Risk Processes Perturbed by α-stable Lévy Motion

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Abstract

The classical model of collective risk theory is extended in that an α-stable Lévy Motion is added to the compound Poisson process. The convolution formula for the probability of ruin is derived. We then investigate the asymptotic behaviour of the ruin probability as the initial capital becomes large.

Keywords: Risk theory, Ruin probability, α-stable Lévy motion, Mittag–Leffler function, Subexponential distributions.

1 Introduction

Suppose that an insurance company has initial capital x, and receives premium income at constant rate c. Claims occur at epochs of a Poisson process of rate λ, the claims being independent (of each other and the Poisson process) with law F on $[0, \infty)$. Write μ for the mean of F. It is easy to see that eventual ruin of the company is certain unless $c > \lambda\mu$. Dufresne and Gerber (1991) extended the above classical Poisson model of collective risk theory by adding a diffusion component to the compound Poisson process. The diffusion term expresses either an additional uncertainty of the aggregate claims or of the premium income. They established an explicit convolution formula for the infinite–time ruin probability. In that formula the perturbation is represented by the distribution function of the exponential law.

Gaussian distributions and processes have long since been studied and their utility in stochastic modelling is well–accepted. However, they do not allow for large fluctuations and may sometimes be not adequate for modelling high variability. We therefore add instead of a Brownian component an α-stable Lévy motion to the classical risk process. An α-stable Lévy motion is a random element whose finite–dimensional distributions are α-stable. The tails of non–Gaussian stable laws decrease like a power function. The rate of decay mainly depends on a parameter α which takes values

between 0 and 2. The smaller α, the slower the decay and the heavier the tails. We allow the α-stable Lévy motion only to have downward jumps. This can be achieved by choosing the so–called skewness parameter β accordingly.
In section 2 we derive an analoguous convolution formula for the infinite–time ruin probability when the classical risk process is perturbed by α-stable Lévy motion. The convolution formula now contains the Mittag–Leffler function, a generalization of the exponential function to which it reduces in the Gaussian case $(\alpha = 2)$. Section 3 finally is devoted to the asymptotic behaviour of the ruin probability as the initial capital becomes large.

2 Description of the Model and Main Result

The main objects we have to deal with are stable distributions and stable processes. A first definition of a univariate stable distribution concerns the "stability" property: the family of stable distributions is preserved under convolution. More precisely, a random variable X is said to have a stable distribution if for any positive numbers a and b, there is a positive number c and a real number d such that

$$aX' + bX'' \stackrel{d}{=} cX + d\,,$$

where X' and X'' are independent copies of X and "$\stackrel{d}{=}$" means equality in law. Explicit formulas for the densities of stable distributions exist only for $\alpha = 2$ (Gaussian distribution), for the symmetric stable distribution with $\alpha = 1$ (Cauchy distribution), and for the one–sided stable distribution with $\alpha = 1/2$. Their characteristic functions however can always be written down explicitly. The following formulation is to be found in Zolotarev (1986), representation (B).

Proposition 1 *A random variable X has a stable distribution if and only if the logarithm of its characteristic function g can be represented in the form*

$$\log g(\theta) = -\sigma^{\alpha}|\theta|^{\alpha}\omega(\theta,\alpha,\beta) + im\theta\,,$$

where

$$\omega(\theta,\alpha,\beta) = \begin{cases} \exp\{-i\beta\,\mathrm{sign}(\theta)\,\pi K(\alpha)/2\} & \alpha \neq 1\,, \\ \pi/2 + i\beta\log|\theta|\,\mathrm{sign}(\theta) & \alpha = 1\,, \end{cases}$$

$0 < \alpha \leq 2$, $-1 \leq \beta \leq 1$, $\sigma > 0$, $m \in \mathbb{R}$ *and* $K(\alpha) = \alpha - 1 + \mathrm{sign}(1-\alpha)$.

Since univariate stable distributions are characterized by four parameters, we denote stable laws by $S_\alpha(\sigma,\beta,m)$ and write $X \sim S_\alpha(\sigma,\beta,m)$ to indicate that X has the stable distribution $S_\alpha(\sigma,\beta,m)$. Because m and σ merely determine location and scale we shall mostly consider stable distributions with $m = 0$ and $\sigma = 1$. Note that by doing so we are excluding the degenerate case $\sigma = 0$. We introduce the notation $G(\,\cdot\,;\alpha,\beta)$, $g(\,\cdot\,;\alpha,\beta)$ for the distribution function and density function of a stable law with parameters α, β, $\sigma = 1$ and $m = 0$, respectively.

Definition 1 A stochastic process $Z_\alpha = (Z_\alpha(t) : t \geq 0)$ is called *(standard)* α*-stable Lévy motion* if

(i) $Z_\alpha(0) = 0$ a.s.

(ii) Z_α has independent increments.

(iii) $Z_\alpha(t) - Z_\alpha(s) \sim S_\alpha((t-s)^{1/\alpha}, \beta, 0)$ for any $0 \leq s < t < \infty$ and for some $0 < \alpha \leq 2$, $|\beta| \leq 1$.

See Figure 1 for some simulated sample paths in the case $\alpha = 1.2$, $\beta = 0$. Observe that the process Z_α has stationary increments. It is Brownian motion when $\alpha = 2$. For a comprehensive survey of properties of α-stable random variables and α-stable Lévy motions we refer to Janicki and Weron (1994) or Samorodnitsky and Taqqu (1994).

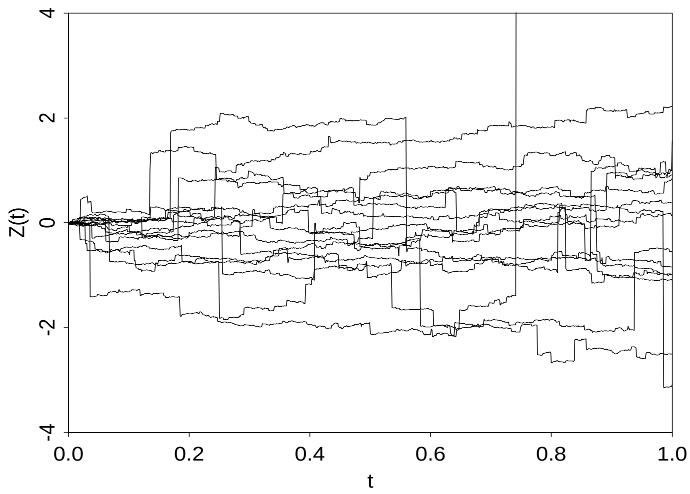

Figure 1: Simulations of α-stable Lévy motion with $\alpha = 1.2$, $\beta = 0$.

The classical risk process $R = (R(t) : t \geq 0)$ is given by

$$R(t) = x + ct - \sum_{k=1}^{N(t)} Y_k\,, \quad \left(\sum_{k=1}^{0} Y_k \stackrel{\text{def}}{=} 0 \right)\,, \tag{1}$$

where $x \geq 0$, $N = (N(t) : t \geq 0)$ is a homogeneous Poisson process with intensity λ, $(Y_k : k \in \mathbb{N})$ is a sequence of independent and identically distributed (iid) random variables independent of N with distribution function F on $[0, \infty)$ and finite mean μ. The premium rate c is given by $c = (1+\theta)\lambda\mu$, where the relative safety loading θ is assumed to be positive. We now consider a process $Q = (Q(t) : t \geq 0)$ defined by $Q(t) = R(t) + \eta Z_\alpha(t)$, where R is given in (1), i.e.

$$Q(t) = x + ct - \sum_{k=1}^{N(t)} Y_k + \eta Z_\alpha(t) \ , \qquad t \geq 0 \, . \tag{2}$$

Here η is a positive number and Z_α is α-stable Lévy motion with $1 < \alpha < 2$ and $\beta = -1$, independent of R. The condition $\beta = -1$ ensures that there are no upward jumps of Z_α and the condition $\alpha > 1$ is needed to have a finite mean. It will turn out that the above conditions on the parameters of Z_α are crucial in some of the derivations below. From a modelling point of view, one could view the downward jumps of Z_α as certain extra random payments either involving the income side or the claim payment side. Figure 2 shows one simulated sample path of the process Q together with the underlying classical risk process R.

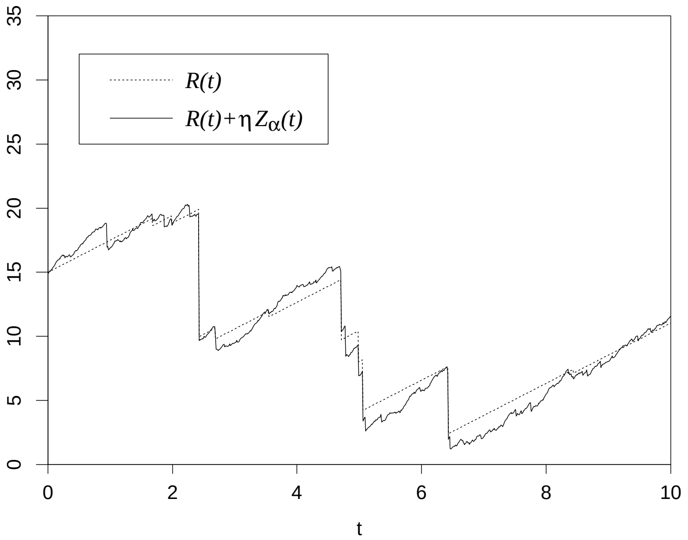

Figure 2: Simulation of a classical risk process R (initial capital $x = 15$, premium rate $c = 2.5$, intensity $\lambda = 1$ and exponentially distributed claims with mean $\mu = 2$) and of its perturbed version $Q = R + \eta Z_\alpha$. Here $\alpha = 1.5$ and $\eta = 0.75$.

The functional

$$\Psi(x) = P[\inf_{t \geq 0} Q(t) < 0 \,|\, Q(0) = x] \ , \qquad x \geq 0 \, , \tag{3}$$

the so–called ruin probability, is the object of this paper.
Under the above conditions, the process $(Q(t)-x : t \geq 0)$ belongs to the class $\Xi = \{(Y(t) : t \geq 0), Y(0)=0\}$ of homogeneous processes with independent increments, not having positive jumps. Such a process is characterized by

$$E[e^{sY(t)}] = e^{t\xi(s)} \ , \qquad \mathrm{Re}\{s\} \geq 0 \ , \tag{4}$$

where $e^{\xi(s)}$ is the Lévy–Khintchine representation of the characteristic function of an infinitely divisible distribution, i.e. $\xi(s)$ can be written in the form

$$\xi(s) = cs + \frac{\sigma^2 s^2}{2} + \int_{-\infty}^{0} (e^{su}-1)\,\Pi_1(du) + \int_{-\infty}^{0} (e^{su}-1-su)\,\Pi_2(du) \ , \tag{5}$$

where $c \in \mathbb{R}$, $\sigma \geq 0$ and Π_1, Π_2 are measures defined on the negative half line such that

$$\int_{-1}^{0} |u|\,\Pi_1(du) < \infty \ , \qquad \int_{-1}^{0} u^2\,\Pi_2(du) < \infty \ .$$

In addition, in the case when Π_2 can not be made identically zero by varying Π_1 and c, we have $\int_{-1}^{0} |u|\,\Pi_2(du) = \infty$, $\int_{-\infty}^{-1} |u|\,\Pi_2(du) < \infty$.
The following theorem from Zolotarev (1964) turns out to be crucial.

Proposition 2 *Let $(Y(t) : t \geq 0) \in \Xi$ and $\gamma = E[Y(1)] \geq 0$. Define $\Psi(x) = P[\inf_{t\geq 0} Y(t) < -x]$ for $x \geq 0$. Then the function Ψ is determined from the Lévy exponent $\xi(s)$ by the relation*

$$s\int_0^{\infty} e^{-sx}\Psi(x)\,dx = 1 - \frac{\gamma s}{\xi(s)} \ , \qquad \mathrm{Re}\{s\} \geq 0 \ . \tag{6}$$

The idea now is to apply Proposition 2 to the process $(Q(t)-x : t \geq 0)$ and to solve for $\Psi(x)$ in (6). For the sake of simplicity we assume in the sequel $\eta = 1$. The formula we shall derive in Theorem 1 can easily be generalized for arbitrary positive η, see formula (16) in connection with Theorem 1. Denote by I_1, I_2 the integrals in (5) with respect to Π_1 and Π_2. For the process $(Q(t)-x : t \geq 0)$ we have $\sigma = 0$,

$$\Pi_1(u,0) = \lambda(1-F(-u)) \ , \quad u \leq 0 \ , \tag{7}$$

$$\Pi_2(du) = \frac{q}{|u|^{1+\alpha}} I_{(-\infty,0)}(u)\,du \quad \text{with} \quad q = \frac{\alpha(\alpha-1)}{\Gamma(2-\alpha)} \ . \tag{8}$$

The representation for Π_2 is given in Zolotarev (1986), p. 68.
Set $\overline{F}(x) = 1-F(x)$. We define the integrated tail distribution F_I as

$$F_I(x) = \frac{1}{\mu}\int_0^x \overline{F}(y)\,dy \ . \tag{9}$$

The Laplace–Stieltjes transform $\hat{f}_I(s)$ of F_I is given by

$$\hat{f}_I(s) = \int_0^\infty e^{-su}\, dF_I(u) = \frac{1}{\mu}\int_0^\infty e^{-su}\overline{F}(u)\, du \ . \tag{10}$$

With the aid of (10), the integral I_1 can be expressed as

$$I_1 = -\lambda\mu s \hat{f}_I(s) \ . \tag{11}$$

Set $\tilde{\alpha} = \alpha - 1$ and note that $0 < \tilde{\alpha} < 1$, $q = \alpha\tilde{\alpha}/\Gamma(1-\tilde{\alpha})$. Using (8), we immediately find for I_2

$$I_2 = s^\alpha \ . \tag{12}$$

Combining (11) and (12) yields

$$\begin{aligned} \xi(s) &= cs + I_1 + I_2 \\ &= s(c - \lambda\mu\hat{f}_I(s) + s^{\tilde{\alpha}}) \ . \end{aligned}$$

Consider now $\gamma s/\xi(s)$, where $\gamma = E[Q(1) - x] = c - \lambda\mu$ is assumed to be strictly positive. Define $\rho = \lambda\mu/c$. Then we can write

$$\begin{aligned} \frac{\gamma s}{\xi(s)} &= \frac{\gamma}{c - \lambda\mu\hat{f}_I(s) + s^{\tilde{\alpha}}} \\ &= (1-\rho)\frac{c}{c + s^{\tilde{\alpha}} - \lambda\mu\hat{f}_I(s)} \\ &= (1-\rho)\frac{\dfrac{c}{c+s^{\tilde{\alpha}}}}{1 - \rho\hat{f}_I(s)\dfrac{c}{c+s^{\tilde{\alpha}}}} \ . \end{aligned}$$

The function $\hat{u}(s) = c/(c + s^{\tilde{\alpha}})$ with $\hat{u}(0) = 1$ is completely monotone, i.e. possesses derivatives $\hat{u}^{(n)}(s)$ of all orders and $(-1)^n\hat{u}^{(n)}(s) \geq 0$. Hence $\hat{u}(s)$ is the Laplace–Stieltjes transform of a probability distribution U on $\mathbb{R}$ (Feller (1971), p. 439). The following lemma characterizes the law U in terms of the density of extremal stable laws ($\beta = 1$) or the so-called Mittag–Leffler functions $E_\sigma(x) = \sum_{n=0}^\infty x^n/\Gamma(1+\sigma n)$, $\sigma > 0$. If f and g are two functions, we mean by $f(x) \sim g(x)$ that $\lim_{x\to\infty} f(x)/g(x) = 1$. Recall that for a stable random variable $X \sim S_\alpha(1, \beta, 0)$ we denote by $G(\,\cdot\,; \alpha, \beta)$, $g(\,\cdot\,; \alpha, \beta)$ its distribution function and density function, respectively.

Lemma 1 *Let* $X \sim S_{\tilde{\alpha}}(1, 1, 0)$ *with index* $0 < \tilde{\alpha} < 1$. *Then for* $c > 0$, $s \geq 0$ *one has* *(i)* $\hat{u}(s) = \dfrac{c}{c+s^{\tilde{\alpha}}} = \displaystyle\int_0^\infty e^{-sy} u(y)\, dy$, *where* $u(y) = \displaystyle\int_0^\infty \frac{g(y/u^{1/\tilde{\alpha}}; \tilde{\alpha}, 1)}{u^{1/\tilde{\alpha}}}\, c\, e^{-cu}\, du$.

(ii) $U(x) = \displaystyle\int_0^x u(y)\, dy = 1 - \sum_{n=0}^{\infty} \frac{(-c)^n}{\Gamma(1+\tilde{\alpha}n)}\, x^{\tilde{\alpha}n} = 1 - E_{\tilde{\alpha}}(-cx^{\tilde{\alpha}})$,

where $E_\sigma(x) = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{\Gamma(1+\sigma n)}$, $\sigma > 0$.

(iii) The distribution U *has a regularly varying tail:*

$$\overline{U}(x) = 1 - U(x) \sim \frac{1}{c\,\Gamma(1-\tilde{\alpha})}\, x^{-\tilde{\alpha}} , \quad x \to \infty .$$

(iv) For $0 \le \delta < \tilde{\alpha}$ *we have*

$$\int_0^\infty x^\delta u(x)\, dx = \frac{\Gamma(1+\delta/\tilde{\alpha})\,\Gamma(1-\delta/\tilde{\alpha})}{c^{\delta/\tilde{\alpha}}\,\Gamma(1-\delta)} .$$

Remarks.

1. The Mittag–Leffler function $E_\sigma(x) = \sum_{n=0}^{\infty} x^n/\Gamma(1+\sigma n)$, $\sigma > 0$ is a generalization of the exponential to which it reduces when $\sigma = 1$.

2. As $\tilde{\alpha}$ decreases from 1 to 0, the tails of the distribution U become thicker, see Figure 3 below.

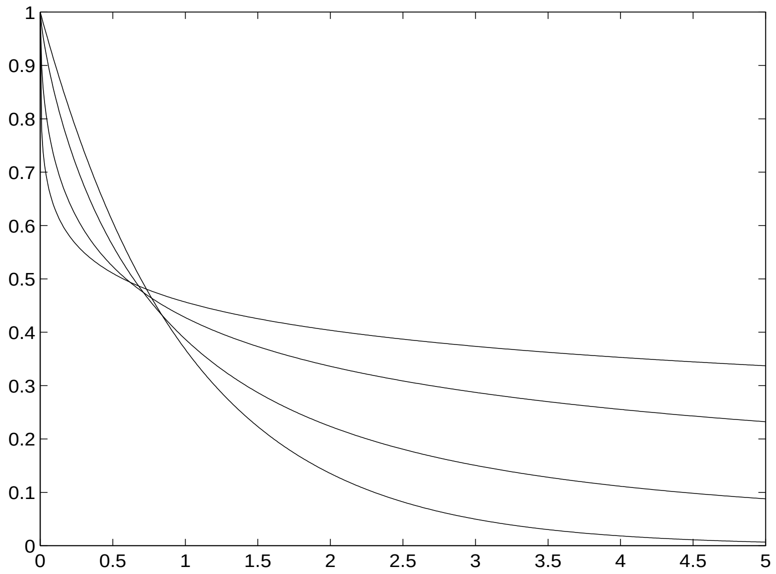

Figure 3: Function $\overline{U}(x)$ for $c = 1$ and $\tilde{\alpha} \in \{0.2, 0.5, 0.8, 1\}$

PROOF. (i) Consider the function $\hat{v}(s) = c/(c+s)$. Then $\hat{v}(s)$ is the Laplace transform of the exponential law, i.e. $\hat{v}(s) = \int_0^\infty e^{-sy} v(y)\, dy$ with $v(y) = ce^{-cy}$. Now

$$\hat{u}(s) = \frac{c}{c + s^{\tilde{\alpha}}} = \hat{v}(s^{\tilde{\alpha}}) = \int_0^\infty e^{-s^{\tilde{\alpha}} y} v(y)\, dy \,. \tag{13}$$

Since $0 < \tilde{\alpha} < 1$ and $\beta = 1$, the Laplace transform of X exists for $s \geq 0$ and is given by $E[e^{-sX}] = \exp\{-s^{\tilde{\alpha}}\}$, see Zolotarev (1986), p. 112. It follows that

$$\begin{aligned} e^{-s^{\tilde{\alpha}} y} &= e^{-(s y^{1/\tilde{\alpha}})^{\tilde{\alpha}}} \\ &= E\left[e^{-(s y^{1/\tilde{\alpha}}) X}\right] \\ &= \int_0^\infty e^{-(s y^{1/\tilde{\alpha}}) u}\, g(u; \tilde{\alpha}, 1)\, du \\ &= \int_0^\infty e^{-sv}\, \frac{g(v/y^{1/\tilde{\alpha}}; \tilde{\alpha}, 1)}{y^{1/\tilde{\alpha}}}\, dv \,. \end{aligned} \tag{14}$$

With (14) we obtain for (13)

$$\begin{aligned} \hat{u}(s) &= \int_0^\infty \left(\int_0^\infty e^{-su}\, \frac{g(u/y^{1/\tilde{\alpha}}; \tilde{\alpha}, 1)}{y^{1/\tilde{\alpha}}}\, du \right) c\, e^{-cy}\, dy \\ &= \int_0^\infty e^{-sy} \left(\int_0^\infty \frac{g(y/u^{1/\tilde{\alpha}}; \tilde{\alpha}, 1)}{u^{1/\tilde{\alpha}}}\, c\, e^{-cu}\, du \right) dy \end{aligned}$$

which proves (i).

(ii) Straightforward calculation yields

$$\begin{aligned} U(x) = \int_0^x u(y)\, dy &= \int_0^x \left(\int_0^\infty \frac{g(y/u^{1/\tilde{\alpha}}; \tilde{\alpha}, 1)}{u^{1/\tilde{\alpha}}}\, c\, e^{-cu}\, du \right) dy \\ &= \int_0^\infty c\, e^{-cu} \left(\int_0^{x/u^{1/\tilde{\alpha}}} g(v; \tilde{\alpha}, 1)\, dv \right) du \\ &= 1 - \int_0^\infty e^{-cu} \left(\frac{x}{\tilde{\alpha} u} \frac{g(x/u^{1/\tilde{\alpha}}; \tilde{\alpha}, 1)}{u^{1/\tilde{\alpha}}} \right) du \\ &= 1 - \sum_{n=0}^\infty \frac{(-c)^n}{\Gamma(1 + \tilde{\alpha} n)}\, x^{\tilde{\alpha} n} \,. \end{aligned}$$

The last equality establishes the connection between extremal stable laws and the Mittag–Leffler function and follows from Feller (1971), p. 453.

(iii) Note that for $k \in \mathbb{N}_0$ we have

$$s \int_0^\infty e^{-su}\, \frac{(-c\, u^{\tilde{\alpha}})^k}{\Gamma(1 + \tilde{\alpha} k)}\, du = \left(-\frac{c}{s^{\tilde{\alpha}}} \right)^k \,.$$

Thus we obtain by interchanging the order of integration and summation

$$s \int_0^\infty e^{-su} E_{\tilde{\alpha}}(-c u^{\tilde{\alpha}})\, du = \frac{s^{\tilde{\alpha}}}{c} \frac{c}{c + s^{\tilde{\alpha}}}$$

and conclude that

$$s\int_0^\infty e^{-su}E_{\tilde\alpha}(-cu^{\tilde\alpha})\,du \sim \frac{s^{\tilde\alpha}}{c}\;, \quad s\downarrow 0\;.$$

Applying Karamata's Tauberian Theorem (Bingham et al. (1987), p. 37) we obtain

$$1-U(x)=E_{\tilde\alpha}(-cx^{\tilde\alpha}) \sim \frac{1}{c\,\Gamma(1-\tilde\alpha)}\,x^{-\tilde\alpha}\;, \quad x\to\infty\;.$$

(iv) The proof of the last statement follows by direct calculation and is omitted. □

We now return to the calculation of the function $\Psi(x)$. Recall that $\rho=\lambda\mu/c$. Hence

$$\begin{aligned}\int_0^\infty e^{-sx}\,d(1-\Psi(x)) &= 1-s\int_0^\infty e^{-sx}\Psi(x)\,dx\\ &= \frac{\gamma s}{\xi(s)}\\ &= (1-\rho)\frac{\hat u(s)}{1-\rho\hat f_I(s)\,\hat u(s)}\\ &= (1-\rho)\hat u(s)\sum_{n=0}^\infty\left(\rho\hat f_I(s)\,\hat u(s)\right)^n\;.\end{aligned}$$

Inverting the last expression yields

$$1-\Psi(x)=(1-\rho)\sum_{n=0}^\infty\rho^n(F_I^{*n}*U^{*(n+1)})(x)\;,$$

where F^{*n}, $n\ge 1$ denotes the n-fold convolution of F with itself and F^{*0} is the distribution function corresponding to the Dirac measure at zero. We summarize our result in the following theorem.

Theorem 1 *Consider a classical risk process perturbed by α-stable Lévy motion with $1<\alpha<2$ and skewness parameter $\beta=-1$*

$$Q(t)=x+ct-\sum_{k=1}^{N(t)}Y_k+Z_\alpha(t)\;, \qquad t\ge 0\;,$$

where $x\ge 0$, $c=(1+\theta)\lambda\mu$, $(N(t):t\ge 0)$ is a homogeneous Poisson process with intensity λ, $(Y_k:k\in\mathbb{N})$ is a sequence of iid random variables with distribution function F on $[0,\infty)$ and mean μ. Then the probability of ruin $\Psi(x)$ defined in (3) satisfies

$$1-\Psi(x)=(1-\rho)\sum_{n=0}^\infty\rho^n(F_I^{*n}*U^{*(n+1)})(x)\;, \tag{15}$$

where $\rho=\lambda\mu/c$, $F_I(x)=\frac{1}{\mu}\int_0^x\overline{F}(y)\,dy$ and $\overline{U}(x)=\sum_{n=0}^\infty\frac{(-c)^n}{\Gamma(1+\tilde\alpha n)}x^{\tilde\alpha n}$ with $\tilde\alpha=\alpha-1$.

REMARKS.

1. Formula (15) generalizes formula (3.4) of Dufresne and Gerber (1991) to which it reduces when $\alpha = 2$. In that case $U(x) = 1 - e^{-cx}$ is the distribution function of the exponential law.

2. If we consider a perturbation term of the form $\eta Z_\alpha(t)$ instead of $Z_\alpha(t)$ for some positive number η, it follows that the distribution function U has the form

$$U(x) = 1 - \sum_{n=0}^{\infty} \frac{(-c/\eta^\alpha)^n}{\Gamma(1+\tilde{\alpha}n)} x^{\tilde{\alpha}n} . \tag{16}$$

3. Intuitively we expect that the ruin probability is a decreasing function of α. The smaller α, the more "dramatic" the stable Lévy motion behaves. Let $1 < \alpha_1 \leq \alpha_2 < 2$. It is tempting to conjecture that for fixed $x > 0$ the following inequality holds

$$\Psi(\alpha_1, x) \geq \Psi(\alpha_2, x) . \tag{17}$$

However, the question is still open whether or not (17) is true.

4. When the intensity λ of the claim arrival process equals 0, the process $(Q(t) : t \geq 0)$ reduces to a stable Lévy motion with drift:

$$Q(t) = x + ct + \eta Z_\alpha(t) , \quad t \geq 0 .$$

Such a process can be viewed as a weak approximation of a classical risk process when the claim size distribution has infinite variance, see Furrer et al. (1997). Formula (15) then becomes

$$1 - \Psi(x) = U(x) = 1 - \sum_{n=0}^{\infty} \frac{(-c/\eta^\alpha)^n}{\Gamma(1+\tilde{\alpha}n)} x^{\tilde{\alpha}n}$$

and can be interpreted as an approximation for the infinite–time ruin probability of a risk process with infinite claim size variance.

3 Asymptotic Behaviour of the Ruin Probability

The purpose of this section is the investigation of the asymptotic behaviour of the ruin probability $\Psi(x)$ as x becomes large. Notice that we can write $\varphi(x) := 1 - \Psi(x) = K * U(x)$, where K is the distribution function of the random geometric sum $X_1 + \cdots + X_N$, all X_i having distribution $F_I * U$. Roughly speaking the tail behaviour of φ is then related to that of F_I and/or U. Intuitively we can think of a balance, putting on each scale the tails of F_I and U, respectively. Both weights then contribute to the tail behaviour of φ when holding in equilibrium, see Theorem 2. If the mass of one tail exceeds the one of the other, the equilibrium is disturbed and it is solely the dominant distribution that affects the asymptotic behaviour of φ. See Theorem 3 when we allow for "large" claims such that the tail of F_I dominates and Theorem 4 when the perturbation U is the relevant quantity.
In the following we shall carry out the above heuristic reasoning in more mathematical detail. An appropriate tool for modelling the possibility of large claims is the class $\mathcal{S}$ of subexponential distribution functions. Typical examples are Pareto, lognormal, distribution functions with regularly varying tails. The definition of $\mathcal{S}$ is as follows.

Definition 2 If F is a distribution function on $[0,\infty)$ with unbounded support, then we say that F is a *subexponential distribution function* $(F \in \mathcal{S})$ if and only if

$$\frac{1 - F^{*2}(x)}{1 - F(x)} \to 2 \,, \quad x \to \infty \,.$$

The name subexponential comes from the following property: if $F \in \mathcal{S}$, then the right tail of F decreases slower than any exponential, i.e. $\lim_{x\to\infty} e^{\epsilon x}\,\overline{F}(x) = \infty$ for all $\epsilon > 0$. The class $\mathcal{S}$ has the following closure properties under convolution operations.

Proposition 3 *Let $H = F_1 * F_2$ be the convolution of two distribution functions on $[0,\infty)$.*

a) If $F_2 \in \mathcal{S}$ and $\overline{F}_1(x) = o(\overline{F}_2(x))$ as $x \to \infty$, then $H \in \mathcal{S}$. Moreover, $\overline{H}(x) \sim \overline{F}_2(x)$ as $x \to \infty$.

b) If $H \in \mathcal{S}$ and $\overline{F}_1(x) = o(\overline{H}(x))$, then $F_2 \in \mathcal{S}$ and indeed $\overline{F}_2(x) \sim \overline{H}(x)$ as $x \to \infty$.

For a proof see Embrechts et al. (1979). The next proposition can also be found in the same paper.

Proposition 4 *Suppose $\rho \in (0,1)$ and H a proper distribution function on $[0,\infty)$. If $K(x) = (1-\rho)\sum_{n=o}^{\infty} \rho^n H^{*n}(x)$, then the following assertions are equivalent:*

i) $K \in \mathcal{S}$.

ii) $H \in \mathcal{S}$.

iii) $\dfrac{\overline{K}(x)}{\overline{H}(x)} \to \dfrac{\rho}{1-\rho}\,, \quad x \to \infty\,.$

We will also need the following proposition from Feller (1971), p. 278 which also shows that $\mathcal{S}$ contains the class of distribution functions with regularly varying tails. Denote by RV_δ, $\delta \in \mathbb{R}$, the class of regularly varying functions, i.e. a positive measurable function h defined on $[0,\infty)$ is regularly varying with index δ if for all $t > 0$,

$$\lim_{x\to\infty} \frac{h(tx)}{h(x)} = t^\delta \ .$$

The case $\delta = 0$ corresponds to the class of slowly varying functions, therefore $h \in RV_\delta$ can be written as $h(x) = x^\delta L(x)$, $L \in RV_0$.

Proposition 5 *If* F_1 *and* F_2 *are two distribution functions such that* $\overline{F}_i(x) \sim x^{-\delta} L_i(x)$, $x \to \infty$, *with* $L_i \in RV_0$, *then the convolution* $G = F_1 * F_2$ *has a regularly varying tail such that* $\overline{G}(x) \sim x^{-\delta}(L_1(x) + L_2(x))$, $x \to \infty$.

3.1 $\overline{F} \in RV_{-\alpha}$

We first consider the case $\overline{F} \in RV_{-\alpha}$, where α equals the index of stability of the stable Lévy motion. Observe that in this case F belongs to the domain of attraction of a stable law with index α and skewness parameter $\beta = 1$, see for instance Ibragimov and Linnik (1971), Theorem 2.6.1. We prove the following theorem.

Theorem 2 *Suppose that* $1 < \alpha < 2$ *and* $\overline{F} \in RV_{-\alpha}$, *i.e.* $\overline{F}(x) = x^{-\alpha} L(x)$ *for some slowly varying function* L. *Then*

$$\begin{aligned} \Psi(x) &\sim \overline{F}_I(x)/\theta + \frac{1}{\gamma\,\Gamma(1-\tilde\alpha)}\, x^{-\tilde\alpha}\,, \quad x \to \infty \\ &\sim \left(\frac{L(x)}{\theta\mu\tilde\alpha} + \frac{1}{\gamma\,\Gamma(1-\tilde\alpha)}\right) x^{-\tilde\alpha}\,, \quad x \to \infty\,, \end{aligned} \tag{18}$$

where $\theta = c/(\lambda\mu) - 1$, $\gamma = c - \lambda\mu$ *and* $\tilde\alpha = \alpha - 1$.

Remarks.

1. When $\alpha \nearrow 2$ the second summand in (18) tends to 0 and hence $\Psi(x) \sim \overline{F}_I(x)/\theta$, a result which can be found in Veraverbeke (1993) for distribution functions $F_I \in \mathcal{S}$.

2. Whereas monotonicity in α for $\Psi(x)$ is still an open problem, the desired property holds for the above asymptotic form of $\Psi(x)$. Define $W(\alpha, x) = \left(\frac{L(x)}{\theta\mu\tilde{\alpha}} + \frac{1}{\gamma\,\Gamma(1-\tilde{\alpha})}\right) x^{-\tilde{\alpha}}$. Then for $1 < \alpha_1 \leq \alpha_2 < 2$ and $x \geq 1$ one has $W(\alpha_1, x) \geq W(\alpha_2, x)$.

Proof. With $\varphi = 1 - \Psi$ we can write

$$
\begin{aligned}
\varphi &= (1-\rho) \sum_{n=0}^{\infty} \rho^n (F_I^{*n} * U^{*(n+1)}) \\
&= (1-\rho) \sum_{n=0}^{\infty} \rho^n H^{*n} * U \\
&= K * U \;,
\end{aligned}
$$

where $H = F_I * U$ and $K = (1-\rho)\sum_{n=0}^{\infty} \rho^n H^{*n}$. Since $\overline{F} \in RV_{-\alpha}$, we conclude that the tail of F_I behaves as

$$
\overline{F}_I(x) \sim \frac{L(x)}{\tilde{\alpha}\mu}\, x^{-\tilde{\alpha}} \;, \quad x \to \infty \tag{19}
$$

(Feller (1971), p. 281). From Proposition 5 and Lemma 1, (iii) we obtain

$$
\overline{H}(x) = 1 - F_I * U(x) \sim \left(\frac{L(x)}{\tilde{\alpha}\mu} + \frac{1}{c\,\Gamma(1-\tilde{\alpha})}\right) x^{-\tilde{\alpha}} \;, \quad x \to \infty
$$

and therefore $H \in \mathcal{S}$. Consequently, by Proposition 4, $\lim_{x\to\infty} \overline{K}(x)/\overline{H}(x) = \rho/(1-\rho)$ or

$$
\begin{aligned}
\overline{K}(x) &\sim \left(\frac{\rho}{1-\rho}\right)\left(\frac{L(x)}{\tilde{\alpha}\mu} + \frac{1}{c\,\Gamma(1-\tilde{\alpha})}\right) x^{-\tilde{\alpha}} \;, \quad x \to \infty \\
&= A(x)\, x^{-\tilde{\alpha}} \;,
\end{aligned}
$$

say. Finally,

$$
\begin{aligned}
\Psi(x) &= 1 - K * U(x) \\
&\sim \left(A(x) + \frac{1}{c\,\Gamma(1-\tilde{\alpha})}\right) x^{-\tilde{\alpha}} \;, \quad x \to \infty \\
&= \left(\left(\frac{\rho}{1-\rho}\right)\frac{L(x)}{\tilde{\alpha}\mu} + \frac{1}{\gamma\,\Gamma(1-\tilde{\alpha})}\right) x^{-\tilde{\alpha}}
\end{aligned}
$$

or equivalentely, because of (19),

$$
\Psi(x) \sim \overline{F}_I(x)/\theta + \frac{1}{\gamma\,\Gamma(1-\tilde{\alpha})}\, x^{-\tilde{\alpha}} \;, \quad x \to \infty \,.
$$

$\square$

The following corollary is an immediate consequence of Theorem 2. We consider the case where $\overline{F} \in RV_{-\alpha}$ and assume that the slowly varying function L tends to a finite positive constant as $x \to \infty$. A typical example where this condition is fulfilled are Pareto distributed claims.

Corollary 1 *Suppose that* $\overline{F}(x) = x^{-\alpha}L(x)$, $1 < \alpha < 2$ *and* $\lim_{x\to\infty} L(x) = k$ *with* $0 < k < \infty$. *Then*

$$\Psi(x) \sim \frac{1+\rho b}{\gamma\,\Gamma(1-\tilde{\alpha})}\,x^{-\tilde{\alpha}}\,, \quad x \to \infty\,,$$

where $b = lim_{x\to\infty}\overline{F}_I(x)/\overline{U}(x) = kc\Gamma(1-\tilde{\alpha})/(\tilde{\alpha}\mu)$.

Remark. In the proof of Theorem 2 we basically used the knowledge of the functions φ and U. However, the same result can be obtained without knowing those functions explicitly. Recall that

$$\xi(s) = cs + \int_{-\infty}^{0} (e^{su} - 1)\,\Pi_1(du) + \int_{-\infty}^{0} (e^{su} - 1 - su)\,\Pi_2(du)\ ,$$

where Π_1, Π_2 are defined in (7) and (8), respectively. Note that $\int_{-\infty}^{0} u\,\Pi_1(du) = -\lambda\mu$. Set $\Pi = \Pi_1 + \Pi_2$ and keep in mind that under the net profit condition $\gamma = c - \lambda\mu > 0$. Introduce the notation $\hat{h}(s) = \int_0^\infty e^{-sx}h(x)\,dx$, where $h(x) = \int_{-\infty}^{-x}\Pi(y)\,dy$. Consequently,

$$\begin{aligned}\xi(s) &= \gamma s + \int_{-\infty}^{0} (e^{su} - 1 - su)\,\Pi(du)\\ &= \gamma s + s^2\hat{h}(s)\ ,\end{aligned}$$

where the last equality follows by two–fold integration by parts. We conclude that

$$1 - \frac{\gamma s}{\xi(s)} \sim \frac{1}{\gamma}\,s\hat{h}(s)\ , \quad s \downarrow 0$$

and therefore

$$\begin{aligned}s\int_0^\infty e^{-sx}\Psi(x)\,dx &= 1 - \frac{\gamma s}{\xi(s)}\\ &\sim \frac{1}{\gamma}\,s\hat{h}(s)\ , \quad s \downarrow 0\\ &= \frac{1}{\gamma}\,s\int_0^\infty e^{-sx}\left(\int_{-\infty}^{-x}\Pi(y)\,dy\right)\,dx\ .\end{aligned}$$

Because of the definition of Π_2 and the assumption on F it is obvious that the function Π belongs to the class $RV_{-\tilde{\alpha}}$. An argument of Abelian and Tauberian type then yields

$$\Psi(x) \sim \frac{1}{\gamma}\int_{-\infty}^{-x}\Pi(y)\,dy\ , \quad x \to \infty$$

$$
\begin{aligned}
&= \frac{1}{\gamma}\int_x^\infty \lambda\overline{F}(y)\,dy + \frac{1}{\gamma}\int_x^\infty \frac{q}{\alpha y^\alpha}\,dy \\
&= \overline{F}_I(x)/\theta + \frac{1}{\gamma\,\Gamma(1-\tilde{\alpha})}\,x^{-\tilde{\alpha}} \\
&\sim \left(\frac{1}{\theta}\,\frac{L(x)}{\tilde{\alpha}\mu} + \frac{1}{\gamma\,\Gamma(1-\tilde{\alpha})}\right)x^{-\tilde{\alpha}}\,, \quad x\to\infty\,.
\end{aligned}
$$

3.2 $F_I \in \mathcal{S}$, $\overline{U}(x) = o(\overline{F}_I(x))$

We next consider the case where $F_I \in \mathcal{S}$ and $\overline{U}(x) = o(\overline{F}_I(x))$. The above conditions are satisfied for instance when $\overline{F} \in RV_{-\delta}$ with $\delta < \alpha$.

Theorem 3 *Suppose that* $F_I \in \mathcal{S}$ *and that* $\overline{U}(x) = o(\overline{F}_I(x))$ *. Then*

$$\Psi(x) \sim \overline{F}_I(x)/\theta\,, \quad x\to\infty\,,$$

with $\theta = c/(\lambda\mu) - 1$ *.*

Remark. The above asymptotic version of $\Psi(x)$ is independent of the perturbation Z_α .

Proof. Again we can write $\varphi = K * U$ with $K = (1-\rho)\sum_{n=0}^\infty \rho^n H^{*n}$ and $H = F_I * U$. From the assumptions and Propostions 3 and 4 we conclude that $H \in \mathcal{S}$, $\overline{H} \sim \overline{F}_I$ and $K \in \mathcal{S}$. Moreover,

$$\overline{K}(x) \sim \overline{F}_I(x)/\theta\,, \quad x\to\infty\,.$$

Consequently, since $\overline{U}(x) = o(\overline{F}_I(x))$, we have $\overline{U}(x) = o(\overline{K}(x))$. Together with $K \in \mathcal{S}$ we conclude that $\varphi = K * U \in \mathcal{S}$ and $\Psi(x) = \overline{\varphi}(x) \sim \overline{K}(x)$ as $x\to\infty$ (Proposition 3) which ends the proof. $\square$

Remark. One may also consider the case where $\overline{F}_I(x) = o(\overline{U}(x))$, which means that the perturbation law has heavier tails than the claim size law. This condition is fulfilled for instance when the claim size distribution F is exponential or when $\overline{F} \in RV_\delta$ with $\delta > \alpha$. However, from a modelling point of view, this assumption may not be very relevant. The following theorem is added for the sake of completeness. The proof is based on the same arguments as the proof of Theorem 3.

Theorem 4 *Suppose that* $\overline{F}_I(x) = o(\overline{U}(x))$. *Then one has*

$$\Psi(x) \sim \frac{1}{\gamma\,\Gamma(1-\tilde{\alpha})}\, x^{-\tilde{\alpha}} \;, \quad x \to \infty \,,$$

where $\gamma = c - \lambda\mu > 0$ *and* $\tilde{\alpha} = \alpha - 1$.

Acknowledgement

The author would like to thank Paul Embrechts and Hanspeter Schmidli for useful discussions which lead to an improvement of the paper.

References

[1] Baxter, G. and Donsker, M. D. (1957). On the distribution of the supremum functional for processes with stationary independent increments. *Trans. Amer. Math. Soc.*, **85** (1), 73–87.

[2] Billingsley, P. (1986). *Probability and Measure.* Second Edition. John Wiley, New York.

[3] Bingham, N. H., Goldie, C. M., Teugels, J. L. (1987). *Regular Variation.* Cambridge University Press, Cambridge.

[4] Dufresne, F. and Gerber, H. U. (1991). Risk theory for the compound Poisson process that is perturbed by diffusion. *Insurance: Mathematics and Economics*, **10**, 51–59.

[5] Embrechts, P., Goldie, C. M. and Veraverbeke, N. (1979). Subexponentiality and Infinite Divisibility. *Z. Wahrsch. Verw. Geb.*, **49**, 335–347.

[6] Embrechts, P. and Schmidli, H. (1994). Modelling of extremal events in insurance and finance. *Zeitschrift für Operations Research,* **39**, 1–34.

[7] Feller, W. (1971). *An Introduction to Probability Theory and its Applications.* Vol. II, Second ed., John Wiley, New York.

[8] Furrer, H. J., Michna, Z. and Weron, A. (1997). Stable Lévy motion approximation in collective risk theory. *Insurance: Mathematics and Economics.* To appear.

[9] Furrer, H. J. and Schmidli, H. (1994). Exponential inequalities for ruin probabilities of risk processes perturbed by diffusion. *Insurance: Mathematics and Economics*, **15**, 23–36.

[10] Gerber, H. U. (1970). An extension of the renewal equation and its application in the collective theory of risk. *Scand. Actuarial J.*, 205–210.

[11] Grandell, J. (1991). *Aspects of Risk Theory.* Springer–Verlag, New York.

[12] Ibragimov, I. A. and Linnik, Y. V. (1971). *Independent and stationary sequences of random variables.* Wolters–Noordhoff Publishing, Groningen, The Netherlands.

[13] Janicki, A. and Weron, A. (1994). *Simulation and Chaotic Behavior of Stable Processes.* Marcel Dekker, New York.

[14] Samorodnitsky, G. and Taqqu, M. S. (1994). *Stable Non–Gaussian Random Processes.* Chapman and Hall, London.

[15] Veraverbeke, N. (1993). Asymptotic estimates for the probability of ruin in a Poisson model with diffusion. *Insurance: Mathematics and Economics*, **13**, 57–62.

[16] Zolotarev, V. M. (1964). The first passage time of a level and the behaviour at infinity for a class of processes with independent increments. *Theory Probab. Appl.*, **9** (1), 653–661.

[17] Zolotarev, V. M. (1986). *One–dimensional Stable Distributions,* Vol. 65 of "Translations of mathematical monographs", American Mathematical Society.