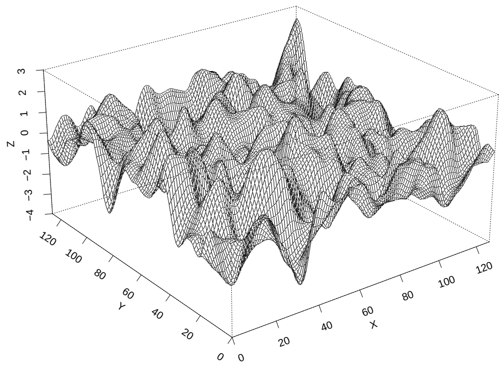
The Term Structure of Interest Rates as a Random Field. Applications to Credit Risk



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presented by
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The graph on the cover depicts a simulated Gaussian random field with correlation function

$$c(x,y) = \begin{cases} \frac{1}{2^{\gamma-1}\Gamma(\gamma)} \left(\frac{\|x-y\|}{\theta}\right)^{\gamma} J_\gamma(\|x-y\|/\theta)\,, & \|x-y\| \neq 0, \\ 1\,, & \|x-y\| = 0, \end{cases} \tag{0.0}$$

where we set $\gamma = 5$ and $\theta = 2$. The function J_γ denotes a modified Bessel function of the third kind of order γ.

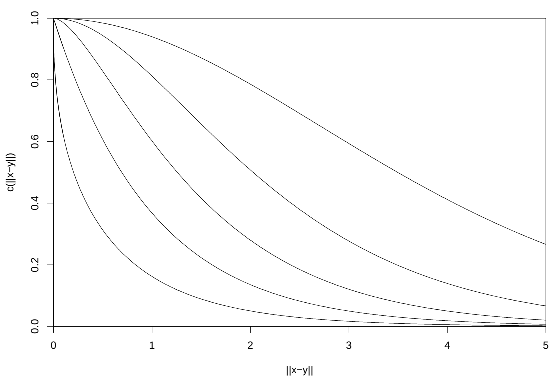

Figure 0: Graph of the correlation function c in (0.0) for $\gamma \in \{5, 2, 1, 0.5, 0.2\}$, from top to bottom, all with $\theta = 2$.

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THE TERM STRUCTURE OF INTEREST RATES AS A RANDOM FIELD. APPLICATIONS TO CREDIT RISK

HANSJÖRG FURRER

Abstract. The principal aim of this paper is the modeling of the term structure of interest rates as a positive-valued random field. Special emphasis is given to chi-squared fields which can be generated from a finite number of Gaussian fields. In particular, we introduce a short rate model which is based on the square of an Ornstein-Uhlenbeck process. It is shown that bond prices can be expressed as an exponential-affine function of the short rate. We extend our approach to the credit risk area and model the default intensities of a class of obligors as a two-parameter positive-valued random field. Finally, random fields will be applied to the modeling of firm values in a structural credit risk framework.

1. Introduction

Spatial patterns occur in a wide variety of scientific disciplines such as hydrology, meteorology, climatology, neurology, geo-statistics, soil science, and many more. To cover models of spatial variation, the theory of stochastic processes turns out to be useful provided the index set of the processes is defined in a general manner. If $Z(\mathbf{x})$, $\mathbf{x} \in \mathscr{T}$, labels topographical height for example, then $\mathscr{T}$ is commonly assumed to be a subset of $\mathbb{R}^2$. In the event the dynamic aspect is of importance too, the index set may include an additional component capturing the development in time. This leads to so-called spatio-temporal models. The extension of single-parameter stochastic processes to multi-parameter processes is commonly designated as random field. To specify a random field it suffices to give the joint distribution of any finite subset of $(Z(\mathbf{x}_1), \ldots, Z(\mathbf{x}_n))$ in a consistent way. Key characteristics of a random field are its mean and covariance function. In general, however, the distribution of a random field is *not* fully specified by the mean and covariance function alone. This is only the case for an important subclass of random fields known as Gaussian fields. In most modeling strategies, a specific parametric form for the covariance function R is assumed. Care has to be taken in order to derive a non-negative definite function. Correlation models that are not non-negative definite can lead to negative variances inconsistent with theory and intuition. For spatio-temporal processes, the main difficulty lies in specifying an appropriate space-time covariance structure. One simple way would be to multiply a time covariance function with a space covariance function. Such separable covariance models, however, neglect any space-time interactions, and are thus mainly used for convenience rather than for their ability to fit the data well.

Random fields in the realm of term structure models were introduced by Kennedy [16], [17]. He proposed a model where the instantaneous forward rates $f(t,T)$ are modeled as a two-parameter Gaussian random field. The first parameter of the forward rate surface corresponds to the current time, the second to the maturity date. In more technical terms, Kennedy sets $f(t,T) = \mu(t,T) + Z(t,T)$, and derives a necessary and sufficient restriction on the drift function $\mu(t,T)$ to ensure that discounted zero-bond prices are martingales under the risk-neutral measure. A consequence of this modeling approach is that bond prices of different maturities are no longer perfectly correlated. Indeed, the correlation structure can be chosen in an arbitrary way. Goldstein [13] concentrates on the dynamics of the forward rates $df(t,T) = \mu(t,T)\,dt + \sigma(t,T)\,dZ(t,T)$, and generalizes Kennedy's drift restriction result to non-Gaussian fields. Collin-Dufresne and Goldstein [3] specify the bond price dynamics as a Gaussian random field, and consider a generalized affine framework where the log-bond prices themselves are taken as state variables. Thus, in contrast to the traditional affine framework, the state vector is of infinite dimension.

Our principal aim in this paper is the modeling of the term structure of interest rates as a positive-valued random field. Starting from an arbitrary Gaussian field, we obtain a positive-valued field by means of a positive transformation. Of particular importance is the Ornstein-Uhlenbeck process (OU process) with parameter β and size $c > 0$, characterized by the covariance function $R(s,t) = c\exp\{-\beta|s-t|\}$. Observe that the two processes one-dimensional Brownian motion and OU process are both zero-mean Gaussian processes. However, only the latter is stationary. We consider short rate models which include either the square of Brownian motion or the square of an OU process, and show that bond prices are exponential-affine functions of the short rate. Our single-parameter models will then be carried forward to the credit risk area, thereby extending the parameter set to include an additional spatial component. In doing so, we arrive at a model for the default intensities of a family of obligors. Alternatively, a two-parameter spatio-temporal framework can also be used to model the obligors' market value of assets.

The rest of this paper is organized as follows. Section 2 is of expository nature and is devoted to a brief overview of the theory of random fields. The material in this section is mainly taken from the books of Adler [1] and Matérn [19]. As a prelude to the more specific random fields, we present in Section 2.1 a short summary of the theory of isotropic fields. These are processes that are, from a distributional viewpoint, invariant under rotations about the origin. Consequently, the correlation of an isotropic field solely depends on the radial distance from the origin. In Section 2.2 we introduce the important class of Gaussian random fields. Because of the simplicity in working with the multivariate normal distribution, Gaussian random fields have gained a lot of attraction. An important result in the realm of Gaussian fields is the Karhunen-Loève expansion. In essence, the Karhunen-Loève expansion provides a representation of a Gaussian field in terms of an infinite series. We illustrate its use in the Brownian case by deriving the so-called Cameron-Martin formula. There is no doubt that Gaussian random fields are quite useful in modeling various phenomena. In some applications, however, the use of a Gaussian field is inappropriate because Gaussian fields can attain negative values with positive probability. The so-called chi-squared random fields which we introduce in Section 2.3 are positive-valued fields which are generated from a finite number of stationary Gaussian fields.

In Section 3 we start dealing with specific short rate models. We first present a framework where the spot rate is based on the square of standard Brownian motion, see Section 3.1. The square of a Brownian motion, however, does not qualify as a chi-squared field since Brownian motion is not a stationary process. Notwithstanding this, the proposed model has its merits since bond prices are expressible as exponential-affine functions of the short rate. Spurred by the arguable lack of stationarity, we extend our search for appropriate short rate models, and present in Section 3.2 a framework where the random field component is the square of an OU process. Indeed, the square of an OU process is a chi-squared field on the real line, and yet the affine structure remains intact. As an application, we calculate the price of a European call option written on a zero-bond. Section 3.3 is an excursion into the area of forward rate random field models. As such, the material presented here is not new, and we follow Goldstein's [13] approach to modeling the forward rate dynamics as $df(t,T) = \mu(t,T)\,dt + \sigma(t,T)\,dZ(t,T)$, where $Z(t,T)$ denotes a random field with deterministic correlation structure $c(t;T_1,T_2)$. We restate the result and proof in which the form of the drift function $\mu(t,T)$ is specified in order to preclude arbitrage under the risk neutral measure. Section 3.3 must be seen in preparation for Section 4.4 where we derive the drift restriction in a defaultable framework under the recovery of market value assumption.

Section 4 starts with a brief overview of the ideas and the terminology inherent to the pricing of defaultable securities. Defaultable pricing models are commonly classified either as reduced form models or structural models. The former category assumes that default occurs at a random time τ which is governed by a risk-neutral intensity process $\lambda = \{\lambda(t) : t \geq 0\}$. Structural models, on the other hand, are based on modeling the stochastic evolution of the assets of the issuer with default occurring at the first time the assets fall below a certain level. In Section 4.1 we consider reduced-form debt pricing models more thoroughly. This class of models is quite flexible in the sense that it allows for non-zero recovery. Duffie and Singleton [11] consider reduced form models under the so-called recovery of market value (RMV) assumption. This means that, if default occurs at time t, a fraction, say L_t, of the market value is lost. This leads to a pricing rule based on a default-adjusted short rate process $\bar{r} = r+s$, where $s = \lambda L$ denotes the short spread. Modeling both processes r and s as in Section 3.2, thereby allowing for non-zero correlation between r and s, we show that defaultable zero-coupon bond prices are of exponential-affine form. For simplicity, we will assume $L \equiv 1$, implying that $s = \lambda$. In Section 4.2 we consider a family of obligors and model the short spread as a two-parameter random field, one parameter referring to the current time and the other labeling the obligor. This setup corresponds to a spatio-temporal model, and hence we are confronted with the determination of appropriate covariance structures. Section 4.3 aims to model the firm value of a family of obligors as a two-parameter random field. Special emphasis is given to the fraction of obligors who will default at a fixed maturity date T. Section 4.4 returns to the theme of modeling the instantaneous forward rates as a random field. This time, however, we deal with defaultable securities, and derive the form of the drift function and the form of the credit spread necessary to preclude arbitrage. Section 5 concludes the article.

2. An overview of random fields

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space on which all random objects will be defined. A filtration $\{\mathcal{F}_t : t \geq 0\}$ of σ-algebras, satisfying the usual conditions, is fixed and defines the information available at each time t.

Definition 2.1. (**Random field**). *A (real-valued) random field is a family of random variables $Z(\mathbf{x})$ indexed by $\mathbf{x} \in \mathbb{R}^d$ together with a collection of distribution functions of the form $F_{\mathbf{x}_1,\ldots,\mathbf{x}_n}$ which satisfy*

$$F_{\mathbf{x}_1,\ldots,\mathbf{x}_n}(b_1, \ldots, b_n) = \mathbb{P}[Z(\mathbf{x}_1) \leq b_1, \ldots, Z(\mathbf{x}_n) \leq b_n]\,,$$

$b_1, \ldots, b_n \in \mathbb{R}$.

The mean function of Z is $m(\mathbf{x}) = \mathbb{E}[Z(\mathbf{x})]$ whereas the *covariance function* and the *correlation function* are respectively defined as

$$R(\mathbf{x}, \mathbf{y}) = \mathbb{E}[Z(\mathbf{x})Z(\mathbf{y})] - m(\mathbf{x})m(\mathbf{y})$$
$$c(\mathbf{x}, \mathbf{y}) = \frac{R(\mathbf{x}, \mathbf{y})}{\sqrt{R(\mathbf{x}, \mathbf{x})R(\mathbf{y}, \mathbf{y})}}\,.$$

Notice that the covariance function of a random field Z is a non-negative definite function on $\mathbb{R}^d \times \mathbb{R}^d$, that is if $\mathbf{x}_1, \ldots, \mathbf{x}_k$ is any collection of points in $\mathbb{R}^d$, and $\xi_1, \ldots, \xi_k$ are arbitrary real constants, then

$$\sum_{\ell=1}^{k} \sum_{j=1}^{k} \xi_\ell \xi_j\, R(\mathbf{x}_\ell, \mathbf{x}_j) = \sum_{\ell=1}^{k} \sum_{j=1}^{k} \xi_\ell \xi_j \mathbb{E}[Z(\mathbf{x}_\ell)Z(\mathbf{x}_j)]$$
$$= \mathbb{E}\Big[\Big(\sum_{j=1}^{k} \xi_j Z(\mathbf{x}_j)\Big)^2\Big] \geq 0\,.$$

Without loss of generality, we assumed $m = 0$. The property of non-negative definiteness characterizes covariance functions. Hence, given any function $m : \mathbb{R}^d \to \mathbb{R}$ and a non-negative definite function $R : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$, it is always possible to construct a random field for which m and R are the mean and covariance function, respectively. Non-negative definite functions can be characterized in the following way, see Bochner [2].

Theorem 2.1. (**Bochner's Theorem**) *A continuous function R from $\mathbb{R}^d$ to the complex plane is non-negative definite if and only if it is the Fourier-Stieltjes transform of a measure F on $\mathbb{R}^d$, that is the representation*

$$R(\mathbf{x}) = \int_{\mathbb{R}^d} e^{i\mathbf{x}\cdot\boldsymbol{\lambda}}\, dF(\boldsymbol{\lambda})$$

holds for $\mathbf{x} \in \mathbb{R}^d$. Here $\mathbf{x} \cdot \boldsymbol{\lambda}$ denotes the scalar product $\sum_{k=1}^{d} x_k \lambda_k$ and F is a bounded, real valued function satisfying $\int_{\mathcal{A}} dF(\boldsymbol{\lambda}) \geq 0$ for all measurable $\mathcal{A} \subset \mathbb{R}^d$. □

Occasionally we shall consider two random fields Z_1, Z_2 simultaneously. The cross covariance function is defined as

$$R_{12}(\mathbf{x}, \mathbf{y}) = \mathbb{E}[Z_1(\mathbf{x})Z_2(\mathbf{y})] - m_1(\mathbf{x})m_2(\mathbf{y}),$$

where m_1 and m_2 are the respective mean functions. Obviously, $R_{12}(\mathbf{x}, \mathbf{y}) = R_{21}(\mathbf{y}, \mathbf{x})$. A family of processes Z_ι with ι belonging to some index set $\mathscr{I}$ can be considered as a process in the product space $(\mathbb{R}^d, \mathscr{I})$. From this representation, consistency conditions for cross covariance functions can be deduced, see Cramér [4].

A central concept in the study of random fields is that of homogeneity or stationarity. We call a random field *homogeneous* or (second-order) *stationary* if $\mathbb{E}[Z(\mathbf{x})^2]$ is finite for all $\mathbf{x}$ and

- $m(\mathbf{x}) \equiv m$ is independent of $\mathbf{x} \in \mathbb{R}^d$
- $R(\mathbf{x}, \mathbf{y})$ solely depends on the difference $\mathbf{x} - \mathbf{y}$.

Thus we may consider

$$R(\mathbf{h}) = \mathrm{Cov}(Z(\mathbf{x}), Z(\mathbf{x}+\mathbf{h})) = \mathbb{E}[Z(\mathbf{x})Z(\mathbf{x}+\mathbf{h})] - m^2, \quad \mathbf{h} \in \mathbb{R}^d,$$

and denote R the covariance function of Z. In this case, the following correspondence exists between the covariance and correlation function, respectively:

$$c(\mathbf{h}) = \frac{R(\mathbf{h})}{R(\mathbf{0})},$$

i.e. $c(\mathbf{h}) \propto R(\mathbf{h})$. For this reason, the attention is confined to either c or R. Two stationary random fields Z_1, Z_2 are *stationarily correlated* if their cross covariance function $R_{12}(\mathbf{x}, \mathbf{y})$ depends on the difference $\mathbf{x} - \mathbf{y}$ only. The two random fields are *uncorrelated* if R_{12} vanishes identically.

2.1. **Isotropic random fields.** An interesting special class of homogeneous random fields that often arise in practice is the class of isotropic fields. These are characterized by the property that the covariance function R depends only on the length $\|\mathbf{h}\|$ of the vector $\mathbf{h}$:

$$R(\mathbf{h}) = R(\|\mathbf{h}\|).$$

In many applications, random fields are considered as functions of "time" and "space". In this case, the parameter set is most conveniently written as $(t, \mathbf{x})$ with $t \in \mathbb{R}_+$ and $\mathbf{x} \in \mathbb{R}^d$. Such processes are often homogeneous in $(t, \mathbf{x})$ and isotropic in $\mathbf{x}$ in the sense that

$$\mathbb{E}[Z(t, \mathbf{x})Z(t+h, \mathbf{x}+\mathbf{y})] = R(h, \|\mathbf{y}\|),$$

where R is a function from $\mathbb{R}^2$ into $\mathbb{R}$. In such a situation, the covariance function can be written as

$$R(t, \|\mathbf{x}\|) = \int_{\mathbb{R}} \int_{\lambda=0}^{\infty} e^{itu} H_d(\lambda\|\mathbf{x}\|)\, dG(u, \lambda),$$

where

$$H_d(r) = \left(\frac{2}{r}\right)^{(d-2)/2} \Gamma(d/2) J_{(d-2)/2}(r)$$

and J_m is the Bessel function of the first kind of order m and G is a multiple of a distribution function on the half plane $\{(\lambda, u) | \lambda \geq 0, u \in \mathbb{R}\}$. For a proof, see for instance Adler [1], Section 2.5.

To ensure non-negative definiteness of R one often specifies R to belong to a parametric family whose members are known to be positive definite. One attempt is to use separable covariances

$$R(t, \mathbf{x}) = R^{(1)}(t) R^{(2)}(\mathbf{x}) ,$$

where $R^{(1)}$ is positive definite on $\mathbb{R}_+$ and $R^{(2)}$ is a positive definite function on $\mathbb{R}^d$. A simple example would be $R^{(1)}(t) = \exp\{-\theta_1 t\}$ and $R^{(2)}(\mathbf{x}) = \exp\{-\theta_2 \|\mathbf{x}\|\}$, yielding $R(t, \mathbf{x}) = \exp\{-\theta_1 t - \theta_2 \|\mathbf{x}\|\}$, $\theta_1 > 0$, $\theta_2 > 0$. However, the class of separable correlation functions is limited as it neglects any space-time interactions. Cressie and Huang [5] therefore discuss classes of non-separable stationary covariance functions. See also Chapter 5 of Gneiting [12] where isotropic correlation functions in the space domain are discussed.

Example 2.1 (Cressie and Huang [5]). *Three parameter non-separable space-time stationary covariance function*

$$R(t, \mathbf{x}) = \frac{\sigma^2}{(a^2 t^2 + 1)^{d/2}} \exp\Big\{ -\frac{b^2 \|\mathbf{x}\|^2}{a^2 t^2 + 1} \Big\} , \tag{2.1}$$

where $a \geq 0$ is the scaling parameter of time, $b \geq 0$ is the scaling parameter of space, and $\sigma^2 = R(0, \mathbf{0})$.

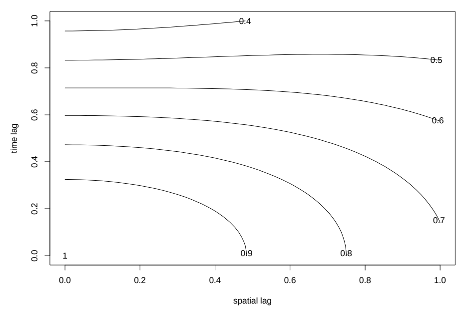

Figure 1: Contour plot of the covariance function (2.1) with $a = b = \sigma^2 = 1$ and $d = 1$.

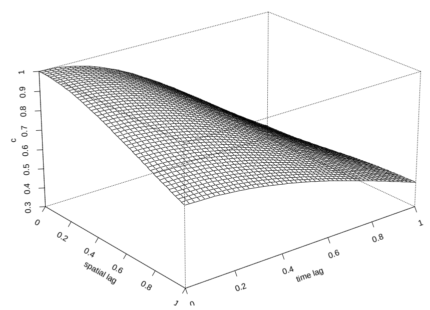

Figure 2: Perspective plot of the covariance function (2.1) with $a = b = \sigma^2 = 1$ and $d = 1$.

2.2. **Gaussian random fields.** An important special class of random fields is that of Gaussian fields. A Gaussian random field is a random field where all the finite-dimensional distributions $F_{\mathbf{x}_1, \ldots, \mathbf{x}_n}$ are multivariate normal. As a consequence, a Gaussian random field is completely determined by specifying the mean and covariance function, respectively. A key result in the theory of Gaussian fields is the so-called Karhunen-Loève expansion. In essence, the Karhunen-Loève expansion provides a representation of a Gaussian random field through a type of eigenfunction expansion. Before stating the result, we introduce some terminology and notation. Let $\mathcal{U}$ be a compact interval in $\mathbb{R}^d$ and let R be a continuous covariance function on $\mathcal{U} \times \mathcal{U}$. A nonzero number λ for which there exists a function ϕ satisfying

$$\int_{\mathcal{U}} R(\mathbf{x}, \mathbf{y})\, \phi(\mathbf{y})\, d\mathbf{y} = \lambda\, \phi(\mathbf{x}) \tag{2.2}$$

and $\int_{\mathcal{U}} |\phi(\mathbf{y})|^2\, d\mathbf{y} < \infty$ is called an *eigenvalue* of R and the corresponding function ϕ an *eigenfunction*. In general, there exists a sequence $(\lambda_i)_{i\in\mathbb{N}}$ of eigenvalues and corresponding eigenfunctions $(\phi_i)_{i\in\mathbb{N}}$ fulfilling (2.2). One can assume that the eigenfunctions ϕ_k form an orthonormal sequence, i.e.

$$\int_{\mathcal{U}} \phi_i(\mathbf{y})\phi_k(\mathbf{y})\, d\mathbf{y} = \begin{cases} 1, & i = k \\ 0, & i \neq k\,. \end{cases} \tag{2.3}$$

A fundamental result in the theory of integral equations is the following theorem, see for instance Adler [1], Theorem 3.3.1.

Theorem 2.2. (**Mercer's Theorem**). *Let $R(\mathbf{x}, \mathbf{y})$ be continuous and non-negative definite on the compact interval $\mathcal{U} \times \mathcal{U} \subset \mathbb{R}^{2d}$, with eigenvalues λ_j and eigenfunctions ϕ_j satisfying $\int_{\mathcal{U}} R(\mathbf{x}, \mathbf{y})\, \phi(\mathbf{y})\, d\mathbf{y} = \lambda\, \phi(\mathbf{x})$. Then*

$$R(\mathbf{x}, \mathbf{y}) = \sum_{j=1}^{\infty} \lambda_j \phi_j(\mathbf{x})\phi_j(\mathbf{y})\,,$$

where the series converges absolutely and uniformly on $\mathcal{U} \times \mathcal{U}$. □

The Karhunen-Loève expansion of a Gaussian random field is given in the following theorem, see Adler [1], Theorem 3.3.3 for more details.

Theorem 2.3. (**Karhunen-Loève expansion**) *Let $Z(\mathbf{x})$ be a real-valued, zero-mean, Gaussian random field with continuous covariance function R which has Mercer expansion $R(\mathbf{x}, \mathbf{y}) = \sum_j \lambda_j \phi_j(\mathbf{x})\phi_j(\mathbf{y})$. Then, under some regularity conditions,*

$$Z(\mathbf{x}) = \sum_{i=1}^{\infty} \sqrt{\lambda_i}\, \phi_i(\mathbf{x})\xi_i\,, \tag{2.4}$$

in L^2 and a.s., where $(\xi_i)_{i\in\mathbb{N}}$ is a sequence of independent and identically standard normally distributed random variables. □

We shall occasionally consider integrals of the type

$$I_{\mathcal{U}} = \int_{\mathcal{U}} \zeta(\mathbf{y}) Z(\mathbf{y})\, d\mathbf{y}\,,$$

where the deterministic function ζ is continuous over the subset $\mathcal{U} \subset \mathbb{R}^d$, and the random field Z is stationary with covariance function R. Such integrals can be defined in analogy with the Riemann integral by considering limits of finite linear forms of the form $Z(\mathbf{x}_k)$. The first two moments of $I_{\mathcal{U}}$ are then given by

$$\mathbb{E}[I_{\mathcal{U}}] = \int_{\mathcal{U}} \zeta(\mathbf{y}) m(\mathbf{y})\, d\mathbf{y}\,, \quad \mathbb{C}\mathrm{ov}(I_{\mathcal{U}}, I_{\mathcal{V}}) = \int_{\mathcal{U}}\int_{\mathcal{V}} \zeta(\mathbf{x})\zeta(\mathbf{y}) R(\mathbf{x} - \mathbf{y})\, d\mathbf{x} d\mathbf{y}\,.$$

If it is assumed that $Z(\mathbf{x})$ is a Gaussian field then $I_{\mathcal{U}}$ is a Gaussian random variable with mean value $\int_{\mathcal{U}} \zeta(\mathbf{y}) m(\mathbf{y})\, d\mathbf{y}$ and variance $\mathbb{V}\mathrm{ar}(I_{\mathcal{U}}) = \iint_{\mathcal{U}\times\mathcal{U}} \zeta(\mathbf{x})\zeta(\mathbf{y}) R(\mathbf{x}-\mathbf{y})\, d\mathbf{x} d\mathbf{y}$.

In the following Proposition we derive the so-called Cameron-Martin formula. We present a proof which is based on the Karhunen-Loève expansion for Brownian motion W. Note that the process W^2 is *not* a chi-squared field in the sense of Definition 2.2 because W is not a stationary process. The Cameron-Martin formula can be used for bond pricing purposes when the short rate process is described in terms of W^2, see Section 3.1. An alternative proof of the Cameron-Martin formula can be found in Revuz and Yor [22], p. 425.

Proposition 2.1. *Let W denote standard one-dimensional Brownian motion. Then, for $\alpha \in \mathbb{R}_+$,*

$$\mathbb{E}\Big[e^{-\alpha \int_0^T W^2(s)\, ds}\Big] = \Big(\cosh\big(T\sqrt{2\alpha}\,\big)\Big)^{-1/2}, \tag{2.5}$$

where $\cosh(x) = (\exp\{x\} + \exp\{-x\})/2$ *denotes the hyperbolic cosine function.*

Proof. The usefulness of the Karhunen-Loève expansion hinges on the ability to solve the integral equation (2.2). Recall that the covariance function R in the Brownian case is given by $R(s,t) = s \wedge t$, where $a \wedge b = \min\{a,b\}$. Equation (2.2) therefore reads

$$\lambda\phi(x) = \int_0^T (x \wedge y)\phi(y)\, dy = \int_0^x y\phi(y)\, dy + x\int_x^T \phi(y)\, dy\,.$$

From this, it follows that $\phi(0) = 0$. Taking the derivative with respect to x, we obtain $\lambda\dot\phi(x) = \int_x^T \phi(y)\, dy$. Setting $x = T$, we get $\dot\phi(T) = 0$. Taking the derivative once more with respect to x, it follows that

$$\lambda\ddot\phi(x) = -\phi(x)\,. \tag{2.6}$$

The general solution of (2.6) is given by

$$\phi(x) = A\sin\big(x/\sqrt{\lambda}\,\big) + B\cos\big(x/\sqrt{\lambda}\,\big)\,.$$

The initial condition $\phi(0) = 0$ yields $B = 0$. On the other hand, from $\dot\phi(T) = 0$ we conclude that $A\cos(T/\sqrt{\lambda}) = 0$, whence $T/\sqrt{\lambda} = (2k+1)\pi/2$ for $k \in \mathbb{N}_0$, or equivalently

$$\lambda_k = \Big(\frac{2T}{(2k+1)\pi}\Big)^2\,. \tag{2.7}$$

The value of the constant A can be obtained from the orthonormality condition (2.3). We have that

$$\begin{aligned} 1 = \int_0^T \phi_k^2(y)\, dy &= A^2 \int_0^T \sin^2\big(y/\sqrt{\lambda_k}\,\big)\, dy \\ &= A^2 \int_0^T \sin^2\Big(\frac{y\pi(2k+1)}{2T}\Big)\, dy \end{aligned}$$

$$= \frac{2TA^2}{(2k+1)\pi} \int_0^{(2k+1)\pi/2} \sin^2(u)\, du = A^2T/2\,.$$

In the last equality we used $\int \sin^2(u)\,du = u/2 - \sin(2u)/4$. Solving for A thus yields $A = \sqrt{2/T}$. Summarizing, we have that Brownian motion can be defined as the infinite sum (in $L^2([0,T])$ and a.s.)

$$W(t) = \sum_{k=1}^{\infty} \frac{2\sqrt{2T}}{\pi(2k+1)} \sin\Big(\frac{\pi t(2k+1)}{2T}\Big)\, \xi_k\,, \qquad t \in [0,T],$$

with independent standard normal random variables ξ_k. We are now in a position to calculate the Laplace transform of $\int_0^T W^2(s)\,ds$.

$$\begin{aligned}\mathbb{E}\Big[e^{-\alpha\int_0^T W^2(s)\,ds}\Big] &= \mathbb{E}\Big[e^{-\alpha\int_0^T \left(\sum_k \sqrt{\lambda_k}\phi_k(s)\xi_k\right)^2 ds}\Big] \\ &= \mathbb{E}\Big[e^{-\alpha\int_0^T \sum_{k,j} \sqrt{\lambda_k\lambda_j}\phi_k(s)\phi_j(s)\xi_k\xi_j\,ds}\Big] \\ &= \mathbb{E}\Big[e^{-\alpha\sum_{k,j} \sqrt{\lambda_k\lambda_j}\xi_k\xi_j \int_0^T \phi_k(s)\phi_j(s)\,ds}\Big]\,.\end{aligned}$$

Using the orthonormality of the sequence (ϕ_i) and the independence of the standard normal variates ξ_i we get

$$\mathbb{E}\Big[e^{-\alpha\int_0^T W^2(s)\,ds}\Big] = \mathbb{E}\Big[e^{-\alpha\sum_{j\ge 0}\lambda_j\xi_j^2}\Big] = \prod_{j\ge 0}\mathbb{E}\Big[e^{-\alpha\lambda_j\xi^2}\Big]\,.$$

Let $Z \sim \mathcal{N}(0,1)$. Note that $\mathbb{E}\big[e^{-\gamma Z^2}\big] = \int_{\mathbb{R}} e^{-\gamma z^2}\varphi(z)\,dz = \big(1+2\gamma\big)^{-1/2}$, where $\varphi(x) = 1/\sqrt{2\pi}\exp\{-x^2/2\}$, $x \in \mathbb{R}$, denotes the density function of the standard normal distribution. Consequently,

$$\mathbb{E}\Big[e^{-\alpha\int_0^T W^2(s)\,ds}\Big] = \prod_{j\ge 0}\big(1+2\alpha\lambda_j\big)^{-1/2} = \Big(\prod_{j\ge 0}\big(1+2\alpha\lambda_j\big)\Big)^{-1/2}\,.$$

It remains to show that $\prod_{j\ge 0}(1+2\alpha\lambda_j) = \cosh\big(T\sqrt{2\alpha}\big)$. From the identity $\cosh(y) = \cos(iy)$ and the infinite product expansion of the cosine function $\cos(x) = \prod_{n\ge 1}\big(1-4x^2/(\pi(2n-1))^2\big)$ we find the infinite product expansion of the hyperbolic cosine function:

$$\cosh(y) = \prod_{k\ge 0}\Big(1+\frac{4y^2}{\pi^2(2k+1)^2}\Big)\,.$$

Now observe that, with λ_k as given in (2.7),

$$\prod_{k\ge 0}\big(1+2\alpha\lambda_k\big) = \prod_{k\ge 0}\Big(1+\frac{4\theta^2}{\pi^2(2k+1)^2}\Big) = \cosh(\theta)$$

with $\theta = T\sqrt{2\alpha}$, which completes the proof. □

2.3. **Chi-squared random fields.** The amenities in working with Gaussian random fields are mainly a consequence of the relatively easy form of the (multivariate) Gaussian distribution. Nature, however, does not always produce Gaussian fields. For instance, the tails of the distribution of a random field may be fatter than normal. From a modeling point of view, Gaussian fields may also be inappropriate because they allow for negative values. In the credit risk context for example it makes little sense to stipulate model where an obligor's default intensity is modeled as a Gaussian random field. This would imply that negative intensities occur with positive probability. This is inconsistent with theory, since, for a deterministic intensity λ, the risk-neutral probability of an obligor's default during the infinitesimal time interval $[t, t + dt]$, conditional on survival up to t, is given by λdt. The so-called chi-squared fields are a possible way out. A chi-squared field is a positive-valued random field generated from a finite number of stationary Gaussian random fields. Chapter 7 of Adler [1] provides an introduction to non-Gaussian fields, including chi-squared fields.

Definition 2.2. (**Chi-squared random field**). *Let $Z_1, \dots, Z_n$ be independent, stationary real-valued Gaussian random fields with mean function $m(\mathbf{x}) = \mathbb{E}[Z_i(\mathbf{x})] = 0$, $i = 1, \dots, n$, common covariance function $R(\mathbf{y}) = \mathbb{E}[Z(\mathbf{x})Z(\mathbf{x}+\mathbf{y})]$ and variance $\sigma^2 = R(\mathbf{0})$. For $\mathbf{x} \in \mathbb{R}^d$, the process*

$$Y(\mathbf{x}) := Z_1^2(\mathbf{x}) + \cdots + Z_n^2(\mathbf{x})$$

is called a chi-squared field with parameter n.

The name chi-squared field with parameter n stems from the fact that the random variable $Y(\mathbf{x})$ has a scaled one-dimensional chi-squared distribution with n degrees of freedom. Recall that the density function of a scaled chi-squared distributed random variable Y with r degrees of freedom is given by

$$f_{\chi^2(r,\sigma)}(u) = \frac{1}{\Gamma(r/2)\,(2\sigma)^{r/2}}\, u^{r/2-1}\, e^{-u/(2\sigma)}\,, \qquad u \geq 0. \tag{2.8}$$

Notice that $\mathbb{E}[Y] = r\sigma^2$ and $\mathbb{V}\mathrm{ar}(Y) = 2r\sigma^4$.

Since a chi-squared field Y is generated from a finite number of stationary (Gaussian) fields, it follows that Y is stationary too. To derive the covariance function R^* of a chi-squared field Y we need the well-known property that the conditional distributions of a multivariate normal distribution are again multivariate normal. In particular, for a bivariate normal random vector $\mathbf{Z} = (Z_1, Z_2)'$ with mean vector $\boldsymbol{\mu} = (\mu_1, \mu_2)'$ and covariance matrix

$$\Sigma = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}, \quad \sigma_1, \sigma_2 \geq 0\,, \quad |\rho| \leq 1\,,$$

the conditional distribution of Z_1 given Z_2 is normal with mean $\mathbb{E}[Z_1|Z_2] = \mu_1 + \rho\sigma_1\sigma_2^{-1}(Z_2 - \mu_2)$ and variance $\mathbb{V}\mathrm{ar}(Z_1|Z_2) = \sigma_1^2(1-\rho^2)$. From this, we find that

$$\begin{aligned} \mathbb{E}\big[Z_1^2 Z_2^2\big] &= \int_{\mathbb{R}} \mathbb{E}\big[Z_1^2 Z_2^2 | Z_2 = z\big]\, \varphi\Big(\frac{z-\mu_2}{\sigma_2}\Big)/\sigma_2\, dz \\ &= \int_{\mathbb{R}} z^2\, \mathbb{E}\big[Z_1^2 | Z_2 = z\big]\, \varphi\Big(\frac{z-\mu_2}{\sigma_2}\Big)/\sigma_2\, dz \\ &= (\sigma_2^2 + \mu_2^2)\Big(\mu_1^2 + \sigma_1^2(1-\rho^2)\Big) \\ &\qquad\qquad + 4\mu_1\mu_2\sigma_1\sigma_2 + \big(\rho\sigma_1\mu_2\big)^2 + 3\big(\rho\sigma_1\sigma_2\big)^2\,. \end{aligned} \tag{2.9}$$

Assume that $\sigma^2 = R(\mathbf{x},\mathbf{x}) = 1$. By independence of the fields Z_i, Z_k, $i \neq k$, we get

$$
\begin{aligned}
R^*(\mathbf{x},\mathbf{y}) &= \mathbb{E}\big[Y(\mathbf{x})Y(\mathbf{y})\big] - m_Y(\mathbf{x})m_Y(\mathbf{y}) \\
&= \mathbb{E}\Big[\Big(\sum_{i=1}^n Z_i^2(\mathbf{x})\Big)\Big(\sum_{k=1}^n Z_k^2(\mathbf{y})\Big)\Big] - n^2 \\
&= \mathbb{E}\Big[\sum_{i=1}^n Z_i^2(\mathbf{x})Z_i^2(\mathbf{y}) + \sum_{i\neq k} Z_i^2(\mathbf{x})Z_k^2(\mathbf{y})\Big] - n^2 \\
&= \sum_{i=1}^n \mathbb{E}\big[Z_i^2(\mathbf{x})Z_i^2(\mathbf{y})\big] + \sum_{i\neq k}\mathbb{E}\big[Z_i^2(\mathbf{x})\big]\mathbb{E}\big[Z_k^2(\mathbf{y})\big] - n^2 \\
&= \sum_{i=1}^n \big\{(1-R^2(\mathbf{x},\mathbf{y})) + 3R^2(\mathbf{x},\mathbf{y})\big\} + n(n-1) - n^2 \\
&= 2nR^2(\mathbf{x},\mathbf{y}),
\end{aligned}
\tag{2.10}
$$

where R denotes the common covariance function of the fields Z_i. In the second-last equality we used (2.9).

3. Default-free models

The theory of interest rate modeling is commonly based on the assumption of a specific one-dimensional dynamic for the instantaneous spot rate process r. We take as given an arbitrage free setting in which all securities are priced in terms of r and an equivalent martingale measure $\mathbb{Q}$. Modeling directly such dynamics is very convenient because all fundamental quantities such as forward rates or bonds are readily defined. For instance, the time-t price of a zero-coupon bond with maturity T, $t \leq T$, is characterized by

$$
P(t,T) = \mathbb{E}_{\mathbb{Q}}\Big[e^{-\int_t^T r_s\,ds}\big|\mathcal{F}_t\Big]. \tag{3.1}
$$

From this expression it is clear that whenever we can determine the distribution of $\exp\{-\int_t^T r_s\,ds\}$ in terms of a specific dynamic for r, bond prices can be calculated. In the Vasicek model, the short rate process r is modeled as a mean reverting Ornstein-Uhlenbeck process (OU process) under the risk-neutral measure $\mathbb{Q}$, that is $dr_t = \kappa(m - r_t)dt + \sigma dW(t)$, where κ, m, and σ are positive constants and W denotes standard Brownian motion. The above stochastic differential equation has the solution

$$
r_t = r_s e^{-\kappa(t-s)} + m\big(1 - e^{-\kappa(t-s)}\big) + \sigma\int_s^t e^{-\kappa(t-u)}\,dW(u),
$$

and it is well-known that r_t conditional on $\mathcal{F}_s$, $s \leq t$, has a Gaussian distribution. Moreover, $\int_s^t r_u\,du$ is itself normally distributed, implying that bonds can be priced by directly computing the expectation (3.1).

The main drawback of the Vasicek model is that the short rate r_t can attain negative values with positive probability. In the Cox, Ingersoll and Ross (CIR) framework, a "square root" term in the diffusion coefficient of the instantaneous short rate dynamics is introduced:

$$dr_t = \kappa(m - r_t)dt + \sigma\sqrt{r_t}\, dW(t)\,. \tag{3.2}$$

Not only provides the CIR model positive[1] instantaneous short rates but also remains analytically tractable.

Motivated by the general theory of random fields, we shall subsequently consider short rate models where the evolution of r is specified in integral rather than in differential form. In such a model, the stochastic component will be represented by a random field Z. It is tempting to postulate a model where the random component is a Gaussian field. The ease of analytical tractability provoked by the normal distribution, however, comes again at the cost of possible negative values for r_t. To circumvent this problem, we will adhere to positive-valued random fields. Observe that a positive-valued field can be generated from an arbitrary random field Z by means of a transformation $g(Z)$, where g denotes some positive function. Specifically, in Section 3.1 we consider a short rate model based on the square of Brownian motion W. Recall that one-dimensional Brownian motion is a zero-mean Gaussian process with covariance function $R(s,t) = s \wedge t$, implying that neither W nor W^2 are stationary processes. The full merits of this model lie in its tractability though, culminating in an affine term structure for zero-coupon bond prices, see Proposition 3.1. In Section 3.2 we advocate a short rate model where the random field component is the square of an OU process. It is well-known that the OU process is a stationary zero-mean Gaussian process, and hence its square is a chi-squared field on the line. Given these model specifications, the analytical tractability and the affine structure for the zero-coupon bond prices are preserved, see Proposition 3.2. As an application, we derive the price of a European call option written on a zero-coupon bond. We conclude the essay on default-free models with an excursion into forward rate random field models, see Section 3.3.

3.1. The short rate process as a squared Brownian motion.

For $t \geq 0$, we consider the following model setup:

$$\begin{aligned} r_t &= r_0 + \alpha(t) + \sigma Y(t) \\ Y(t) &= W^2(t)\,, \\ W &: \text{ standard Brownian motion.} \end{aligned} \tag{3.3}$$

We assume that r_0 and σ are non-negative constants, and that the deterministic function α is positive and differentiable with $\alpha(0) = 0$. Observe that the first two moments of r are given by

$$\begin{aligned} \mathbb{E}[r_t] &= r_0 + \alpha(t) + \sigma t \\ \mathbb{V}\mathrm{ar}(r_t) &= 2\sigma^2 t^2 \\ \mathbb{C}\mathrm{ov}(r_s, r_t) &= 2\sigma^2 (s \wedge t)^2\,. \end{aligned}$$

[1]The solution r of (3.2) will never reach zero from a strictly positive initial value provided $\kappa m > \sigma^2/2$, which is sometimes known as Feller condition.

In the last equality we used the fact that $\mathbb{E}\big[W(s)^2W(t)^2\big] = 2(s\wedge t)^2 + st$. This follows from (2.9) since $\mathbb{C}\mathrm{ov}\big(W(s),W(t)\big) = s\wedge t$, implying that $\rho = \mathbb{C}\mathrm{orr}\big(W(s),W(t)\big) = (s\wedge t)/\sqrt{st}$.

Proposition 3.1. *The short-rate model* (3.3) *provides an affine term structure model in the sense that zero-coupon bond prices can be written as*

$$P(t,T) = e^{A(t,T)-B(t,T)r_t}\,, \qquad \textit{where}$$

$$\begin{aligned} A(t,T) &= \big(r_0+\alpha(t)\big)B(t,T) - \int_t^T \big(r_0+\alpha(s)\big)\,ds \\ &\qquad\qquad + \log\Big(\cosh\big(\sqrt{2\sigma}(T-t)\big)\Big)^{-1/2}, \\ B(t,T) &= \frac{1}{\sqrt{2\sigma}}\,\tanh\big(\sqrt{2\sigma}(T-t)\big)\,. \end{aligned}$$

Remark. Notice that

$$\begin{aligned} P(0,T) &= e^{A(0,T)-B(0,T)r_0} \\ &= e^{-\int_0^T \big(r_0+\alpha(s)\big)\,ds}\,\Big(\cosh\big(T\sqrt{2\sigma}\,\big)\Big)^{-1/2}\,, \end{aligned}$$

a result which can directly be obtained from Proposition 2.1. □

Proof. Recall that $Y = W^2$ is the square of a one-dimensional Bessel process started at 0, i.e. Y is the strong solution of the SDE

$$dY(t) = dt + 2\sqrt{Y(t)}\,dW(t)\,, \qquad Y(0) = 0.$$

Equivalently, $W^2(t) = 2\int_0^t W(s)\,dW(s) + t$. Now observe that the dynamics of r can be written as

$$\begin{aligned} dr_t &= \dot\alpha(t)\,dt + \sigma\,dY(t) \\ &= \dot\alpha(t)\,dt + \sigma\big(dt + 2\sqrt{Y(t)}\,dW(t)\big) \\ &= \big(\dot\alpha(t)+\sigma\big)\,dt + 2\sigma\sqrt{Y(t)}\,dW(t) \\ &= \mu(t,r)\,dt + \sigma(t,r)\,dW(t)\,, \end{aligned}$$

say, with $\mu(t,r) = \dot\alpha(t)+\sigma$ and $\sigma(t,r) = 2\sigma\sqrt{Y(t)}$. It follows that

$$\begin{aligned} \frac{\sigma^2(t,r)}{2} = 2\sigma^2 Y &= 2\sigma(r_t - r_0 - \alpha(t)) \\ &= \sigma(t) + \tau(t)r_t\,, \end{aligned}$$

say, with $\sigma(t) = -2\sigma(r_0+\alpha(t))$ and $\tau(t) = 2\sigma$. Similarly, we can write $\mu(t,r) = \mu(t) + \eta(t)r_t$, where $\mu(t) = \dot\alpha(t)+\sigma$ and $\eta(t) = 0$. It follows from the fundamental PDE for the zero-coupon bond price that the functions $A(t,T)$ and $B(t,T)$ satisfy the following system of differential equations

$$\begin{aligned} -\partial A_t &= \sigma(t)B^2(t,T) - \mu(t)B(t,T)\,, \qquad && A(t,t) = 0\,, \\ \partial B_t &= \tau(t)B^2(t,T) - \eta(t)B(t,T) - 1\,, && B(t,t) = 0\,. \end{aligned}$$

Thus, the function B satisfies $\partial B_t = 2\sigma B^2(t,T) - 1$, from where it follows that

$$B(t,T) = \frac{1}{\sqrt{2\sigma}} \tanh\bigl(\sqrt{2\sigma}(T-t)\bigr)\,.$$

For the function A we get, by means of partial integration,

$$\begin{aligned} A(t,T) &= -\int_t^T \bigl(\partial B_t(u,T)+1\bigr)(r_0+\alpha(u))\,du - \int_t^T \bigl(\dot\alpha(u)+\sigma\bigr)B(u,T)\,du \\ &= \bigl(r_0+\alpha(t)\bigr)B(t,T) - \int_t^T \bigl(r_0+\alpha(s)\bigr)\,ds - \sigma\int_t^T B(u,T)\,du \end{aligned}$$

Since $\int \tanh(x)\,dx = \log\bigl(\cosh(x)\bigr)$, we conclude

$$\begin{aligned} \sigma\int_t^T B(u,T)\,du &= \frac{\sigma}{\sqrt{2\sigma}}\int_t^T \tanh\bigl(\sqrt{2\sigma}(T-u)\bigr)\,du \\ &= \log\Bigl(\cosh\bigl(\sqrt{2\sigma}(T-t)\bigr)\Bigr)/2\,, \end{aligned}$$

which completes the proof. □

Remark. To generate a positive valued spot rate process r originating from linear Brownian motion we could also postulate a model that includes $|W|$ instead of W^2. The default-free bond prices at time $t=0$ can then be calculated via the formula

$$\mathbb{E}\Bigl[e^{-\alpha\int_0^1 |W(s)|\,ds}\Bigr] = \sum_{j=0}^{\infty}\theta_j e^{-\delta_j\alpha^{2/3}}\ ,$$

where δ_j are the positive roots of the derivative of

$$P(y) = \sqrt{2y}\Bigl(J_{-1/3}\bigl((2y)^{3/2}/3\bigr) + J_{1/3}\bigl((2y)^{3/2}/3\bigr)\Bigr)/3\,.$$

Here J_α are Bessel functions of order α and $\theta_j = \bigl(1+3\int_0^{\delta_j}P(y)\,dy\bigr)/(3\delta_j)$, see Kac [14]. □

3.2. **The short rate process as a squared OU process.** As mentioned before, the process W^2 is not a chi-squared process in the sense of Definition 2.2. For $\beta>0$, $\varrho>0$, we shall now consider a process Z given by

$$dZ(t) = -\beta Z(t)\,dt + \varrho\,dW(t)\,,\quad t\ge 0, \tag{3.4}$$

where W denotes standard linear Brownian motion. Equation (3.4) is known as *Langevin's* equation, and has the well-known solution

$$Z(t) = Z(0)e^{-\beta t} + \varrho\int_0^t e^{-\beta(t-s)}\,dW(s)\,,\quad t\ge 0.$$

If the initial variable $Z(0)$ has a normal distribution with mean zero and variance $\varrho^2/(2\beta)$, then Z is called an OU process with parameter β and size $c=\varrho^2/(2\beta)>0$. Note that such a process is a stationary zero-mean Gaussian process with covariance function $R(s,t) = c\exp\{-\beta|t-s|\}$, see for example Karatzas and Shreve [15] p. 358. The eigenvalues and eigenfunctions of R can be derived as in the Brownian case. Let ω_k denote the roots of $\omega + \beta\tan(\omega T) = 0$ and δ_j be the roots of $\beta - \delta\tan(\delta T) = 0$. Then the integral equation $\int_0^T c\, e^{-\beta|x-y|}\phi(y)\,dy = \lambda\phi(x)$ has the eigenvalues

$$\frac{2\beta c}{\omega_1^2+\beta^2}, \ \frac{2\beta c}{\delta_1^2+\beta^2}, \ \frac{2\beta c}{\omega_2^2+\beta^2}, \ \frac{2\beta c}{\delta_2^2+\beta^2}, \ldots$$

and the corresponding eigenfunctions ϕ are given by

$$\frac{\sin(\omega_k x)}{\sqrt{T/2 - \sin(2T\omega_k)/(4\omega_k)}}, \qquad \frac{\cos(\delta_j x)}{\sqrt{T/2 + \sin(2T\delta_k)/(4\delta_k)}},$$

see Appendix A for a derivation of this result.

By analogy with (3.3), we consider the following alternative short rate random field model. For $t \geq 0$, define

$$\begin{aligned} & r_t = \hat{r}_0 + \alpha(t) + \sigma Y(t) \\ (3.5) \qquad & Y(t) = Z^2(t)\,, \\ & Z: \ \text{OU process with parameter } \beta \text{ and size } c = \varrho^2/(2\beta) > 0. \end{aligned}$$

We suppose $\hat{r}_0$ and σ are non-negative constants, and the deterministic function α is positive and differentiable with $\alpha(0) = 0$. Note that Y is a chi-squared field on the real line with parameter 1 and covariance function $R^*(s,t) = 2c^2\exp\{-2\beta|t-s|\}$, see (2.10). Hence, the first two moments of r are given by

$$\begin{aligned} \mathbb{E}[r_t] &= \hat{r}_0 + \alpha(t) + \sigma c \\ \mathbb{V}\mathrm{ar}(r_t) &= 2\sigma^2 c^2 \\ \mathbb{C}\mathrm{ov}(r_s, r_t) &= 2\sigma^2 c^2 \exp\{-2\beta|t-s|\}\,. \end{aligned}$$

Proposition 3.2. *The short-rate model* (3.5) *provides an affine term structure model in the sense that zero-coupon bond prices can be written as*

$$P(t,T) = e^{A(t,T) - B(t,T) r_t}\,, \qquad (3.6)$$

where

$$\begin{aligned} A(t,T) = {} & \big(\hat{r}_0 + \alpha(t)\big)B(t,T) - \int_t^T \big(\hat{r}_0 + \alpha(s)\big)\,ds + \beta(T-t)/2 \\ & + \log\left(\cosh\!\Big(\sqrt{\Delta}(T-t) + \operatorname{atanh}\big(\beta/\sqrt{\Delta}\,\big)\Big)\sqrt{1-\beta^2/\Delta}\right)^{-1/2}, \end{aligned}$$

$$B(t,T) = \frac{-\beta + \sqrt{\Delta}\tanh\left(\sqrt{\Delta}(T-t) + \operatorname{atanh}\big(\beta/\sqrt{\Delta}\,\big)\right)}{2\varrho^2\sigma},$$

with $\Delta = \beta^2 + 2\varrho^2\sigma$.

Remark. When $\beta = 0$ and $\varrho = 1$, the random field component Y is the linear Brownian motion. In that case, the above functions A and B coincide with the corresponding functions in Proposition 3.1. □

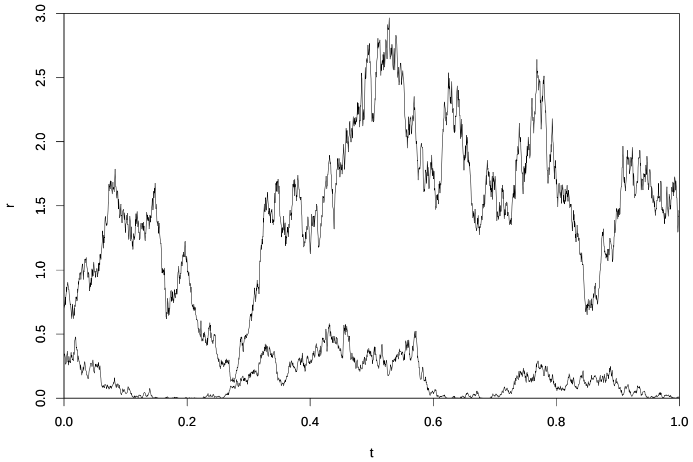

Figure 3: Two simulated sample paths of the short rate process r specified in (3.5) with $\hat{r}_0 = 0$, $\alpha \equiv 0$, and $\beta = \varrho = \sigma = 1$. We used the Euler discretization of (3.4) to obtain simulated versions of the process Z.

Proof. Applying Itô's formula to the function $f(x) = x^2$, we find that

$$\begin{aligned} dY(t) &= 2Z(t)dZ(t) + \varrho^2\,dt \\ &= 2Z(t)\big(-\beta Z(t)\,dt + \varrho dW(t)\big) + \varrho^2\,dt \\ &= \big(\varrho^2 - 2\beta Y(t)\big)\,dt + 2\varrho\sqrt{Y(t)}\,dW(t) \\ Y(0) &= Z^2(0)\,. \end{aligned} \tag{3.7}$$

The dynamics of r then reads

$$\begin{aligned} dr_t &= \dot{\alpha}(t)\,dt + \sigma\,dY(t) \\ &= \dot{\alpha}(t)\,dt + \sigma\big(\varrho^2 - 2\beta Y(t)\big)dt + 2\varrho\sigma\sqrt{Y(t)}\,dW(t) \\ &= \big(\dot{\alpha}(t) + \sigma(\varrho^2 - 2\beta Y(t))\big)\,dt + 2\varrho\sigma\sqrt{Y(t)}\,dW(t) \\ &= \mu(t,r)\,dt + \sigma(t,r)\,dW(t)\,, \end{aligned}$$

say, with $\mu(t,r) = \dot{\alpha}(t) + \sigma\big(\varrho^2 - 2\beta Y(t)\big)$ and $\sigma(t,r) = 2\varrho\sigma\sqrt{Y(t)}$. It follows that

$$\begin{aligned} \frac{\sigma^2(t,r)}{2} = 2\varrho^2\sigma^2 Y &= 2\varrho^2\sigma(r_t - \hat{r}_0 - \alpha(t)) \\ &= \sigma(t) + \tau(t)r_t\,, \end{aligned}$$

say, with $\sigma(t) = -2\varrho^2\sigma(\hat{r}_0 + \alpha(t))$ and $\tau(t) = 2\varrho^2\sigma$. Similarly, we can write $\mu(t,r) = \mu(t) + \eta(t)r_t$, where $\mu(t) = \dot{\alpha}(t) + \sigma\varrho^2 + 2\beta(\hat{r}_0 + \alpha(t))$ and $\eta(t) = -2\beta$. The functions $A(t,T)$ and $B(t,T)$ are the solutions to the system

$$-\partial A_t = \sigma(t)B^2(t,T) - \mu(t)B(t,T)\,, \qquad A(t,t) = 0\,,$$
$$\partial B_t = \tau(t)B^2(t,T) - \eta(t)B(t,T) - 1\,, \qquad B(t,t) = 0\,.$$

With τ and η as above, the Riccati-type equation for B reads

$$\partial B_t = 2\varrho^2\sigma B^2(t,T) + 2\beta B(t,T) - 1, \qquad B(t,t) = 0\,,$$

whence

$$B(t,T) = \frac{-\beta + \sqrt{\Delta}\,\tanh\Big(\sqrt{\Delta}(T-t) + \operatorname{atanh}\big(\beta/\sqrt{\Delta}\,\big)\Big)}{2\varrho^2\sigma}\,, \tag{3.8}$$

where $\Delta = \beta^2 + 2\varrho^2\sigma$. For the function A we obtain

$$\begin{aligned} A(t,T) &= -\int_t^T \big(\partial B_t(u,T) + 1\big)(\hat{r}_0 + \alpha(u))\,du - \int_t^T \big(\dot{\alpha}(u) + \varrho^2\sigma\big)B(u,T)\,du \\ &= \big(\hat{r}_0 + \alpha(t)\big)B(t,T) - \int_t^T \big(\hat{r}_0 + \alpha(s)\big)\,ds - \varrho^2\sigma\int_t^T B(u,T)\,du\,. \end{aligned}$$

Recall that $\int \tanh(x)\,dx = \log\big(\cosh(x)\big)$ and that $\cosh(\operatorname{atanh}(x)) = (1-x^2)^{-1/2}$, hence

$$\begin{aligned} &\int_t^T B(u,T)\,du \\ &\quad= \frac{-\beta(T-t) + \sqrt{\Delta}\int_t^T \tanh\Big(\sqrt{\Delta}(T-u) + \operatorname{atanh}\big(\beta/\sqrt{\Delta}\,\big)\Big)\,du}{2\varrho^2\sigma} \\ &\quad= \frac{-\beta(T-t) + \log\left(\cosh\Big(\sqrt{\Delta}(T-t) + \operatorname{atanh}\big(\beta/\sqrt{\Delta}\,\big)\Big)\sqrt{1-\beta^2/\Delta}\right)}{2\varrho^2\sigma}\,. \end{aligned}$$

This leads to

$$\begin{aligned} A(t,T) &= \big(\hat{r}_0 + \alpha(t)\big)B(t,T) - \int_t^T \big(\hat{r}_0 + \alpha(s)\big)\,ds + \beta(T-t)/2 \\ &\qquad + \log\left(\cosh\Big(\sqrt{\Delta}(T-t) + \operatorname{atanh}\big(\beta/\sqrt{\Delta}\,\big)\Big)\sqrt{1-\beta^2/\Delta}\right)^{-1/2}\,, \end{aligned}$$

which completes the proof. □

Assuming now that for all contingent claims X there exists a suitable self-financing strategy that replicates the claim, the time-t price of such a claim is given by

$$\pi_t(X) = \mathbb{E}_{\mathbb{Q}}\Big[e^{-\int_t^T r_s\,ds}X(T)\big|\mathcal{F}_t\Big]\,. \tag{3.9}$$

In particular, the case of a European call option with maturity S, strike price K and written on a zero-coupon bond with maturity $T > S$ leads to the formula

$$\pi_t = \mathbb{E}_{\mathbb{Q}}\Big[e^{-\int_t^S r_u\,du}\big(P(S,T)-K\big)^+\big|\mathcal{F}_t\Big]\,.$$

For simplicity, we consider the time-0 price of such an option. Let $B(t)$ denote the bank account numéraire, that is $B(t)=\exp\{\int_0^t r_s\,ds\}$. Then

$$\begin{aligned}\pi_0 &= \mathbb{E}_{\mathbb{Q}}\Big[\big(P(S,T)-K\big)^+/B(S)\Big]\\ &= \mathbb{E}_{\mathbb{Q}}\Big[\big(e^{A(S,T)-B(S,T)r_S}-K\big)^+/B(S)\Big]\\ &= \mathbb{E}_{\mathbb{Q}}\Big[e^{A(S,T)-B(S,T)r_S}\mathbb{1}_{\{r_S<r^*\}}\tfrac{1}{B(S)}\Big]-K\,\mathbb{E}_{\mathbb{Q}}\Big[\mathbb{1}_{\{r_S<r^*\}}\tfrac{1}{B(S)}\Big]\,.\end{aligned}$$

where $r^*=\big(A(S,T)-\log(K)\big)/B(S,T)$. We apply the change of numéraire technique to change from the risk neutral measure $\mathbb{Q}$ to the T-forward measure $\mathbb{Q}_T$. The latter is defined by the Radon-Nikodym derivative

$$\frac{d\mathbb{Q}_T}{d\mathbb{Q}}\bigg|_{\mathcal{F}_t}=\frac{P(t,T)}{P(0,T)B(t)}\,,\qquad 0\le t\le T\,,$$

so that the price at time 0 of the above claim is given by

$$\pi_0 = P(0,T)\mathbb{E}_{\mathbb{Q}_T}\big[\mathbb{1}_{\{r_S<r^*\}}\big]-K\,P(0,S)\mathbb{E}_{\mathbb{Q}_S}\big[\mathbb{1}_{\{r_S<r^*\}}\big]\,. \tag{3.10}$$

In what follows, we determine the distribution of the short rate r_S under the forward measures $\mathbb{Q}_T$ and $\mathbb{Q}_S$, respectively. To begin with, recall that the bond price dynamics can easily be obtained via Itô's formula. In an affine model, this leads to

$$\frac{dP(t,T)}{P(t,T)}=r_t\,dt-B(t,T)\sigma(t,r)\,dW(t)\,,$$

where $\sigma(t,r)$ denotes the volatility coefficient of the short rate dynamics. Furthermore, under $\mathbb{Q}_T$,

$$W_T(t)=W(t)-\int_0^t\lambda_s^T\,ds \tag{3.11}$$

is a $\mathbb{Q}_T$-Brownian motion, where $\lambda_t^T=-B(t,T)\sigma(t,r)$. From equations (3.7) and (3.11), we obtain the dynamics of Y under the measure $\mathbb{Q}_T$:

$$\begin{aligned}dY(t) &= \Big(\varrho^2-2Y(t)\big(\beta+2\varrho^2\sigma B(t,T)\big)\Big)\,dt+2\varrho\sqrt{Y(t)}\,dW_T(t)\\ &= \Big(\varrho^2-2Y(t)\beta(t,T)\Big)\,dt+2\varrho\sqrt{Y(t)}\,dW_T(t)\,,\end{aligned}$$

say, where

$$\beta(t,T)=\beta+2\varrho^2\sigma B(t,T). \tag{3.12}$$

Under the measure $\mathbb{Q}_T$, the process Y can be thought of as the square of a process Z whose dynamics is given by

$$dZ(t)=-\beta(t,T)Z(t)+\varrho\,dW_T(t)\,. \tag{3.13}$$

The stochastic differential equation (3.13) has the solution

$$Z(t) = e^{-(\vartheta(t,T)-\vartheta(s,T))} Z(s) + \varrho \int_s^t e^{-(\vartheta(t,T)-\vartheta(u,T))}\, dW_T(u)\,, \quad 0 \le s \le t,$$

where $\vartheta(t,T) = \int_0^t \beta(u,T)\,du$. It follows that, under the T-forward measure $\mathbb{Q}_T$, $Z(t)$ conditional on $Z(s)$ is normally distributed with mean and variance given respectively by

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}_T}[Z(t)|Z(s)] &= Z(s) e^{-\int_s^t \beta(u,T)\,du}\,, \\ \mathbb{V}\mathrm{ar}_{\mathbb{Q}_T}(Z(t)|Z(s)) &= \varrho^2 \int_s^t e^{-2\int_u^t \beta(v,T)\,dv}\,du\,. \end{aligned} \tag{3.14}$$

Before we proceed with the derivation of the bond option price, we notice that the expressions on the right hand side of (3.14) can be calculated explicitly. Indeed, with $B(t,T)$ and $\beta(t,T)$ as given in (3.8) and (3.12), respectively, a straightforward computation shows that, for $\theta \ge 0$,

$$e^{-\theta\int_s^t \beta(v,T)\,dv} = \left(\frac{\cosh\big(\sqrt{\Delta}(T-t)+C\big)}{\cosh\big(\sqrt{\Delta}(T-s)+C\big)}\right)^{\theta}\,, \tag{3.15}$$

where we set for convenience $C = \operatorname{atanh}(\beta/\sqrt{\Delta})$. We obtain the conditional variance of $Z(t)$ given $Z(s)$ if we set $\theta = 2$ in (3.15) and integrate from s to t. This leads to

$$\begin{aligned} \int_s^t e^{-2\int_u^t \beta(v,T)\,dv}\,du &= \cosh^2\big(\sqrt{\Delta}(T-t)+C\big)\int_s^t \cosh^{-2}\big(\sqrt{\Delta}(T-u)+C\big)\,du \\ &= \frac{\cosh^2\big(\sqrt{\Delta}(T-t)+C\big)}{\sqrt{\Delta}} \times \\ &\qquad \Big(\tanh\big(\sqrt{\Delta}(T-s)+C\big) - \tanh\big(\sqrt{\Delta}(T-t)+C\big)\Big)\,. \end{aligned}$$

Now set $x = (r^* - \hat{r}_0 - \alpha(S))/\sigma$. Recall that, under both measures $\mathbb{Q}$ and $\mathbb{Q}_T$, the random variable $Z(0)$ has a normal distribution with mean zero and variance $\varrho^2/(2\beta)$. Consequently, we have that

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}_T}\big[\mathbb{1}_{\{r_S<r^*\}}\big] &= \mathbb{Q}_T\big[r_S < r^*\big] \\ &= \mathbb{Q}_T\big[\hat{r}_0 + \alpha(S) + \sigma Z^2(S) < r^*\big] \\ &= \mathbb{Q}_T\big[Z^2(S) < x\big] \\ &= \int_{\mathbb{R}} \mathbb{Q}_T\big[Z^2(S) < x | Z(0) = z\big] \varphi\big(z\sqrt{2\beta}/\varrho\big)\sqrt{2\beta}/\varrho\,dz\,. \end{aligned}$$

Now introduce the notation

$$\mu_{(U,V,W)}(z) = z e^{-\int_U^V \beta(\omega,W)\,d\omega}\,, \quad \sigma^2_{(U,V,W)} = \varrho^2 \int_U^V e^{-2\int_\tau^V \beta(\omega,W)\,d\omega}\,d\tau\,.$$

Using this formulation, it follows from (3.14) that $Z(S)|Z(0) = z$ has a normal distribution with mean and variance given respectively by $\mu_{(0,S,T)}(z)$ and $\sigma^2_{(0,S,T)}$. We conclude that, under the measure $\mathbb{Q}_T$,

$$Z^2(S)|Z(0) = z \quad \stackrel{\mathcal{L}}{=} \quad \sigma^2_{(0,S,T)}\,\chi^2\left(1, \frac{\mu_{(0,S,T)}(z)}{\sigma_{(0,S,T)}}\right),$$

where $\chi^2(r,\delta)$ denotes a non-centrally chi-squared distributed random variable with r degrees of freedom and non-centrality parameter δ. The density function of such a random variable is given by

$$f_{\chi^2}(x;r,\delta) = \sum_{j=0}^{\infty} \frac{e^{-\delta/2}(\delta/2)^j}{j!}\, f_{\chi^2(r+2j)}(x)\,. \tag{3.16}$$

Here $f_{\chi^2(\rho)}(x)$ is the density function of the (central) chi-squared distribution with ρ degrees of freedom and scaling parameter $\sigma = 1$, see (2.8). Combining the above we deduce that

$$\mathbb{E}_{\mathbb{Q}_T}\left[\mathbb{1}_{\{r_S<r^*\}}\right] = \int_{\mathbb{R}} F_{\chi^2}\left(\tfrac{r^*-\hat{r}_0-\alpha(S)}{\sigma\sigma^2_{(0,S,T)}};1,\tfrac{\mu_{(0,S,T)}(z)}{\sigma_{(0,S,T)}}\right)\varphi\big(z\sqrt{2\beta}/\varrho\big)\sqrt{2\beta}/\varrho\,dz\,,$$

where $F_{\chi^2}(x;r,\delta)$ is the cumulative distribution function of a non-central chi-squared variate with r degrees of freedom and non-centrality parameter δ. Similarly, we can determine the law of r_S under the S-forward measure $\mathbb{Q}_S$, yielding

$$\mathbb{E}_{\mathbb{Q}_S}\left[\mathbb{1}_{\{r_S<r^*\}}\right] = \int_{\mathbb{R}} F_{\chi^2}\left(\tfrac{r^*-\hat{r}_0-\alpha(S)}{\sigma\sigma^2_{(0,S,S)}};1,\tfrac{\mu_{(0,S,S)}(z)}{\sigma_{(0,S,S)}}\right)\varphi\big(z\sqrt{2\beta}/\varrho\big)\sqrt{2\beta}/\varrho\,dz\,.$$

Putting things together, we arrive at the European call option price as specified in (3.10).

Remarks. 1) It is clear that bond prices can be calculated if we can characterize the distribution of $\exp\{-\int_t^T Y(s)\,ds\}$, where $Y = \{Y(t) : t \geq 0\}$ denotes the solution of a process with the dynamics of (3.7). Pitman and Yor [20] consider the stochastic differential equation

$$dh(t) = \big(\delta + 2\alpha h(t)\big)dt + 2\sqrt{h(t)}\,dW(t)\,, \qquad t \geq 0, \tag{3.17}$$

and study the law of $\exp\{-\gamma\int_0^t h(s)\mu(ds)\}$ for non-negative measures $\mu(ds)$ on $\mathbb{R}_+$.

2) The formulation in (3.5) can be generalized in the following way:

$$\begin{aligned} &r_t = \hat{r}_0 + \alpha(t) + \sigma Y(t), \\ &Y(t) = \sum_{i=1}^{n} Z_i^2(t)\,, \\ &Z_i: \ \text{independent OU processes with parameter } \beta_i \\ &\qquad \text{and size } c_i = \varrho_i^2/(2\beta_i) > 0. \end{aligned}$$

We can write $r_t = x(t) + y_2(t) + \cdots + y_n(t)$, where $x(t) = \hat{r}_0 + \alpha(t) + \sigma Z_1^2(t)$ and $y_k(t) = \sigma Z_k^2(t)$, $k = 2,\ldots,n$. Due to independence of the processes Z_k, we have that

$$P(t,T) = \mathbb{E}\Big[e^{-\int_t^T x(s)\,ds}\big|\mathcal{F}_t\Big]\prod_{k=2}^{n}\mathbb{E}\Big[e^{-\int_t^T y_k(s)\,ds}\big|\mathcal{F}_t\Big]$$
$$= P_1\big(t,T;x(t),\Theta_1\big)\prod_{k=2}^{n}P_1\big(t,T;y_k(t),\Theta\big)\,,$$

where $\Theta_1 = (\hat r_0, \alpha(t))$ and $\Theta = (0,0)$. The function $P_1(t,T;x(t),\Theta_1)$ is the bond price formula (3.6) with r_t replaced by $x(t)$. Analogously, the function $P_1(t,T;y_k(t),\Theta)$ denotes the bond price formula (3.6) with r_t replaced by $y_k(t)$ and $\hat r_0$, $\alpha(t)$ replaced by $\Theta = (0,0)$. This implies that the bond prices $P(t,T)$ are expressible as exponential-affine functions of the n factors x, $y_2,\ldots,y_n$. □

3.3. **Forward rate random field models.** Up to now, we confined ourselves to single-parameter random fields. In Sections 3.1 and 3.2 we introduced short rate models which are indexed by time as their sole parameter. Bond prices are then characterized by means of the fundamental relation (3.1). An alternative and widely used approach to the bond price modeling is based on an exogenous specification of a family $f(t,T)$ of forward rates, where $0 \le t \le T \le T^*$ and T^* denotes a fixed horizon date. That is, $f(t,T)$ is the forward interest rate at date $t \le T$ for instantaneous risk-free borrowing or lending over the infinitesimal period $[T, T+\Delta T]$. Given such a family $f(t,T)$, bond prices are then characterized by the relation

$$P(t,T) = \exp\Big\{-\int_t^T f(t,v)\,dv\Big\}\,.$$

Heath, Jarrow and Morton (HJM) assumed that, for a fixed maturity T, the forward rate $f(t,T)$ evolves according to the diffusion process

$$df(t,T) = \alpha(t,T)\,dt + \sigma(t,T)\,dW(t)\,, \qquad f(0,T) = f^{\mathrm{M}}(0,T)\,, \tag{3.18}$$

with $T \mapsto f^{\mathrm{M}}(0,T)$ denoting the market curve at time $t = 0$. The advantage of modeling forward rates as in (3.18) is that the initial term structure of interest rates is, by construction, an input of the model. The dynamics in (3.18), however, is not necessarily arbitrage-free. Heath, Jarrow and Morton proved that, in order for an equivalent martingale measure to exist, the function α can not be chosen arbitrarily, but must satisfy

$$\alpha(t,T) = \sigma(t,T)\int_t^T \sigma(t,v)\,dv\,.$$

Kennedy [16], [17] interprets the forward rate $f(t,T)$ as a two-parameter random field. Specifically, for a Gaussian random field Z, he considers the forward rate surface $f(t,T) = \mu(t,T) + Z(t,T)$, and derives a necessary and sufficient restriction on the drift function $\mu(t,T)$ to ensure that discounted zero-bond prices are martingales under the risk-neutral measure. Goldstein [13] concentrates on the dynamics of the forward rate, $df(t,T) = \mu(t,T)\,dt + \sigma(t,T)\,dZ(t,T)$, and generalizes Kennedy's drift restriction result to non-Gaussian fields. The purpose of this section is to set the scene for Section 4.4. Proposition 4.1 of that section generalizes Proposition 1 in Goldstein [13] to credit risk models under the recovery of market value assumption.

Let $Z(t,T)$, $t \le T$, be a random field labeled with two time indices. The first parameter refers to the current time, the second to the maturity date. For all dates t, the random field describes a realization of a random function $T \mapsto Z(t,T)$. The correlation between the quantities $Z(t_1,T_1)$ and $Z(t_2,T_2)$ is expressed as $\mathbb{C}\mathrm{ov}\big(Z(t_1,T_1),Z(t_2,T_2)\big) = R(t_1,t_2,T_1,T_2)$, where the function R is non-negative definite in (t_i,T_i), $i \in \{1,2\}$. Specifically, we suppose

$$\mathbb{C}\mathrm{ov}\big(Z(t_1,T_1),Z(t_2,T_2)\big) = R(t_1 \wedge t_2, T_1, T_2)\,. \tag{3.19}$$

The fact that the covariance function R is specified as a function of $t_1 \wedge t_2$ ensures that the increments of the random field in the t-direction are uncorrelated. Indeed, for t, $h \in \mathbb{R}_+$ such that $t+h \le T$ and $u \le t \wedge S$ we have that

$$\begin{aligned}
&\mathbb{C}\mathrm{ov}\big(Z(t+h,T) - Z(t,T), Z(u,S)\big)\\
&\qquad = \mathbb{C}\mathrm{ov}\big(Z(t+h,T), Z(u,S)\big) - \mathbb{C}\mathrm{ov}\big(Z(t,T), Z(u,S)\big)\\
&\qquad = R(u,T,S) - R(u,T,S)\\
&\qquad = 0.
\end{aligned}$$

If the random field Z is Gaussian, this property is referred to as the independent increments property in the t-direction.

By analogy with the Itô calculus, the correlation structure (3.19) may also be expressed via the cross variation

$$d\langle Z(\cdot,T_1), Z(\cdot,T_2)\rangle_t = c(t,T_1,T_2)\,dt$$

for a deterministic function c which is symmetric in T_1 and T_2 and non-negative definite in (t,T_1) and (t,T_2).

Lemma 3.1. *Define the forward rate dynamics as*

$$df(t,T) = \mu(t,T)\,dt + \sigma(t,T)\,dZ(t,T)\,,$$

where $Z(t,T)$ is a random field with deterministic correlation structure c specified by $d\langle Z(\cdot,T_1), Z(\cdot,T_2)\rangle_t = c(t,T_1,T_2)\,dt$. We suppose that $\mu(t,T)$ and $\sigma(t,T)$ satisfy the technical regularity conditions imposed by the HJM framework. Define $I_t = \int_t^T f(t,v)\,dv$, whence $P(t,T) = \exp\{-I_t\}$. Then

(a) $$dI_t = \mu^*(t,T)\,dt + \int_t^T dv\,\sigma(t,v)\,dZ(t,v) - r_t\,dt$$

(b) $$\frac{dP(t,T)}{P(t,T)} = -\mu^*(t,T)\,dt - \int_t^T dv\,\sigma(t,v)\,dZ(t,v) + r_t\,dt + \frac{1}{2}\int_t^T \sigma(t,u)\sigma^*(t,T,u)\,du\,dt\,,$$

where $r_t = f(t,t)$ and

$$\mu^*(t,T) = \int_t^T \mu(t,s)\,ds\,, \qquad \sigma^*(t,T,S) = \int_t^T \sigma(t,v)c(t,S,v)\,dv.$$

Proof. (a)

$$\begin{aligned} I_t &= \int_t^T f(t,v)\,dv \\ &= \int_t^T \Big(f(0,v) + \int_0^t \mu(s,v)\,ds + \int_0^t \sigma(s,v)\,dZ(s,v)\Big)\,dv \\ &= \int_t^T f(0,v)\,dv + \int_t^T\int_0^t \mu(s,v)\,ds\,dv + \int_t^T\int_0^t \sigma(s,v)\,dZ(s,v)\,dv\,. \end{aligned}$$

Applying the stochastic version of Fubini's theorem, see for example Protter [21], Theorem 45 p. 159, we can write

$$\begin{aligned} I_t &= \int_0^T f(0,v)\,dv + \int_0^t\int_s^T \mu(s,v)\,dv\,ds + \int_0^t\int_s^T \sigma(s,v)\,dv\,dZ(s,v) \\ &\quad - \int_0^t f(0,v)\,dv - \int_0^t\int_s^t \mu(s,v)\,dv\,ds - \int_0^t\int_s^t \sigma(s,v)\,dv\,dZ(s,v) \\ &= I_0 + \int_0^t \mu^*(s,T)\,ds + \int_0^t\int_s^T dv\,\sigma(s,v)\,dZ(s,v) \\ &\quad - \int_0^t \Big(f(0,v) + \int_0^v \mu(s,v)\,ds + \int_0^v \sigma(s,v)\,dZ(s,v)\Big)\,dv \\ &= I_0 + \int_0^t \mu^*(s,T)\,ds + \int_0^t\int_s^T dv\,\sigma(s,v)\,dZ(s,v) - \int_0^t f(v,v)\,dv \\ &= I_0 + \int_0^t \mu^*(s,T)\,ds + \int_0^t\int_s^T dv\,\sigma(s,v)\,dZ(s,v) - \int_0^t r_v\,dv\,. \end{aligned}$$

The differential form of I_t thus reads

$$dI_t = \mu^*(t,T)\,dt + \int_t^T dv\,\sigma(t,v)\,dZ(t,v) - r_t\,dt\,,$$

which leaves (a).

(b) By definition, $P(t,T) = \exp\{-I_t\}$. Applying Itô's formula to $f(x) = \exp\{-x\}$ yields

$$\begin{aligned} \frac{dP(t,T)}{P(t,T)} &= -dI_t + \frac{1}{2} d\langle I.\rangle_t \\ &= -\mu^*(t,T)\,dt - \int_t^T dv\,\sigma(t,v)\,dZ(t,v) + r_t\,dt \\ &\quad + \frac{1}{2}\int_t^T \sigma(t,u)\sigma^*(t,T,u)\,du\,dt. \end{aligned} \tag{3.20}$$

In the last summand on the right side of (3.20) we used the multiplication formalism $dZ(t,u)\,dZ(t,v) = c(t,u,v)\,dt$. □

Proposition 3.3. *Let the dynamics of the forward rates be specified as in Lemma 3.1. Then the risk-neutral drift restriction is given by*

$$\mu(t,T) = \sigma(t,T)\sigma^*(t,T,T)\,. \tag{3.21}$$

Proof. Under the measure $\mathbb{Q}$, bond prices discounted at the short rate process r are $\mathbb{Q}$-martingales. Set $Z^*(t,T) = P(t,T)/B(t)$, where $B(t)$ denotes the bank account numéraire, i.e. $B(t) = \exp\{\int_0^t r_s\,ds\}$. Recall that $dB(t) = B(t)\,r_t\,dt$, hence the dynamics of $Z^*(t,T)$ reads

$$\begin{aligned} dZ^*(t,T) &= \frac{dP(t,T)}{B(t)} + P(t,T)\,d\Big(\frac{1}{B(t)}\Big) \\ &= \frac{dP(t,T)}{B(t)} + P(t,T)\,\Big(-\frac{1}{B^2(t)}\,dB(t)\Big) \\ &= \frac{dP(t,T)}{B(t)} - \frac{P(t,T)}{B(t)}\,r_t\,dt \\ &= Z^*(t,T)\frac{dP(t,T)}{P(t,T)} - Z^*(t,T)\,r_t\,dt\,. \end{aligned}$$

Consequently, by Lemma 3.1,

$$\begin{aligned} \frac{dZ^*(t,T)}{Z^*(t,T)} &= \frac{dP(t,T)}{P(t,T)} - r_t\,dt \\ &= -\mu^*(t,T)\,dt - \int_t^T dv\,\sigma(t,v)\,dZ(t,v) \\ &\qquad + \frac{1}{2}\int_t^T \sigma(t,u)\sigma^*(t,T,u)\,du\,dt\,. \end{aligned}$$

In order $Z^*(t,T)$ be a $\mathbb{Q}$-martingale, the dt-terms must vanish, whence

$$\mu^*(t,T) = \frac{1}{2}\int_t^T \sigma(t,u)\sigma^*(t,T,u)\,du\,. \tag{3.22}$$

Differentiating (3.22) with respect to T yields (3.21), which completes the proof. □

4. Models with credit risk

When pricing default-free securities, equation (3.9) provides the fundamental pricing rule. If, however, the issuer defaults before maturity T, then both the amount and timing of the payoff to the investor are uncertain. In this case, it is often convenient to interpret a zero-bond as a portfolio comprising two securities: a security that pays one unit at maturity T if and only if the issuer survives up to time T and a security that pays the (random) amount Z received at default if default happens before T. Let τ denote the default time and let $\Lambda_t = \mathbb{1}_{\{\tau\le t\}}$ be the default indicator process. Consequently, $1-\Lambda_t = \mathbb{1}_{\{\tau>t\}}$ is the survival indicator which has outcome 1 if the issuer has not defaulted prior to t and zero otherwise. With this notation, the price of a zero-coupon bond is then

$$\overline{P}(t,T) = \mathbb{E}_{\mathbb{Q}}\Big[e^{-\int_t^T r_s\,ds}(1-\Lambda_T)\Big|\mathcal{F}_t\Big] + \mathbb{E}_{\mathbb{Q}}\Big[e^{-\int_t^T r_s\,ds} Z\Lambda_T\Big|\mathcal{F}_t\Big]\,. \tag{4.1}$$

Under zero recovery, i.e. $Z=0$, (4.1) reduces to

$$\overline{P}(t,T) = \mathbb{E}_{\mathbb{Q}}\Big[e^{-\int_t^T r_s\,ds}(1-\Lambda_T)\big|\mathcal{F}_t\Big]\,. \tag{4.2}$$

4.1. **Reduced form models.** In a reduced-form pricing framework, it is assumed that a risk-neutral default intensity process $\lambda = \{\lambda(t) : t \geq 0\}$ is associated to the default time τ. That is, τ is the first jump time of a counting process with intensity λ. We suppose for the moment that there is no recovery at default whatsoever. An implication of these assumptions, as shown by Lando [18], is that the bond price (4.2) can be calculated as

$$\overline{P}(t,T) = \mathbb{E}_{\mathbb{Q}}\Big[e^{-\int_t^T (r_s+\lambda(s))\,ds}\big|\mathcal{F}_t\Big]\,, \tag{4.3}$$

provided that default has not occurred by time t. In the special case in which the default time τ and the short rate process r are independent, the bond price formula (4.3) can be decomposed into

$$\overline{P}(t,T) = \mathbb{E}_{\mathbb{Q}}\Big[e^{-\int_t^T r_s\,ds}\big|\mathcal{F}_t\Big]\mathbb{E}_{\mathbb{Q}}\Big[e^{-\int_t^T \lambda(s)\,ds}\big|\mathcal{F}_t\Big]\,. \tag{4.4}$$

Under the assumption that default has not already occurred by time t,

$$p(t,T) = \mathbb{E}_{\mathbb{Q}}\big[\mathbb{1}_{\{\tau>T\}}|\mathcal{F}_t\big] = \mathbb{E}_{\mathbb{Q}}\Big[e^{-\int_t^T \lambda(s)\,ds}\big|\mathcal{F}_t\Big] \tag{4.5}$$

is the risk-neutral conditional survival probability.

The existing reduced-form models can be extended to allow for non-zero recovery. The current bond price literature encompasses a variety of recovery models. By and large, they all assume that, conditional on the occurrence of default in the next instant, the bond under consideration has a given expected fractional recovery. A prominent class among these models, introduced by Duffie and Singleton [11] and referred to as recovery of market value (RMV) model, takes recovery to be a fraction of the market value of the bond just prior to default. To be more precise, the assumption is that, at each time t, the claim pays $(1-L_t)V_{t-}$, where $V_{t-} = \lim_{s\uparrow t} V_s$ is the price of the claim just before default, and L_t is the random variable describing the fractional loss of market value of the claim at default. It is assumed that the loss process $L = \{L_t : t \geq 0\}$ is bounded by 1 and predictable. Under technical conditions, Duffie and Singleton [11] derive the following pricing rule for a zero-coupon bond at any time t before default

$$\overline{P}(t,T) = \mathbb{E}_{\mathbb{Q}}\Big[e^{-\int_t^T \bar{r}_s\,ds}\big|\mathcal{F}_t\Big]\,, \tag{4.6}$$

where $\bar{r}$ denotes the default-adjusted short rate, i.e.

$$\bar{r}_t = r_t + \lambda(t)L_t\,, \qquad t \geq 0. \tag{4.7}$$

From here, one can now proceed in various directions. For example, one can model $\bar{r}$ directly or rather its components r and s, where $s(t) = \lambda(t)L_t$ denotes the short spread.

Henceforth, we shall assume that $L \equiv 1$, implying $s(t) = \lambda(t)$. Inspired by the setup introduced in Section 3, we postulate a random field model for both processes r and λ:

$$r_t = \hat{r}_0 + \alpha(t) + \sigma_r Y(t)\,,$$
$$\lambda(t) = \lambda_0 + \mu(t) + \sigma_\lambda U(t)\,,$$

where $\hat{r}_0$, λ_0, σ_r, σ_λ are non-negative constants, α and μ positive and differentiable functions with $\alpha(0) = \mu(0) = 0$. Furthermore, we assume that Y and U are chi-squared fields with parameters n and m, respectively. That is, $Y(t) = Z_1^2(t) + \cdots + Z_n^2(t)$ and $U(t) = V_1^2(t) + \cdots + V_m^2(t)$ for independent and stationary, zero-mean, Gaussian processes Z_k, $k = 1, \ldots, n$, and V_j, $j = 1, \ldots, m$.

Assuming independence between the fields Y and U, the conditional survival probabilities (4.5) can be calculated explicitly if we use the formulation of Section 3.2 to model the default intensity and short rate process. Empirical evidence, however, suggest that default intensities vary with the business cycle. During recessions, when interest rates are low, default rates tend to be higher. A (negative) correlation between the processes r and λ can be captured through the joint dependence of r and λ on some of the factors Z_k and V_j. As an example, we consider the following setup. Let δ be a real number with $|\delta| \leq 1$. Then define

$$r_t = \hat{r}_0 + \alpha(t) + \sigma_r Y(t)\,, \tag{4.8}$$
$$\lambda(t) = \lambda_0 + \mu(t) + \sigma_\lambda\Big(\delta Y(t) + \sqrt{1-\delta^2}\,\widetilde{Y}(t)\Big)\,, \tag{4.9}$$

where Y and $\widetilde{Y}$ are independent chi-squared fields with common parameter n. Obviously, we have that $\mathbb{C}\mathrm{ov}\big(r_t, \lambda(t)\big) = 2n\delta\sigma_r\sigma_\lambda$ and consequently $\mathbb{C}\mathrm{orr}\big(r_t, \lambda(t)\big) = \delta$. The degree of correlation between r_t and $\lambda(t)$ is thus specified by the constant δ. Here we assumed that $\mathbb{V}\mathrm{ar}(Z_k(t)) = R_Z(0) = 1$.

Next, we would like to calculate the defaultable bond price $\overline{P}(t,T)$ in (4.6). Adding up (4.8) and (4.9), we can write

$$\begin{aligned}\bar{r}_t &= r_t + \lambda(t)\\ &= \bar{r}_0 + \theta(t) + \varsigma Y(t) + \widetilde{\sigma}\widetilde{Y}(t)\,,\\ &= x(t) + y(t)\,,\end{aligned}$$

say, where $x(t) = \bar{r}_0 + \theta(t) + \varsigma Y(t)$ and $y(t) = \widetilde{\sigma}\widetilde{Y}(t)$ with $\bar{r}_0 = \hat{r}_0 + \lambda_0$, $\theta(t) = \alpha(t) + \mu(t)$, $\varsigma = \sigma_r + \delta\sigma_\lambda$, and $\widetilde{\sigma} = \sigma_\lambda\sqrt{1-\delta^2}$.

As an example, we concentrate again on the model specifications (3.5). That is, Y and $\widetilde{Y}$ are assumed to be the squares of independent OU processes. Due to the independence of the fields Y and $\widetilde{Y}$, the price at time t of the defaultable zero-bond is given by

$$\begin{aligned}\overline{P}(t,T) &= \mathbb{E}_{\mathbb{Q}}\Big[e^{-\int_t^T x(s)\,ds}\big|\mathcal{F}_t\Big]\,\mathbb{E}_{\mathbb{Q}}\Big[e^{-\int_t^T y(s)\,ds}\big|\mathcal{F}_t\Big]\\ &= P_1(t,T;x(t),\Theta_1)\,P_1(t,T;y(t),\Theta_2)\,,\end{aligned} \tag{4.10}$$

say, where $\Theta_1 = (\bar{r}_0, \theta(t), \varsigma)$ and $\Theta_2 = (0, 0, \widetilde{\sigma})$. The function $P_1(t, T; x(t), \Theta_1)$ is the bond price formula (3.6) with r_t replaced by $x(t)$ and $\hat{r}_0$, $\alpha(t)$, σ replaced by $\Theta_1 = (\bar{r}_0, \theta(t), \varsigma)$. The function $P_1(t, T; y(t), \Theta_2)$ is defined analogously. Equation (4.10) thus tells us that defaultable bond prices are expressible as exponential-affine functions of the factors x and y.

4.2. **A reduced form random field model.** In the following setup we take as given a short rate process r modeled by a non-negative valued process. In addition, we consider a family of obligors indexed by $\iota \in \mathscr{I}$. We allow the index set $\mathscr{I}$ be countably or uncountably infinite. We denote $\lambda(t, \iota)$ the default intensity at time t of obligor $\iota \in \mathscr{I}$ and assume that $\lambda(t, \iota)$ is a two-parameter positive-valued random field:

$$
\begin{aligned}
r_t &= \hat{r}_0 + \alpha(t) + \sigma_r Y(t)\,, && t \geq 0, \\
\lambda(t, \iota) &= \lambda_0^\iota + \mu(t, \iota) + \sigma_\lambda(\iota) U(t, \iota)\,, && t \geq 0,\ \iota \in \mathscr{I}.
\end{aligned}
\tag{4.11}
$$

We suppose $\hat{r}_0$, λ_0^ι σ_r, $\sigma_\lambda(\cdot)$ are non-negative constants and the deterministic functions α, μ are non-negative and differentiable with $\alpha(0) = \mu(0, \cdot) = 0$. In the sequel, we assume that $\mathscr{I} \subseteq [0, 1]$ and that Y, U are chi-squared random fields with parameters n and m, respectively. The field U thus admits the representation $U(t, \iota) = V_1^2(t, \iota) + \cdots + V_m^2(t, \iota)$, where V_k, $k = 1, \ldots, m$, are independent and stationary, zero-mean space-time Gaussian random fields with deterministic covariance function R_V:

$$
R_V(t, \kappa) = \mathbb{C}\mathrm{ov}\big(V(s, \iota), V(s + t, \iota + \kappa)\big)\,,
$$

for ι, $\kappa \in \mathscr{I}$ with $|\iota + \kappa| \leq 1$. We interpret t as the temporal lag and ι as the spatial lag of the field V_k. Observe that, for s, $t \geq 0$ and ι, $\kappa \in \mathscr{I}$

$$
\begin{aligned}
\mathbb{C}\mathrm{ov}\big(\lambda(s, \iota), \lambda(t, \kappa)\big) &= \sigma_\lambda(\iota)\sigma_\lambda(\kappa)\mathbb{C}\mathrm{ov}\big(U(s, \iota), U(t, \kappa)\big) \\
&= 2m\, \sigma_\lambda(\iota)\, \sigma_\lambda(\kappa)\, R_V^2\left(|s - t|, \|\iota - \kappa\|\right).
\end{aligned}
$$

Assuming that default of obligor $\iota \in \mathscr{I}$ has not yet occurred by time t, the risk-neutral conditional survival probability $p(t, T; \iota)$ is given by

$$
p(t, T; \iota) = \mathbb{E}_{\mathbb{Q}}\Big[e^{-\int_t^T \lambda(s, \iota)\, ds} \big| \mathcal{F}_t\Big]\,. \tag{4.12}
$$

Notice that

$$
\int_t^T \lambda(s, \iota)\, ds = \lambda_0^\iota (T - t) + \int_t^T \mu(s, \iota)\, ds + \sigma_\lambda(\iota) \int_t^T U(s, \iota)\, ds
$$

and that the first two moments of $J(t, T; \iota) = \int_t^T U(s, \iota) ds$ are given by

$$
\begin{aligned}
\mathbb{E}\big[J(t, T; \iota)\big] &= m R_V(0, 0)(T - t)\,, \\
\mathbb{C}\mathrm{ov}\big(J(t, T; \iota), J(t, T; \kappa)\big) &= 2m \int_t^T\!\!\int_t^T R_V^2\left(|u - v|, \|\iota - \kappa\|\right) du\, dv\,.
\end{aligned}
$$

However, knowledge of the first two moments of $J(t, T; \iota)$ in general is not enough to calculate the survival probability (4.12). To make use of the results in Sections 3.2 and 4.1, we employ a separable correlation structure.

Assume that R_V is of the form

$$R_V\big(|s-t|,\|\iota-\kappa\|\big) = R_V^{(1)}\big(|s-t|\big)\, R_V^{(2)}\big(\|\iota-\kappa\|\big)\,.$$

Specifically, let $m=1$ and $R_V^{(1)}\big(|s-t|\big) = c\exp\{-\beta|s-t|\}$. Then, for each $\iota\in\mathscr{I}$, the default intensity $\lambda(\cdot,\iota)$ of obligor ι evolves as the square of an OU process. The spatial correlation can be captured by an arbitrary permissible covariance structure, e.g. $R_V^{(2)}(\|\iota-\kappa\|) = 1-\vartheta\|\iota-\kappa\|$ with $\vartheta\le 2$, or $R_V^{(2)}(\|\iota-\kappa\|) = \exp\{-\theta\|\iota-\kappa\|\}$ for $\theta>0$. Under these assumptions, the conditional probability $p(t,T;\iota)$ at time t that issuer ι survives to time T is of the form

$$p(t,T;\iota) = P_1(t,T;\lambda(t,\iota),\Theta)\,, \tag{4.13}$$

where $P_1(t,T;\lambda(t,\iota),\Theta)$ denotes the bond price formula (3.6) with r_t replaced by $\lambda(t,\iota)$ and r_0, $\alpha(t)$, σ replaced by $\Theta = (\lambda_0^\iota,\mu(t,\iota),\sigma_\lambda(\iota))$. Assuming independence between the default-free short rate process r and the intensity surface $\lambda(t,\iota)$, the zero-coupon bond prices $\overline{P}(t,T;\iota)$ under the zero recovery assumption are given by

$$\overline{P}(t,T;\iota) = \mathbb{E}_{\mathbb{Q}}\Big[e^{-\int_t^T r_s\,ds}\big|\mathcal{F}_t\Big]\, p(t,T;\iota)\,,$$

with $p(t,T;\iota)$ as in (4.13). If moreover r is an affine function of the variable Y as outlined in Section 3.2, we deduce that $\overline{P}(t,T;\iota)$ is an exponential-affine function of the factors r and $\lambda(\cdot,\iota)$.

4.3. **A simplified firm's value random field model.** In this section we apply the random field methodology to structural credit risk models. In doing so, we consider a family of obligors $\mathscr{I}\subseteq[0,1]$ and assume that default of each obligor ι is triggered by the change in value of the assets of its firm, see Schönbucher [25] p. 305 for more details on a simplified firm's value approach. Specifically, denote $V(t,\iota)$ the market value at time t of the assets of obligor ι and assume that $V(t,\iota)$ is of the form

$$V(t,\iota) = \mu(t,\iota) + \sigma(t,\iota)Y(t,\iota)\,,\qquad t\ge 0.$$

We suppose μ and σ are deterministic functions and Y is a two-parameter, spatio-temporal random field which is assumed to be homogeneous in t and isotropic in ι with covariance function R. Furthermore, we assume that obligor ι's default occurs at maturity date T in the event that its asset value falls below a prespecified deterministic barrier δ, i.e. if $V(T,\iota)<\delta(\iota)$.

In the following we are interested in the proportion $N(T)$ of obligors whose asset value V at time T is below the barrier δ. Define

$$N(t) = \frac{1}{\mu(\mathscr{I})}\int_{\mathscr{I}} \mathbb{1}_{\{V(t,x)<\delta(x)\}}\,dx, \tag{4.14}$$

where $\mu(\mathcal{A})$ denotes the Lebesgue measure of the set $\mathcal{A}$. Let $\mathscr{I}=[0,1]$, hence $\mu(\mathscr{I})=1$. For the mean value of $N(T)$, it is immediate that

(4.15)
$$\begin{aligned}\mathbb{E}[N(T)] &= \int_0^1 \mathbb{E}\Big[\mathbb{1}_{\{V(T,x)<\delta(x)\}}\Big]\,dx\\ &= \int_0^1 \mathbb{P}\Big[Y(T,x) < \tfrac{\delta(x)-\mu(T,x)}{\sigma(T,x)}\Big]\,dx\\ &= \int_0^1 \mathbb{P}\Big[Y(0,0) < \tfrac{\delta(x)-\mu(T,x)}{\sigma(T,x)}\Big]\,dx\,.\end{aligned}$$

The last equality follows from the homogeneity of the field Y. Similarly, we can write down an expression for the second moment of N.

(4.16)
$$\begin{aligned}\mathbb{E}\big[N^2(T)\big] &= \int_0^1\!\!\int_0^1 \mathbb{E}\Big[\mathbb{1}_{\{V(T,x)<\delta(x)\}\cap\{V(T,y)<\delta(y)\}}\Big]\,dx\,dy\\ &= \int_0^1\!\!\int_0^1 \mathbb{P}\Big[Y(T,x) < \tfrac{\delta(x)-\mu(T,x)}{\sigma(T,x)}, Y(T,y) < \tfrac{\delta(y)-\mu(T,y)}{\sigma(T,y)}\Big]\,dx\,dy\,.\end{aligned}$$

Now assume that Y is a zero-mean Gaussian random field with unit variance and covariance function $R(t,x)$. Let us set $\psi(z) = \big(\delta(z) - \mu(T,z)\big)/\sigma(T,z)$. Equation (4.15) simplifies to

$$\mathbb{E}[N(T)] = \int_0^1 \Phi\big(\psi(x)\big)\,dx\,,$$

where $\Phi(x) = \int_{-\infty}^{x} \exp\{-u^2/2\}du/\sqrt{2\pi}$ is the standard normal distribution function. Denoting $\phi(x,y;\rho)$ the bivariate Gaussian density function

$$\phi(x,y;\rho) = \frac{1}{2\pi\sqrt{1-\rho^2}}\exp\Big\{-\tfrac{x^2+2\rho xy+y^2}{2(1-\rho^2)}\Big\}$$

the second moment of $N(T)$ reads

$$\mathbb{E}\big[N^2(T)\big] = \int_0^1\!\!\int_0^1 \int_{-\infty}^{\psi(x)}\int_{-\infty}^{\psi(y)} \phi\big(v,w;R(0,|x-y|)\big)dv\,dw\,dx\,dy\,.$$

Suppose for the moment that $\psi \equiv 0$. In that case, the above integral can be simplified since

$$\int_{-\infty}^{0}\!\!\int_{-\infty}^{0} \phi\big(v,w;\rho\big)\,dv\,dw = \frac{\arccos(\rho)}{2\pi}\,.$$

In the general case $\psi(z) \neq 0$, however, it appears that closed form solutions for the (higher) moments of $N(T)$ do not exist, nor does it seem possible to obtain exact distribution results for N. Hence, the moments of $N(T)$ must be determined numerically.

Let us for the moment abandon the assumption of stationarity. Instead, we shall consider a particular separable model which is invariant in time but not stationary in space. For the spatial correlation, we recourse to the Brownian covariance structure. That is, we suppose the covariance function R of the random field $Y(t,\iota)$ has the form:

$$R\big((t,x),(s,y)\big) = R^{(1)}(|s-t|)\,(x\wedge y)\,,$$

where $R^{(1)}$ is an admissible temporal covariance function on the real line. Suppose again $\psi \equiv 0$, whence $N(T) = \int_0^1 \mathbb{1}_{\{Y(T,x)<0\}}\,dx$. Since T is fixed, $x \mapsto Y(T,x)$ is one-dimensional Brownian motion starting from 0 (except for the scaling factor $\sqrt{R^{(1)}(0)}$). The distribution of $N(T)$ is thus specified by Paul Lévy's Arcsine law for the occupation time of $(0,\infty)$:

$$\mathbb{P}[N(T) \le x] = \int_0^x \frac{1}{\pi\sqrt{u(1-u)}}\,du = \frac{2}{\pi}\arcsin\big(\sqrt{x}\,\big)\,, \quad 0 \le x \le 1,$$

see for instance Karatzas and Shreve [15], p. 273. In particular, we have that $\mathbb{E}[N(T)] = 1/2$ and $\mathbb{V}\mathrm{ar}(N(T)) = 1/8$.

So far, we confined ourselves to model the obligors' firm values as a Gaussian random field. The drawback of this approach is that firm values can attain negative values with positive probability. To avoid negative outcomes we can model the firm values as a chi-squared field:

$$V(t,\iota) = \mu(t,\iota) + \sigma(t,\iota)Y(t,\iota)\,,$$

where

$$Y(t,\iota) = Z_1^2(t,\iota) + \cdots + Z_n^2(t,\iota)\,.$$

Here Z_k, $k = 1,\ldots,n$ are independent and stationary zero-mean Gaussian random fields with covariance function R_Z and variance $\sigma^2 = R_Z(0,0)$. It is obvious that we can write down expressions for the moments of N also in this case. The mean fraction of obligors that will default by time T is given by

$$\mathbb{E}[N(T)] = \int_0^1 F_{\chi^2(n,\sigma)}(\psi(x))\,dx\,,$$

where $F_{\chi^2(r,\sigma)}$ denotes the cumulative distribution function of a central chi-squared distributed random variable with r degrees of freedom and scaling parameter σ, see (2.8) for the definition of the corresponding density function. To derive the second moment, we suppose for simplicity $n = 1$. Recall from Section 2.3 that $Z(T,x)$ conditional on $Z(T,y)$ has a normal law with mean value $R_Z(0,|x-y|)Z(T,y)/\sigma^2$ and variance $\sigma^2 - R_Z^2(0,|x-y|)/\sigma^2$, respectively. Hence

$$\begin{aligned}\mathbb{E}\big[N^2(T)\big] &= \int_0^1\!\!\int_0^1 \mathbb{P}\big[Z^2(T,x) < \psi(x), Z^2(T,y) < \psi(y)\big]\,dx\,dy \\ &= \int_0^1\!\!\int_0^1 \int_{-\infty}^{\sqrt{\psi(y)}} \mathbb{P}\big[X^2 < \psi(x)\big]\,\varphi\Big(\frac{z}{\sigma}\Big)/\sigma\,dz\,dx\,dy\,,\end{aligned}$$

say, where X is normally distributed with mean value $\tilde\mu = \tilde\mu(x,y,z) = z\,R_Z(0,|x-y|)/\sigma^2$ and variance $\tilde\sigma^2 = \tilde\sigma^2(x,y) = \sigma^2 - R_Z^2(0,|x-y|)/\sigma^2$. Let $\widetilde{X}$ denote a standard normal variate. We then have

$$\mathbb{E}\big[N^2(T)\big] = \int_0^1\!\!\int_0^1 \int_{-\infty}^{\sqrt{\psi(y)}} \mathbb{P}\Big[\big(\tilde\mu/\tilde\sigma + \widetilde{X}\big)^2 \le \psi(x)/\tilde\sigma^2\Big]\,\varphi\Big(\frac{z}{\sigma}\Big)/\sigma\,dz\,dx\,dy$$

$$= \int_0^1 \int_0^1 \int_{-\infty}^{\sqrt{\psi(y)}} F_{\chi^2}\Big(\psi(x)/\tilde{\sigma}^2; 1, \tilde{\mu}/\tilde{\sigma}\Big)\varphi\Big(\frac{z}{\sigma}\Big)/\sigma \, dz \, dx \, dy.$$

Here $F_{\chi^2}(x; r, \delta)$ denotes the cumulative distribution function of a non-central chi-squared distributed random variable with r degrees of freedom and non-centrality parameter δ. The corresponding density function is defined in (3.16).

In a situation like this where the observations should be restricted to be positive values, we could alternatively take the logarithms of the firm values as fundamental modeling quantities. The modeling then proceeds assuming that the log-transformed data have a Gaussian distribution and will be modeled as a Gaussian random field:

$$V(t, \iota) = e^{L(t,\iota)} \,,$$

where, for a Gaussian random field Y,

$$L(t, \iota) = \mu(t, \iota) + \sigma(t, \iota) Y(t, \iota) \,.$$

4.4. **Defaultable forward rate models.** In this section we resume the setup of forward rate random field models introduced in Section 3.3. We suppose the defaultable bond price prior to default is described through the following equations:

$$\bar{P}(t, T) = \exp\Big\{ - \int_t^T \bar{f}(t, v) \, dv \Big\} \,, \tag{4.17}$$

where

$$d\bar{f}(t, T) = \bar{\mu}(t, T) \, dt + \bar{\sigma}(t, T) \, dZ(t, T) \,. \tag{4.18}$$

It is assumed that the functions $\bar{\mu}$ and $\bar{\sigma}$ satisfy the technical regularity conditions imposed by HJM. Here $Z(t, T)$ denotes a two-parameter random field with correlation structure specified by $d\langle Z(\cdot, T_1), Z(\cdot, T_2)\rangle_t = c(t, T_1, T_2) \, dt$, see Section 3.3. Observe that the defaultable forward rates $\bar{f}(t, T)$ are only defined up to the time of default. Proposition 4.1 below shows that, with an exogenously given short spread $s(t) = \lambda(t) L_t$, one has the usual HJM drift restriction in order to preclude arbitrage under the risk-neutral measure $\mathbb{Q}$. Since λ is the intensity process associated to the default time τ, the default indicator process $\Lambda_t = \mathbb{1}_{\{\tau \le t\}}$ which is zero before default and 1 afterward can be written in the form

$$d\Lambda_t = (1 - \Lambda_t)\lambda(t) \, dt + dM(t) \,, \tag{4.19}$$

where $M = \{M(t) : t \ge 0\}$ is a martingale under $\mathbb{Q}$. This follows from the fact that the process $M(t) := \Lambda_t - \int_0^{t \wedge \tau} \lambda(s) \, ds$ is a martingale under $\mathbb{Q}$.

Proposition 4.1. *Let $Z(t,T)$, $t \le T$, be a random field with deterministic correlation structure c specified by $d\langle Z(\cdot,T_1), Z(\cdot,T_2)\rangle_t = c(t,T_1,T_2)\,dt$. Suppose the defaultable zero-bond price $\overline{P}(t,T)$ is modeled as*

$$\overline{P}(t,T) = \exp\Big\{-\int_t^T \bar{f}(t,v)\,dv\Big\}\,,$$

where

$$d\bar{f}(t,T) = \bar{\mu}(t,T)\,dt + \bar{\sigma}(t,T)\,dZ(t,T)\,.$$

We suppose $\bar{\mu}$, $\bar{\sigma}$ satisfy the technical regularity conditions imposed by the HJM framework. Then, with given processes λ and L for the risk-neutral default intensity and fractional loss quota L, respectively, we have the risk neutral drift restriction and short spread condition

$$\begin{aligned} \bar{\mu}(t,T) &= \bar{\sigma}(t,T)\bar{\sigma}^*(t,T,T)\,, \\ \bar{r}_t - r_t &= \lambda(t)\,L_t\,, \end{aligned} \tag{4.20}$$

respectively, where $\bar{r}_t = \bar{f}(t,t)$ and $\bar{\sigma}^*(t,T,S) = \int_t^T \bar{\sigma}(t,v)c(t,S,v)\,dv$.

Proof. We use the fact that bond prices discounted at the risk-free short rate process r have to be $\mathbb{Q}$-martingales. Let $B(t) = \exp\{\int_0^t r_s\,ds\}$ denote the bank account numéraire. Then

$$Z^*(t,T) = \frac{\overline{P}(t,T)}{B(t)}\,(1-\Lambda_t) + \int_0^t (1-L_s)\,\frac{\overline{P}(s-,T)}{B(s)}\,d\Lambda_s\,, \tag{4.21}$$

where Λ_t is the default indicator process with the dynamics specified in (4.19). Recall that $dB(t)^{-1} = -B(t)^{-1}r_t\,dt$. The differential form of (4.21) thus reads

$$\begin{aligned} dZ^*(t,T) &= d\Big(\frac{\overline{P}(t,T)}{B(t)}\Big)\,(1-\Lambda_t) - \frac{\overline{P}(t,T)}{B(t)}\,d\Lambda_t + (1-L_t)\,\frac{\overline{P}(t-,T)}{B(t)}\,d\Lambda_t \\ &= \Big(\frac{d\overline{P}(t,T)}{B(t)} - \frac{\overline{P}(t,T)}{B(t)}\,r_t\,dt\Big)(1-\Lambda_t) \\ &\qquad - \frac{\overline{P}(t,T)}{B(t)}\,d\Lambda_t + (1-L_t)\,\frac{\overline{P}(t-,T)}{B(t)}\,d\Lambda_t\,. \end{aligned}$$

It remains to specify the dynamics of $\overline{P}(t,T)$. By analogy with the default-free case, we define $\bar{I}_t = \int_t^T \bar{f}(t,v)\,dv$, whence $\overline{P}(t,T) = \exp\{-\bar{I}_t\}$. Applying Itô's lemma, we obtain

$$\begin{aligned} \frac{d\overline{P}(t,T)}{\overline{P}(t,T)} &= -d\bar{I}_t + \frac{1}{2}d\langle \bar{I}_\cdot\rangle_t \\ &= \bar{r}_t\,dt - \bar{\mu}^*(t,T)\,dt - \int_t^T dv\,\bar{\sigma}(t,v)\,dZ(t,v) \\ &\qquad + \frac{1}{2}\int_t^T \bar{\sigma}(t,u)\bar{\sigma}^*(t,T,u)\,du\,dt\,, \end{aligned} \tag{4.22}$$

where $\bar{r}_t = \bar{f}(t,t)$ and $\bar{\mu}^*(t,T) = \int_t^T \bar{\mu}(t,s)\,ds$, see Lemma 3.1. In order $Z^*(t,T)$ be a $\mathbb{Q}$-martingale, the drift term in the dynamics of $Z^*(t,T)$ must be equal to zero, whence

$$0 = \bar{r}_t - \bar{\mu}^*(t,T) + \frac{1}{2}\int_t^T \bar{\sigma}(t,u)\bar{\sigma}^*(t,T,u)\,du - r_t - \lambda(t)\,L_t\,. \tag{4.23}$$

Equation (4.23) was obtained by inserting the dynamics of $\overline{P}(t,T)$ given in (4.22) into the expression for $dZ^*(t,T)$, thereby collecting the dt-terms. Taking partial derivatives in (4.23) with respect to T leaves the drift restriction in (4.20). □

Remark. Schmidt [24], Theorem 3.1 derives an analogous result by means of an eigenvalue expansion of the covariance operator D of a D-Wiener process. □

5. Conclusions

The theory of random fields proves successful to the modeling of both the term structure of interest rates as well as the various quantities inherent to credit risk models. Our focus was on non-negative valued fields. We understand that interest rate or default intensity models which allow for negative values with positive probability are inconsistent with theory and intuition. We first introduced a short rate model that is based on the square of one-dimensional Brownian motion. This setup provides an affine term structure in the sense that zero-coupon bond prices are expressible as exponential-affine functions of the short rate. The main criticism of this approach is that neither Brownian motion nor its square are stationary processes. It is mainly for this reason that we then proposed a model in which (the square of) Brownian motion is replaced by (the square of) an OU process. The OU process is a stationary zero-mean Gaussian process. Therefore, its square is a so-called chi-squared process on the line. The corresponding short rate model is still analytically tractable and the affine structure also remains intact.
Inspired by the chi-squared processes on the line, we extended the index set of the processes by an additional spatial parameter. We interpret the spatial component as a label indexing a family of obligors. The two-parameter fields can be used either for the modeling of default intensities in a reduced form credit risk model or for mimicking the issuers' firm values. The difficulty with spatio-temporal models lies in the specification of an appropriate covariance structure. It is convenient to work with separable covariance models since then the single-index term structure models can nicely be integrated into the two-parameter framework.

In this paper we dealt with simple credit risk events. Yet we believe that the random field theory can be exploited further to capture these and other events more thoroughly. For instance, we could be interested in the 'fraction' of obligors whose firm value falls below a certain level at *any* time $t \in [0,T]$. The object of interest would then be $\{(t,\iota)|V(t,\iota) < \delta\} \subset [0,T] \times \mathscr{I}$. This then leads to the study of level crossings and excursion sets of random fields, see Adler [1] for a discussion of these topics. Alternatively, we can think of the theory of stochastic set functions as an appropriate tool for modeling credit risk events. For a region $\mathcal{U} \in \mathbb{R}^d$, the quantity $Z(\mathcal{U})$ would represent

a cluster of defaulted obligors in $\mathcal{U}$, where $Z(\mathcal{U})$ is a stationary stochastic set function, see for instance Matérn [19], Section 2.6.
Before proposing more sophisticated models, however, one should attempt to calibrate the simple ones as discussed in this paper to historical data. Only then do we know whether or not these models capture the complexity of interest and credit risk markets.

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Appendix A. Karhunen-Loève expansion for the OU process

The derivation of the Karhunen-Loève expansion for the OU process goes along the same lines as the proof of Proposition 2.1. Inserting the covariance function $R(x,y) = c\exp\{-\beta|x-y|\}$ into the integral equation (2.2) yields

$$\begin{aligned}\lambda\phi(x) &= \int_0^T ce^{-\beta|x-y|}\phi(y)\,dy \\ &= \int_0^x ce^{-\beta(x-y)}\phi(y)\,dy + \int_x^T ce^{-\beta(y-x)}\phi(y)\,dy\,.\end{aligned}$$

Differentiating this equation twice with respect to x we obtain

$$\lambda\ddot{\phi}(x) = \big(-2\beta c + \lambda\beta^2\big)\phi(x)\,.$$

This can be rewritten in the form

$$\ddot{\phi}(x) + \omega^2\phi(x) = 0\,, \tag{A.1}$$

where $\omega^2 = (2\beta c - \lambda\beta^2)/\lambda$. The general solution of (A.1) is given by

$$\phi(x) = A\cos(\omega x) + B\sin(\omega x)$$

with the two boundary conditions

$$\begin{aligned}\beta\phi(0) - \dot{\phi}(0) &= 0 \\ \beta\phi(T) + \dot{\phi}(T) &= 0\end{aligned} \tag{A.2}$$

whence

$$\begin{aligned}A\beta - B\omega &= 0 \\ A\big(\beta - \omega\tan(\omega T)\big) + B\big(\omega + \beta\tan(\omega T)\big) &= 0\,.\end{aligned} \tag{A.3}$$

The system (A.3) has a non-trivial solution if and only if the determinant of

$$\begin{pmatrix}\beta & -\omega \\ \beta - \omega\tan(\omega T) & \omega + \beta\tan(\omega T)\end{pmatrix}$$

is zero, whence

$$\omega + \beta \tan(\omega T) = 0 \qquad \text{and} \qquad \beta - \omega \tan(\omega T) = 0.$$

Let ω_k denote the roots of the equation $\omega + \beta \tan(\omega T) = 0$. It follows that $A = 0$ and therefore $\phi_k(x) = B \sin(\omega_k x)$. The constant B can be determined using the orthonormality condition:

$$1 = \int_0^T \phi_k^2(y)\, dy = B^2 \int_0^T \sin^2(\omega_k y)\, dy = B^2 \left(\frac{T}{2} - \frac{\sin(2T\omega_k)}{4\omega_k} \right).$$

Solving for B yields $B = \big(T/2 - \sin(2T\omega_k)/(4\omega_k)\big)^{-1/2}$. In the same way we can calculate the constant A when considering the roots δ_j of the equation $\beta - \delta \tan(\delta T) = 0$:

$$1 = \int_0^T \phi_j^2(y)\, dy = A^2 \int_0^T \cos^2(\delta_j y)\, dy = A^2 \left(\frac{T}{2} + \frac{\sin(2T\delta_j)}{4\delta_j} \right),$$

yielding $A = (T/2 + \sin(2T\delta_j)/(4\delta_j))^{-1/2}$.

Remark. The integral equation (2.2) can be solved analytically only in special cases like the ones we considered. Where this is not possible, numerical methods are required to determine the eigenvalues and eigenfunctions of the covariance function R. □

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