection is trivial. Now consider the inclusion $C^\infty([0,1]) \to C([0,1])$. It is a Banach algebra homomorphism and hence continuous as $C([0,1])$ is semisimple. Thus there exists a constant $c > 0$ such that $\|f\|_\infty \le c\|f\|$. Now, consider the differentiation operator $D : C^\infty([0,1]) \to C^\infty([0,1])$, $f \mapsto f'$. Let $(f_n, f_n') \to (f, g)$. Then $\|f_n - f\|_\infty \le c\|f_n - f\| \to 0$ and $\|f_n' - g\|_\infty \le c\|f_n' - g\| \to 0$. Thus

$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty}\left(f_n(0) + \int_0^x f_n'(t)dt\right) = f_n(0) + \int_0^x g(t)dt$$

for every $x \in [0,1]$ so $g = f'$. It follows from the closed graph theorem that $D$ is continuous so there exists a constant $C > 0$ such that $\|f'\| \le C\|f\|$ for every $f \in C^\infty([0,1])$. But for every $\alpha > 0$, $\alpha\|e^{\alpha x}\| = \|\frac{d}{dx}e^{\alpha x}\| \le C\|e^{\alpha x}\|$ so $\alpha \le C$. But $\alpha$ is arbitrary so this can not hold. ■

**Exercise 3.4.** Let $X$ be a locally compact Hausdorff space. Show that $X \to \widehat{C_0(X)}$ is a homeomorphism.

*Solution.* Let us denote the bijection by $\varphi$. Recall that for $x \in X$, $\varphi(x) : C_0(X) \to \mathbb{C}$, $\varphi(x)(f) = f(x)$. As the Gelfand topology is the coarsest topology which makes all the maps $\hat{f} : \widehat{C_0(X)} \to \mathbb{C}$, $\chi \mapsto \chi(f)$, where $f \in C_0(X)$, it suffices to verify that $x \mapsto \hat{f} \circ \varphi(x) = \varphi(x)(f) = f(x)$ is continuous for every $f$. But that simply follows from the fact that all the functions $f \in C_0(X)$ are continuous.

Let us now show that $\varphi$ is open. So let $U \subseteq X$ be open and let $x_0 \in U$. Let $f \in C_0(X)$ be a function such that $0 \le f \le 1$, $f(x_0) = 1$ and $f = 0$ on $X \setminus U$. Consider $U_0\{\varphi(x) \mid |\varphi(x)(f) - \varphi(x_0)(f)| < 1\}$. This is an open set which contains $\varphi(x_0)$. Further, if $\varphi(x) \in U_0$ then $|f(x) - 1| < 1$ so $x \in U$ as $f(x) = 0$ on $X \setminus U$. This shows that $U$ contains an open neighborhood of $x_0$ and as $x_0$ was arbitrary we conclude that $U$ is open. ■

**Exercise 4.3.** Find an example of a Banach algebra such that its Gelfand transform is not surjective.

*Solution.* Consider the Banach algebra $C^1([0,1])$. For every $x \in X$ we get a character $\varphi(x)$ given by $\varphi(x)(f) = f(x)$. It is straightforward to verify that this map $\varphi : [0,1] \to \widehat{C^1([0,1])}$ is an embedding. Every character on $C^1([0,1])$ is continuous when $C^1([0,1])$ is endowed with the supremum norm, since that is coarser than the norm on $C^1([0,1])$. But $C^1([0,1])$ is dense in $C([0,1])$ so every character of $C^1([0,1])$ extends to a character of $C([0,1])$. We know that all the characters on $C([0,1])$ have the form $\varphi(x)$ so we conclude that $\varphi$ is actually a homeomorphism. Now, let $f$ be a continuous function on $[0,1]$ which is not differentiable. That function induces a continuous function $C(\widehat{C^1([0,1])})$ given by $f \circ \varphi^{-1}$. This function is not the Gelfand transform of any function in $C^1([0,1])$. ■

**Exercise 5.1.** A square matrix $A$ with coefficients in $\mathbb{C}$ represents a normal operator if and only if there exists a unitary matrix $u$ such that $uAu^{-1}$ is diagonal.

*Solution.* Let $e$ be an eigenvector of $A$ with eigenvalue $\lambda$ and let $v$ be orthogonal to $e$. Recall that $e$ is a eigenvector of $A^*$ with eigenvalue $\overline{\lambda}$ so $\langle e, Av\rangle = \langle A^*e, v\rangle \overline{\lambda}\langle e, v\rangle = 0$ and similarly, $\langle e, A^*v\rangle = 0$. This shows that $A$ restricts to a normal operator on the orthogonal complement of $e$. Hence we can use induction on the dimension to find an orthonormal basis of eigenvectors. ■

**Exercise 6.2.** Let $X$ be a compact Hausdorff space. Show that for $f, g \in \mathcal{B}^\infty(X)$ it holds that

(a) $\|f\|_\infty \leq \|f\|$.

(b) $\|f+g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$

(c) $\|f \cdot g\|_\infty \leq \|f\|_\infty \cdot \|g\|_\infty$.

*Solution.* (a) We show that $\mathrm{EssIm}(f) \subseteq \overline{f(X)}$ for every $f$ and then the claim follows. We have that $f^{-1}(\mathbb{C} \setminus \overline{f(X)}) = \emptyset$ so $E(f^{-1}(V)) = 0$. As $\mathbb{C} \setminus \mathrm{EssIm}(f)$ is the largest open set such that $E(f^{-1}(\mathbb{C} \setminus \mathrm{EssIm}(f))) = 0$ we conclude that $\mathbb{C} \setminus \overline{f(X)} \subseteq \mathbb{C} \setminus \mathrm{EssIm}(f)$ and thus $\mathrm{EssIm}(f) \subseteq \overline{f(X)}$.

(b) Let $\varepsilon > 0$ and $\lambda \in \mathrm{EssIm}(f+g)$. We know that $E((f+g)^{-1}(B(\lambda,\varepsilon))) \neq 0$. Now,

$$(f+g)^{-1}(B(\lambda,\varepsilon) \subseteq \bigcup_{\lambda' \in \mathbb{Q}+i\mathbb{Q}} \left(f^{-1}(B(\lambda',\varepsilon) \cap g^{-1}(B(\lambda-\lambda',\varepsilon))\right).$$

As $E$ of the left hand side is non-zero and there are only countably many constituents of the right hand side, we conclude that there exists $\lambda' \in \mathbb{Q}+i\mathbb{Q}$ such that

$$E\left(f^{-1}(B(\lambda',\varepsilon) \cap g^{-1}(B(\lambda-\lambda',\varepsilon))\right) \neq 0.$$

But then

$$|\lambda| \leq |\lambda'| + |\lambda-\lambda'| \leq \|f\|_\infty + \|g\|_\infty + 2\varepsilon$$

so $\|f+g\|_\infty \leq \|f\|_\infty + \|g\|_\infty + 2\varepsilon$. As $\varepsilon > 0$ is arbitrary, we conclude that $\|f+g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$.

(c) Let $C > 0$ be such that $\|f\|, \|g\| \leq C$. Let $0 \neq \lambda \in \mathrm{EssIm}(fg)$ and let $0 < \varepsilon \leq |\lambda|/2$. It is routine to verify that

$$(f \cdot g)^{-1}(B(\lambda,\varepsilon)) \subseteq \bigcup_{\substack{\lambda' \in \mathbb{Q}+i\mathbb{Q} \\ |\lambda'| \geq |\lambda|/2C}} \left(f^{-1}(B(\lambda',\varepsilon) \cap g^{-1}(B(\lambda/\lambda', 3\varepsilon C/|\lambda|))\right).$$

Now, a similar argument as in (b) gives the result. ■

**Exercise 6.3.** For every $f \in \mathcal{B}^\infty(X)$ and every $g \in N$, show that $\|f+g\|_\infty = \|f\|_\infty$.

*Solution.* First, $\|f+g\|_\infty \le \|f\|_\infty + \|g\|_\infty = \|f\|_\infty$. Then we note that

$$\|f\|_\infty = \|f+g-g\|_\infty \le \|f+g\|_\infty + \|g\|_\infty = \|f+g\|_\infty$$

so $\|f+g\|_\infty = \|f\|_\infty$. ■

**Exercise 6.4.** Show that the quotient norm on $\mathcal{B}^\infty(X)/N$ is given by $\|[f]\| = \|f\|_\infty$.

*Solution.* First,

$$\|[f]\| = \inf_{g\in N} \|f+g\| \ge \inf_{g\in N} \|f+g\|_\infty = \inf_{g\in N} \|f\|_\infty = \|f\|_\infty.$$

Let us now show that $\|[f]\| \le \|f\|_\infty$. Take $r > \|f\|_\infty$. Then $E(f^{-1}(\mathbb{C} \setminus B(0,r))) = 0$. Let $g := f \cdot \chi_{f^{-1}(\mathbb{C}\setminus B(0,r))}$. Then $\|g\|_\infty = 0$ so $g \in N$. Further $\|f-g\| \le r$ so $\|[f]\| \le r$. Since $r > \|f\|_\infty$ was arbitrary, we conclude that $\|[f]\| \le \|f\|_\infty$. ■

**Exercise 6.5.** Show that $L^\infty(E) := \mathcal{B}^\infty(X)/N$ endowed with the quotient norm is a $C^*$-algebra and for every $[f] \in L^\infty(E)$, the spectrum of $[f]$ coincides with the essential image $\operatorname{EssIm}(f)$ of any representation of $[f]$.

*Solution.* We verify the latter part. Let $\lambda \notin \mathrm{Sp}_{L^\infty(E)}([f])$. Then there exists $g \in \mathcal{B}^\infty(X)$ such that $(f-\lambda)g = 1 + n$ where $n \in N$ i.e. $E(\{(f-\lambda)g \neq 1\}) = 0$ and therefore $E(f^{-1}(B(\lambda, \frac{1}{\|g\|_\infty}))) = 0$. This shows that $\lambda \notin \operatorname{EssIm}(f)$. Conversely, assume that $\lambda \notin \operatorname{EssIm}(f)$. Then there exists $\varepsilon > 0$ such that $E(f^{-1}(B(\lambda,\varepsilon))) = 0$. Let $g := \frac{1}{f-\lambda}\chi_{f^{-1}(\mathbb{C}\setminus B(\lambda,\varepsilon))}$. Then $(f-\lambda)g = 1+n$ where $n \in N$ so $\lambda \notin \mathrm{Sp}_{L^\infty(E)}([f])$. ■

**Exercise 7.1.** In the context of Theorem 4.10, show that for $f \in C(\mathrm{Sp}_A(x))$ we have $\mathrm{Sp}_A(f(x)) = f(\mathrm{Sp}_A(x))$.

*Solution.* Here $A$ is a unital $C^*$-algebra and $x \in A$ is a normal element. Then

$$B := \overline{\{P(x,x^*) \mid P \in \mathbb{C}[X,Y]\}}$$

is a unital abelian $C^*$-subalgebra and $\mathrm{Sp}_A(b) = \mathrm{Sp}_B(b)$ for every $b \in B$ by Proposition 4.9. Now, theorem 4.10 gives that $C(\mathrm{Sp}_A(x)) \to B$, $f \mapsto f(x)$ is a $C^*$-algebra isomorphism sending $\mathbf{1}$ to $e$ and id to $x$. Hence we get:

$$\mathrm{Sp}_A(f(x)) = \mathrm{Sp}_B(f(x)) = \mathrm{Sp}_{C(\mathrm{Sp}_A(x))}(f) = f(\mathrm{Sp}_A(x))$$

where in the last step we used that if $X$ is a compact Hausdorff space and $f \in C(X)$ then $\mathrm{Sp}_{C(X)}(f) = f(X)$. ■

**Exercise 7.2.** Let $T \in \mathcal{L}(\mathcal{H})$ be a normal operator and $E$ the associated resolution of the identity. For $f \in C(\mathrm{Sp}(T))$ define a resolution of the identity on $f(\mathrm{Sp}(T)) = \mathrm{Sp}(f(T))$ by $f_\# E(\omega) := E(f^{-1}(\omega))$ for every Borel set $\omega \subseteq \mathrm{Sp}(f(T))$. Show that $f_\# E$ is the resolution of the idenity on $\mathrm{Sp}(f(T))$ associated with $f(T)$.

*Solution.* Let $f \in C(\mathrm{Sp}(T))$ and $x \in \mathcal{H}$. Using that $f_\# E_{x,x} = (f_\# E)_{x,x}$, Exercise 1 and the fundamental property of push-forward measures, we get

$$\langle f(T)x, x\rangle = \int_{\mathrm{Sp}(T)} f(\lambda) dE_{x,x}(\lambda) = \int_{f(\mathrm{Sp}(T))} \lambda d f_\#(E_{x,x})(\lambda) = \int_{\mathrm{Sp}(f(T))} \lambda d(f_\# E)_{x,x}(\lambda).$$

Now it follows from the uniqueness part of Corollary 5.20 that $f_\# E$ is the resolution of the identity associated with $f(T)$. ■

**Exercise 7.4.** Let $T \in \mathcal{L}(\mathcal{H})$ be a normal operator, $E$ its associated resolution of the identity and $f \in C(\mathrm{Sp}(T))$. If $0 \in \mathrm{Sp}(f(T))$ then $\operatorname{Ker} f(T) = \operatorname{Im} E(f^{-1}(0))$.

*Solution.* For $x \in \mathcal{H}$ we have

$$\|f(T)x\|^2 = \langle f(T)^* f(T)x, x\rangle = \int_{\mathrm{Sp}(T)} |f|^2(\lambda) dE_{x,x}$$

so $f(T)x = 0$ if and only if $E_{x,x}(\mathrm{Sp}(T) \setminus f^{-1}(0)) = 0$ which is if and only if $E_{x,x}(f^{-1}(0)) = \|x\|^2$ as $E_{x,x}(\mathrm{Sp}(T)) = \|x\|^2$. Now, $E_{x,x}(f^{-1}(0)) = \|x\|^2$ if and only if $E(f^{-1}(0))x = x$ as $E_{x,x}(f^{-1}(0)) = \langle E(f^{-1}(0))x, x\rangle$ and $E(f^{-1}(0))$ is a self adjoint projection. But a projection fixes a vector if and only if it is in its image so we conclude that $f(T)x = 0$ if and only if $x$ is in the image of $E(f^{-1}(0))$. ■

**Exercise 7.5.** In the same context as in Exercise 4, show that if $\lambda_0 \in \mathrm{Sp}(T)$, then $\operatorname{Ker}(T - \lambda_0 \,\mathrm{id}) = \operatorname{Im} E(\{\lambda_0\})$. Conclude that $\lambda_0$ is an eigenvalue of $T$ if and only if $E(\{\lambda_0\}) \neq 0$. Show that if $\lambda_0 \in \mathrm{Sp}(T)$ is an isolated point of $\mathrm{Sp}(T)$ then $\lambda_0$ is an eigenvalue.

*Solution.* The first part follows immediately from Exercise 4 with $f(\lambda) = \lambda - \lambda_0$. By definition $\lambda_0$ is an eigenvalue of $T$ if and only if $\operatorname{Ker}(T - \lambda_0 \,\mathrm{id}) \neq 0$ so $\lambda_0$ is an eigenvalue if and only if $\operatorname{Im} E(\{\lambda_0\}) \neq 0$ i.e. $E(\{\lambda_0\}) \neq 0$. Now, if $\lambda_0 \in \mathrm{Sp}(T)$ is an isolated point then $\{\lambda_0\}$ is open in $\mathrm{Sp}(T)$ so by the spectral theorem, $E(\{\lambda_0\}) \neq 0$ and hence is $\lambda_0$ an eigenvalue of $T$. ■

**Exercise 7.6.** Let $T \in \mathcal{L}(\mathcal{H})$ be a normal operator. Show that $\|T\| = \sup\{|\langle Tx, x\rangle| \mid \|x\| \leq 1\}$.

*Solution.* It follows from Cauchy-Schwarz that $\sup\{|\langle Tx, x\rangle| \mid \|x\| \leq 1\} \leq \|T\|$. Now, since $T$ is normal, $\|T\| = \|T\|_{\mathrm{sp}}$ and hence as $\mathrm{Sp}(T)$ is compact, there exists $\lambda_0 \in \mathrm{Sp}(T)$ such that $|\lambda_0| = \|T\|$. Let $0 < \varepsilon < |\lambda_0|$ (the result is clear if $\lambda_0 = 0$. Then $E(B(\lambda_0, \varepsilon)) \neq 0$ so there exists a unit vector $x \in \mathcal{H}$ such that $E(B(\lambda_0, \varepsilon))x = x$. Then $E_{x,x}$ is concentrated on $B(\lambda_0, \varepsilon)$ so we get

$$|\langle Tx, x\rangle| = \left| \int_{\mathrm{Sp}(T)} \lambda dE_{x,x}(\lambda) \right| = \left| \int_{\mathrm{Sp}(T) \cap B(\lambda_0, \varepsilon)} \lambda dE_{x,x}(\lambda) \right| \geq |\lambda_0| - \varepsilon.$$

Now we conclude that $\|T\| \leq \sup\{|\langle Tx, x\rangle| \mid \|x\| \leq 1\}$. ■

**Exercise 7.7.** Let $V \in \mathcal{L}(\mathcal{H})$ be a unitary operator and $P$ the orthogonal projection of $\mathcal{H}$ onto the eigenspace of $V$ associated with the eigenvalue 1. Use the Spectral theorem to show that for every $x \in \mathcal{H}$ it holds that

$$\lim_{n \to \infty} \frac{1}{n}(x + Vx + \cdots + V^{n-1}x) = P(x).$$

*Solution.* We begin by noting that if $x$ belongs to the 1-eigenspace of $V$ then the result is clear. Hence it suffices to show that if $x$ is orthogonal to the 1-eigenspace then $\lim_{n\to\infty}\frac{1}{n}\left(x+Vx+\cdots+V^{n-1}x\right)=0$. So for $n\geq 1$ let $f_n:\mathrm{Sp}(V)\to\mathbb{C}$, $f_n(\lambda):=\frac{1}{n}\sum_{k=0}^{n-1}\lambda^k$. Since $V$ is unitary, $|\lambda|=1$ for every $\lambda\in\mathrm{Sp}(V)$. Hence we note that $f_n\to\chi_{\{1\}}$ pointwise as $n\to\infty$ since $f_n(1)=1$ for every $n$ and

$$|f_n(\lambda)|=\frac{1}{n}\left|\frac{1-\lambda^n}{1-\lambda}\right|\leq\frac{2}{n}\frac{1}{|1-\lambda|}\longrightarrow 0$$

as $n\to\infty$. Further, $|f_n(\lambda)|\leq 1$ for every $n$ by the triangle inequality. Now we can apply the dominated convergence theorem to get

$$\|f_n(V)x\|^2=\int_{\mathrm{Sp}(V)}|f_n(\lambda)|^2dE_{x,x}(\lambda)\longrightarrow\int_{\mathrm{Sp}(V)}\chi_{\{1\}}(\lambda)dE_{x,x}(\lambda)=\langle E(\{1\})x,x\rangle=0$$

if $x$ is orthogonal to the 1-eigenspace of $V$ because by Exercise 5, $E(\{1\})$ is the projection onto the 1-eigenspace of $V$. ■

In the next two exercises $(X,\mu)$ will be a probability space and $\psi:X\to X$ a measure preserving transformation i.e. $\psi_{\#}\mu=\mu$. Note that if $f\in L^2(X,\mu)$ then $f\in L^1(X,\mu)$ and $\|f\|_1\leq\|f\|_2$ by Cauchy-Schwarz. Hence if $f_n\to f$ in $L^2$ then $f_n\to f$ in $L^1$.

**Exercise 7.8.** Show that for every $f\in L^2(X,\mu)$ it holds that $\lim_{n\to\infty}\frac{1}{n}(f+f\circ\psi+\cdots+f\circ\psi^{n-1})=g$ in $L^2$ where $g\in L^2(X,\mu)$ is a function satisfying $g\circ\psi=g$. Moreover, show that the only $\psi$-invariant $L^2$-functions are constants if and only if for every $f\in L^2$ the above limit equals $\int_X fd\mu$.

*Solution.* Let $V:L^2(X,\mu)\to L^2(X,\mu)$, $Vf=f\circ\psi$. Then $V$ is unitary since

$$\langle Vf,Vg\rangle=\int_X f\circ\psi\cdot\overline{g\circ\psi}d\mu=\int_X f\cdot\overline{g}d\mu=\langle f,g\rangle$$

where the second step followed from the fact that $\psi$ is measure preserving. Now, it follows from Exercise 7 that $\lim_{n\to\infty}\frac{1}{n}\left(f+f\circ\psi+\cdots+f\circ\psi^{n-1}\right)=P(f)=:g$ in $L^2$ where $P$ is the projection onto the 1-eigenspace of $V$. But then $Vg=g$ i.e. $g\circ\psi=g$. Now,

$$\begin{aligned}\int_X gd\mu&=\int_X\lim_{n\to\infty}\frac{1}{n}\left(f+f\circ\psi+\cdots+f\circ\psi^{n-1}\right)d\mu\\&=\lim_{n\to\infty}\int_X\frac{1}{n}\left(f+f\circ\psi+\cdots+f\circ\psi^{n-1}\right)d\mu\\&=\int_X fd\mu\end{aligned}$$

where interchange of integral and limit is justified by the fact that we have convergence in $L^2$ and hence also in $L^1$ since $(X,\mu)$ is a probability space and the last step follows from the fact that $\psi$ is measure preserving. Now, if the only $\psi$-invariant $L^2$-functions are constants then $g$ is a constant which is equal to $\int_X gd\mu=\int_X f\mu$ since $\mu$ is a probability measure. Conversely, assume that it holds for every $f\in L^2(X,\mu)$ that

$$\lim_{n\to\infty}\frac{1}{n}\left(f+f\circ\psi+\cdots+f\circ\psi^{n-1}\right)=\int_X fd\mu$$

in $L^2$. Then $P(f)=\int_X fd\mu$ for every $f\in L^2(X,\mu)$. Let $f$ be a $\psi$-invariant function. Then $f=P(f)=\int_X fd\mu$ so $f$ is constant. ■

**Exercise 7.9.** Show that the transformation $\psi : X \to X$ is ergodic if (i.e. the only $\psi$-invariant functions are constants) if and only if for every $f, g \in L^2(X, \mu)$ it holds that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \langle f \circ \psi^k, g \rangle = \langle f, \mathbf{1} \rangle \langle \mathbf{1}, g \rangle. \tag{2}$$

*Solution.* Just note that the right hand side is equal to $\langle \int_X f d\mu, g \rangle$ and then use the previous exercise and the fact that a vector in a Hilbert space is determined by its inner products with the other vectors of the space. ■

**Exercise 7.10.** Let $\mathcal{H}$ be a separable Hilbert space and $A \subseteq \mathcal{L}(\mathcal{H})$ an abelian $C^*$-subalgebra. Assume that there exists $v \in \mathcal{H}$ with $\overline{Av} = \mathcal{H}$. Let $E$ be the resolution of the identity associated with $A$. Show the following:

(a) For any Borel set $\omega \subseteq \hat{A}$ we have $E(\omega) = 0$ if and only if $E_{v,v}(\omega) = 0$. Conclude that $L^\infty(E) = L^\infty(\hat{A}, E_{v,v})$.

(b) Use the isomorphism $\Lambda : \mathcal{H} \to L^2(\hat{A}, E_{v,v})$ to show that the subalgebra

$$Z_{\mathcal{L}(\mathcal{H})}(A) := \{T \in \mathcal{L}(\mathcal{H}) \mid Ta = aT \text{ for every } a \in A\}$$

is isomorphic to $L^\infty(\hat{A}, E_{v,v})$.

(c) Conclude that the image of the $C^*$-subalgebra homomorphism $\Phi : L^\infty(E) \to \mathcal{L}(\mathcal{H})$ is $Z_{\mathcal{L}(\mathcal{H})}(A)$ which hence is abelian.

(d) Give an example of $A \subseteq \mathcal{L}(\mathcal{H})$ for which $Z_{\mathcal{L}(\mathcal{H})}(A)$ is not abelian.

*Solution.* (a) It is clear that if $E(\omega) = 0$ then $E_{v,v}(\omega) = 0$ so assume that $E_{v,v}(\omega) = 0$. Then

$$0 = E_{v,v}(\omega) = \langle E(\omega)v, v \rangle = \langle E(\omega)v, E(\omega)v \rangle = \int_{\hat{A}} |\widehat{E(\omega)}|^2(\chi) dE_{v,v}(\chi)$$

so we conclude that $\widehat{E(\omega)}(\chi) = 0$ for $E_{v,v}$-almost every $\chi$. Hence, we get

$$\|E(\omega)av\|^2 = \int_{\hat{A}} |\hat{a}|^2 |\widehat{E(\omega)}|^2 dE_{v,v} = 0$$

for every $a \in A$. It follows from density of $Av$ that $E(\omega) = 0$. Now let $f \in \mathcal{B}^\infty(X)$ and assume that $f(\chi) = 0$ for $E_{v,v}$-almost every $\chi$ i.e.$E_{v,v}(f^{-1}(\mathbb{C} \smallsetminus \{0\})) = 0$. Then $E(f^{-1}(\mathbb{C} \smallsetminus \{0\})) = 0$ so $\mathrm{EssIm}(f) = \{0\}$ and hence $\|f\|_\infty = 0$. This shows that $L^\infty(E) = L^\infty(\hat{A}, E_{v,v})$.

(b) Let $T : L^2(\hat{A}, E_{v,v}) \to L^2(\hat{A}, E_{v,v})$ be a bounded linear operator such that $TM_f g = M_f T g$ for every $f \in C(\hat{A})$ and every $g \in L^2(\hat{A}, E_{v,v})$. Then we have

$$(Tf)(\chi) = (TM_f)(\mathbf{1})(\chi) = (M_f T)(\mathbf{1})(\chi) = f(\chi) T(\mathbf{1})(\chi)$$

which shows that $T$ is given by multiplication with $T(\mathbf{1})$ as the continuous functions are dense in $L^2(\hat{A}, E_{v,v})$. Let us show that $T(\mathbf{1}) \in L^\infty(\hat{A}, E_{v,v})$. For a contradiction, let us assume that for every $n \geq 1$, $E_{v,v}(\{|f| > n\}) > 0$. Let

$$f_n := \frac{1}{E_{v,v}(\{|f| > n\})^{1/2}} \chi_{\{|f|>n\}}.$$

Then $\|f_n\|_2 = 1$ but $\|Tf_n\|_2 > n$ which contradicts the boundedness of $T$. Hence $T(\mathbf{1}) \in L^\infty(\hat{A}, E_{v,v})$. Now

$$Z_{\mathcal{L}(H)}(A) \longrightarrow L^\infty(\hat{A}, E_{v,v}), \qquad T \longmapsto (T \circ \Lambda^{-1})(\mathbf{1})$$

is the desired isomorphism.

(c) It suffices to show that $\Lambda \circ \Phi(f) = M_f \circ \Lambda$ for every $f \in L^\infty(E) = L^\infty(\hat{A}, E_{v,v})$. So let $a \in A$ and $f \in L^\infty$. Then

$$\begin{aligned}\langle \Phi(f)av, av\rangle_{\mathcal{H}} &= \int_{\hat{A}} |\hat{a}|^2 f dE_{v,v} = \langle M_f \hat{a}, \hat{a}\rangle_{L^2(\hat{A}, E_{v,v})} = \langle (M_f \circ \Lambda)av, \Lambda(av)\rangle_{L^2(\hat{A}, E_{v,v})} \\ &= \langle (\Lambda^{-1} \circ M_f \circ \Lambda)av, av\rangle_{\mathcal{H}}.\end{aligned}$$

From the polarization identity and density of $Av$ it follows that $\Phi(f) = \Lambda^{-1} \circ M_f \circ \Lambda$ i.e. $\Lambda \circ \Phi(f) = M_f \circ \Lambda$.

(d) One can take as $\mathcal{H}$ a Hilbert space which is neither trivial nor one dimensional, and as $A$ the $C^*$-subalgebra of $\mathcal{L}(\mathcal{H})$ generated by the identity. Then $Z_{\mathcal{L}(\mathcal{H})}(A) = \mathcal{L}(\mathcal{H})$ which is not abelian. Note that $\overline{Av} = \mathbb{C}v \neq \mathcal{H}$ for any $v \in \mathcal{H}$. ■

**Exercise 8.1.** Let $X$ be a compact Hausdorff space and $\mu$ a positive regular measure on $X$. For every $f \in C(X)$ show that the multiplication operator defined on $L^2(X, \mu)$ by $M_f g(x) := f(x)g(x)$, $g \in L^2(X, \mu)$, has norm $\|M_f\| = \|f\|_\infty$. Define also

$$A := \{M_f \mid f \in C(X)\} \subseteq \mathcal{L}(L^2(X, \mu)).$$

(a) Show that $A$ is a commutative $C^*$-subalgebra of $\mathcal{L}(L^2(X, \mu))$.

(b) Taking into account that $\widehat{C(X)}$ can by identified with $X$, determine $\hat{A}$.

(c) Determine the resolution of identity on $\hat{A}$ given by the spectral theorem.

*Solution.* We get directly that $\|M_f g\|_2 \leq \|f\|_\infty \|g\|_2$ so $\|M_f\| \leq \|f\|_\infty$. Let $r < \|f\|_\infty$. Then $\mu(\{|f| \geq r\}) > 0$ so we can define

$$g := \frac{1}{\mu(\{|f| \geq r\})^{1/2}} \chi_{\{|f| \geq r\}}.$$

Then $\|g\|_2 = 1$ and $\|M_f g\|_2 \geq r$ so we see that $\|M_f\| \geq r$. As $r < \|f\|_\infty$ was arbitrary, we conclude that $\|M_f\| \geq \|f\|_\infty$ and hence $\|M_f\| = \|f\|_\infty$.

(a) This follows directly from the fact that $A$ is the image of $C^*$-algebra under an isometric $C^*$-algebra embedding.

(b) The isomorphism between $C(X)$ and $A$ induces an homeomorphism between $\widehat{C(X)}$ and $\hat{A}$ given by

$$\widehat{C(X)} \longrightarrow \hat{A}, \qquad \chi \longmapsto (M_f \mapsto \chi(f)).$$

Since $\widehat{C(X)}$ is homeomorphic to $X$ it follows that $\hat{A}$ is also homeomorphic to $X$.

(c) Now,

$$\langle M_f g, g\rangle = \int_X f|g|^2 d\mu = \int_X f dE_{g,g}$$

where $dE_{g,g}(x) = |g|^2(x)d\mu(x)$. For a Borel set $\omega \subseteq X$ we have

$$\langle E(\omega)g, g\rangle = E_{g,g}(\omega) = \int_\omega |g|^2 d\mu = \int_X \chi_\omega g \cdot \overline{g} d\mu$$

so $E_\omega(g) = \chi_\omega g$. Now one uses the homeomorphism $X \to \hat{A}$, $x \mapsto (M_f \mapsto f(x))$ to move this resolution of the identity on $X$ to a the desired resolution of the identity on $\hat{A}$. ■

**Exercise 10.2.** Let $G$ be a locally compact Hausdorff abelian group, $f \in L^1(G)$ and $\varepsilon > 0$. Show that there exists an open set $V$ containing the identity such that if $u : G \to [0, \infty)$ is a Borel function which vanishes outside $V$ and satisfies $\int_G u(x)d\mu(x) = 1$ then

$$\|f - f * u\|_1 < \varepsilon.$$

*Solution.* By uniform continuity of the left translation, there exists an open set $V$ containing $e$ such that whenever $g, h \in G$ are such that $gh^{-1} \in V$ then $\int_G |f(xg^{-1}) - f(xh^{-1})|d\mu(x) < \varepsilon$. By possibly replacing $V$ by $V \cap V^{-1}$ we may assume that $V^{-1} = V$. Now, let $u : G \to [0, \infty)$ be a Borel function with support in $V$ and $\int_G u(x)d\mu(x) = 1$. Then we get

$$\begin{aligned}
\|f - f * u\|_1 &= \int_G \left| f(x) - \int_G f(xy^{-1})u(y)d\mu(y) \right| d\mu(x) \\
&= \int_G \left| \int_G (f(x) - f(xy^{-1})u(y)d\mu(y) \right| d\mu(x) \\
&\le \int_G u(y) \int_G |f(x) - f(xy^{-1})|d\mu(x)d\mu(y) \\
&= \int_V u(y) \int_G |f(x) - f(xy^{-1})|d\mu(x)d\mu(y) \\
&< \int_V u(y)\varepsilon d\mu(y) \\
&= \varepsilon
\end{aligned}$$

and we are done. ■

**Remark.** Here we summarize some facts about the $p$-adic numbers. Let $p$ be a prime number. One can look at the $p$-adic numbers $\mathbb{Z}_p$ as the ring of formal power series $x = \sum_{n=0}^\infty a_n p^n$ where $a_n \in \mathbb{Z}/p\mathbb{Z}$ for every $n$. Then $\mathbb{Q}_p$, the associated field of fractions, is the field of formal series of the form $x = \sum_{n=m}^\infty a_n p^n$ where $m \in \mathbb{Z}$. The valuation $\nu(x)$ is the least integer $m$ such that $a_m \neq 0$. We have an injective homomorphism (not continuous!) of $\mathbb{Q}$ into $\mathbb{Q}_p$ which can be described as follows: Let $r = p^k \frac{a}{b}$ be a rational number, $a$, $b$ coprime integers which are not divisible by $p$. Then $r$ is mapped to $\sum_{n=k}^\infty a_n p^n$ where the coefficients are computed inductively as follows: Let $a_n = 0$ for every $n < k$ and $r_k \coloneqq r$. Having defined $a_{k+l-1}$ and $r_{k+l}$, let $a_{k+l}$ be the unique number in $\{0, 1, \ldots, p-1\}$ such that $r_{k+l+1} \coloneqq r_k - a_{k+1}p^k$ has the form $r_{k+l+1} = p^{k+l+1}a'/b'$ where $a'$, $b'$ are coprime integers which are are not divisible by $p$.

**Exercise 10.3.** Show that $(\mathbb{A}_\mathbb{Q}, +)$ is a locally compact Hausdorff abelian group. Moreover, show that there exists an injection $\mathbb{Q} \to \mathbb{A}_\mathbb{Q}$, $x \mapsto (x, x, x, \ldots)$. Finally, show that $i(\mathbb{Q})$ is a discrete subgroup of $\mathbb{A}_\mathbb{Q}$ and $\mathbb{A}_\mathbb{Q}/i(\mathbb{Q})$ is compact.

*Solution.* Let $x, y \in \mathbb{A}_\mathbb{Q}$ be distinct. Then there exists a finite $S \subseteq \mathbb{P}$ such that $x, y \in \mathbb{A}_\S$. Since $\mathbb{A}_S$ is Hausdorff, because each of its factors is, there exist disjoint neighborhoods $U$ and $V$ of $x$ and $y$, respectively, which are both contained in $\mathbb{A}_S$. But $\mathbb{A}_S$ is open in $\mathbb{A}_\mathbb{Q}$ so $U$ and $V$ are also open in $\mathbb{A}_\mathbb{Q}$. This shows that $\mathbb{A}_\mathbb{Q}$ is Hausdorff.

For local compactness, it suffices to show that 0 has a basis of neighborhoods with compact closure. So let $V$ be a neighborhood of 0. Then $V \cap \mathbb{A}_\emptyset$ is an open subset of $\mathbb{A}_\emptyset = \mathbb{R} \times \prod_{p\in\mathbb{P}} \mathbb{Z}_p$ containing 0 so it contains a set of the form $V_\infty \times \prod_{p\in\mathbb{P}} V_p$ which contains 0 and is such that $V_p = \mathbb{Z}_p$ for all but finitely many $p$. Let $U_p = V_p$ for all $p$ and $U_\infty \subseteq V_\infty$ contain 0 and be with compact closure. Then $U \coloneqq U_\infty \times \prod_{p\in\mathbb{P}} U_p \subseteq V$ has compact closure in $\mathbb{A}_\emptyset$ (because $\mathbb{Z}_p$ is compact and each $U_p \subseteq \mathbb{Z}_p$) and hence has compact closure in $\mathbb{A}_\mathbb{Q}$ because $\mathbb{A}_\emptyset$ embeds continuously into $\mathbb{A}_\mathbb{Q}$.

It is clear that $\mathbb{A}_\mathbb{Q}$ is abelian and it is clear that the map $i$ is injective. To show that $i(\mathbb{Q}$ is discrete, it suffices to show that 0 is open in $i(\mathbb{Q})$. So consider the open set $U \coloneqq (-1, 1) \times \prod_{p\in\mathbb{P}} \mathbb{Z}_p$ which contains 0. Let $r$ be a rational number such that $i(r) \in U$. Then $r$ maps to $\mathbb{Z}_p$ for every $p$ and hence is an integer. But since $r \in (-1, 1)$ it follows that $r = 0$. This shows that $i(\mathbb{Q})$ is discrete.

For compactness of $\mathbb{A}_\mathbb{Q}/i(\mathbb{Q})$, consider the compact set $C \coloneqq [0, 1] \times \prod_{p\in\mathbb{P}} \mathbb{Z}_p$. We claim that for every $a \in \mathbb{A}_\mathbb{Q}$, there exist an $r \in \mathbb{Q}$ such that $a - i(r) \in C$. So let $a \in \mathbb{A}_\mathbb{Q}$. For an element $x = \sum_{n=m}^{\infty} x_n p^n$ in $\mathbb{Q}_p$ we let $b_p(x) \coloneqq \sum_{n=m}^{-1} x_n p^n$ and $c_p(x) \coloneqq \sum_{n=0}^{\infty} x_n p^n$. Write $a = (a_\infty, a_2, a_3, \ldots)$. As there are only finitely many primes $p$ such that $b_p(a_p) \neq 0$, it is not difficult to see that there exists a rational number $r$ such that $b_p(r) = b_p(a_p)$ for every $p$. (Let $S$ be the set of primes $p$ such that $b_p(a_p) \neq 0$. One takes $r = \sum_{p\in S} b_p(a_p)$ where one evaluates the expression for $b_p(a_p)$, considering the coefficients as elements of $\{0, 1, \ldots, p-1\}$.) By possibly adding a suitable integer to $r$, it will hold that $a - i(r) \in C$. This shows that $\mathbb{A}_\mathbb{Q}/i(\mathbb{Q})$ is compact as it is the image of a compact set under a continuous map. ■