Valuing Options Embedded in Life Insurance Policies

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Market-consistent Actuarial Valuation

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Course material

- Slides

- Longstaff, F. A. and Schwartz, E. S. (2001). Valuing American Options by Simulation: A Simple Least-Squares Approach. *The Review of Financial Studies*, Vol. 14, No. 1, pp. 113-147.

The above two documents can be downloaded from

`www.math.ethz.ch/~hjfurrer/teaching/`

A. Motivation and Introduction

- The balance sheet equation asserts that

$$A(t) = L(t) = D(t) + E(t)$$

where A: total value of assets; L: total value of liabilities; D: value of debt (insurance liabilities); E: value of equity

- An insurer is solvent at time t if $E(t) \geq 0$. To work out whether this is the case requires a **valuation** of both assets $A(t)$ and debt liabilities $D(t)$, where the latter poses a significant challenge.

- Liabilities stemming from policies written shall take the form

$$\mathrm{BEL} + \mathrm{MVM}$$

where BEL denotes the best estimate value to cover expected cash flows and MVM is a risk margin to cover the uncertainty of cash flows, see e.g. Article 77 of the Solvency II Framework Directive [4].

Capital and Risk Measures (1/3)

All notions of capital embody the idea of a loss-bearing buffer that ensures that the financial institution remains solvent.

- Regulatory capital: this is the capital an institution should hold according to regulatory rules (Basel II/III for banks, SST and Solvency II for insurers in Switzerland and the EU, respectively).

- Economic capital: this is an internal capital requirement in order to control the probability of becoming insolvent, typically over a one-year horizon:

- To ensure solvency in 1 year's time with high probability α ($\alpha = 0.99$, say), a company may require extra capital x_0.

Capital and Risk Measures (2/3)

- Let $E(t) = A(t) - D(t)$ denote a company's equity capital (eligible own funds in Solvency II, risk-bearing capital in SST). The capital requirement then reads

$$\begin{aligned} x_0 &= \inf\big\{x : \mathbb{P}[E(t+1) + x(1+r) \geq 0] = \alpha\big\} \\ &= \inf\big\{x : \mathbb{P}[-E(t+1) \leq x(1+r)] = \alpha\big\} \\ &= \inf\big\{x : \mathbb{P}[E(t) - E(t+1)/(1+r) \leq E(t) + x] = \alpha\big\}. \end{aligned}$$

  Here r denotes the one-year risk-free interest rate.

- This shows that the sum of the available capital $E(t)$ plus the amount x_0 can be taken as the solvency capital requirement; it is a quantile of the distribution of $\Delta E(t+1) = E(t) - E(t+1)/(1+r)$, i.e.

$$E(t) + x_0 = q_\alpha\big(\Delta E(t+1)\big)$$

Capital and Risk Measures (3/3)

- This is the Value-at-Risk idea but, more generally, capital can be computed by applying risk measures to the distribution of $\Delta E(t+1)$

- In case expected shortfall (or Tail-Value-at-Risk) is used as risk measure:

$$\mathsf{ES}_\alpha(\Delta E(t+1)) = \frac{1}{1-\alpha} \int_\alpha^1 q_u\big(\Delta E(t+1)\big)\, du.$$

What does the Solvency II Directive say

- The Solvency II Framework Directive 2009/138/EC (Level 1) is structured as follows:

Rules relating to technical provisions	Articles 76 to 86
Determination (and classification) of own funds	Articles 87 to 99
Solvency capital requirements	Articles 100 to 127
Minimum capital requirement	Articles 128 to 131

- **Solvency II roadmap:** Solvency II is likely to be set into force in mid 2013, followed by a six-month phase-in period (→ 01/01/2014). Before that, the "Omnibus II Directive", the "Implementing measures" (Level 2), and the implementing technical standards (ITS) must be adopted.

Article 77 of Directive 2009/138/EC

(1) The value of technical provisions shall be equal to the sum of a best estimate and a risk margin [...]

(2) The **best estimate** shall correspond to the probability-weighted average of future cash-flows, taking account of the time value of money (expected present value of future cash-flows), using the relevant risk-free interest rate term structure [...]

(3) The **risk margin** shall be such as to ensure that the value of the technical provisions is equivalent to the amount that insurance and reinsurance undertaktings would be expected to require in order to take over and meet the insurance and reinsurance obligations.

(4) Insurance and reinsurance undertakings shall value the best estimate and the risk margin separately. However, where future cash flows associated with insurance or reinsurance obligations can be replicated reliably using financial instruments for which a reliable market value is observable the value of technical provisions [...] shall be determined on the basis of the market value for those instruments.

Relevant risk-free interest rate term structure

- The (Draft) Level 2 Implementing measures stipulate that the rates of the relevant risk-free interest rate term structure shall be calculated as the sum of

  (a) the rates of a basic risk-free interest rate term strucure;
  (b) where applicable, a counter-cyclical premium (CCP);
  (c) where applicable, a matching premium (MP).

- On 21 March 2012, the ECON (European Parliament's Economic and Monetary Affairs Committee) vote on the Omnibus II Directive passed. This will allow (re-) insurers to use the CCP and MP in stressed situations!

- From an actuarial perspective, the concept of a CCP and MP seems to be in contradiction with Article 76 of the Directive 2009/138/EC:

- **Article 76 of 2009/138/EC:** The calculation of technical provisions shall make use of and be consistent with information provided by the financial markets [...]

Article 79 of Directive 2009/138/EC

- **Valuation of options and financial guarantees:**

  When calculating technical provisions, insurance and reinsurance undertakings shall take account of the value of financial guarantees and any contractual options included in insurance and reinsurance policies.

  Any assumptions made by insurance and reinsurance undertakings with respect to the likelihood that policy holders will exercise contractual options, including lapses and surrenders, shall be realistic and based on current and credible information. The assumptions shall take account, either explicitly or implicitly, of the impact that future changes in financial and non-financial conditions may have on the exercise of those options.

Market-consistency and risk neutrality (1/2)

- Market-consistent valuation of technical provisions (and assets) has to be done on a mark-to-model basis because there are no relevant quoted prices on ADLT (active, deep, liquid and transparent) markets.

- Let $L(t)$ denote the mark-to-model value of an insurance obligation. Mark-to-model valuation is typically done according to

$$L(t) = f\big(t, \mathbf{Z}(t)\big)$$

where $\mathbf{Z}(t)$ denotes the (observable) risk factors such as interst rates, stock prices, mortality rates, ...

Market-consistency and risk neutrality (2/2)

- The function f is derived as an expectation of future discounted cash flows in a pricing model under a risk-neutral measure $\mathbb{Q}$:

$$L(t) = f\big(t, \mathbf{Z}(t)\big) = \mathbb{E}_{\mathbb{Q}}\Big[\text{future discounted cash flows} \;\Big|\; \mathcal{F}_t\Big]$$

  Here $\mathcal{F}_t$ denotes the information available at time t.

- Likewise, the value of $L(t+1)$ is given by

$$L(t+1) = f\big(t+1, \mathbf{Z}(t+1)\big) = \mathbb{E}_{\mathbb{Q}}\Big[\text{future discounted cash flows} \;\Big|\; \mathcal{F}_{t+1}\Big]$$

- The problem is how to estimate these conditional expectations?!

Estimating conditional expectations (1/2)

Nested simulations

- Assuming that the valuation models embodied in f do not admit closed form solutions, then a nested simualtion approach requires two rounds of simulation; an outer simulation followed by inner simulation:

  1. **Outer simulation:** sampling of $\mathbf{Z}(t+1)$ under a plausible model for real-world dynamics of risk factors specified by a measure $\mathbb{P}$.

  2. **Inner simulation:** Monte Carlo approximation of $\mathbb{E}_{\mathbb{Q}}$ by generating paths for risk factors $(\mathbf{Z}(s))_{s \geq t+1}$ under $\mathbb{Q}$ and evaluating cash flows.

- **Note:** the amount of simulations and calculations required to proceed in this way is often too demanding computationally (a set of inner scenarios branching out from each outer scenario needs to be generated).

Estimating conditional expectations (2/2)

Least-squares Monte Carlo simulation (LSMC)

- This alternative approach uses a form of analytic approximation involving regressing for the liability value $L(t+1)$ on some key economic variables

- LSMC uses least-squares to obtain an approximation for the conditional expectation $\mathbb{E}_{\mathbb{Q}}$ at time $t+1$. It is assumed that $\mathbb{E}_{\mathbb{Q}}[\,\cdot\,|\mathcal{F}_{t+1}]$ can be given as a linear combination of a countable set of $\mathcal{F}_{t+1}$-measurable basis functions

- With this LSMC approach to the liability valuation, the number of inner scenarios required for each outer scenario projection can be reduced significantly (perhaps just one single inner scenario)

B. Options in life insurance contracts

- Products offered by life insurance companies such as "variable annuities" (VA) for instance often incorporate sophisticated guarantee mechanisms and embeddded options such as

  - maturity guarantees
  - rate of return guarantee (interest rate guarantee)
  - 'cliquet' or 'rachet' guarantees (guaranteed amounts are re-set regularly)
  - mortality aspects (guaranteed annuity options)
  - surrender possibilities
  - . . .

Dreadful past experience

- Such issued guarantees and written options constitute liabilities to the insurer, and subsequently represent a value which in adverse circumstances may jeopardize e.g. the company's solvency position

- Historically, there was no proper valuation, reporting or risk management of these contract elements

- Many (UK domiciled) life insurance companies were unable to meet their obligations when the issued (interest rate) guarantees moved from being far out of the money (at policy inception) to being very much in the money

- As a result, many companies have experienced severe solvency problems.

C. Valuing American options by LSM

- **Definition**: An American option is a contract between two parties giving the buyer the right to, say, purchase one unit of a security for the amount K at any time on or before maturity T

- Recall: a European option, in contrast, can only be exercised at a fixed date

- **General facts:**

  - an American option can only be exercised once
  - the buyer of the option has the choice when to stop
  - exercise decision can only be based on price information up to the present moment ($\rightarrow$ filtration, stopping times)
  - American options are more valuable than their European counterparts
  - price of an American call option equals price of the European call option ($\rightarrow$ it is optimal to wait until the option expires)

Valuation framework (1/2)

- $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{Q})$ filtered probability space supporting all sources of financial randomness

- The probability measure $\mathbb{Q}$ is a risk-neutral probability measure (i.e. discounted price processes are $\mathbb{Q}$-martingales)

- $Y = \{Y(t) : 0 \leq t \leq T\}$ with $Y(t)$ representing the payoff from exercise at time t. Example: $Y(t) = (K - S(t))^+$

- $B = \{B(t) : 0 \leq t \leq T\}$ with $B(t) = \exp\{\int_0^t r_u \, du\}$ money market account and $\{r_t : 0 \leq t \leq T\}$ instantaneous short rate process

- $U = \{U(t) : 0 \leq t \leq T\}$ price process

Valuation framework (2/2)

- Valuing an American option means
  - finding the optimal exercise rule (exercise time)
  - computing the expected discounted payoff under this rule
- If the option seller knew in advance which stopping time τ_0 the investor will use:

$$U(0) = \mathbb{E}_{\mathbb{Q}}\left[\frac{Y(\tau_0)}{B(\tau_0)}\right], \qquad Y(t) = \big(K - S(t)\big)^{+}$$

- Since τ is *not* known, the option seller should prepare for the worst possible case, and charge the maximum value

$$U(0) = \sup_{\tau \in \mathcal{T}} \mathbb{E}_{\mathbb{Q}}\left[\frac{Y(\tau)}{B(\tau)}\right],$$

where $\mathcal{T}$ are the stopping times taking values in $[0, T]$

Main result

Proposition. Suppose there is $\mathbb{Q} \sim \mathbb{P}$ and define $Z = \{Z(t) : 0 \leq t \leq T\}$ by

$$Z(t) = \sup_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}_{\mathbb{Q}} \left[\frac{Y(\tau)}{B(\tau)} \,\middle|\, \mathcal{F}_t \right] B(t) \,. \tag{1}$$

Then $Z(t)/B(t)$ is the smallest $\mathbb{Q}$-supermartingale satisfying $Z(t) \geq Y(t)$. Moreover, the supremum in (1) is achieved by an optimal stopping time $\tau(t)$ that has the form

$$\tau(t) = \inf\big\{ s \geq t : Z(s) = Y(s) \big\} \tag{2}$$

In other words, $\tau(t)$ maximises the right hand side of (1):

$$\mathbb{E}_{\mathbb{Q}} \left[\frac{Y(\tau(t))}{B(\tau(t))} \,\middle|\, \mathcal{F}_t \right] = \sup_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}_{\mathbb{Q}} \left[\frac{Y(\tau)}{B(\tau)} \,\middle|\, \mathcal{F}_t \right] .$$

View the pricing problem through option values: dynamic programming formulation

- **Idea**: to work backwards in time
- Explicit construction of $Z(t)$ by means of dynamic programming:

$$V(t) := \begin{cases} Y(t)\,, & t = T \\ \max\left\{ Y(t), \quad \underbrace{\mathbb{E}_{\mathbb{Q}}\left[\frac{V(t+1)}{B(t+1)} \,\middle|\, \mathcal{F}_t \right] B(t)}_{\text{expected payoff from continuation}} \right\}, & t \leq T-1 \end{cases} \tag{3}$$

- $V = \{V(t) : 0 \leq t \leq T\}$ is called snell envelope. It is the smallest super-martingale dominating Y. Thus, $Z = V$.

View the pricing problem through stopping times

- Dynamic programming rules (3) focus on option values
- Now we want to view the pricing problem through stopping rules
- Make restriction to options that can be exercised only at a fixed set of dates $t_1 < t_2 < \cdots < t_m$. Restriction is regarded as an approximation to a contract allowing continuous exercise
- **Stopping rule**: at any exercise time, compare payoff from immediate exercise with the value of continuation. Exercise if the immediate payoff is higher
- **Continuation value**: value of holding rather than exercising the option:

$$C(t_i) = \mathbb{E}_{\mathbb{Q}}\left[\frac{V(t_{i+1})}{B(t_{i+1})}\,\middle|\,\mathcal{F}_{t_i}\right] B(t_i) \tag{4}$$

Regression-based methods: the LSM algorithm

- **Note**: estimating the conditional expectations in (4) is the main difficulty in pricing American options by simulation

- **Idea**: use regression methods to estimate the continuation values from simulated sample paths:
  - each continuation value $C(t_i)$ is the regression of the (discounted) option value $V(t_{i+1})$ on the current state $S(t_i)$

Regression in practice

- **Step 1**: approximate $C(t_i)$ by a linear combination of known functions of the current state $S(t_i)$:

$$C(t_i) = \sum_{j=0}^{\infty} \alpha_{ij} \, L_j\big(S(t_i)\big) \, ,$$

  where $\alpha_{ij} \in \mathbb{R}$ and $L_j(x)$ are basis functions (e.g. Laguerre, Legendre, Hermite polynomials)

- **Step 2**: use regression to estimate the coefficients α_{ij} in this approximation. The coefficients α_{ij} are estimated from pairs

$$\big(S(t_i, \omega), V(t_{i+1}, \omega)\big)$$

  consisting of the value of the underlying at time t_i and the corresponding option value at time t_{i+1}

Comments

- The accuracy depends on the choice of basis functions
- Obviously, a finite sum will have to do it:

$$C(t_i) = \sum_{j=0}^{M} \alpha_{ij} \, L_j\big(S(t_i)\big)$$

- The coefficients α_{ij} are determined by means of least-squares $\to \hat{\alpha}_{ij}$
- The LSM algorithm is a fast and broadly applicable algorithm (beyond classical American put options)

Pricing algorithm (1/2)

(i) Simulate n independent paths

$$\big(S(t_1,\omega_k), S(t_2,\omega_k), \ldots, S(t_m,\omega_k)\big), \quad k = 1, 2, \ldots, n$$

under the risk neutral measure $\mathbb{Q}$

(ii) At terminal nodes, set

$$\hat{V}(t_m,\omega_k) = Y(t_m,\omega_k)$$

Pricing algorithm (2/2)

(iii) Apply backward induction: for $i = m-1, \dots, 1$

- Given estimated values $\hat{V}(t_{i+1}, \omega_k)$, use regression to calculate $\hat{\alpha}_{i1}, \dots, \hat{\alpha}_{iM}$

- Set

$$\hat{V}(t_i, \omega_k) = \begin{cases} Y(t_i, \omega_k), & Y(t_i, \omega_k) \geq \hat{C}(t_i, \omega_k), \\ \hat{V}(t_{i+1}, \omega_k), & Y(t_i, \omega_k) < \hat{C}(t_i, \omega_k), \end{cases}$$

  with $\hat{C}(t_i) = \sum_{j=0}^{M} \hat{\alpha}_{ij} \, L_j\big(S(t_i)\big)$

(iv) Set

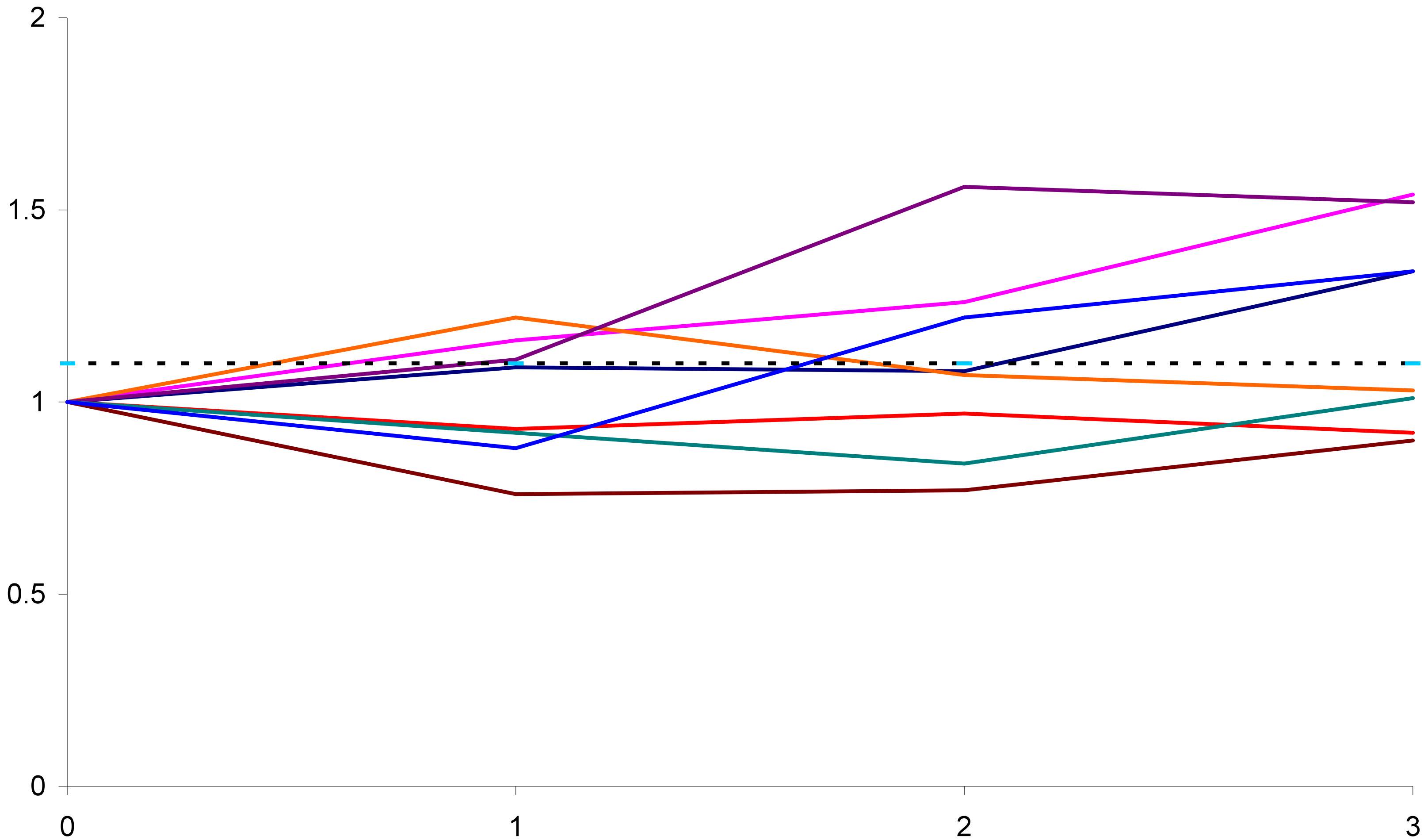
$$\hat{V}(0) = \frac{1}{n} \sum_{k=1}^{n} \hat{V}(t_1, \omega_k) \, .$$

Numerical example

$\Omega = \{\omega_1, \ldots, \omega_8\}$, $K = 1.1$ and $S(t_i, \omega_k)$ as follows:

	$t_0 = 0$	$t_1 = 1$	$t_2 = 2$	$t_3 = 3$
ω_1	1	1.09	1.08	1.34
ω_2	1	1.16	1.26	1.54
ω_3	1	1.22	1.07	1.03
ω_4	1	0.93	0.97	0.92
ω_5	1	1.11	1.56	1.52
ω_6	1	0.76	0.77	0.90
ω_7	1	0.92	0.84	1.01
ω_8	1	0.88	1.22	1.34

Stock price evolution

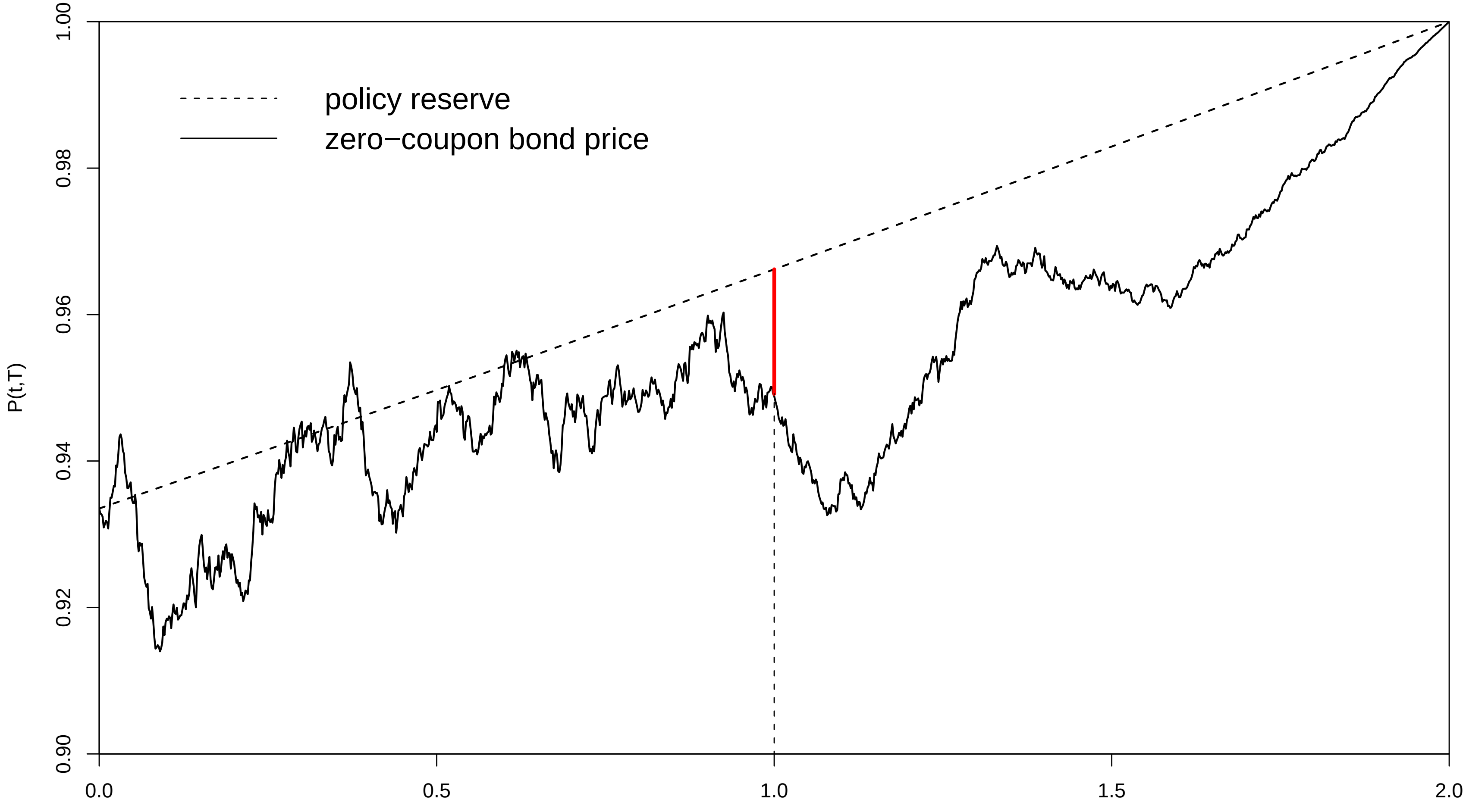


D. Surrender option in a pure endowment contract

A pure endowment contract of duration n provides for payment of the sum insured only if the policy holder survives to the end of the contract period.

- Illustrative example:
  - net single premium payment made at time $t = 0$ is invested in a zero-coupon bond with the same maturity T as the policy.
  - guaranteed interest rate r_G (technical interest rate), e.g. $r_G = 3.5\%$
  - no profit sharing
  - contract shall provide for a terminal guarantee (at $t = T$) and surrender benefit (at $t < T$), contingent on survival
  - we assume that the surrender value equals the book value of the mathematical reserves (no surrender penalty)

Visualization of the surrender option in a pure endowment contract of duration $n = 2$



Dynamic lapse rule

- Book value may be higher or lower than the market value $\Rightarrow$ policy holder can use the American option to improve the value of the contract by surrendering at the right time

- **Dynamic lapse rule**: when market interest rates exceed the guaranteed minimum interest rate the policy holder is assumed to terminate the contract at time $t = 1$ and to take advantage of the higher yields available in the financial market.

- Hence, the dynamic lapse rule is based on spread

$$\text{market yield} - \text{technical interest rate}$$

- From the viewpoint of asset pricing theory, surrender options equal American put options (Bermudan options).

General framework and notation (1/4)

- $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{Q})$ filtered probability space supporting all sources of financial and demographic randomness

- $\mathbb{Q}$: risk-neutral probability measure (i.e. discounted price processes are $\mathbb{Q}$-martingales)

- $B = \{B(t) : 0 \leq t \leq T\}$ with $dB(t) = r(t)B(t)\,dt$: money market account and $\{r(t) : 0 \leq t \leq T\}$ instantaneous short rate process, i.e.

$$B(t) = \exp\left(\int_0^t r(u)\,du\right)$$

General framework and notation (2/4)

- $D(s,t)$: discount factor from time t to s ($s \leq t$):

$$D(s,t) = \frac{B(s)}{B(t)} = \exp\left(- \int_s^t r(u)\, du \right) .$$

- $r = \{r(t) : 0 \leq t \leq T\}$: dynamics of the term structure of interest rates; Vasicek model:

$$dr(t) = (b - ar(t))\, dt + \sigma dW(t), \qquad r(0) = r_0, \tag{5}$$

  with $a, b, \sigma > 0$ and $W = \{W(t) : 0 \leq t \leq T\}$ standard $\mathbb{Q}$-Brownian motion.

- $U = \{U(t) : 0 \leq t \leq T\}$ option price process; $U(t)$ is the value of the surrender option at time t, assuming the option has not previously been exercised.

General framework and notation (3/4)

- $Z(t_1), Z(t_2), \ldots, Z(t_n)$: succession of cash flows emanating from the life insurance contract, where payment $Z(t_k)$ occurs at time t_k

- $L = \{L(t) : 0 \leq t \leq T\}$ market-consistent value process of the life insurance contract where

$$L(t) = B(t)\,\mathbb{E}_{\mathbb{Q}}\left[\sum_{i}^{n} \frac{Z(t_i)\mathbf{1}_{\{t<t_i\}}}{B(t_i)}\middle|\mathcal{F}_{t_i}\right], \tag{6}$$

- $V(t)$: book value of the policy reserve; given by $V(t) = V(0)(1+r_G)^t$ with deterministic technical interest rate r_G (e.g. $r_G = 3.5\%$) and $V(T) = 1$.

- ${}_t p_x$: probability that an individual currently aged-x survives for t more years.

General framework and notation (4/4)

- $\tau(x)$ or τ: future lifetime of a life aged x
- biometric risk assumed to be independent of the financial risk
- $Y(t)$: payoff from exercise at time t, i.e. $Y(t) = (V(t) - P(t,T))^+$

Closed-form expression for the price of the surrender option

Definition of the cash flows:

- At maturity $t = T = 2$:

$$Z(2) = \mathbf{1}_{\{V(1) \leq P(1,2)\} \cap \{\tau > 2\}} \tag{7}$$

- **Interpretation**:

  - $Z(2) = V(2) = 1$ if the policy holder is alive at time $t = 2$ $(\tau > 2)$ and has not terminated the contract at time $t = 1$. The policyholder opts for continuation at $t = 1$ if the surrender value $V(1)$ is less than the value $P(1,2)$ of the reference portfolio.

  - $Z(2) = 0$ if the policy holder died before $t = 2$ or exercised the surrender option at time $t = 1$.

Definition of the cash flows (cont'd)

- At time $t = 1$:

$$Z(1) = V(1)\,\mathbf{1}_{\{V(1)>P(1,2)\}\cap\{\tau>1\}} \tag{8}$$

- **Interpretation**:
  - $Z(1) = V(1)$ in case the policyholder is alive at $t = 1$ and surrenders, thus cashing in the amount $V(1)$. Surrender occurs if the policy reserve $V(1)$ exceeds the value of the reference portfolio $P(1,2)$.
  - $Z(1) = 0$ if the policyholder died before $t = 1$ or does not exercise the surrender option. The financial rational policy holder will not exercise the surrender option as long as the policy reserve $V(1)$ is smaller than the reference portfolio value $P(1,2)$.

Time-0 valuation (1/3)

By means of (6) we have that

$$\begin{aligned}
L(0) &= B(0)\,\mathbb{E}_{\mathbb{Q}}\left[\tilde{Z}(1)+\tilde{Z}(2)\middle|\mathcal{F}_0\right] \\
&= \mathbb{E}_{\mathbb{Q}}\left[\tilde{Z}(1)+\tilde{Z}(2)\right] \\
&= \mathbb{E}_{\mathbb{Q}}\left[\frac{Z(1)}{B(1)}\right]+\mathbb{E}_{\mathbb{Q}}\left[\frac{Z(2)}{B(2)}\right] \\
&= \mathbb{E}_{\mathbb{Q}}\left[\frac{V(1)}{B(1)}\,\mathbf{1}_{\{V(1)>P(1,2)\}\cap\{\tau>1\}}\right]+\mathbb{E}_{\mathbb{Q}}\left[\frac{1}{B(1)}\,\mathbf{1}_{\{V(1)\le P(1,2)\}\cap\{\tau>2\}}\right] \\
&= \ {}_1p_x\,\mathbb{E}_{\mathbb{Q}}\left[\frac{V(1)}{B(1)}\,\mathbf{1}_{\{V(1)>P(1,2)\}}\right]+\ {}_2p_x\,\mathbb{E}_{\mathbb{Q}}\left[\frac{1}{B(2)}\,\mathbf{1}_{\{V(1)\le P(1,2)\}}\right] \qquad (9)
\end{aligned}$$

Time-0 valuation (2/3)

Rewriting the first term on the right-hand side of (9) yields

$$L(0) = {}_1p_x\,\mathbb{E}_{\mathbb{Q}}\left[\frac{(V(1) - P(1,2))^+}{B(1)}\right] + {}_1p_x\,\mathbb{E}_{\mathbb{Q}}\left[\frac{P(1,2)}{B(1)}\,\mathbf{1}_{\{V(1)>P(1,2)\}}\right]$$

$$+ {}_2p_x\,\mathbb{E}_{\mathbb{Q}}\left[\frac{1}{B(2)}\,\mathbf{1}_{\{V(1)\le P(1,2)\}}\right] .$$

Add and subtract ${}_2p_x\mathbb{E}_{\mathbb{Q}}[\mathbf{1}_{\{V(1)>P(1,2)\}}/B(2)]$ and observe that

$$\begin{aligned} {}_2p_xP(0,2) &= {}_2p_x\mathbb{E}_{\mathbb{Q}}\left[\frac{1}{B(2)}\right] \\ &= {}_2p_x\mathbb{E}_{\mathbb{Q}}\left[\frac{\mathbf{1}_A}{B(2)}\right] + {}_2p_x\mathbb{E}_{\mathbb{Q}}\left[\frac{\mathbf{1}_{A^C}}{B(2)}\right] \end{aligned}$$

Time-0 valuation (3/3)

$$
\begin{aligned}
L(0) = {} & {}_1p_x \, \mathbb{E}_{\mathbb{Q}} \left[\frac{(V(1) - P(1,2))^+}{B(1)} \right] \\
& + {}_1p_x \, \mathbb{E}_{\mathbb{Q}} \left[\frac{P(1,2)}{B(1)} \, \mathbf{1}_{\{V(1) > P(1,2)\}} \right] - {}_2p_x \, \mathbb{E}_{\mathbb{Q}} \left[\frac{1}{B(2)} \, \mathbf{1}_{\{V(1) > P(1,2)\}} \right] \\
& + {}_2p_x P(0,2) \\
= {} & {}_1p_x \, \mathbb{E}_{\mathbb{Q}} \left[\frac{(V(1) - P(1,2))^+}{B(1)} \right] \\
& + ({}_1p_x - {}_2p_x) \, \mathbb{E}_{\mathbb{Q}} \left[\frac{P(1,2)}{B(1)} \, \mathbf{1}_{\{V(1) > P(1,2)\}} \right] \\
& + {}_2p_x P(0,2) \, .
\end{aligned}
$$

Decomposition of the liability value $L(0)$ into three components

We conclude that

$$L(0) = l_1 + l_2 + l_3,$$

where

$$l_1 = {}_2p_x P(0,2)\,, \tag{10}$$

$$l_2 = {}_1p_x\, \mathbb{E}_{\mathbb{Q}}\left[\frac{(V(1) - P(1,2))^+}{B(1)}\right]\,, \tag{11}$$

$$l_3 = ({}_1p_x - {}_2p_x)\, \mathbb{E}_{\mathbb{Q}}\left[\frac{P(1,2)}{B(1)}\,\mathbf{1}_{\{V(1)>P(1,2)\}}\right]\,. \tag{12}$$

Decomposed liability value gives most valuable and risk management-relevant information

Interpretation of the three different components:

- **First term** (10): market-consistent liability value of an identical contract without surrender option.
- **Second term** (11): surrender option premium; equal to the price of a European put option with strike $K = V(1)$, time-to-maturity $T = 1$ written on a pure discount bond maturing at time $S = 2$ (providing protection against rising interest rates)
- **Third term** (12): residual term (difference of two 'neighbouring' survival probabilities and thus negligible)

Numerical example

- $x = 45$ with ${}_1p_x = 0.998971$ and ${}_2p_x = 0.997860$
- $r_G = 3.5\%$, hence $V(0) = (1 + 0.035)^{-2} = 0.9335$
- Vasicek short rate dynamics specified by the parameters $a = 0.36$, $b = 0.0216$, $\sigma \in \{0.05, 0.25, 0.5\}$ and $r_0 = (A(0,2) - \log V(0))/B(0,2) = 0.0255$, yielding $P(0,2) = V(0) = 0.9335$
- For the calculation of l_2, we use the explicit formulae for European bond options in a Vasicek short rate dynamics (see Appendix)

Liability component	Standard deviation of the Vasicek dynamics					
	$\sigma = 5\%$		$\sigma = 25\%$		$\sigma = 50\%$	
l_1	0.932	97.8%	0.932	92.7%	0.932	87.1%
l_2	0.021	2.2%	0.073	7.3%	0.139	12.9%
$l_1 + l_2$	0.953	100%	1.005	100%	1.071	100%

LSM algorithm for pricing the surrender option

- **Recall**: LSM approach is based on
  - Monte Carlo simulation
  - Least squares regression

- Decision whether to surrender at time t or not is made by comparing the payoff from immediate exercise with the continuation value. The continuation value is determined by a least square regression of the option value $U(t_{i+1})$ on the current values of state variables

- Idea is to work backwards in time, starting from the contract maturity date T.

- **Note**: following algorithm is formulated for time-0 discounted payoffs and value estimates. Thus, with a slight abuse of notation, $U(t)$ stands for $D(0,t)U(t)$.

Pricing algorithm

(i) Simulate n independent paths

$$\big(P(t_1, T; \omega_k), P(t_2, T; \omega_k), \ldots, P(t_m, T; \omega_k)\big)\,, \quad k = 1, 2, \ldots, n$$

under the risk neutral measure $\mathbb{Q}$ where $t_j = jT/m$ for $j = 0, 1, \ldots, m$

(ii) At terminal nodes (policy expiry date), set

$$\hat{U}(T; \omega_k) = Y(T; \omega_k) \quad (= 0)$$

with $Y(t) = D(0,t)\,(V(t) - P(t,T))^+$ and $V(T) = P(T,T) = 1$. Choice of exercising or not at contract maturity T is irrelevant since – by assumption – market value of the contract equals the book value.

(iii) Apply backward induction: for $i = m-1, \dots, 1$

- Given estimated values $\hat{U}(t_{i+1};\omega_k)$, use OLS regression over all simulated sample paths to calculate the weights $\hat{\alpha}_{i1}, \dots, \hat{\alpha}_{iM}$, i.e. find how the values $\hat{U}(t_{i+1};\omega_k)$ depend on the state variables $P(t_i, T;\omega_k)$ known at time t_i

- Set

$$\hat{U}(t_i;\omega_k) = \begin{cases} Y(t_i;\omega_k), & Y(t_i;\omega_k) \geq \hat{C}(t_i;\omega_k), \\ \hat{U}(t_{i+1};\omega_k), & Y(t_i;\omega_k) < \hat{C}(t_i;\omega_k), \end{cases}$$

  with

$$\hat{C}(t_i;\omega_k) = \sum_{j=0}^{M} \hat{\alpha}_{ij}\, L_j\big(P(t_i, T;\omega_k)\big)$$

  for some basis functions $L_j(x)$.

(iv) Set

$$\hat{U}(0) = \frac{1}{n} \sum_{k=1}^{n} \hat{U}(t_1;\omega_k) \qquad \square$$

Remarks

- Accuracy of the LSM approach (like any regression-based methods) depends on the choice of the basis functions
- Polynomials are a popular choice
- Above pricing algorithm is formulated in discounted figures: payoffs and value estimates are denominated in time-0 units of currency. In practice, however, payoffs and value estimates are denominated in time-t units. This requires explicit discounting in the algorithm:
  - regress $D(t_i, t_{i+1})U(t_{i+1}; \omega_k)$ (instead of $U(t_{i+1}; \omega_k)$) against the state variables $L_j\big(P(t_i, T; \omega_k)$ to obtain the regression weights and the continuation values.
- Glasserman [6] p. 115 presents an algorithm for the joint simulation of the pair (r, D) at times $t_1, \ldots, t_m$ without discretization error.

Extracts from R-Codes

```
T= 5                       # contract maturity date
t= seq(from=0,to=T,by=1)   # time instants when the contract can be surrendered
n = 100000                 # number of simulated sample paths

r= matrix(0,nrow=n,ncol=T+1)
I= matrix(0,nrow=n,ncol=T+1)      # I(t) = int_0^t r(u)du
D= matrix(0,nrow=n,ncol=T+1)      # D(t) = exp(-I(t))
r[,1] = r0
Z1    = matrix(rnorm((T-1)*n,mean=0,sd=1),nrow=n,ncol=T)
Z2    = matrix(rnorm((T-1)*n,mean=0,sd=1),nrow=n,ncol=T)

#joint simulation of (r(t),D(t)), cf. Glasserman p. 115:
for (k in 2:(T+1)){
r[,k]= exp(-kappa*(t[k]-t[k-1]))*r[,k-1] + m*(1-exp(-kappa*(t[k]-t[k-1])))
          +sigma*sqrt(1/(2*kappa)*(1-exp(-2*kappa*(t[k]-t[k-1]))))*Z1[,k-1]
⋮
I[,k]= I[,k-1]+mu.I[,k]+sqrt(sigma2.I[,k])*(rho.r.I[,k]*Z1[,k-1]+sqrt(1-(rho.r.I[,k])^2)*Z2[,k-1])
D[,k]= exp(-I[,k])
}

# corresponding bond prices:
PtT = matrix(0,nrow=n,ncol=T)
for (k in (1:T)){
  btT = (1-exp(-kappa*(T-t[k])))/kappa
  atT = (m-sigma^2/(2*kappa^2))*(btT-(T-t[k]))-sigma^2/(4*kappa)*(btT)^2
  PtT[,k] = exp(atT-btT*r[,k])
}
PtT = cbind(PtT,1)
```

Extracts from R-Codes (cont'd)

```
#surrender value price process:
U   = matrix(0,nrow=n,ncol=T)                       # surrender option value process
DU  = matrix(0,nrow=n,ncol=T)                       # one-step back discounted value process
U[,T-1] = (V[,T-1]-PtT[,T-1])*(V[,T-1]>PtT[,T-1]) # can start at T-1 because book value=market value at t=T
C       = matrix(0,nrow=n,ncol=T-1)                 # continuation values
Y       = matrix(0,nrow=n,ncol=T-1)                 # payoffs from immediate exercise

M       = 3                                         # number of basis functions  [f(x) = 1, f(x) = x, f(x) = x^2]
alpha   = matrix(0,nrow=M,ncol=T-1)                 # regression weights

for (i in ((T-2):1)){
   P1 = PtT[,i]
   P2 = (PtT[,i])^2
   DU[,i+1] = U[,i+1]*D[,i+1]/D[,i]
   out = lm(DU[,i+1]~ P1 + P2)
   alpha[,i]= out$coeff                             # not explicitly used
   C[,i]    = out$fitted.values
   Y[,i]    = (V[,i]-PtT[,i])*(V[,i]>PtT[,i])
   U[,i]    = Y[,i]*(Y[,i]>C[,i]) + D[,i+1]/D[,i]*U[,i+1]*(Y[,i]<C[,i])
}

# surrender option price:
U0 = mean(U[,1]*D[,1])
round(U0,3)
```

Results

Surrender option values (absolute figures and expressed as a percentage of the initial mathematical reserve $V(0) = (1 + r_G)^{-T}$):

Contract maturity	Technical interest rate					
	$r_G = 1.5\%$		$r_G = 3.5\%$		$r_G = 5.5\%$	
$T = 2$	0.018	1.8%	0.015	1.7%	0.013	1.5%
$T = 5$	0.078	8.4%	0.059	6.9%	0.044	5.7%
$T = 10$	0.194	22.5%	0.113	16.0%	0.063	10.7%
$T = 15$	0.327	40.8%	0.151	25.3%	0.062	13.9%

Conclusions

- We have evaluated the surrender option of a single premium pure endowment contract by means of (i) closed-form formulae and (ii) Monte Carlo simulation methods

- For the LSM algorithm we used polynomial basis functions in combination with the reference portfolio values as state variables

- Surrender option becomes more valuable with e.g.

  + increasing contract maturity date
  + decreasing guaranteed interest rate r_G
  + increasing volatility of the short rate dynamics
  + lower mortality rates

- model can be extended to include exogeneous surrender decisions (beyond continuation values falling below surrender values)

E. Appendix: Vasicek model

Affine term structure: The term structure for the Vasicek model, i.e. the family of bond price processes, is given in the following result, see for instance Björk [3], Proposition 22.3, p. 334.

Proposition: In the Vasicek model, bond prices are given by

$$P(t,T) = e^{A(t,T)-B(t,T)r(t)}, \tag{13}$$

where

$$B(t,T) = \frac{1}{a}\left(1 - e^{-a(T-t)}\right),$$

$$A(t,T) = \frac{(B(t,T) - T + t)(ab - \sigma^2/2)}{a^2} - \frac{\sigma^2 B^2(t,T)}{4a}.$$

□

European bond options (1/2)

Reference: Björk [3], Proposition 22.9, p 338.

Proposition: For the Vasicek model, the price for a European call option with time to maturity T and strike price K on an S-bond is as follows:

$$\mathrm{ZBC}(t,T,K,S) = P(t,S)\Phi(d) - P(t,T)K\Phi(d-\sigma_p), \tag{14}$$

where

$$d = \frac{1}{\sigma_p}\log\left(\frac{P(t,S)}{P(t,T)K}\right) + \frac{\sigma_p}{2},$$

$$\sigma_p = \frac{1}{a}\left(1 - e^{-a(S-T)}\right)\sqrt{\frac{\sigma^2}{2a}\left(1 - e^{-2a(T-t)}\right)}.$$

□

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