Valuation of the Surrender Option in Life Insurance Policies

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Market-consistent Actuarial Valuation

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Contents

Course material

- Slides

- Longstaff, F. A. and Schwartz, E. S. (2001). Valuing American Options by Simulation: A Simple Least-Squares Approach. *The Review of Financial Studies*, Vol. 14, No. 1, pp. 113-147.

The above two documents can be downloaded from

`www.math.ethz.ch/~hjfurrer/teaching/`

A. Motivation and introduction

- Products offered by (life) insurance companies become more and more complex and often incorporate sophisticated guarantee mechanisms and embeddded options such as
  - maturity guarantees
  - rate of return guarantee (interest rate guarantee)
  - 'cliquet' or 'rachet' guarantees (guaranteed amounts are re-set regularly)
  - mortality aspects (guaranteed annuity options)
  - surrender possibilities
  - . . .

Dreadful past experience

- Such issued guarantees and written options constitute liabilities to the insurer, and subsequently represent a value which in adverse circumstances may jeopardize the company's financial position

- Historically, there was no proper valuation, reporting or risk management of these contract elements

- Many companies were unable to meet their obligations when the issued (interest rate) guarantees moved from being far out of the money (at policy inception) to being very much in the money

- As a result, many companies have experienced severe solvency problems.

Solvency II Directive requires proper modeling of options and guarantees

- **Article 79** of the new Solvency II Directive [2] stipultates that financial guarantees and contractual options included in insurance and reinsurance contracts have to be valued properly:

  When calculating technical provisions, insurance and reinsurance undertakings shall take account of the value of financial guarantees and any contractual options included in insurance and reinsurance policies.

  Any assumptions made by insurance and reinsurance undertakings with respect to the likelihood that policy holders will exercise contractual options, including lapses and surrenders, shall be realistic and based on current and credible information. The assumptions shall take account, either explicitly or implicitly, of the impact that future changes in financial and non-financial conditions may have on the exercise of those options.

Surrender option in life insurance contracts

- **Surrender option**: possibility to terminate the insurance contract before maturity and to receive a (guaranteed) surrender value

- Driving surrender factors:
  - deterioration/improvement of policyholder's health
  - mis-selling
  - financial market conditions (e.g. poor equity performance or higher market yields)

Goals of this lecture

- Pricing the surrender option by means of arbitrage pricing techniques using

  (i) closed-form solution
  (ii) Monte Carlo simulation (LSM algorithm)

- The surrender option pricing problem corresponds to the valuation of a contingent claim for the insurer where the contingency will be related to the level of interest rates ($\triangleq$ dynamic lapse rule)

- **Note**: the objective of this lecture is not to provide a complete insurance risk management framework for life insurance policies with embedded options and guarantees but rather to provide a conceptual analysis of the problem

Surrender option in a pure endowment contract

A pure endowment contract of duration n provides for payment of the sum insured only if the policy holder survives to the end of the contract period.

- Illustrative example:
  - net single premium payment made at time $t = 0$ is invested in a zero-coupon bond with the same maturity T as the policy.
  - guaranteed interest rate r_G (technical interest rate), e.g. $r_G = 3.5\%$
  - no profit sharing
  - contract shall provide for a terminal guarantee (at $t = T$) and surrender benefit (at $t < T$), contingent on survival
  - we assume that the surrender value equals the book value of the mathematical reserves (no surrender penalty)

Dynamic lapse rule

- Book value may be higher or lower than the market value $\Rightarrow$ policy holder can use the American option to improve the value of the contract by surrendering at the right time

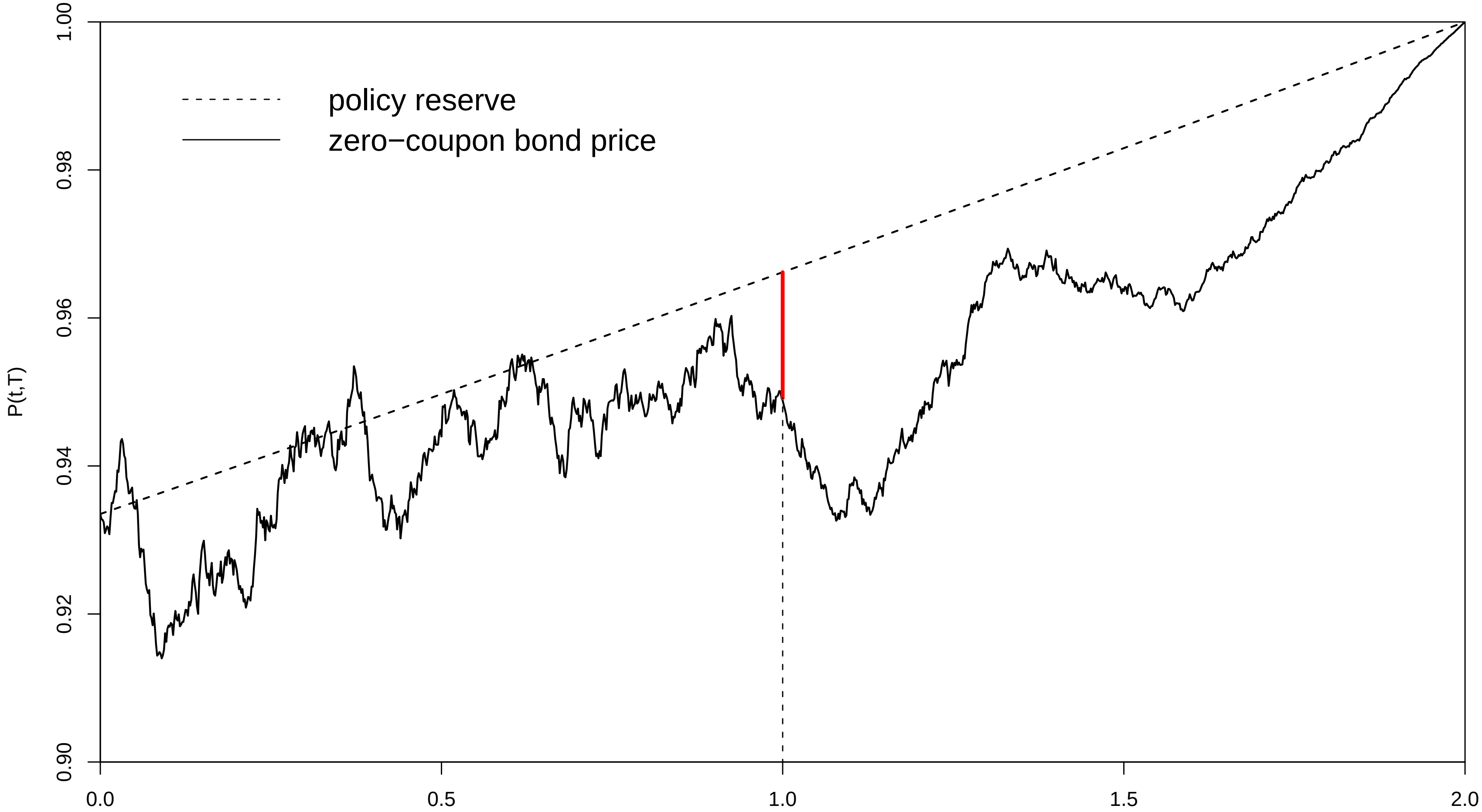
- **Dynamic lapse rule**: when market interest rates exceed the guaranteed minimum interest rate the policy holder is assumed to terminate the contract at time $t = 1$ and to take advantage of the higher yields available in the financial market.

- Hence, the dynamic lapse rule is based on spread

$$\text{market yield} - \text{technical interest rate}$$

- From the viewpoint of asset pricing theory, surrender options equal American put options (Bermudan options).

Visualization of the surrender option in a pure endowment contract of duration $n = 2$



General framework and notation (1/4)

- $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{Q})$ filtered probability space supporting all sources of financial and demographic randomness

- $\mathbb{Q}$: risk-neutral probability measure (i.e. discounted price processes are $\mathbb{Q}$-martingales)

- $B = \{B(t) : 0 \leq t \leq T\}$ with $dB(t) = r(t)B(t)\,dt$: money market account and $\{r(t) : 0 \leq t \leq T\}$ instantaneous short rate process, i.e.

$$B(t) = \exp\left(\int_0^t r(u)\,du\right)$$

General framework and notation (2/4)

- $D(s,t)$: discount factor from time t to s $(s \leq t)$:

$$D(s,t) = \frac{B(s)}{B(t)} = \exp\left(-\int_s^t r(u)\,du\right).$$

- $r = \{r(t) : 0 \leq t \leq T\}$: dynamics of the term structure of interest rates; Vasicek model:

$$dr(t) = (b - ar(t))\,dt + \sigma dW(t), \qquad r(0) = r_0, \tag{1}$$

  with $a, b, \sigma > 0$ and $W = \{W(t) : 0 \leq t \leq T\}$ standard $\mathbb{Q}$-Brownian motion.

- $U = \{U(t) : 0 \leq t \leq T\}$ option price process; $U(t)$ is the value of the surrender option at time t, assuming the option has not previously been exercised.

General framework and notation (3/4)

- $Z(t_1), Z(t_2), \ldots, Z(t_n)$: succession of cash flows emanating from the life insurance contract, where payment $Z(t_k)$ occurs at time t_k

- $L = \{L(t) : 0 \leq t \leq T\}$ market-consistent value process of the life insurance contract where

$$L(t) = B(t)\,\mathbb{E}_{\mathbb{Q}}\left[\sum_{i}^{n} \frac{Z(t_i)\mathbf{1}_{\{t<t_i\}}}{B(t_i)}\Big|\mathcal{F}_{t_i}\right], \tag{2}$$

- $V(t)$: book value of the policy reserve; given by $V(t) = V(0)(1 + r_G)^t$ with deterministic technical interest rate r_G (e.g. $r_G = 3.5\%$) and $V(T) = 1$.

- ${}_tp_x$: probability that an individual currently aged-x survives for t more years.

General framework and notation (4/4)

- $\tau(x)$ or τ: future lifetime of a life aged x
- biometric risk assumed to be independent of the financial risk
- $Y(t)$: payoff from exercise at time t, i.e. $Y(t) = (V(t) - P(t,T))^+$

B. Closed-form expression for the price of the surrender option

Definition of the cash flows:

- At maturity $t = T = 2$:

$$Z(2) = \mathbf{1}_{\{V(1) \leq P(1,2)\} \cap \{\tau > 2\}} \tag{3}$$

- **Interpretation**:

  - $Z(2) = V(2) = 1$ if the policy holder is alive at time $t = 2$ $(\tau > 2)$ and has not terminated the contract at time $t = 1$. The policyholder opts for continuation at $t = 1$ if the surrender value $V(1)$ is less than the value $P(1,2)$ of the reference portfolio.

  - $Z(2) = 0$ if the policy holder died before $t = 2$ or exercised the surrender option at time $t = 1$.

Definition of the cash flows (cont'd)

- At time $t = 1$:

$$Z(1) = V(1)\,\mathbf{1}_{\{V(1)>P(1,2)\}\cap\{\tau>1\}} \qquad (4)$$

- **Interpretation**:
  - $Z(1) = V(1)$ in case the policyholder is alive at $t = 1$ and surrenders, thus cashing in the amount $V(1)$. Surrender occurs if the policy reserve $V(1)$ exceeds the value of the reference portfolio $P(1,2)$.
  - $Z(1) = 0$ if the policyholder died before $t = 1$ or does not exercise the surrender option. The financial rational policy holder will not exercise the surrender option as long as the policy reserve $V(1)$ is smaller than the reference portfolio value $P(1,2)$.

Time-0 valuation (1/3)

By means of (2) we have that

$$
\begin{aligned}
L(0) &= B(0)\,\mathbb{E}_{\mathbb{Q}}\left[\tilde{Z}(1)+\tilde{Z}(2)\middle|\mathcal{F}_0\right] \\
&= \mathbb{E}_{\mathbb{Q}}\left[\tilde{Z}(1)+\tilde{Z}(2)\right] \\
&= \mathbb{E}_{\mathbb{Q}}\left[\frac{Z(1)}{B(1)}\right]+\mathbb{E}_{\mathbb{Q}}\left[\frac{Z(2)}{B(2)}\right] \\
&= \mathbb{E}_{\mathbb{Q}}\left[\frac{V(1)}{B(1)}\,\mathbf{1}_{\{V(1)>P(1,2)\}\cap\{\tau>1\}}\right]+\mathbb{E}_{\mathbb{Q}}\left[\frac{1}{B(1)}\,\mathbf{1}_{\{V(1)\le P(1,2)\}\cap\{\tau>2\}}\right] \\
&= {}_1p_x\,\mathbb{E}_{\mathbb{Q}}\left[\frac{V(1)}{B(1)}\,\mathbf{1}_{\{V(1)>P(1,2)\}}\right]+{}_2p_x\,\mathbb{E}_{\mathbb{Q}}\left[\frac{1}{B(2)}\,\mathbf{1}_{\{V(1)\le P(1,2)\}}\right] \qquad (5)
\end{aligned}
$$

Time-0 valuation (2/3)

Rewriting the first term on the right-hand side of (5) yields

$$L(0) = {}_1p_x\,\mathbb{E}_{\mathbb{Q}}\left[\frac{(V(1)-P(1,2))^+}{B(1)}\right] + {}_1p_x\,\mathbb{E}_{\mathbb{Q}}\left[\frac{P(1,2)}{B(1)}\,\mathbf{1}_{\{V(1)>P(1,2)\}}\right]$$

$$+\ {}_2p_x\,\mathbb{E}_{\mathbb{Q}}\left[\frac{1}{B(2)}\,\mathbf{1}_{\{V(1)\leq P(1,2)\}}\right]\ .$$

Add and subtract ${}_2p_x\mathbb{E}_{\mathbb{Q}}[\mathbf{1}_{\{V(1)>P(1,2)\}}/B(2)]$ and observe that

$$\begin{aligned}
{}_2p_xP(0,2) &= {}_2p_x\mathbb{E}_{\mathbb{Q}}\left[\frac{1}{B(2)}\right] \\
&= {}_2p_x\mathbb{E}_{\mathbb{Q}}\left[\frac{\mathbf{1}_A}{B(2)}\right] + {}_2p_x\mathbb{E}_{\mathbb{Q}}\left[\frac{\mathbf{1}_{A^C}}{B(2)}\right]
\end{aligned}$$

Time-0 valuation (3/3)

$$
\begin{aligned}
L(0) = & \; {}_1p_x \, \mathbb{E}_{\mathbb{Q}} \left[\frac{(V(1) - P(1,2))^+}{B(1)} \right] \\
& + \; {}_1p_x \, \mathbb{E}_{\mathbb{Q}} \left[\frac{P(1,2)}{B(1)} \, \mathbf{1}_{\{V(1) > P(1,2)\}} \right] - \; {}_2p_x \, \mathbb{E}_{\mathbb{Q}} \left[\frac{1}{B(2)} \, \mathbf{1}_{\{V(1) > P(1,2)\}} \right] \\
& + \; {}_2p_x P(0,2) \\
= & \; {}_1p_x \, \mathbb{E}_{\mathbb{Q}} \left[\frac{(V(1) - P(1,2))^+}{B(1)} \right] \\
& + \left({}_1p_x - \; {}_2p_x \right) \mathbb{E}_{\mathbb{Q}} \left[\frac{P(1,2)}{B(1)} \, \mathbf{1}_{\{V(1) > P(1,2)\}} \right] \\
& + \; {}_2p_x P(0,2) \, .
\end{aligned}
$$

Decomposition of the liability value $L(0)$ into three components

We conclude that

$$L(0) = l_1 + l_2 + l_3,$$

where

$$l_1 = {}_2p_x P(0,2)\,, \tag{6}$$

$$l_2 = {}_1p_x \, \mathbb{E}_{\mathbb{Q}} \left[\frac{(V(1) - P(1,2))^+}{B(1)} \right] , \tag{7}$$

$$l_3 = ({}_1p_x - {}_2p_x) \, \mathbb{E}_{\mathbb{Q}} \left[\frac{P(1,2)}{B(1)} \mathbf{1}_{\{V(1) > P(1,2)\}} \right] . \tag{8}$$

Decomposed liability value gives most valuable and risk management-relevant information

Interpretation of the three different components:

- **First term** (6): market-consistent liability value of an identical contract without surrender option.
- **Second term** (7): surrender option premium; equal to the price of a European put option with strike $K = V(1)$, time-to-maturity $T = 1$ written on a pure discount bond maturing at time $S = 2$ (providing protection against rising interest rates)
- **Third term** (8): residual term (difference of two 'neighbouring' survival probabilities and thus negligible)

Numerical example

- $x = 45$ with ${}_1p_x = 0.998971$ and ${}_2p_x = 0.997860$
- $r_G = 3.5\%$, hence $V(0) = (1 + 0.035)^{-2} = 0.9335$
- Vasicek short rate dynamics specified by the parameters $a = 0.36$, $b = 0.0216$, $\sigma \in \{0.05, 0.25, 0.5\}$ and $r_0 = (A(0,2) - \log V(0))/B(0,2) = 0.0255$, yielding $P(0,2) = V(0) = 0.9335$
- For the calculation of l_2, we use the explicit formulae for European bond options in a Vasicek short rate dynamics (see Appendix)

Liability component	Standard deviation of the Vasicek dynamics					
	$\sigma = 5\%$		$\sigma = 25\%$		$\sigma = 50\%$	
l_1	0.932	97.8%	0.932	92.7%	0.932	87.1%
l_2	0.021	2.2%	0.073	7.3%	0.139	12.9%
$l_1 + l_2$	0.953	100%	1.005	100%	1.071	100%

C. LSM algorithm for pricing the surrender option

- **Recall**: LSM approach is based on
  - Monte Carlo simulation
  - Least squares regression

- Decision whether to surrender at time t or not is made by comparing the payoff from immediate exercise with the continuation value. The continuation value is determined by a least square regression of the option value $U(t_{i+1})$ on the current values of state variables

- Idea is to work backwards in time, starting from the contract maturity date T.

- **Note**: following algorithm is formulated for time-0 discounted payoffs and value estimates. Thus, with a slight abuse of notation, $U(t)$ stands for $D(0,t)U(t)$.

Pricing algorithm

(i) Simulate n independent paths

$$\big(P(t_1,T;\omega_k), P(t_2,T;\omega_k), \ldots, P(t_m,T;\omega_k)\big)\,, \quad k = 1,2,\ldots,n$$

under the risk neutral measure $\mathbb{Q}$ where $t_j = jT/m$ for $j = 0,1,\ldots,m$

(ii) At terminal nodes (policy expiry date), set

$$\hat{U}(T;\omega_k) = Y(T;\omega_k) \quad (= 0)$$

with $Y(t) = D(0,t)\,(V(t) - P(t,T))^+$ and $V(T) = P(T,T) = 1$. Choice of exercising or not at contract maturity T is irrelevant since – by assumption – market value of the contract equals the book value.

(iii) Apply backward induction: for $i = m-1, \dots, 1$

- Given estimated values $\hat{U}(t_{i+1}; \omega_k)$, use OLS regression over all simulated sample paths to calculate the weights $\hat{\alpha}_{i1}, \dots, \hat{\alpha}_{iM}$, i.e. find how the values $\hat{U}(t_{i+1}; \omega_k)$ depend on the state variables $P(t_i, T; \omega_k)$ known at time t_i

- Set

$$\hat{U}(t_i; \omega_k) = \begin{cases} Y(t_i; \omega_k), & Y(t_i; \omega_k) \geq \hat{C}(t_i; \omega_k), \\ \hat{U}(t_{i+1}; \omega_k), & Y(t_i; \omega_k) < \hat{C}(t_i; \omega_k), \end{cases}$$

with

$$\hat{C}(t_i; \omega_k) = \sum_{j=0}^{M} \hat{\alpha}_{ij} \, L_j\big(P(t_i, T; \omega_k)\big)$$

for some basis functions $L_j(x)$.

(iv) Set

$$\hat{U}(0) = \frac{1}{n} \sum_{k=1}^{n} \hat{U}(t_1; \omega_k)$$

□

Remarks

- Accuracy of the LSM approach (like any regression-based methods) depends on the choice of the basis functions

- Polynomials are a popular choice

- Above pricing algorithm is formulated in discounted figures: payoffs and value estimates are denominated in time-0 units of currency. In practice, however, payoffs and value estimates are denominated in time-t units. This requires explicit discounting in the algorithm:
  - regress $D(t_i, t_{i+1})U(t_{i+1}; \omega_k)$ (instead of $U(t_{i+1}; \omega_k)$) against the state variables $L_j\big(P(t_i, T; \omega_k)$ to obtain the regression weights and the continuation values.

- Glasserman [4] p. 115 presents an algorithm for the joint simulation of the pair (r, D) at times $t_1, \ldots, t_m$ without discretization error.

Extracts from R-Codes

```
T= 5                      # contract maturity date
t= seq(from=0,to=T,by=1)  # time instants when the contract can be surrendered
n = 100000                # number of simulated sample paths

r= matrix(0,nrow=n,ncol=T+1)
I= matrix(0,nrow=n,ncol=T+1)      # I(t) = int_0^t r(u)du
D= matrix(0,nrow=n,ncol=T+1)      # D(t) = exp(-I(t))
r[,1] = r0
Z1    = matrix(rnorm((T-1)*n,mean=0,sd=1),nrow=n,ncol=T)
Z2    = matrix(rnorm((T-1)*n,mean=0,sd=1),nrow=n,ncol=T)

#joint simulation of (r(t),D(t)), cf. Glasserman p. 115:
for (k in 2:(T+1)){
r[,k]= exp(-kappa*(t[k]-t[k-1]))*r[,k-1] + m*(1-exp(-kappa*(t[k]-t[k-1])))
          +sigma*sqrt(1/(2*kappa)*(1-exp(-2*kappa*(t[k]-t[k-1]))))*Z1[,k-1]
⋮
I[,k]= I[,k-1]+mu.I[,k]+sqrt(sigma2.I[,k])*(rho.r.I[,k]*Z1[,k-1]+sqrt(1-(rho.r.I[,k])^2)*Z2[,k-1])
D[,k]= exp(-I[,k])
}

# corresponding bond prices:
PtT = matrix(0,nrow=n,ncol=T)
for (k in (1:T)){
  btT = (1-exp(-kappa*(T-t[k])))/kappa
  atT = (m-sigma^2/(2*kappa^2))*(btT-(T-t[k]))-sigma^2/(4*kappa)*(btT)^2
  PtT[,k] = exp(atT-btT*r[,k])
}
PtT = cbind(PtT,1)
```

Extracts from R-Codes (cont'd)

```
#surrender value price process:
U   = matrix(0,nrow=n,ncol=T)                     # surrender option value process
DU  = matrix(0,nrow=n,ncol=T)                     # one-step back discounted value process
U[,T-1] = (V[,T-1]-PtT[,T-1])*(V[,T-1]>PtT[,T-1]) # can start at T-1 because book value=market value at t=T
C       = matrix(0,nrow=n,ncol=T-1)               # continuation values
Y       = matrix(0,nrow=n,ncol=T-1)               # payoffs from immediate exercise

M       = 3                                       # number of basis functions  [f(x) = 1, f(x) = x, f(x) = x^2]
alpha   = matrix(0,nrow=M,ncol=T-1)               # regression weights

for (i in ((T-2):1)){
   P1 = PtT[,i]
   P2 = (PtT[,i])^2
   DU[,i+1] = U[,i+1]*D[,i+1]/D[,i]
   out = lm(DU[,i+1]~ P1 + P2)
   alpha[,i]= out$coeff                           # not explicitly used
   C[,i]    = out$fitted.values
   Y[,i]    = (V[,i]-PtT[,i])*(V[,i]>PtT[,i])
   U[,i]    = Y[,i]*(Y[,i]>C[,i]) + D[,i+1]/D[,i]*U[,i+1]*(Y[,i]<C[,i])
}

# surrender option price:
U0 = mean(U[,1]*D[,1])
round(U0,3)
```

Results

Surrender option values (absolute figures and expressed as a percentage of the initial mathematical reserve $V(0) = (1 + r_G)^{-T}$):

	Technical interest rate					
Contract maturity	$r_G = 1.5\%$		$r_G = 3.5\%$		$r_G = 5.5\%$	
$T = 2$	0.018	1.8%	0.015	1.7%	0.013	1.5%
$T = 5$	0.078	8.4%	0.059	6.9%	0.044	5.7%
$T = 10$	0.194	22.5%	0.113	16.0%	0.063	10.7%
$T = 15$	0.327	40.8%	0.151	25.3%	0.062	13.9%

Conclusions

- We have evaluated the surrender option of a single premium pure endowment contract by means of (i) closed-form formulae and (ii) Monte Carlo simulation methods

- For the LSM algorithm we used polynomial basis functions in combination with the reference portfolio values as state variables

- Surrender option becomes more valuable with e.g.
  - + increasing contract maturity date
  - + decreasing guaranteed interest rate r_G
  - + increasing volatility of the short rate dynamics
  - + lower mortality rates

- model can be extended to include exogeneous surrender decisions (beyond continuation values falling below surrender values)

D. Appendix: Vasicek model

Affine term structure: The term structure for the Vasicek model, i.e. the family of bond price processes, is given in the following result, see for instance Björk [1], Proposition 22.3, p. 334.

Proposition: In the Vasicek model, bond prices are given by

$$P(t,T) = e^{A(t,T)-B(t,T)r(t)}, \tag{9}$$

where

$$B(t,T) = \frac{1}{a}\left(1 - e^{-a(T-t)}\right),$$

$$A(t,T) = \frac{(B(t,T) - T + t)(ab - \sigma^2/2)}{a^2} - \frac{\sigma^2 B^2(t,T)}{4a}.$$

□

European bond options (1/2)

Reference: Björk [1], Proposition 22.9, p 338.

Proposition: For the Vasicek model, the price for a European call option with time to maturity T and strike price K on an S-bond is as follows:

$$\mathrm{ZBC}(t,T,K,S) = P(t,S)\Phi(d) - P(t,T)K\Phi(d-\sigma_p), \tag{10}$$

where

$$d = \frac{1}{\sigma_p}\log\left(\frac{P(t,S)}{P(t,T)K}\right) + \frac{\sigma_p}{2},$$

$$\sigma_p = \frac{1}{a}\left(1 - e^{-a(S-T)}\right)\sqrt{\frac{\sigma^2}{2a}\left(1 - e^{-2a(T-t)}\right)}.$$

□

References

[1] Björk, T. (2004). *Arbitrage Theory in Continuous Time*. Second edition. Oxford University Press.

[2] European Parliament (2009). Directive 2009/138/EC of the European Parliament and of the Council of 25 November 2009 on the taking-up and pursuit of the business of Insurance and Reinsurance (Solvency II). http://eur-lex.europa.eu/LexUriServ/LexUriServ.do?uri=OJ:L:2009:335:0001:0155:en:PDF

[3] Furrer, H.J. (2009). Market-Consistent Valuation and Interest Rate Risk Measurement of Policies with Surrender Options. Preprint.

[4] Glasserman, P. (2004). *Monte Carlo Methods in Financial Engineering.* Springer.

[5] Longstaff, F. A. and Schwartz, E. S. (2001). Valuing American Options by Simulation: A Simple Least-Squares Approach. *The Review of Financial Studies*, Vol. 14, No. 1, pp. 113-147.