Leonhard Euler and Financial Risk Management

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A. Main events of his career

- Leonhard Euler: one of the greatest mathematicians of all time
- created new branches of mathematics such as
  - calculus of variations
  - graph theory
  - topology
  - modern differential and integral calculus
- more than 850 papers published!
- in 1907, the Swiss Academy of Sciences established the **Euler Commission** with the charge of publishing all of Euler's papers, manuscripts and correspondence. The project – known as *Opera Omnia* – began in 1911 and is still in progress!

Time line (1/3)

1707 Born in Basel on April 15. Son of Margreth Bruckner and Paul Euler, a pastor. Soon after his birth, the family moves to Riehen (canton Basel-Stadt)

1721 At the age of 14, Euler matriculates at the University of Basel studying first philosophy and then theology. Johann I Bernoulli intervenes

1724 Euler completes his university studies and receives his Masters of Arts degree from the University of Basel

1725 Catherine I, widow of czar Peter I (the Great), establishes the St. Petersburg Academy of Sciences following Peter's plans. She hires Euler's friends Daniel and Niklaus II Bernoulli (sons of Johann I)

1726 Euler participates in a competition for a professorship in physics at Basel. To this end, he writes a monograph on the propagation of sound (*Dissertatio physica de sono*). Does not succeed in getting the position

Time line (2/3)

1727 Euler moves to St. Petersburg and becomes an adjunct in mathematics

1733 Euler takes over the chair in mathematics after Daniel Bernoulli returns to Basel. Gets married to Katharina Gsell. They have 13 children, of whom only 3 boys and 2 girls survive childhood

1735 Solves the problem ("Basel problem") of finding the sum $\zeta(2)$, where $\zeta(\kappa) = \sum_{n\geq 1} \frac{1}{n^\kappa}$, and acquires international reputation ($\zeta(2) = \pi^2/6$)

1738 Euler loses the vision in his right eye after a serious illness

1741 Political turmoil in Russia after death of the then czarina Anna. Euler gets an offer from King Friedrich II, and leaves Russia to join the Academy of Sciences in Berlin, Prussia. Euler to become the director of mathematics in the Prussian Academy of Science

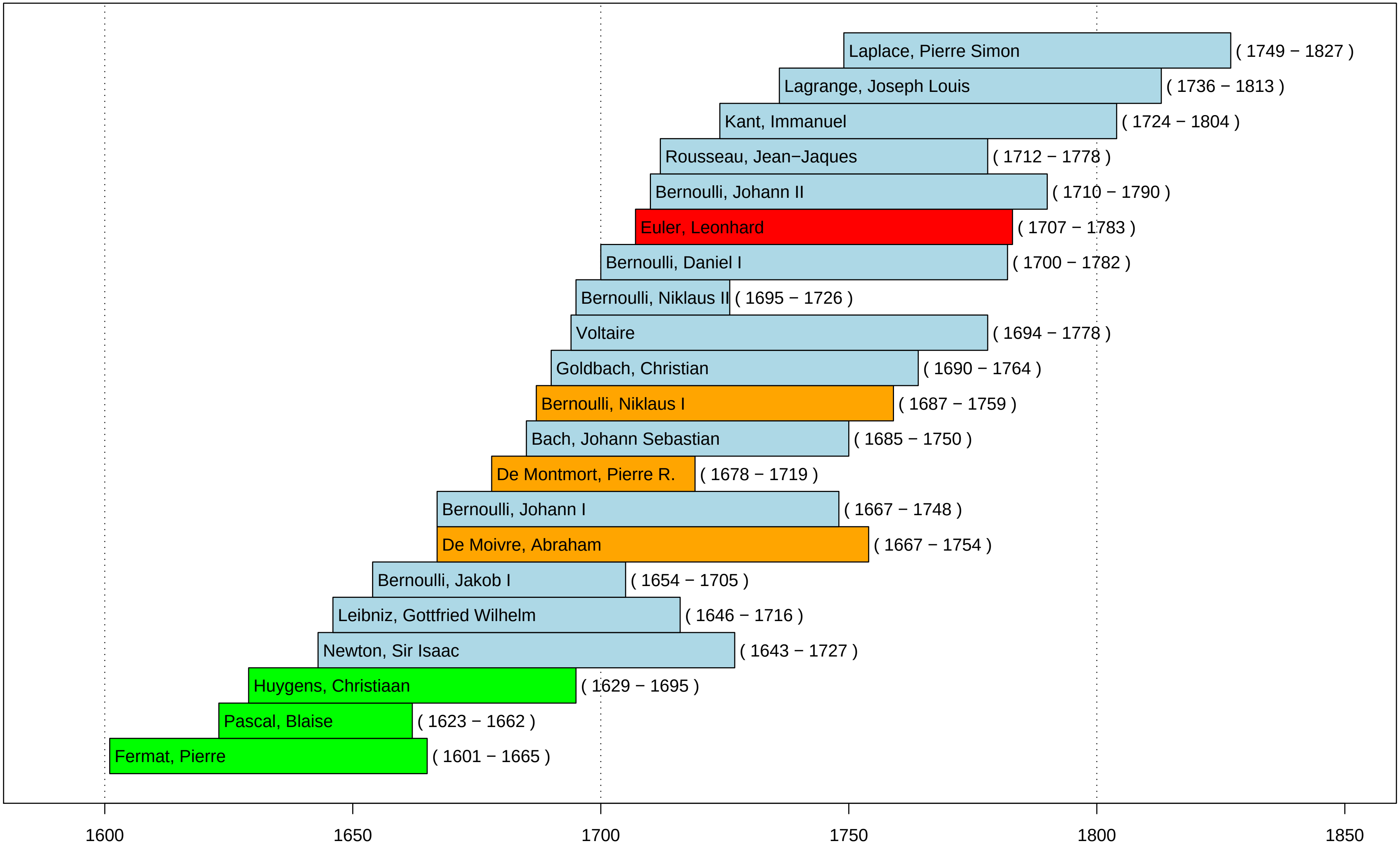
Time line (3/3)

1762 Catherine II (the Great) becomes czarina in Russia and starts the efforts to get Euler back

1766 Euler returns to St. Petersburg. His eyesight begins to deteriorate

1771 St. Petersburg is struck by a disastrous fire. Euler barely escapes with his life. Rescued by Peter Grimm, a family servant

1783 Euler dies in St. Petersburg on September, 18.


Laplace, Pierre Simon (1749 − 1827)
Lagrange, Joseph Louis (1736 − 1813)
Kant, Immanuel (1724 − 1804)
Rousseau, Jean−Jaques (1712 − 1778)
Bernoulli, Johann II (1710 − 1790)
Euler, Leonhard (1707 − 1783)
Bernoulli, Daniel I (1700 − 1782)
Bernoulli, Niklaus II (1695 − 1726)
Voltaire (1694 − 1778)
Goldbach, Christian (1690 − 1764)
Bernoulli, Niklaus I (1687 − 1759)
Bach, Johann Sebastian (1685 − 1750)
De Montmort, Pierre R. (1678 − 1719)
Bernoulli, Johann I (1667 − 1748)
De Moivre, Abraham (1667 − 1754)
Bernoulli, Jakob I (1654 − 1705)
Leibniz, Gottfried Wilhelm (1646 − 1716)
Newton, Sir Isaac (1643 − 1727)
Huygens, Christiaan (1629 − 1695)
Pascal, Blaise (1623 − 1662)
Fermat, Pierre (1601 − 1665)
1600
1650
1700
1750
1800
1850

Euler: the great universalist

Topics Euler had worked on:

- Differential and integral calculus
- Trigonometric functions
- Infinite series and products
- Number theory
- Geometry
- Probability and statistics
- Hydrostatics, hydrodynamics
- Ballistics
- . . .
- Differential equations (ODE, PDE)
- Elliptic functions and integrals
- Algebra
- Topology
- Graph theory
- Calculus of variations
- Applications in music, shipbuilding, . . .
- Lunar and planetary motions
- . . .

Selected topics (1/8)

Discovery	Comments
Euler number	Euler (1728) determines the limit $$e := 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots = 2.71828\ldots$$ Applications in actuarial science: continuously-compounded interest rate $$e^{rt} = \lim_{n\to\infty}\left(1 + \frac{r}{n}\right)^{nt} \qquad (n : \text{number of compounding periods per } t)$$
Euler constant	Jakob Bernoulli discovers that the harmonic series $1 + \frac{1}{2} + \frac{1}{3} + \cdots$ is divergent. Euler establishes the connection between $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ and $\log n$: $$\gamma := \lim_{n\to\infty}\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} - \log n\right) = 0.577215\ldots$$

Selected topics (2/8)

Discovery	Comments
Gamma function	Euler defines $$\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1}\, dt, \quad \alpha > 0.$$ Applications in probability theory: Gamma distribution: $$f_{\alpha,\beta}(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}, \quad x > 0, \alpha, \beta > 0$$ widely used in insurance mathematics: • claim size modeling in non-life insurance • mixed Poisson processes for modeling claim arrivals (Negative Binomial distribution to be obtained by randomizing the Poisson parameter of a homogeneous Poisson process over a Gamma distribution)

Selected topics (3/8)

Discovery	Comments
Euler-MacLaurin formula	Euler (1734) discovers $$\int_0^n f(t)\,dt = \sum_{\nu=0}^{n} f(\nu) - \frac{1}{2}(f(0)+f(n)) + \sum_{\nu\geq 1}\frac{B_{2\nu}}{(2\nu)!}\left(f^{(2\nu-1)}(0) - f^{(2\nu-1)}(n)\right), \qquad (1)$$ where B_r are the Bernoulli numbers ($B_0 = 1$, $B_1 = -1/2$, $B_2 = 1/6$, $B_3 = 0$, $B_4 = -1/30, \ldots$) Applications in actuarial science: approximation of continuous insurance and annuities: $$\bar{a}_{jk}(m, m+n) \stackrel{\text{def}}{=} \int_0^n e^{-rt} p_{jk}(m, m+t)\,dt\,, \qquad (2)$$ where $p_{jk}(s,t) = \mathbb{P}[X(t) = k \mid X(s) = j]$ are the transition probabilities in a Markov model. The approximation of the integral in (2) via (1) is often restricted to the first term in the series including the Bernoulli numbers.

Selected topics (4/8)

Discovery	Comments
Euler equation  	Most remarkable formula in mathematics. To be found in his famous textbook “Introductio in Analysin infinitorum”, published in 1748: $$e^{i\varphi} = \cos\varphi + i\,\sin\varphi$$ $$\boxed{e^{i\pi} + 1 = 0}$$
Cryptology	For $n \in \mathbb{N}$, Euler defines $\varphi(n)$ as the number of positive integers less than n which are relatively prime to n. Examples: • $\varphi(9) = 6$ (1,2,4,5,7,8 are relatively prime to 9) • $\varphi(p) = p - 1$, $\varphi(p^k) = p^{k-1}(p-1)$ for p prime • $n = p_1^{k_1} \cdots p_r^{k_r}$ factorization of n into primes: $$\varphi(n) = \prod_{i=1}^{r} p_i^{k_i - 1}(p_i - 1) = n \prod_{i=1}^{r} \left(1 - \frac{1}{p_i}\right) \qquad (3)$$

Selected topics (5/8)

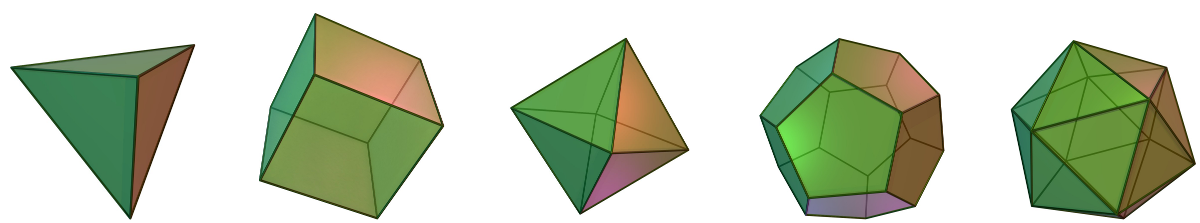
Discovery	Comments
Cryptology	Equation (3) is important in cryptology: it is easy to compute $\varphi(n)$ provided the factorization of n into primes is known. Most important case in cryptology: $m = pq,\ p, q$ (large) primes: $\varphi(m) = m\left(1-\frac{1}{p}\right)\left(1-\frac{1}{q}\right)$ Euler generalizes (1760) Fermat's little theorem to what is now known as Euler's theorem: $a^{\varphi(n)} \equiv 1 \mod n$ for any $n \in \mathbb{N}$ and a relatively prime to n. $\Longrightarrow$ public-key cryptosystems, e.g. RSA-algorithm (cf. [8]) **public key**: m & encryption exponent s relatively prime to $\varphi(m)$ **private key**: m & *decryption exponent* r defined via $\varphi(m)$ **Note**: a hacker would need to factor m into p and q to arrive at $\varphi(m)$. This would allow her to determine r from s. Factoring large keys (1024 bits and beyond), however, is not possible by today's computers

Selected topics (6/8)

Discovery	Comments
Polyhedron formula	For any convex polyhedron, Euler (1758) shows that

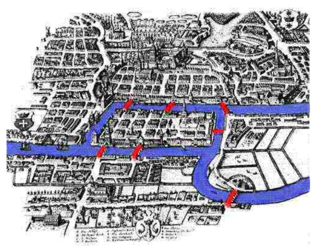
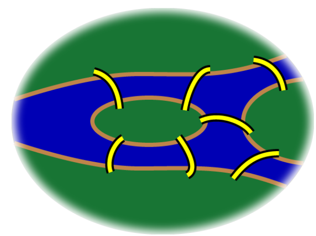
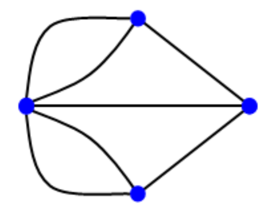
$$\boxed{e - k + f = 2}$$

where e is the number of vertices, k the number of edges and f the number of faces: universal relation between 0, 1 and 2-dimensional objects on a 3-dimensional solid; solely dependent on the solid's topology ($\longrightarrow$ string theory, elementary particle physics,. . .). Examples:



	Tetrahedron	Hexahedron	Octahedron	Dodecahedron	Icosahedron
e	4	8	6	20	12
k	6	12	12	30	30
f	4	6	8	12	20
$e - k + f$	2	2	2	2	2

Selected topics (7/8)

Discovery	**Comments**
Graph theory	The Königsberg Bridge Problem. Königsberg, now Kaliningrad, is set on the Pregel River. In 1735, Euler shows that it is not possible to walk around the city in a way that would involve crossing every bridge exactly once.   
Algebra	author of “Vollständige Anleitung zur Algebra” (1770), first published in Russian. Dictated to one of his assistants (as he had lost the vision in both of his eyes by that time). Contains a proof of Fermat’s conjecture for $n = 3$.

Selected topics (8/8)

Discovery	Comments
Mechanics	Underneath the Viaduc de Millau – currently the highest vehicular bridge in the world – a visitor center offers information about the construction and the technology used to build it. Reference is made to Euler as the calculations with respect to wind flows, oscillations, vibrations and stability are based on his results

B. Leonhard Euler on life insurance

- Most of Euler's papers on probability date from his Berlin years when Friedrich II asked him to do some work on lotteries and on life insurance
- Euler's works related to the calculus of probability, together with his works on life insurance and statistics, are all to be found in *Opera Omnia*, Series I, Volume 7 (1923) Edited L.G. du Pasquier
- Euler's studies around the calculus of probability were inspired by games of chances. The most famous of these problems are:
  - The game of Treize ("Rencontre"); a card game
  - The game of Pharaon; a casino game
  - The St Petersburg problem; a casino game
  - The Genoese lottery: drawing 5 numbers from a list of 90

Four papers on life insurance

- Euler wrote four papers on life insurance:

1. *Recherches générales sur la mortalité et la multiplication du genre humain*
2. *Sur les rentes viagères*
3. *Eclaircissements sur les établissements publics en faveur tant des veuves que des morts...*
4. *Des Herrn Leonhard Eulers nöthige Berechnung zur Einrichtung einer Witwenkasse*, translated from Latin by A. Kästner (1770)

Recherches générales sur la mortalité et la multiplication du genre humain (1/2)

- Euler discusses several questions about the mortality and the multiplication of man based on the published registers of births and deaths at each age
- He recognizes that these registers differ greatly (depending on the towns / provinces where they were taken up), and addresses these questions in a general form based on

(a)	**Hypotheses of mortality**	Euler introduces the notation $(1)N, (2)N, \ldots, (n)N$, where N is the number of newborn infants, and $(k)N$ is the number of those infants who are still alive after k years [current actuarial notation: ${}_kp_0$]
(b)	**Hypotheses of multiplication**	Principle of propagation. Notation used by Euler: M: number of people living at present; mM: number of those who are living next year

Recherches générales sur la mortalité et la multiplication du genre humain (2/2)

- In this paper, Euler derives the fractions $(1), (2), (3), \ldots$, i.e. the *law of mortality*, assuming the following are known:
  - the number of all living people
  - the number births and
  - the number of deaths at each age during the course of a year

- To this end, he assumes that
  - the number of infants born in each year is proportional to the number of all the living
  - the rule of mortality does not change over time
  - the number of living either stays the same or increases / decreases uniformly (i.e. no immigration, no war, no plague etc.)

Sur les rentes viagères (On life annuities)

- This paper is the continuation of the previous one ("Recherches générales. . . ")

- Based on the principle of equivalence (present value of future benefits $\equiv$ present value of future premiums), Euler derives the premium of an immediate life annuity due (single premium payment). Assumed technical interest rate: 5%

- He also derives the premium of an m-year deferred annuity

- Euler remarks the following:

  - *"qu'on a raison de considérer les rentiers comme une espèce plus robuste"* Euler therefore uses an annuity table for the pricing of life annuities (Kersseboum's annuity table)

  - the net premium must be augmented (expenses, safety loading)

Eclaircissements sur les établissements publics. . .

- This is Euler's main contribution in the context of life insurance
- The first part of this writing deals with *widow's pensions* in their general form
- The second part is called *"Sur l'établissement d'une caisse pour les morts"*
- The third and last part (*Plan d'une nouvelle espèce de Tontine, aussi favorable au public qu'utile à l'Etat*) describes how the "tontines" – these are government bonds proposed by the Italian medical doctor Lorenzo Tonti – need to be designed in order to become equally attractive for both the issuer and the investor

C. Capital allocation

Situation:

- Consider a fixed number m of business lines / assets or loans in a portfolio
- Assigned to each of the m investment possibilities are the profits and losses $X_1, X_2, \ldots, X_m$, represented as (possibly dependent) random variables
- Portfolio-wide profit and loss is denoted by

$$X = \sum_{k=1}^{m} X_k$$

- Economic capital (risk capital) required by the portfolio denoted as $\rho(X)$, where ρ is a particular risk measure (e.g. Value-at-Risk (VaR), Tail-VaR)

Capital allocation problem

- **Question:** How much contributes sub-portfolio k to the total risk capital $\rho(X)$?
- **Task:**
  1. calculate the portfolio risk measure $\rho(X)$
  2. allocate the capital $\rho(X)$ to the elements of the portfolio according to some capital allocation principle
- **Notation:** marginal risk contributions are denoted by

$$\rho(X_1|X),\ \rho(X_2|X), \ldots, \rho(X_m|X)$$

Capital allocation problem (cont'd)

- **Desirable property**: an allocation principle satisfies the *full allocation principle* if

$$\sum_{k=1}^{m} \rho(X_k|X) = \rho(X)$$

ie. all of the total risk capital $\rho(X)$ (not more, not less) is allocated to the individual sub-portfolios

- **Performance measurement:** the concept of RORAC (return on risk adjusted capital):

$$\text{RORAC}(X) = \frac{\mathbb{E}[X]}{\rho(X)}, \qquad \text{RORAC}(X_k|X) = \frac{\mathbb{E}[X_k]}{\rho(X_k|X)}$$

Euler's Theorem on homogeneous functions

- **Definition:** A differentiable function $f : \mathbb{R}^m \to \mathbb{R}$ is said to be positive homogeneous of degree $\beta \in \mathbb{R}$ if

$$f(h\mathbf{u}) = h^\beta f(\mathbf{u}), \qquad \mathbf{u} \in \mathbb{R}^m \backslash \{0\},\ h > 0.$$

- **Example:** $f(\mathbf{u}) = f(u_1, u_2) = u_1 u_2$ is positive homogeneous of degree $\beta = 2$:

$$f(h\mathbf{u}) = hu_1 hu_2 = h^2 u_1 u_2 = h^2 f(\mathbf{u})$$

- **Proposition** (Euler's Theorem on homogeneous functions). Let $f : \mathbb{R}^m \to \mathbb{R}$ be continuously differentiable. Then the following are equivalent:

  (i) f is positive homogeneous of degree β

  (ii) $$\beta f(\mathbf{u}) = \sum_{k=1}^{m} u_k \frac{\partial f(\mathbf{u})}{\partial u_k}, \qquad \mathbf{u} = (u_1, \ldots, u_m)' \in \mathbb{R}^m$$

Homogeneity of VaR and ES

- Let X be the change in portfolio value over a time horizon Δt. Denote the distribution function of X by F_X. Value-at-Risk and Tail-Value-at-Risk are defined as follows:

(a) $\mathrm{VaR}_\alpha(X) = \inf\{x : F_X(x) \geq \alpha\} = q_\alpha^+(X)$

(b) $\text{Tail-VaR}_\alpha(X) = \dfrac{1}{\alpha}\displaystyle\int_0^\alpha \mathrm{VaR}_r(X)\,dr.$

- If X has continuous distribution, then

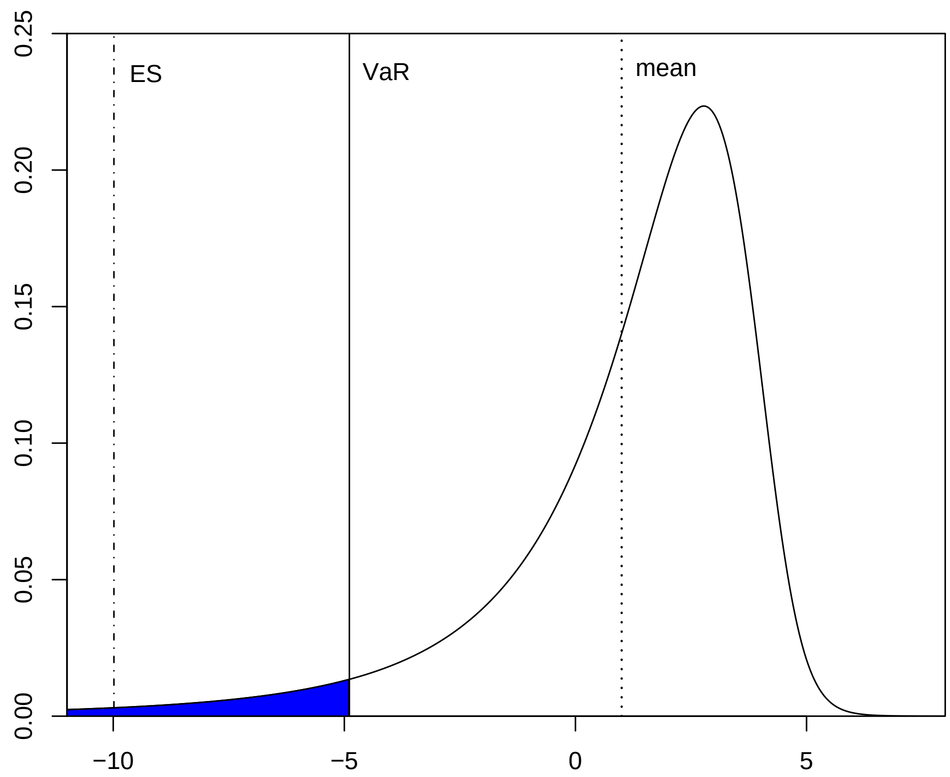
$$\text{Tail-VaR}_\alpha(X) = \mathbb{E}[X \mid X \leq \mathrm{VaR}_\alpha(X)] =: \mathrm{ES}_\alpha(X)$$

- **Note**: Both of the above risk measures are positive homogeneous of degree 1:

$$\mathrm{VaR}_\alpha(\lambda X) = \lambda\,\mathrm{VaR}_\alpha(X) \qquad \text{and} \qquad \mathrm{ES}_\alpha(\lambda X) = \lambda\,\mathrm{ES}_\alpha(X)$$

Illustration of VaR and ES

- Visualization of a loss distribution with the 5%-VaR marked as a vertical line and the ES marked as a dashed-dotted line; the mean loss is shown with a dotted line:



Euler's theorem for homogeneous risk measures

- An application of Euler's theorem to positive homogeneous risk measures ρ yields

$$\boxed{\rho(X) = \sum_{k=1}^{m} \frac{\partial}{\partial h} \rho(X + hX_k)\Big|_{h=0} \stackrel{\text{def}}{=} \sum_{k=1}^{m} \rho_{\text{Euler}}(X_k|X)} \qquad (4)$$

- **Remarks:**

  (a) the Euler principle gives a full allocation of the risk capital $\rho(X)$

  (b) for sub-additive risk measures one has

$$\rho_{\text{Euler}}(X_k|X) \leq \rho(X_k).$$

  That is, the Euler risk contribution of sub-portfolio k will never exceed the portfolio's stand alone risk. Risk contributions of credit assets can not become larger than the face value of the loan

Euler risk contribution for VaR and ES

- In the VaR and ES-case, the directional derivatives (4) exist and are given by (cf. Tasche [9])

Risk measure ρ	**Euler risk contribution** $\rho_{\text{Euler}}(X_k \mid X)$
VaR	$\mathbb{E}\left[X_k \mid X = \mathrm{VaR}_\alpha(X)\right]$
ES	$\mathbb{E}\left[X_k \mid X \leq \mathrm{VaR}_\alpha(X)\right]$

- **Note:** In general, there are no closed-form expressions for $\rho(X)$ and $\rho_{\text{Euler}}(X_k|X)$. Hence, the losses $X_1, \ldots, X_m$ must be generated by means of Monte Carlo simulation

Remarks (1/2)

- The Euler risk contributions for VaR and ES are conditional expectations of the individual sub-portfolios, conditioned on (rare) events $\mathcal{A}$ in the tail of the loss distribution of the full portfolio ($\mathcal{A} = \{X = \mathrm{VaR}_\alpha(X)\}$, $\mathcal{A} = \{X \leq \mathrm{VaR}_\alpha(X)\}$)

- The rarity of the conditioning events $\mathcal{A}$ typically causes practical difficulties when estimating $\mathbb{E}[X_k|\mathcal{A}]$: most simulations will be "wasted" as they lead to portfolio losses smaller than $\mathrm{VaR}_\alpha(X)$

- To address these difficulties around rare event simulation, efficient Monte Carlo methods must be developed such as importance sampling (cf. Glasserman [3])

- For *discrete* risks X one has $\mathbb{P}[X = \mathrm{VaR}_\alpha(X)] > 0$ and hence

$$\mathbb{E}\left[X_k|X = \mathrm{VaR}_\alpha(X)\right] = \frac{\mathbb{E}[X_k 1_{\{X=\mathrm{VaR}_\alpha(X)\}}]}{\mathbb{P}[X = \mathrm{VaR}_\alpha(X)]} \tag{5}$$

Remarks (2/2)

- For *continuous* X, however, (5) does not hold since $\mathbb{P}[X = \mathrm{VaR}_\alpha(X)] = 0$. In that case, *kernel estimation methods* must be applied for measuring marginal risk contributions

- Euler risk contribution for the standard deviation risk measure (covariance principle): if $\rho(X) = \sqrt{\mathrm{var}(X)}$, then

$$\rho_{\mathsf{Euler}}(X_k|X) = \frac{\mathrm{cov}(X_k, X)}{\sqrt{\mathrm{var}(X)}},$$

see for instance McNeil [7], p. 258

Numerical Example

- **Aim:** computing the portfolio risk measure $\rho(X)$ and the marginal risk contributions $\rho_{\text{Euler}}(X_k|X)$ of a credit portfolio in the VaR- and ES-case

- **Assumptions**:

m	number of obligors ($m = 20$)
$\bar{p}_k$	individual default probabilities ($\bar{p}_k \equiv 0.1$)
c_k	loss given default for the k-th obligor
Y_k	default indicator; $\bar{p}_k = \mathbb{P}[Y_k = 1]$
X_k	loss from the k-th obligor; $X_k = c_k Y_k$
L	total loss from defaults, $L = \sum_{k=1}^{m} c_k Y_k$

$$c_k = \begin{cases} 1, & k \in \{1, 2, 3, 4\} \\ 4, & k \in \{5, 6, 7, 8\} \\ 9, & k \in \{9, 10, 11, 12\} \\ 16, & k \in \{13, 14, 15, 16\} \\ 25, & k \in \{17, 18, 19, 20\}. \end{cases}$$

- The default indicators Y_k are assumed to be dependent. Dependence is introduced via a Gauss copula (KMV, CreditMetrics approach) in the form of a so called single factor equicorrelation threshold model

Model setup

- In a threshold model, default of the k-th obligor occurs if $\{Z_k > z_k\}$ for some random variable Z_k (latent variable) and threshold value z_k. It is assumed that

$$\boxed{Z_k = \sqrt{\varrho}\, F + \sqrt{1-\varrho}\, \varepsilon_k} \tag{6}$$

  where F is a systematic risk factor (common risk factor) and ε_k are the idiosyncratic risk factors. The z_k are calibrated to match the individual default probabilities; cf. Glasserman [3] or McNeil [7].

- **Assumptions:** $\varrho = 0.5$, $F \sim \mathcal{N}(0,1)$, $\varepsilon_k \sim \mathcal{N}(0,1)$ (independent of F)

- **Note:** the representation in (6) is a special case of a **factor model** with several *common factors* $F_1, \ldots, F_p$:

$$Z_k = b_{k1}F_1 + \cdots + b_{kp}F_p + a_k\varepsilon_k$$

Estimating VaR and ES risk contributions

Risk measure ρ	**Euler risk contribution** $\rho_{\text{Euler}}(X_k \| L)$	**MC estimate** $\hat{\rho}_{\text{Euler}}(X_k \| L)$
VaR	$\mathbb{E}\left[X_k \| L = \text{VaR}_\alpha(L)\right]$	$\dfrac{\sum_{i=1}^n X_k^{(i)} 1_{\{L^{(i)}=\text{VaR}_\alpha(L)\}}}{\sum_{i=1}^n 1_{\{L^{(i)}=\text{VaR}_\alpha(L)\}}}$
ES	$\mathbb{E}\left[X_k \| L \geq \text{VaR}_\alpha(L)\right]$	$\dfrac{\sum_{i=1}^n X_k^{(i)} 1_{\{L^{(i)}\geq\text{VaR}_\alpha(L)\}}}{\sum_{i=1}^n 1_{\{L^{(i)}\geq\text{VaR}_\alpha(L)\}}}$

n: number of simulations

Monte Carlo simulation algorithm

1. For each of n replications
   (a) generate F from $\mathcal{N}(0,1)$
   (b) for each obligor $k = 1, \ldots, m$, generate an independent variable $\varepsilon_k \sim \mathcal{N}(0,1)$, and set $Z_k = \sqrt{\varrho}\, F + \sqrt{1-\varrho}\, \varepsilon_k$
   (c) for each obligor $k = 1, \ldots, m$, generate the default indicator $Y_k = \{Z_k > z_k\}$, where $z_k = q_{1-\bar{p}_k}(N)$ is the $1-\bar{p}_k$-quantile of a standard normal variate. Set $X_k = c_k Y_k$ and $L = X_1 + \cdots + X_m$
2. Use the empirical quantile $\hat{q}_\alpha = \hat{F}^{\leftarrow}_{L,n}(\alpha)$ to estimate $\mathrm{VaR}_\alpha(L)$ at the probability α, where $\hat{F}_{L,n}$ is the empirical distribution of portfolio losses: $\hat{F}_{L,n}(x) = \frac{1}{n}\sum_{i=1}^{n} 1_{\{L^{(i)} \le x\}}$
3. Calculate the VaR and ES risk contributions using the individual losses $X_k^{(i)}$ and the total portfolio losses $L^{(i)}$ obtained in step (1c), according to

$$\frac{\sum_{i=1}^{n} X_k^{(i)} 1_{\{L^{(i)} = \mathrm{VaR}_\alpha(L)\}}}{\sum_{i=1}^{n} 1_{\{L^{(i)} = \mathrm{VaR}_\alpha(L)\}}}, \qquad \frac{\sum_{i=1}^{n} X_k^{(i)} 1_{\{L^{(i)} \ge \mathrm{VaR}_\alpha(L)\}}}{\sum_{i=1}^{n} 1_{\{L^{(i)} \ge \mathrm{VaR}_\alpha(L)\}}}$$

Results

- For $\alpha = 0.99$ and $n = 250000$ replications, the VaR of the total portfolio loss is estimated to be equal to 167 (expected shortfall estimate: 188.5739)

Obligor	VaR	CI	ES	CI
1-4	0.469	1.27×10^{-1}	0.749	2.07×10^{-2}
5-8	2.404	5.00×10^{-1}	3.073	7.29×10^{-2}
9-12	5.684	1.126	7.164	1.55×10^{-1}
13-16	12.140	1.760	13.448	2.40×10^{-1}
17-20	21.053	2.321	22.709	2.96×10^{-1}

Table 1: Results of plain Monte Carlo estimates for VaR and ES risk contributions, averaged over sets of identical obligors. An asymptotically valid 95%-confidence interval is given by $q_{0.975}(N)\,\hat{\sigma}_k/\sqrt{n}$. For the definition of $\hat{\sigma}_k$, see Glasserman [3], Proposition 1.

```
alpha = .99
c1=1; c2=4; c3=9; c4=16; c5=25
rho = 0.5
exposures = rep(c(c1,c2,c3,c4,c5),each = 4)
m = length(exposures)                           # number of obligors
n = 250000                                      # number of simulations
p.bar = rep(0.1,m)                              # marginal default probabilities
z = rep(qnorm(1-p.bar),m)
#___________________________________________________________________________________________
F = rnorm(n)
epsilon = matrix(rnorm(n * m), nrow = n, ncol = m)

Z = matrix(0,nrow = n, ncol = m)
for (k in 1:m){
 Z[,k] = sqrt(rho)*F + sqrt(1-rho)*epsilon[,k]
}

X = matrix(0,nrow = n, ncol = m)
 for (k in 1:m){
  X[,k] = exposures[k]*(Z[,k] > z[k])
}

PortfolioLoss = apply(X,1,sum)
VaR.L = quantile(PortfolioLoss,alpha,names=FALSE)
ES.L  = mean(PortfolioLoss[PortfolioLoss>VaR.L])
#___________________________________________________________________________________________
# Marginal risk contributions; Value-at-Risk as risk measure
rho.Euler.VaR = rep(0,m)
Numerator   =  rep(0,m)
Denominator =  sum(PortfolioLoss == VaR.L)

for (k in 1:m){
  if (Denominator == 0) rho.Euler.VaR[k] = 0 else
  Numerator[k]      = sum(X[,k]*(PortfolioLoss == VaR.L))
  rho.Euler.VaR[k] = Numerator[k]/Denominator
}
```

Subprime crisis: modeling of credit events

- **Current industry practice:** pricing credit derivatives by means of *copulas* to model the dependence between issuers in a portfolio of defaultable securities

- **Examples:** KMV, CreditMetrics

- **Standard model assumption:** (one-factor) Gaussian copula model (so called Li Gauss copula model)

- Main points of criticism:

  - lack of dynamics (pricing of options on tranches not possible)
  - no theoretical basis for the choice of the copula: the dependence is "forced" into the model
  - Gauss copula has (asymptotically) no joint extremes

- Intensified by the subprime crisis, the Gauss copula model is **harshly criticized** (*"worst ever invention for credit risk analysis"*)

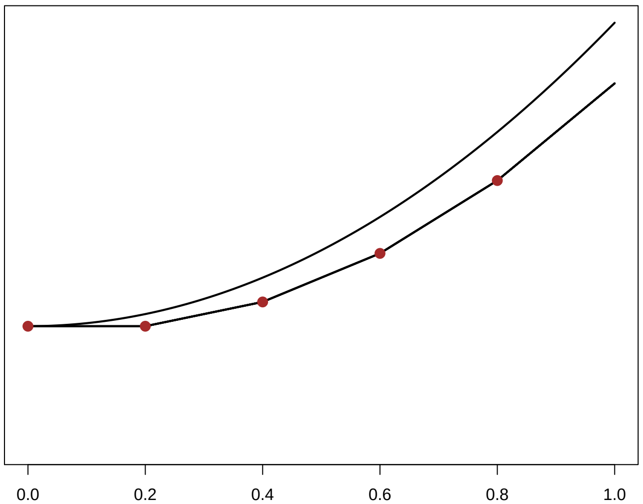
D. Euler scheme for SDE

- Recall the Euler method for ordinary differential equations (ODE):

$$\frac{dx(t)}{dt} = a(t; x(t)) \quad \Longleftrightarrow \quad dx(t) = a(t; x(t)\, dt, \quad x(0) = x_0$$

- Discrete approximation at equidistant grid points $0 = t_0 < t_1 < \ldots < t_N = T$ with $t_i = ih$, $h = T/N$:

$$\boxed{\hat{x}(t_{i+1}) = \hat{x}(t_i) + a(t_i; \hat{x}(t_i))h}$$



Approximating SDE

- The general form of a stochastic differential equation (SDE) is given by

$$dX(t) = a\,(t; X(t))\ dt + b(t; X(t))\, dW(t)\,, \quad X(0) = X_0, \tag{7}$$

where $\{W(t) : t \in [0, T]\}$ is a Brownian motion. The meaning of (7) is that

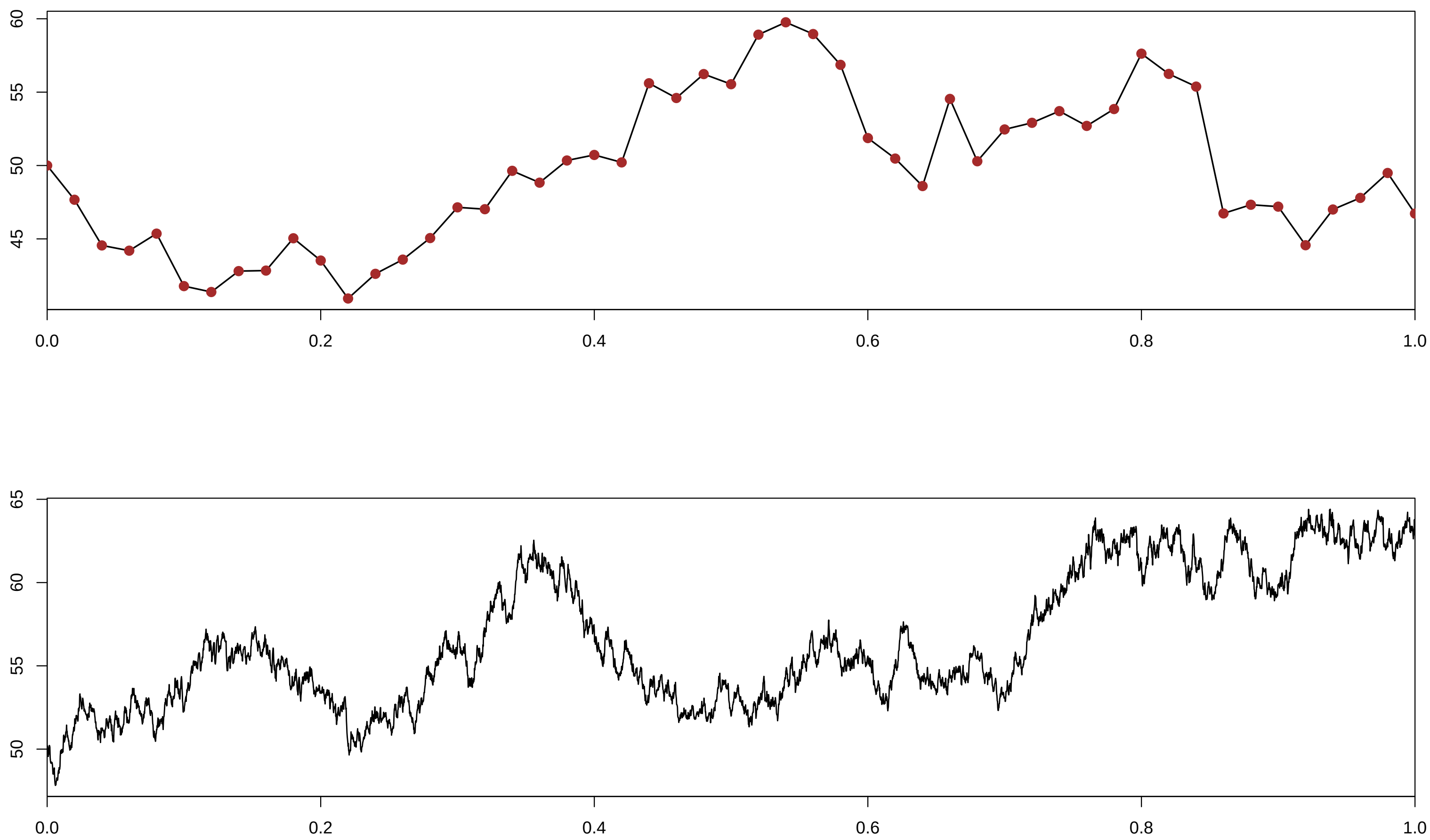
$$X(t) - X(0) = \int_0^t a\,(s; X(s))\ ds + \int_0^t b(s; X(s))\, dW(s) \tag{8}$$

- The Euler scheme for approximating X is $\hat{X}(0) = X(0)$ and

$$\boxed{\hat{X}(t_{i+1}) - \hat{X}(t_i) = a(t; \hat{X}(t_i))(t_{i+1} - t_i) + b(t; \hat{X}(t_i))\sqrt{t_{i+1} - t_i}\, Z_{i+1}}$$

where $\{Z_i : i \in \mathbb{N}\}$ is a sequence of iid standard normal random variables. In between the grid points, $\hat{X}(t)$ is defined by linear interpolation when $t_{i-1} \leq t < t_i$

Euler discretization in practice



Euler discretization of the SDE $dS(t) = S(t)(\mu\, dt + \sigma\, dW(t))$ with $S_0 = 50$, $\mu = 0.05$, $\sigma = 0.3$, and time increments $\Delta t = 1/50$ (upper graph) and $\Delta t = 1/5000$ (lower graph)

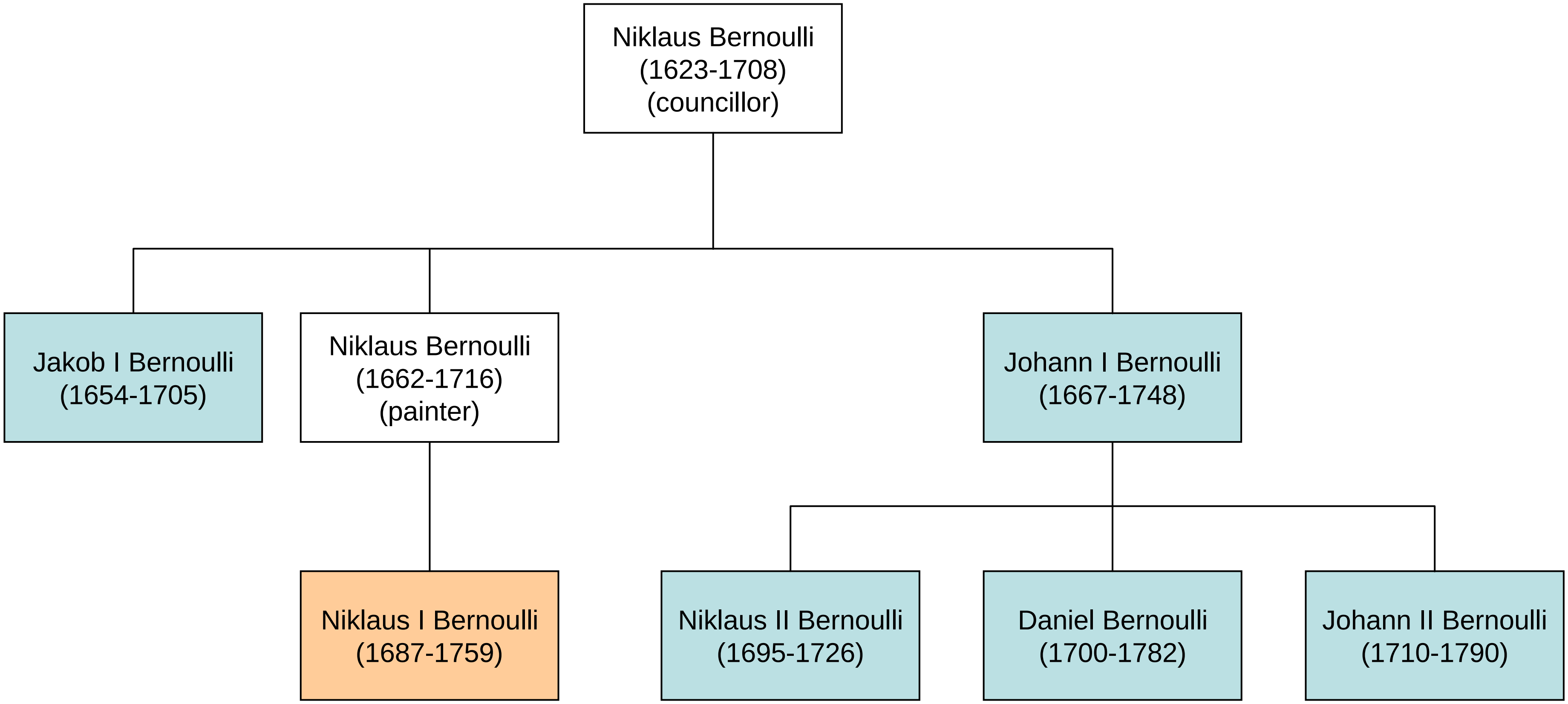
Areas of application

- The Euler scheme is widely used in the context of Monte Carlo simulation. Monte Carlo simulation in turn is required to e.g. price options in case
  - the option payoff is “complicated” (Asian options, barrier options, . . .)
  - the dynamics of the underlying S is beyond plain vanilla geometric Brownian motion (Lévy models, stochastic volatility models, . . .)
- Main source of error:

$$\int_0^h b(s, X(s))\, dW(s) \approx b(0, X(0))W(h)$$

- Improvements: Milstein scheme, Itô-Taylor scheme

Appendix A: Family tree of the Bernoulli's



The Bernoulli dynasty

Jakob I Bernoulli (1654-1705)

Held the mathematics chair at the University of Basel. Paul Euler, Leonhard's father, had attended the lectures of him. Known for his achievements in the area of infinitesimal calculus. Author of the *Ars Conjectandi* (The Art of Conjecture), a book on probability theory. Chapter 5 of Part IV deals with the (weak) law of large numbers.

Johann I Bernoulli (1667-1748)

younger brother of Jakob I Bernoulli. Successor of Jakob I at the University of Basel to the chair of mathematics. Recognized Euler's genius very early. Regular contacts with Euler. Also known for his contributions in infinitesimal calculus.

Niklaus I Bernoulli (1687-1759)

nephew of Jakob I and Johann I Bernoulli [son of Niklaus Bernoulli, 1662-1716]. Cousin of the three brothers Niklaus II, Daniel I and Johann II. Dissertation on *De Usu Artis Conjectandi in Jure*: Applications of probability theory in law.

The Bernoulli Dynasty (cont'd)

Niklaus II Bernoulli	(1695-1726)	son of Johann I Bernoulli. Moved to St. Petersburg and became a member of the newly established Academy of Science there. Died by the time Euler arrives in St. Petersburg
Daniel I Bernoulli	(1700-1782)	son of Johann I Bernoulli. Lifelong friend of Euler. Also moved to St. Petersburg and was a member of the Academy of Science. Proposed a solution to the so called *St. Petersburg Problem.* Left for Switzerland in 1733. Euler succeeded him to the prestigious chair in mathematics at the Academy.
Johann II Bernoulli	(1710-1790)	son of Johann I Bernoulli

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