Valuing Options Embedded in Life Insurance Contracts

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Market-consistent Actuarial Valuation, ETH Zürich

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Outline

Course material

- Slides
- Longstaff, F. A. and Schwartz, E. S. (2001). Valuing American Options by Simulation: A Simple Least-Squares Approach. *The Review of Financial Studies*, Vol. 14, No. 1, pp. 113-147.

The above two documents can be downloaded from

`www.math.ethz.ch/~hjfurrer/teaching/`

Motivation and introduction

- The balance sheet equation asserts that

$$A(t) = L(t) = D(t) + E(t)$$

where A: total value of assets; L: total value of liabilities; D: value of debt (insurance liabilities); E: value of equity.

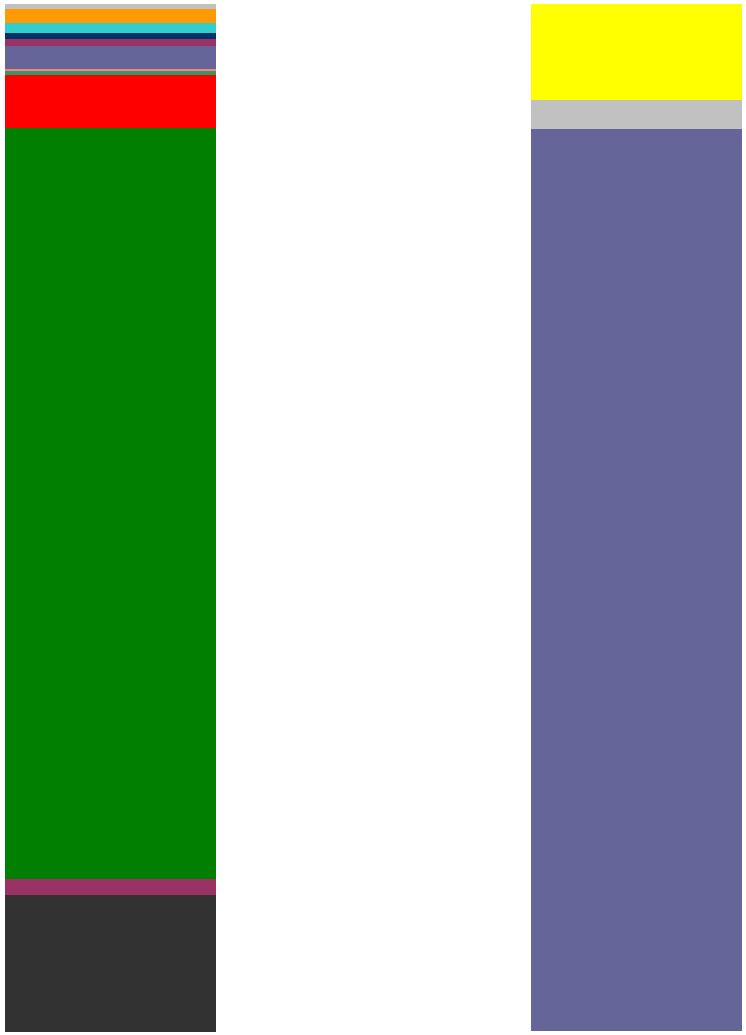
- An insurer is solvent at time t if $E(t) \geq 0$. To work out whether this is the case requires the valuation of both $A(t)$ and $D(t)$, where the latter usually poses a significant challenge.

- Liabilities stemming from policies written shall take the form

$$\text{BEL} + \text{MVM}$$

where BEL denotes the best estimate value to cover expected cash flows and MVM is a risk margin to cover the uncertainty of cash flows, see e.g. Article 77 of the Solvency II Framework Directive [4].

(Market-consistent) Balance sheet of a life insurer



- Anrechenbares Kapital
- MVM
- Bestmöglicher Schätzwert der Verpflichtungen (BEL)
- Sonstige Aktiven
- Flüssige Mittel
- Forderungen aus Versicherungsgeschäft
- Festgelder
- Derivative Instrumente
- Alternative Anlagen (HF, PE, ABS)
- Anteile an Anlagefonds
- Aktien
- Darlehen
- Hypotheken
- Obligationen
- Beteiligungen
- Immobilien

Capital and risk measures (1/4)

All notions of capital embody the idea of a loss-bearing buffer that ensures that the financial institution remains solvent.

- Regulatory capital: this is the capital an institution should hold according to regulatory rules (Basel II/III for banks, SST and Solvency II for insurers in Switzerland and the EU, respectively).
- Economic capital: this is an internal capital requirement in order to control the probability of becoming insolvent, typically over a one-year horizon.
- To ensure solvency in one year's time with high probability $1 - \alpha$ (α small, say $\alpha = 0.01$), a company may require extra capital x_0.

Capital and risk measures (2/4)

- Let $E(t) = A(t) - D(t)$ denote a company's equity capital (eligible own funds in Solvency II, risk-bearing capital in SST). The capital requirement then reads

$$x_0 = \inf\Big\{x : \mathbb{P}[E(t+1) + x(1+r) \geq 0] = 1 - \alpha\Big\}$$

$$= \inf\Big\{x : \mathbb{P}[-E(t+1) \leq x(1+r)] = 1 - \alpha\Big\}$$

$$= \inf\Big\{x : \mathbb{P}[E(t) - E(t+1)/(1+r) \leq E(t) + x] = 1 - \alpha\Big\}.$$

Here r denotes the one-year risk-free interest rate.

Capital and risk measures (3/4)

- Thus the sum of the available capital $E(t)$ plus the amount x_0 can be taken as the solvency capital requirement; it is a quantile of the distribution of $\Delta E(t+1) = E(t) - E(t+1)/(1+r)$, i.e.

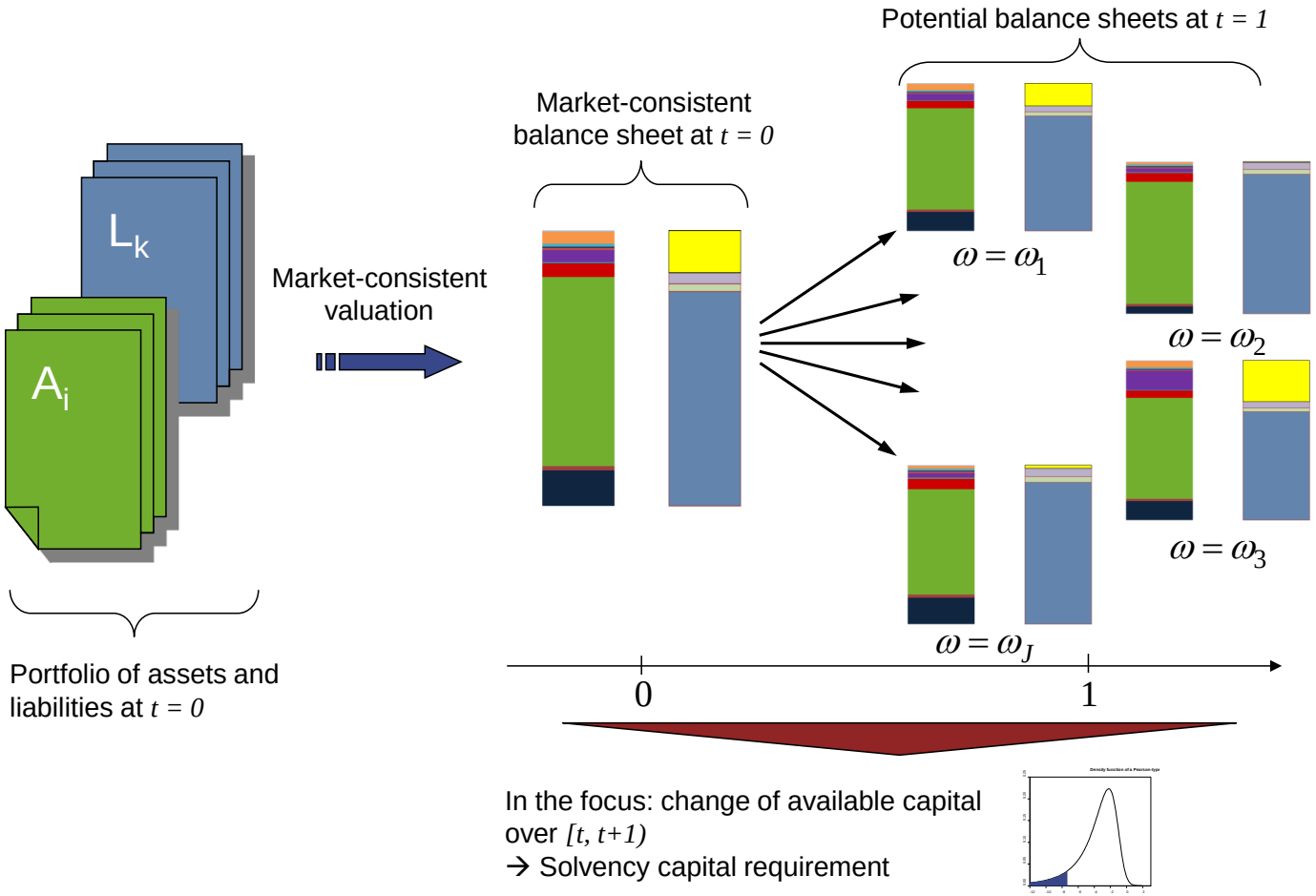
$$E(t) + x_0 = q_{1-\alpha}\big(\Delta E(t+1)\big)$$

  where $q_\alpha(X) = \mathrm{VaR}_\alpha(X) = F_X^{-1}(\alpha) = \inf\{x \in \mathbb{R} : F_X(x) \geq \alpha\}$.
- This is the Value-at-Risk idea but, more generally, capital can be computed by applying risk measures to the distribution of $\Delta E(t+1)$
- In case expected shortfall (or Tail-Value-at-Risk) is used as risk measure:

$$\mathrm{ES}_{1-\alpha}(\Delta E(t+1)) = \frac{1}{\alpha}\int_{1-\alpha}^{1} q_u\big(\Delta E(t+1)\big)\, du.$$

Capital and risk measures (4/4)

The capital requirement is derived as a risk measure of the change in available capital (e.g. own funds, risk-bearing capital):



Solvency II became fully applicable on 01/01/2016

- The Solvency II regime – set into force on 1 January 2016 – aimed to introduce a harmonised, sound and robust prudential framework for insurance firms in the EU; it is based on the risk profile of each individual insurance company.

- The Solvency II framework, like the Basel framework for banks, is divided into three pillars:
  - **Pillar 1** sets out quantitative requirements, including the rules to value assets and liabilities (in particular, technical provisions), in order to calculate regulatory capital requirements;
  - **Pillar 2** sets out qualitative requirements for risk management, governance, as well as the details of the supervisory process with competent authorities;
  - **Pillar 3** addresses transparency, reporting to supervisory authorities and disclosure to the public, thereby enhancing market discipline and increasing comparability, leading to more competition.

Transitional measures

- The **Solvency II Directive** (Directive 2009/138/EC; [4]) – as amended by Directive 2014/51/EU (**Omnibus II Directive)** [5] – replaces 14 existing directives commonly known as 'Solvency I'.
- Furthermore, the **Delegated Act** (implementing rules; [6]) include the rules for the market-consistent valuation of assets and liabilities; in particular, the technical details of the so-called 'long-term guarantee measures'.
- Long-term guarantee measures were introduced to smooth out 'artificial volatility'; they include elements such as matching adjustment, volatility adjustment, extrapolation of the yield curve.
- Solvency II includes a number of measures to ensure a smooth transition from Solvency I, mostly:
  - two measures on the valuation of technical provisions, helping the transition to a market-consistent regime over 16 (!) years
  - tolerance for insurers breaching the SCR (solvency capital requirement) within the first two years.

Market-consistent valuation of liabilities in Solvency II

Article 77 of Directive 2009/138/EC:

- The **value of technical provisions** shall be equal to the sum of a best estimate and a risk margin [...]
- The **best estimate** shall correspond to the probability-weighted average of future cash-flows, taking account of the time value of money (expected present value of future cash-flows), using the relevant risk-free interest rate term structure [...]
- The **risk margin** shall be such as to ensure that the value of the technical provisions is equivalent to the amount that insurance and reinsurance undertaktings would be expected to require in order to take over and meet the insurance and reinsurance obligations.
- Insurance and reinsurance undertakings shall value the best estimate and the risk margin separately. However, where future cash flows associated with insurance or reinsurance obligations can be replicated reliably using financial instruments for which a reliable market value is observable the value of technical provisions [...] shall be determined on the basis of the market value for those instruments.

Market-consistent valuation of liabilities in Solvency II

Article 79 of Directive 2009/138/EC: Valuation of options and guarantees:

- When calculating technical provisions, insurance and reinsurance undertakings shall take account of the value of financial guarantees and any contractual options included in insurance and reinsurance policies.
- Any assumptions made by insurance and reinsurance undertakings with respect to the likelihood that policy holders will exercise contractual options, including lapses and surrenders, shall be realistic and based on current and credible information. The assumptions shall take account, either explicitly or implicitly, of the impact that future changes in financial and non-financial conditions may have on the exercise of those options.

Relevant risk-free interest rate term structure

- Omnibus II introduced the so-called ‘long-term guarantees’ measures, aiming to avoid ’artificial’ balance sheet volatility.
- Thus, the rates of the relevant risk-free interest rate term structure shall be calculated as the sum of

  (a) the rates of a basic risk-free interest rate term strucure;
  (b) where applicable, a matching adjustment (MA);
  (c) where applicable, a volatility adjustment (VA).

- From an actuarial perspective, however, the concept of MA and VA seems to be in contradiction with Article 76 of the Directive 2009/138/EC:
- Article 76 of 2009/138/EC: The calculation of technical provisions shall make use of and be consistent with information provided by the financial markets [...]

Swiss Solvency Test (SST)

- Similar to Solvency II, the Swiss Solvency Test (SST) is a risk-sensitive solvency framework, where the capital requirements are forward-looking (1-year time horizon) and economic.
- The SST framework is laid down in the Insurance Supervision Ordinance (ISO; [10]) and the FINMA-Cirular 2017/3 "SST" [11]
- A revised version of the ISO was set into force on 1 July 2015. Major topics of the revision:
  - discontinuation of Solvency I
  - annual SST reporting for insurance groups (before: semi-annual)
  - group solvency requirements now based on consolidated accounts (before: so-called granular approach as default option)
  - preference of standard models over internal models
  - extended reporting and disclosure requirements (new FINMA-Circulars on ORSA and Public Disclosure).

Lively debates on the relevant risk-free interest rate term structure also in Switzerland I

- From 2013 to 2015, FINMA introduced so called ‘temporary adjustments’ to the SST: Swiss insurance firms were allowed to value their liabilities from in-force business by using interest curves with counterparty credit risk
- That regulation helped the (life) insurance sector to react better to the protracted low interest rate environment
- Starting in 2016, insurance companies must again use risk-free interest rate curves to value their obligations
- The discussions, however, continue on how to construct an appropriate risk-free interest rate term structure:

Lively debates on the relevant risk-free interest rate term structure also in Switzerland II

- which data to use? (swap rates vs. government bond yields; deep, liquid, transparent financial market information)
- method of inter- and extrapolation? (last liquid point (LLP); ultimate forward rate (UFR), rate of convergence)
- measures for 'smoothing-out' volatility? (volatility adjustment; matching adjustment)?
- ...

EU recognises Swiss system as equivalent

- In June 2015, the European Commission decided to recognise the Swiss supervision system as equivalent with Solvency II (with regard to reinsurance, solvency calculation and insurance group supervision)
- This decision was the outcome of a detailed assessment of the Swiss system conducted by EIOPA.
- An important aspect for this outcome had been the revision of the ISO
- As a consequence, internationally active insurers and reinsurers will not experience competitive disadvantage and regulatory duplication.

Options in life insurance contracts

- Products offered by life insurance companies such as 'variable annuities' (VA) for instance often incorporate sophisticated guarantee mechanisms and embedded options such as
  - maturity guarantees
  - rate of return guarantee (interest rate guarantee)
  - 'cliquet' or 'rachet' guarantees (guaranteed amounts are re-set regularly)
  - mortality aspects (guaranteed annuity options)
  - surrender possibilities
  - ...

Dreadful past experience

- Such issued guarantees and written options constitute liabilities to the insurer, and subsequently represent a value which in adverse circumstances may jeopardize e.g. the company's solvency position
- Historically, there was no proper valuation, reporting or risk management of these contract elements
- Many (UK domiciled) life insurance companies were unable to meet their obligations when the issued (interest rate) guarantees moved from being far out of the money (at policy inception) to being very much in the money
- As a result, many companies have experienced severe solvency problems.

Mathematical framework

- As stipulated by the Solvency II or the SST regulation, insurance liabilities must be valued market-consistently.
- Since there are no quoted prices on ADLT markets (active, deep, liquid and transparent) for insurance liabilities, the valuation must be done on a mark-to-model basis.
- Let $L(t)$ denote the mark-to-model value of an insurance obligation. Mark-to-model valuation is typically done according to

$$L(t) = f\big(t, \mathbf{Z}(t)\big)$$

where $\mathbf{Z}(t)$ denotes the (observable) risk factors such as interst rates, mortality rates, lapse rates, ...

Market-consistency and risk neutrality

- The function f is derived as an expectation of future discounted cash flows in a pricing model under a risk-neutral measure $\mathbb{Q}$:

$$L(t) = f\big(t, \mathbf{Z}(t)\big) = \mathbb{E}_{\mathbb{Q}}\Big[\text{future discounted cash flows} \;\Big|\; \mathcal{F}_t\Big]$$

  Here $\mathcal{F}_t$ denotes the information available at time t.

- Likewise, the value of $L(t+1)$ is given by

$$L(t{+}1) = f\big(t{+}1, \mathbf{Z}(t{+}1)\big) = \mathbb{E}_{\mathbb{Q}}\Big[\text{future discounted cash flows} \;\Big|\; \mathcal{F}_{t+1}\Big]$$

- The problem is how to estimate these conditional expectations.

Estimating conditional expectations (1/2)

Nested simulations:

- Assuming that the valuation models embodied in f do not admit closed form solutions, then a nested simualtion approach requires two rounds of simulation; an outer simulation followed by inner simulation:

  1. **Outer simulation:** sampling of $\mathbf{Z}(t+1)$ under a plausible model for real-world dynamics of risk factors specified by a measure $\mathbb{P}$.
  2. **Inner simulation:** Monte Carlo approximation of $\mathbb{E}_{\mathbb{Q}}$ by generating paths for risk factors $(\mathbf{Z}(s))_{s \geq t+1}$ under $\mathbb{Q}$ and evaluating cash flows.

- **Note:** the amount of simulations and calculations required to proceed in this way is often too demanding computationally (a set of inner scenarios branching out from each outer scenario needs to be generated).

Estimating conditional expectations (2/2)

Least-squares Monte Carlo simulation (LSMC):

- This alternative approach uses a form of analytic approximation involving regressing for the liability value $L(t+1)$ on some key economic variables
- LSMC uses least-squares to obtain an approximation for the conditional expectation $\mathbb{E}_{\mathbb{Q}}$ at time $t+1$. It is assumed that $\mathbb{E}_{\mathbb{Q}}[\,\cdot\,|\mathcal{F}_{t+1}]$ can be given as a linear combination of a countable set of $\mathcal{F}_{t+1}$-measurable basis functions
- With this LSMC approach to the liability valuation, the number of inner scenarios required for each outer scenario projection can be reduced significantly (perhaps just one single inner scenario)

Valuing American options

- **Definition**: An American option is a contract between two parties giving the buyer the right to, say, purchase one unit of a security for the amount K at any time on or before maturity T
- Recall: a European option, in contrast, can only be exercised at a fixed date
- **General facts:**
  - an American option can only be exercised once
  - the buyer of the option has the choice when to stop
  - exercise decision can only be based on price information up to the present moment ($\rightarrow$ filtration, stopping times)
  - American options are more valuable than their European counterparts
  - price of an American call option $=$ price of the European call option ($\rightarrow$ it is optimal to wait until the option expires)

Valuation framework (1/2)

- $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{Q})$ filtered probability space supporting all sources of financial randomness
- The probability measure $\mathbb{Q}$ is a risk-neutral probability measure (i.e. discounted price processes are $\mathbb{Q}$-martingales)
- $Y = \{Y(t) : 0 \leq t \leq T\}$ with $Y(t)$ representing the payoff from exercise at time t. Example: $Y(t) = (K - S(t))^+$
- $B = \{B(t) : 0 \leq t \leq T\}$ with $B(t) = \exp\{\int_0^t r_u \, du\}$ money market account and $\{r_t : 0 \leq t \leq T\}$ instantaneous short rate process
- $U = \{U(t) : 0 \leq t \leq T\}$ price process

Valuation framework (2/2)

- Valuing an American option means
  - finding the optimal exercise rule (exercise time)
  - computing the expected discounted payoff under this rule.

- If the option seller knew in advance which stopping time τ_0 the investor will use:

$$U(0) = \mathbb{E}_{\mathbb{Q}}\left[\frac{Y(\tau_0)}{B(\tau_0)}\right], \qquad Y(t) = \left(K - S(t)\right)^+$$

- Since τ is *not* known, the option seller should prepare for the worst possible case, and charge the maximum value

$$U(0) = \sup_{\tau \in \mathcal{T}} \mathbb{E}_{\mathbb{Q}}\left[\frac{Y(\tau)}{B(\tau)}\right],$$

where $\mathcal{T}$ are the stopping times taking values in $[0, T]$

Main result

Proposition. Suppose there is $\mathbb{Q} \sim \mathbb{P}$ and define $Z = \{Z(t) : 0 \leq t \leq T\}$ by

$$Z(t) = \sup_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}_{\mathbb{Q}} \left[\frac{Y(\tau)}{B(\tau)} \middle| \mathcal{F}_t \right] B(t) . \tag{1}$$

Then $Z(t)/B(t)$ is the smallest $\mathbb{Q}$-supermartingale satisfying $Z(t) \geq Y(t)$. Moreover, the supremum in (1) is achieved by an optimal stopping time $\tau(t)$ that has the form

$$\tau(t) = \inf\{s \geq t : Z(s) = Y(s)\} \tag{2}$$

In other words, $\tau(t)$ maximises the right hand side of (1):

$$\mathbb{E}_{\mathbb{Q}} \left[\frac{Y(\tau(t))}{B(\tau(t))} \middle| \mathcal{F}_t \right] = \sup_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}_{\mathbb{Q}} \left[\frac{Y(\tau)}{B(\tau)} \middle| \mathcal{F}_t \right] .$$

Dynamic programming formulation

- **Idea**: to work backwards in time
- Explicit construction of $Z(t)$ by means of dynamic programming:

$$V(t) := \begin{cases} Y(t)\,, & t = T \\ \max\left\{ Y(t),\ \underbrace{\mathbb{E}_{\mathbb{Q}}\left[\frac{V(t+1)}{B(t+1)} \,\middle|\, \mathcal{F}_t \right] B(t)}_{\text{expected payoff from continuation}} \right\}, & t \leq T-1 \end{cases} \tag{3}$$

- $V = \{V(t) : 0 \leq t \leq T\}$ is called snell envelope. It is the smallest supermartingale dominating Y. Thus, $Z = V$.

View the pricing problem through stopping times

- Dynamic programming rules (3) focus on option values
- Now we want to view the pricing problem through stopping rules
- Make restriction to options that can be exercised only at a fixed set of dates $t_1 < t_2 < \cdots < t_m$. Restriction is regarded as an approximation to a contract allowing continuous exercise
- **Stopping rule**: at any exercise time, compare payoff from immediate exercise with the value of continuation. Exercise if the immediate payoff is higher
- Continuation value: value of holding rather than exercising the option:

$$C(t_i) = \mathbb{E}_{\mathbb{Q}} \left[\frac{V(t_{i+1})}{B(t_{i+1})} \,\middle|\, \mathcal{F}_{t_i} \right] B(t_i)\,. \tag{4}$$

Valuing American Options by LSMC

- **Note**: estimating the conditional expectations in (4) is the main difficulty in pricing American options by simulation
- **Idea**: use regression methods to estimate the continuation values from simulated sample paths:
- each continuation value $C(t_i)$ is the regression of the (discounted) option value $V(t_{i+1})$ on the current state $S(t_i)$

Regression in practice

- **Step 1**: approximate $C(t_i)$ by a linear combination of known functions of the current state $S(t_i)$:

$$C(t_i) = \sum_{j=0}^{\infty} \alpha_{ij} L_j(S(t_i)) ,$$

where $\alpha_{ij} \in \mathbb{R}$ and $L_j(x)$ are basis functions (e.g. Laguerre, Legendre, Hermite polynomials)

- **Step 2**: use regression to estimate the coefficients α_{ij} in this approximation. The coefficients α_{ij} are estimated from pairs

$$\big(S(t_i, \omega), V(t_{i+1}, \omega)\big)$$

consisting of the value of the underlying at time t_i and the corresponding option value at time t_{i+1}

Comments

- The accuracy depends on the choice of basis functions
- Obviously, a finite sum will have to do it:

$$C(t_i) = \sum_{j=0}^{M} \alpha_{ij} L_j\big(S(t_i)\big)$$

- The coefficients α_{ij} are determined by means of least-squares $\rightarrow \hat{\alpha}_{ij}$
- The LSMC algorithm is a fast and broadly applicable algorithm (beyond classical American put options).

Pricing algorithm (1/2)

(i) Simulate n independent paths

$$\left(S(t_1,\omega_k), S(t_2,\omega_k), \ldots, S(t_m,\omega_k)\right), \quad k = 1,2,\ldots,n$$

under the risk neutral measure $\mathbb{Q}$

(ii) At terminal nodes, set

$$\hat{V}(t_m,\omega_k) = Y(t_m,\omega_k)$$

Pricing algorithm (2/2)

(iii) Apply backward induction: for $i = m-1, \ldots, 1$

- Given estimated values $\hat{V}(t_{i+1}, \omega_k)$, use regression to calculate $\hat{\alpha}_{i1}, \ldots, \hat{\alpha}_{iM}$
- Set

$$\hat{V}(t_i; \omega_k) = \begin{cases} Y(t_i; \omega_k), & Y(t_i; \omega_k) \geq \hat{C}(t_i; \omega_k), \\ \hat{V}(t_{i+1}; \omega_k), & Y(t_i; \omega_k) < \hat{C}(t_i; \omega_k), \end{cases}$$

with

$$\hat{C}(t_i) = \sum_{j=0}^{M} \hat{\alpha}_{ij} L_j(S(t_i)) .$$

(iv) Set

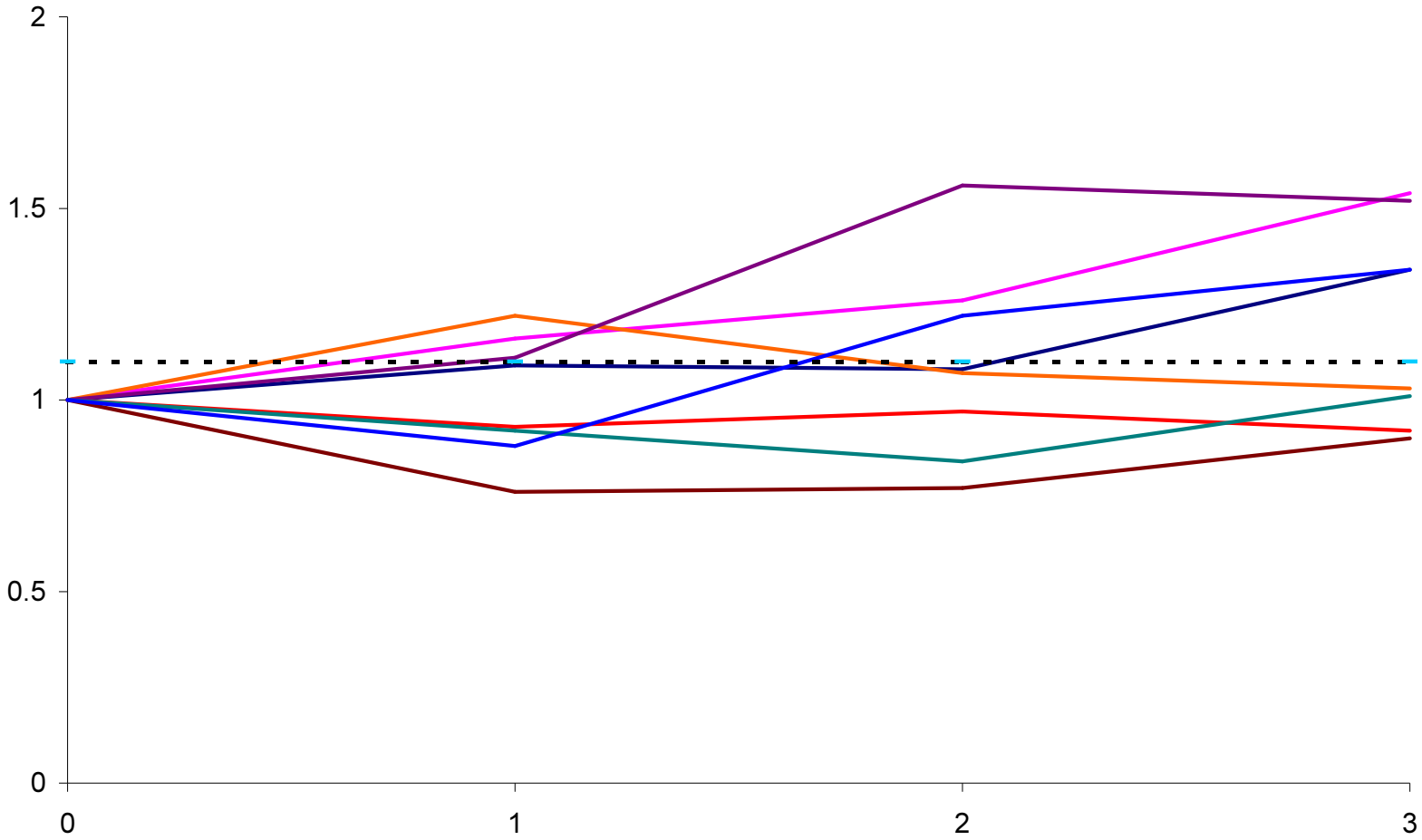
$$\hat{V}(0) = \frac{1}{n} \sum_{k=1}^{n} \hat{V}(t_1, \omega_k) .$$

Numerical example: Valuing an American put option

- $Y(t) = \left(K - S(t)\right)^+$ with $K = 1.1$ and $S(t_i, \omega_k)$, $k = 1, \ldots, 8$, $i = 0, \ldots, 3$ as follows:

	$t_0 = 0$	$t_1 = 1$	$t_2 = 2$	$t_3 = 3$
ω_1	1	1.09	1.08	1.34
ω_2	1	1.16	1.26	1.54
ω_3	1	1.22	1.07	1.03
ω_4	1	0.93	0.97	0.92
ω_5	1	1.11	1.56	1.52
ω_6	1	0.76	0.77	0.90
ω_7	1	0.92	0.84	1.01
ω_8	1	0.88	1.22	1.34

Stock price evolution



Valuing the American put option (1/11)

- At time $t = T$: $V(T) = Y(T) = \big(K - S(T)\big)^+$, where $K = 1.1$
- Cash flows occurring at time $t = T\ (= t_3)$:

	$t_1 = 1$	$t_2 = 2$	$t_3 = 3$
ω_1			0
ω_2			0
ω_3			0.07
ω_4			0.18
ω_5			0
ω_6			0.20
ω_7			0.09
ω_8			0

- Goal: complete the above cash flow matrix!

Valuing the American put option (2/11)

- At time $t = t_2$, there are only five paths where the option is in the money, namely $\omega_1, \omega_3, \omega_4, \omega_6, \omega_7$.
- Decide for which of these paths the option should be exercised.
- Payoff from immediate exercise: $Y(t_2) = \big(K - S(t_2)\big)^+$:

$$
\begin{aligned}
Y(t_2, \omega_1) &= 0.02 \\
Y(t_2, \omega_3) &= 0.03 \\
Y(t_2, \omega_4) &= 0.13 \\
Y(t_2, \omega_6) &= 0.33 \\
Y(t_2, \omega_7) &= 0.26.
\end{aligned}
$$

Valuing the American put option (3/11)

- We shall next determine the continuation values $\hat{C}(t_2)$
- Choose $L_0(x) = 1$, $L_1(x) = x$, $L_2(x) = x^2$ as basis functions.
- Hence: $C(t_2) = \alpha_{20} + \alpha_{21} S(t_2) + \alpha_{22} S^2(t_2)$
- Use regression to estimate the coefficients α_{20}, α_{21} and α_{22}:

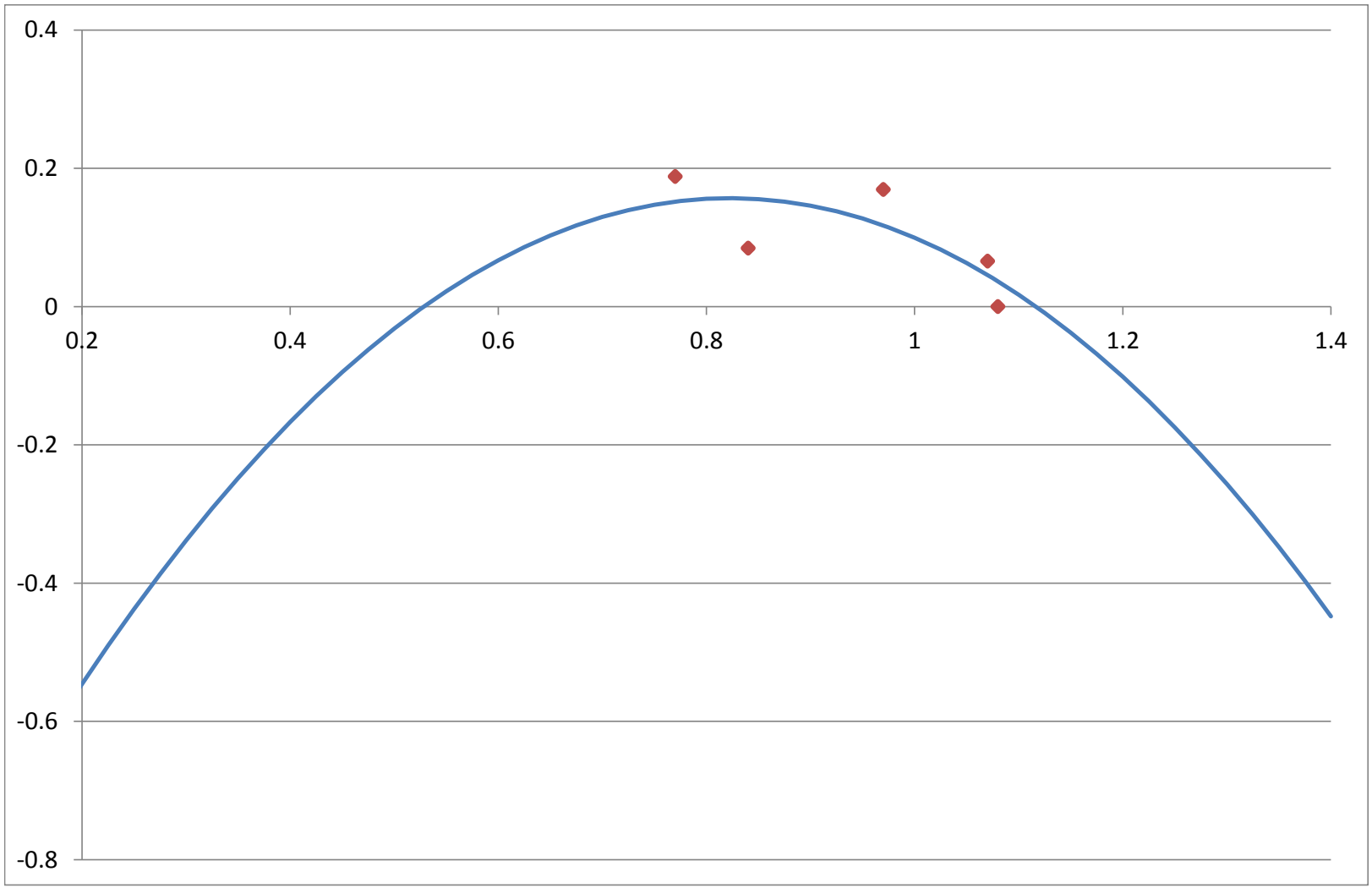
$$
\begin{aligned}
V(t_3, \omega_1)\, e^{-r} &= \alpha_{20} + \alpha_{21} S(t_2, \omega_1) + \alpha_{22} S^2(t_2, \omega_1) \\
V(t_3, \omega_3)\, e^{-r} &= \alpha_{20} + \alpha_{21} S(t_2, \omega_3) + \alpha_{22} S^2(t_2, \omega_3) \\
V(t_3, \omega_4)\, e^{-r} &= \alpha_{20} + \alpha_{21} S(t_2, \omega_4) + \alpha_{22} S^2(t_2, \omega_4) \\
V(t_3, \omega_6)\, e^{-r} &= \alpha_{20} + \alpha_{21} S(t_2, \omega_6) + \alpha_{22} S^2(t_2, \omega_6) \\
V(t_3, \omega_7)\, e^{-r} &= \alpha_{20} + \alpha_{21} S(t_2, \omega_7) + \alpha_{22} S^2(t_2, \omega_7)
\end{aligned}
$$

Valuing the American put option (4/11)

We use R to evaluate the coefficients α_{20}, α_{21} and α_{22}:

```
> S.2 = c(1.08,1.07,0.97,0.77,0.84)
> r = 0.06
> d = exp(-r)
> V = c(0,0.07,0.18,0.2,0.09)*d
> out = lm(V ~ S.2 + I(S.2^2))
> round(out$coefficients,4)
(Intercept)          S.2    I(S.2^2)
    -1.0700       2.9834     -1.8136
> # continuation values (to be compared with the payoffs from immediate exercise at t = 2):
> round(out$fitted.values,4)
     1      2      3      4      5
0.0367 0.0459 0.1175 0.1520 0.1564
>
```

Valuing the American put option (5/11)



Valuing the American put option (6/11)

- We compare the continuation values with the values from immediate exercise:

$$\hat{C}(t_2, \omega_1) = 0.0367 > 0.02 = Y(t_2, \omega_1)$$
$$\hat{C}(t_2, \omega_3) = 0.0459 > 0.03 = Y(t_2, \omega_3)$$
$$\hat{C}(t_2, \omega_4) = 0.1175 < 0.13 = Y(t_2, \omega_4)$$
$$\hat{C}(t_2, \omega_6) = 0.1520 < 0.33 = Y(t_2, \omega_6)$$
$$\hat{C}(t_2, \omega_7) = 0.1564 < 0.26 = Y(t_2, \omega_7)$$

Valuing the American put option (7/11)

- The cash flow matrix at time $t = t_2$ (and $t = t_3$) thus looks as follows:

	$t_1 = 1$	$t_2 = 2$	$t_3 = 3$
ω_1			0
ω_2			0
ω_3			0.07
ω_4		0.13	0
ω_5			0
ω_6		0.33	0
ω_7		0.26	0
ω_8			0

Valuing the American put option (8/11)

- Move one step backwards in time. The payoffs from immediate exercise $Y(t_1) = \big(K - S(t_1)\big)^+$ at time $t = t_1$ are:

$$Y(t_1, \omega_1) = 0.01 \qquad Y(t_1, \omega_6) = 0.34 \qquad Y(t_1, \omega_8) = 0.22$$
$$Y(t_1, \omega_4) = 0.17 \qquad Y(t_1, \omega_7) = 0.18$$

- Use regression to estimate the coefficients α_{10}, α_{11} and α_{12}:

$$\begin{aligned}
V(t_2, \omega_1)\, e^{-r} &= \alpha_{10} + \alpha_{11} S(t_1, \omega_1) + \alpha_{12} S^2(t_1, \omega_1) \\
V(t_2, \omega_4)\, e^{-r} &= \alpha_{10} + \alpha_{11} S(t_1, \omega_4) + \alpha_{12} S^2(t_1, \omega_4) \\
V(t_2, \omega_6)\, e^{-r} &= \alpha_{10} + \alpha_{11} S(t_1, \omega_6) + \alpha_{12} S^2(t_1, \omega_6) \\
V(t_2, \omega_7)\, e^{-r} &= \alpha_{10} + \alpha_{11} S(t_1, \omega_7) + \alpha_{12} S^2(t_1, \omega_7) \\
V(t_2, \omega_8)\, e^{-r} &= \alpha_{10} + \alpha_{11} S(t_1, \omega_8) + \alpha_{12} S^2(t_1, \omega_8)
\end{aligned}$$

Valuing the American put option (9/11)

Again, we use R to evaluate the coefficients α_{10}, α_{11} and α_{12}:

```
> S.1 = c(1.09,0.93,0.76,0.92,0.88)
> r = 0.06
> d = exp(-r)
> V = c(0,0.13,0.33,0.26,0)*d
> out = lm(V ~ S.1 + I(S.1^2))
> round(out$coefficients,4)
(Intercept)         S.1    I(S.1^2)
     2.0375     -3.3354      1.3565
> # continuation values (to be compared with the payoffs from immediate exercise at t = 1):
> round(out$fitted.values,4)
     1      2      3      4      5
0.0135 0.1087 0.2861 0.1170 0.1528
>
```

Valuing the American put option (10/11)

- We compare the continuation values with the values from immediate exercise

$$\hat{C}(t_1, \omega_1) = 0.0135 > 0.01 = Y(t_1, \omega_1)$$
$$\hat{C}(t_1, \omega_4) = 0.1087 < 0.17 = Y(t_1, \omega_4)$$
$$\hat{C}(t_1, \omega_6) = 0.2861 < 0.34 = Y(t_1, \omega_6)$$
$$\hat{C}(t_1, \omega_7) = 0.1170 < 0.18 = Y(t_1, \omega_7)$$
$$\hat{C}(t_1, \omega_8) = 0.1528 < 0.22 = Y(t_1, \omega_8)$$

Valuing the American put option (11/11)

- Ultimate cash flow matrix at time $t = t_1$ (and $t = t_2$ and $t = t_3$):

	$t_1 = 1$	$t_2 = 2$	$t_3 = 3$
ω_1			
ω_2			
ω_3			0.07
ω_4	0.17		
ω_5			
ω_6	0.34		
ω_7	0.18		
ω_8	0.22		

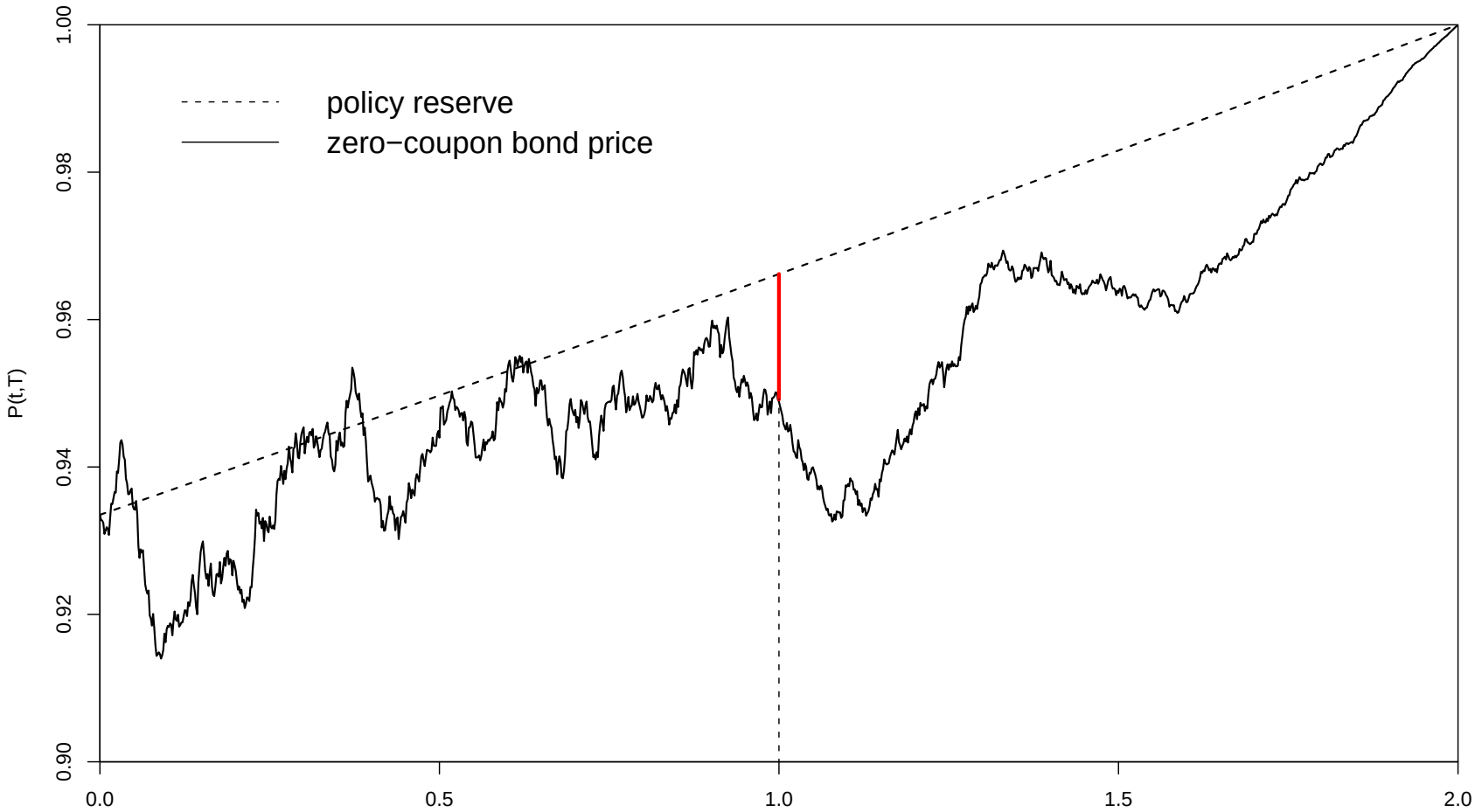
- Value of the American put option at time $t = 0$:

$$V(0) = \frac{0.07\, e^{-3 \cdot 0.06} + (0.17 + 0.34 + 0.18 + 0.22) e^{-0.06}}{8} = 0.1144.$$

Surrender option in a pure endowment contract

- A pure endowment contract of duration n provides for payment of the sum insured only if the policy holder survives to the end of the contract period.
- Illustrative example:
  - net single premium payment made at time $t = 0$ is invested in a zero-coupon bond with the same maturity T as the policy.
  - guaranteed interest rate r_G (technical interest rate), e.g. $r_G = 3.5\%$
  - no profit sharing
  - contract shall provide for a terminal guarantee (at $t = T$) and surrender benefit (at $t < T$), contingent on survival
  - we assume that the surrender value equals the book value of the mathematical reserves (no surrender penalty).

Visualization of the surrender option in a pure endowment contract of duration $n = 2$



Dynamic lapse rule

- Book value may be higher or lower than the market value $\Rightarrow$ policy holder can use the American option to improve the value of the contract by surrendering at the right time
- **Dynamic lapse rule**: when market interest rates exceed the guaranteed minimum interest rate the policy holder is assumed to terminate the contract at time $t = 1$ and to take advantage of the higher yields available in the financial market.
- Hence, the dynamic lapse rule is based on spread

$$\text{market yield} - \text{technical interest rate}$$

- From the viewpoint of asset pricing theory, surrender options equal American put options (Bermudan options).

General framework and notation I

- $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{Q})$ filtered probability space supporting all sources of financial and demographic randomness
- $\mathbb{Q}$: risk-neutral probability measure (i.e. discounted price processes are $\mathbb{Q}$-martingales)
- $B = \{B(t) : 0 \leq t \leq T\}$ with $dB(t) = r(t)B(t)\,dt$: money market account and $\{r(t) : 0 \leq t \leq T\}$ instantaneous short rate process, i.e.

$$B(t) = \exp\left(\int_0^t r(u)\,du\right)$$

- $D(s,t)$: discount factor from time t to s ($s \leq t$):

$$D(s,t) = \frac{B(s)}{B(t)} = \exp\left(-\int_s^t r(u)\,du\right).$$

General framework and notation II

- $r = \{r(t) : 0 \leq t \leq T\}$: dynamics of the term structure of interest rates; Vasicek model:

$$dr(t) = (b - ar(t))\,dt + \sigma dW(t), \qquad r(0) = r_0, \tag{5}$$

  with $a, b, \sigma > 0$ and $W = \{W(t) : 0 \leq t \leq T\}$ standard $\mathbb{Q}$-Brownian motion.

- $U = \{U(t) : 0 \leq t \leq T\}$ option price process; $U(t)$ is the value of the surrender option at time t, assuming the option has not previously been exercised.

- $Z(t_1), Z(t_2), \ldots, Z(t_n)$: succession of cash flows (lump sum payments) emanating from the life insurance contract, where payment $Z(t_k)$ occurs at time t_k

General framework and notation III

- $L = \{L(t) : 0 \leq t \leq T\}$ market-consistent value process of the life insurance contract where

$$L(t) = B(t)\,\mathbb{E}_{\mathbb{Q}}\left[\sum_{i}^{n} \frac{Z(t_i)\mathbf{1}_{\{t<t_i\}}}{B(t_i)}\middle|\mathcal{F}_t\right], \tag{6}$$

- $V(t)$: book value of the policy reserve; given by $V(t) = V(0)(1 + r_G)^t$ with deterministic technical interest rate r_G (e.g. $r_G = 3.5\%$) and $V(T) = 1$.
- ${}_tp_x$: probability that an individual currently aged-x survives for t more years.
- $\tau(x)$ or τ: future lifetime of a life aged x
- biometric risk assumed to be independent of the financial risk
- $Y(t)$: payoff from exercise at time t, i.e. $Y(t) = \big(V(t) - P(t,T)\big)^+$

Closed-form expression for surrender option price

Definition of the cash flows:

- At maturity $t = T = 2$:

$$Z(2) = \mathbf{1}_{\{V(1) \leq P(1,2)\} \cap \{\tau > 2\}} \tag{7}$$

- Interpretation:
  - $Z(2) = V(2) = 1$ if the policy holder is alive at time $t = 2$ ($\tau > 2$) and has not terminated the contract at time $t = 1$. The policyholder opts for continuation at $t = 1$ if the surrender value $V(1)$ is less than the value $P(1, 2)$ of the reference portfolio.
  - $Z(2) = 0$ if the policy holder died before $t = 2$ or exercised the surrender option at time $t = 1$.

Definition of the cash flows (cont'd)

- At time $t = 1$:

$$Z(1) = V(1)\,\mathbf{1}_{\{V(1)>P(1,2)\}\cap\{\tau>1\}} \tag{8}$$

- Interpretation:
  - $Z(1) = V(1)$ in case the policyholder is alive at $t = 1$ and surrenders, thus cashing in the amount $V(1)$. Surrender occurs if the policy reserve $V(1)$ exceeds the value of the reference portfolio $P(1,2)$.
  - $Z(1) = 0$ if the policyholder died before $t = 1$ or does not exercise the surrender option. The financial rational policy holder will not exercise the surrender option as long as the policy reserve $V(1)$ is smaller than the reference portfolio value $P(1,2)$.

Time-**0** valuation (1/3)

By means of (6) we have that

$$
\begin{aligned}
L(0) &= B(0)\,\mathbb{E}_{\mathbb{Q}}\left[\tilde{Z}(1)+\tilde{Z}(2)\middle|\mathcal{F}_0\right] \\
&= \mathbb{E}_{\mathbb{Q}}\left[\tilde{Z}(1)+\tilde{Z}(2)\right] \\
&= \mathbb{E}_{\mathbb{Q}}\left[\frac{Z(1)}{B(1)}\right]+\mathbb{E}_{\mathbb{Q}}\left[\frac{Z(2)}{B(2)}\right] \\
&= \mathbb{E}_{\mathbb{Q}}\left[\frac{V(1)}{B(1)}\,\mathbf{1}_{\{V(1)>P(1,2)\}\cap\{\tau>1\}}\right]+\mathbb{E}_{\mathbb{Q}}\left[\frac{1}{B(1)}\,\mathbf{1}_{\{V(1)\leq P(1,2)\}\cap\{\tau>2\}}\right] \\
&= {}_1p_x\,\mathbb{E}_{\mathbb{Q}}\left[\frac{V(1)}{B(1)}\,\mathbf{1}_{\{V(1)>P(1,2)\}}\right]+{}_2p_x\,\mathbb{E}_{\mathbb{Q}}\left[\frac{1}{B(2)}\,\mathbf{1}_{\{V(1)\leq P(1,2)\}}\right] \qquad (9)
\end{aligned}
$$

Time-**0** valuation (2/3)

Rewriting the first term on the right-hand side of (9) yields

$$L(0) = {}_1p_x\,\mathbb{E}_{\mathbb{Q}}\left[\frac{(V(1)-P(1,2))^+}{B(1)}\right] + {}_1p_x\,\mathbb{E}_{\mathbb{Q}}\left[\frac{P(1,2)}{B(1)}\,\mathbf{1}_{\{V(1)>P(1,2)\}}\right]$$

$$+ {}_2p_x\,\mathbb{E}_{\mathbb{Q}}\left[\frac{1}{B(2)}\,\mathbf{1}_{\{V(1)\leq P(1,2)\}}\right].$$

Add and subtract ${}_2p_x\mathbb{E}_{\mathbb{Q}}[\mathbf{1}_{\{V(1)>P(1,2)\}}/B(2)]$ and observe that

$${}_2p_x P(0,2) = {}_2p_x\mathbb{E}_{\mathbb{Q}}\left[\frac{1}{B(2)}\right]$$

$$= {}_2p_x\mathbb{E}_{\mathbb{Q}}\left[\frac{\mathbf{1}_A}{B(2)}\right] + {}_2p_x\mathbb{E}_{\mathbb{Q}}\left[\frac{\mathbf{1}_{A^C}}{B(2)}\right]$$

Time-**0** valuation (3/3)

$$
\begin{aligned}
L(0) = {} & {}_1p_x\,\mathbb{E}_{\mathbb{Q}}\left[\frac{(V(1)-P(1,2))^+}{B(1)}\right] \\
& + {}_1p_x\,\mathbb{E}_{\mathbb{Q}}\left[\frac{P(1,2)}{B(1)}\,\mathbf{1}_{\{V(1)>P(1,2)\}}\right] - {}_2p_x\,\mathbb{E}_{\mathbb{Q}}\left[\frac{1}{B(2)}\,\mathbf{1}_{\{V(1)>P(1,2)\}}\right] \\
& + {}_2p_x P(0,2) \\
= {} & {}_1p_x\,\mathbb{E}_{\mathbb{Q}}\left[\frac{(V(1)-P(1,2))^+}{B(1)}\right] \\
& + ({}_1p_x - {}_2p_x)\,\mathbb{E}_{\mathbb{Q}}\left[\frac{P(1,2)}{B(1)}\,\mathbf{1}_{\{V(1)>P(1,2)\}}\right] \\
& + {}_2p_x P(0,2)\,.
\end{aligned}
$$

Decomposition of the liability value $L(0)$ into three components

We conclude that

$$L(0) = I_1 + I_2 + I_3,$$

where

$$I_1 = {}_2p_x P(0,2)\,, \tag{10}$$

$$I_2 = {}_1p_x\, \mathbb{E}_{\mathbb{Q}} \left[\frac{(V(1) - P(1,2))^+}{B(1)} \right], \tag{11}$$

$$I_3 = ({}_1p_x - {}_2p_x)\, \mathbb{E}_{\mathbb{Q}} \left[\frac{P(1,2)}{B(1)} \mathbf{1}_{\{V(1) > P(1,2)\}} \right]. \tag{12}$$

Decomposed liability value reveals important risk management information

Interpretation of the three different components:

- First term (10): market-consistent liability value of an identical contract without surrender option.
- Second term (11): surrender option premium; equal to the price of a European put option with strike $K = V(1)$, time-to-maturity $T = 1$ written on a pure discount bond maturing at time $S = 2$ (providing protection against rising interest rates)
- Third term (12): residual term (difference of two 'neighbouring' survival probabilities and thus negligible).

Numerical example

- $x = 45$ with ${}_1p_x = 0.998971$ and ${}_2p_x = 0.997860$
- $r_G = 3.5\%$, hence $V(0) = (1 + 0.035)^{-2} = 0.9335$
- Vasicek short rate dynamics specified by the parameters $a = 0.36$, $b = 0.0216$, $\sigma \in \{0.05, 0.25, 0.5\}$ and $r_0 = (A(0,2) - \log V(0))/B(0,2) = 0.0255$, yielding $P(0,2) = V(0) = 0.9335$
- For the calculation of I_2, we use the explicit formulae for European bond options in a Vasicek short rate dynamics (see Appendix)

	Standard deviation of the Vasicek dynamics					
Liability component	$\sigma = 5\%$		$\sigma = 25\%$		$\sigma = 50\%$	
I_1	0.932	97.8%	0.932	92.7%	0.932	87.1%
I_2	0.021	2.2%	0.073	7.3%	0.139	12.9%
$I_1 + I_2$	0.953	100%	1.005	100%	1.071	100%

LSM algorithm for the pricing of the surrender option

- Recall: LSM approach is based on
  - Monte Carlo simulation
  - Least squares regression
- Decision whether to surrender at time t or not is made by comparing the payoff from immediate exercise with the continuation value. The continuation value is determined by a least square regression of the option value $U(t_{i+1})$ on the current values of state variables
- Idea is to work backwards in time, starting from the contract maturity date T.
- Note: following algorithm is formulated for time-0 discounted payoffs and value estimates. Thus, with a slight abuse of notation, $U(t)$ stands for $D(0,t)U(t)$.

Pricing algorithm I

(i) Simulate n independent paths

$$\left(P(t_1, T; \omega_k), P(t_2, T; \omega_k), \ldots, P(t_m, T; \omega_k)\right), \quad k = 1, 2, \ldots, n$$

under the risk neutral measure $\mathbb{Q}$ where $t_j = jT/m$ for $j = 0, 1, \ldots, m$

(ii) At terminal nodes (policy expiry date), set

$$\hat{U}(T; \omega_k) = Y(T; \omega_k) \quad (= 0)$$

with $Y(t) = D(0, t)\,(V(t) - P(t, T))^+$ and $V(T) = P(T, T) = 1$. Choice of exercising or not at contract maturity T is irrelevant since – by assumption – market value of the contract equals the book value.

(iii) Apply backward induction: for $i = m - 1, \ldots, 1$

Pricing algorithm II

- Given estimated values $\hat{U}(t_{i+1};\omega_k)$, use OLS regression over all simulated sample paths to calculate the regression weights $\hat{\alpha}_{i1},\ldots,\hat{\alpha}_{iM}$, i.e. find how the values $\hat{U}(t_{i+1};\omega_k)$ depend on the state variables $P(t_i,T;\omega_k)$ known at time t_i

- Set

$$\hat{U}(t_i;\omega_k) = \begin{cases} Y(t_i;\omega_k), & Y(t_i;\omega_k) \geq \hat{C}(t_i;\omega_k), \\ \hat{U}(t_{i+1};\omega_k), & Y(t_i;\omega_k) < \hat{C}(t_i;\omega_k), \end{cases}$$

with

$$\hat{C}(t_i;\omega_k) = \sum_{j=0}^{M} \hat{\alpha}_{ij}\, L_j\big(P(t_i,T;\omega_k)\big)$$

for some basis functions $L_j(x)$.

(iv) Set

$$\hat{U}(0) = \frac{1}{n}\sum_{k=1}^{n} \hat{U}(t_1;\omega_k)\,.$$

Remarks

- Accuracy of the LSM approach (like any regression-based methods) depends on the choice of the basis functions
- Polynomials are a popular choice
- Above pricing algorithm is formulated in discounted figures: payoffs and value estimates are denominated in time-0 units of currency. In practice, however, payoffs and value estimates are denominated in time-t units. This requires explicit discounting in the algorithm:
  - regress $D(t_i, t_{i+1})U(t_{i+1}; \omega_k)$ (instead of $U(t_{i+1}; \omega_k)$) against the state variables $L_j\big(P(t_i, T; \omega_k)$ to obtain the regression weights and the continuation values.
- Glasserman [8] p. 115 presents an algorithm for the joint simulation of the pair (r, D) at times $t_1, \ldots, t_m$ without discretization error.

Extracts from R-Codes (1/2)

```
T= 5                        # contract maturity date
t= seq(from=0,to=T,by=1)    # time instants when the contract can be surrendered
n = 100000                  # number of simulated sample paths

r= matrix(0,nrow=n,ncol=T+1)
I= matrix(0,nrow=n,ncol=T+1)       # I(t) = int_0^t r(u)du
D= matrix(0,nrow=n,ncol=T+1)       # D(t) = exp(-I(t))
r[,1] = r0
Z1    = matrix(rnorm((T-1)*n,mean=0,sd=1),nrow=n,ncol=T)
Z2    = matrix(rnorm((T-1)*n,mean=0,sd=1),nrow=n,ncol=T)

#joint simulation of (r(t),D(t)), cf. Glasserman p. 115:
for (k in 2:(T+1)){
   r[,k]= exp(-kappa*(t[k]-t[k-1]))*r[,k-1] + m*(1-exp(-kappa*(t[k]-t[k-1])))
          +sigma*sqrt(1/(2*kappa)*(1-exp(-2*kappa*(t[k]-t[k-1]))))*Z1[,k-1]
   ...
   I[,k]= I[,k-1]+mu.I[,k]+sqrt(sigma2.I[,k])*(rho.r.I[,k]*Z1[,k-1]+sqrt(1-(rho.r.I[,k])^2)*Z2[,k-1])
   D[,k]= exp(-I[,k])
   }
# corresponding bond prices:
PtT = matrix(0,nrow=n,ncol=T)
for (k in (1:T)){
   btT = (1-exp(-kappa*(T-t[k])))/kappa
   atT = (m-sigma^2/(2*kappa^2))*(btT-(T-t[k]))-sigma^2/(4*kappa)*(btT)^2
   PtT[,k] = exp(atT-btT*r[,k])
   }
PtT = cbind(PtT,1)
```

Extracts from R-Codes (2/2)

```
#surrender value price process:
U   = matrix(0,nrow=n,ncol=T)                          # surrender option value process
DU  = matrix(0,nrow=n,ncol=T)                          # one-step back discounted value process
U[,T-1] = (V[,T-1]-PtT[,T-1])*(V[,T-1]>PtT[,T-1])      # can start at T-1 since book value=market value at t=T
C       = matrix(0,nrow=n,ncol=T-1)                    # continuation values
Y       = matrix(0,nrow=n,ncol=T-1)                    # payoffs from immediate exercise

M       = 3                                            # number of basis functions
                                                       # [f(x) = 1, f(x) = x, f(x) = x^2]
alpha   = matrix(0,nrow=M,ncol=T-1)                    # regression weights

for (i in ((T-2):1)){
   P1 = PtT[,i]
 P2 = (PtT[,i])^2
   DU[,i+1] = U[,i+1]*D[,i+1]/D[,i]
   out = lm(DU[,i+1]~ P1 + P2)
   alpha[,i]= out$coeff                                # not explicitly used
   C[,i]    = out$fitted.values
   Y[,i]    = (V[,i]-PtT[,i])*(V[,i]>PtT[,i])
   U[,i]    = Y[,i]*(Y[,i]>C[,i]) + D[,i+1]/D[,i]*U[,i+1]*(Y[,i]<C[,i])
   }

# surrender option price:
U0 = mean(U[,1]*D[,1])
round(U0,3)
```

Results

Surrender option values (absolute figures and expressed as a percentage of the initial mathematical reserve $V(0) = (1 + r_G)^{-T}$):

	Technical interest rate					
Contract maturity	$r_G = 1.5\%$		$r_G = 3.5\%$		$r_G = 5.5\%$	
$T = 2$	0.018	1.8%	0.015	1.7%	0.013	1.5%
$T = 5$	0.078	8.4%	0.059	6.9%	0.044	5.7%
$T = 10$	0.194	22.5%	0.113	16.0%	0.063	10.7%
$T = 15$	0.327	40.8%	0.151	25.3%	0.062	13.9%

Conclusions

- We have evaluated the surrender option of a single premium pure endowment contract by means of (i) closed-form formulae and (ii) Monte Carlo simulation methods
- For the LSM algorithm we used polynomial basis functions in combination with the reference portfolio values as state variables
- Surrender option becomes more valuable with e.g.
  - + increasing contract maturity date
  - + decreasing guaranteed interest rate r_G
  - + increasing volatility of the short rate dynamics
  - + lower mortality rates
- model can be extended to include exogeneous surrender decisions (beyond continuation values falling below surrender values).

Appendix: Vasicek model I

- Affine term structure: The term structure for the Vasicek model, i.e. the family of bond price processes, is given in the following result, see for instance Björk [3], Proposition 22.3, p. 334.
- Proposition: In the Vasicek model, bond prices are given by

$$P(t,T) = e^{A(t,T)-B(t,T)r(t)}, \tag{13}$$

where

$$B(t,T) = \frac{1}{a}\left(1 - e^{-a(T-t)}\right),$$

$$A(t,T) = \frac{(B(t,T) - T + t)(ab - \sigma^2/2)}{a^2} - \frac{\sigma^2 B^2(t,T)}{4a}.$$

□

Appendix: Vasicek model II

- Proposition: For the Vasicek model, the price for a European call option with time to maturity T and strike price K on an S-bond is as follows:

$$\text{ZBC}(t, T, K, S) = P(t, S)\Phi(d) - P(t, T)K\Phi(d - \sigma_p), \qquad (14)$$

where

$$d = \frac{1}{\sigma_p} \log\left(\frac{P(t, S)}{P(t, T)K}\right) + \frac{\sigma_p}{2},$$

$$\sigma_p = \frac{1}{a}\left(1 - e^{-a(S-T)}\right)\sqrt{\frac{\sigma^2}{2a}\left(1 - e^{-2a(T-t)}\right)}.$$

□

- Reference: Björk [3], Proposition 22.9, p. 338.

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