Valuation of options embedded in life insurance contracts

Hansjörg Furrer

Market-consistent Actuarial Valuation, ETH Zürich

22 & 29 April 2024

Course material

- Slides
- Longstaff, F. A. and Schwartz, E. S. (2001). Valuing American Options by Simulation: A Simple Least-Squares Approach. *The Review of Financial Studies*, Vol. 14, No. 1, pp. 113-147.

The above two documents can be downloaded from

`https://people.math.ethz.ch/~hjfurrer/teaching/`

Outline

Motivation and introduction

- Balance sheet equation:

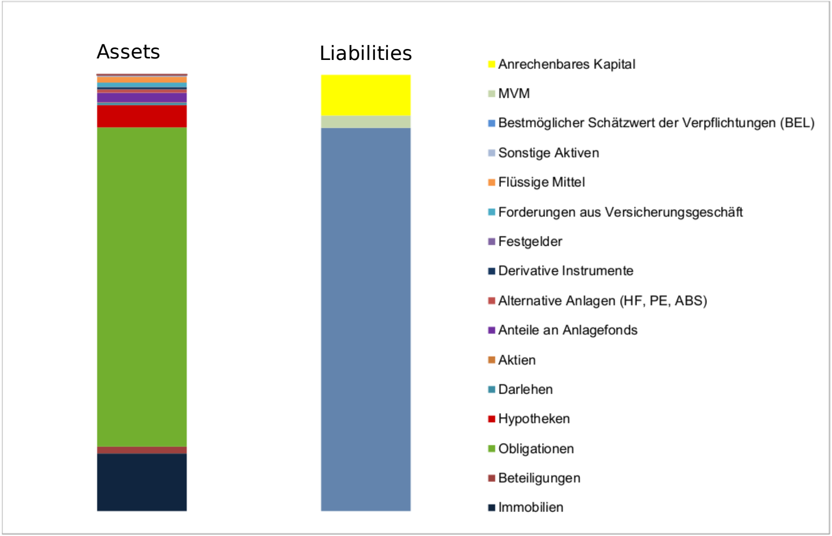
$$A(t) = L(t) = D(t) + E(t)$$

  where A: total value of assets; L: total value of liabilities; D: value of debt (insurance liabilities); E: value of equity.
- An insurer is solvent at time t if $E(t) \geq 0$.
- Insurance liabilities shall take the form

$$\text{BEL} + \text{MVM}$$

  where
  - BEL: best estimate value to cover expected cash flows (CFs);
  - MVM: market value margin to cover uncertainty of CFs
- References:
  - Article 30 No. 2 of the Insurance Supervision Ordinance [10]
  - Article 77 of the Solvency II Framework Directive [4].

(Market-consistent) Balance sheet of a life insurer



Capital and risk measures (1/4)

- All notions of capital embody the idea of a loss-absorbing buffer that ensures that the financial institution remains solvent.
- Regulatory capital: this is the capital an institution should hold according to regulatory rules (Basel II/III for banks, SST and Solvency II for insurers in Switzerland and the EU, respectively).
- Economic capital: this is an internal capital requirement in order to control the probability of becoming insolvent, typically over a one-year horizon.
- To ensure $E(1) \geq 0$ with high probability $1 - \alpha$ (α small, say $\alpha = 0.01$), a company may require extra capital x_0.

Capital and risk measures (2/4)

- Capital requirements(extra amount x_0) are often expressed as a risk measure of the change in the company's available capital

$$\Delta E(t+1) = E(t) - E(t+1)/(1+r)$$

r: one-year risk-free interest rate.

- Value-at-Risk:

$$q_{1-\alpha}\left(\Delta E(t+1)\right),$$

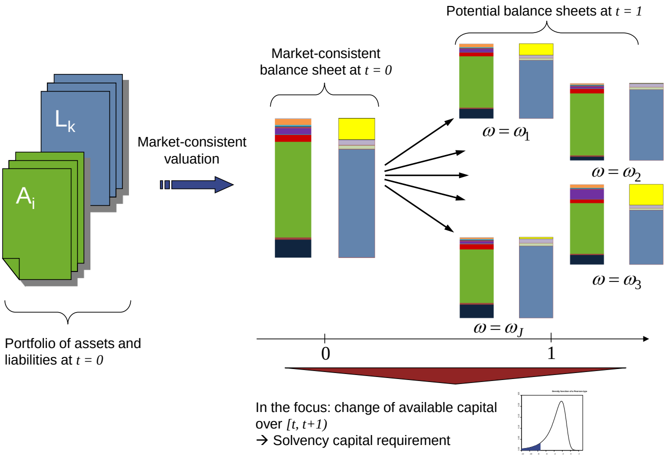
where $q_\alpha(X) = \mathrm{VaR}_\alpha(X) = F_X^{-1}(\alpha) = \inf\{x \in \mathbb{R} : F_X(x) \geq \alpha\}$.

- Expected Shortfall:

$$\mathrm{ES}_{1-\alpha}(\Delta E(t+1)) = \frac{1}{\alpha}\int_{1-\alpha}^{1} q_u\left(\Delta E(t+1)\right) du.$$

Capital and risk measures (3/4)

The capital requirement is derived as a risk measure of the change in available capital:



Capital and risk measures (4/4)

- In the context of calculating capital risk measures, insurers are challenged to revalue their assets and liabilities at the risk horizon t ($t = 1$, say)
- $L(t)$: market-consistent value of an insurance obligation, i.e.

$$L(t) = f\big(t, \mathbf{Z}(t)\big)$$

$\mathbf{Z}(t)$: risk factors (interest rates, mortality rates, lapse rates, . . .).

- The function f is derived as an expectation of future discounted cash flows under a risk-neutral measure $\mathbb{Q}$:

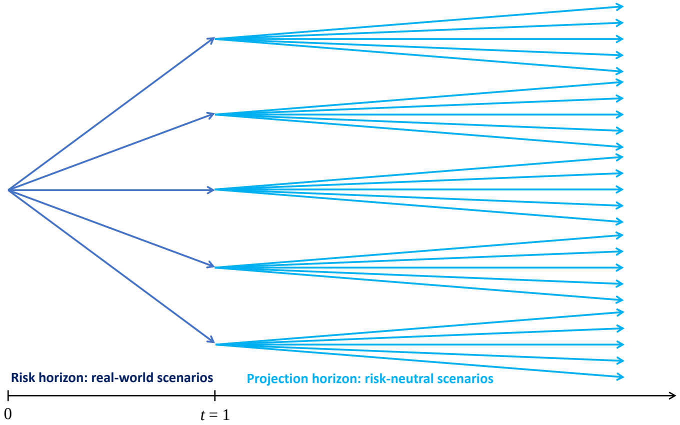
$$\boxed{L(t) = f\big(t, \mathbf{Z}(t)\big) = \mathbb{E}_{\mathbb{Q}}\Big[\text{future discounted cash flows} \;\Big|\; \mathcal{F}_t\Big]}$$

$\mathcal{F}_t$: information available at time t.

- How to estimate conditional expectations?

Estimating conditional expectations

- The empirical distribution of the liability value at the risk horizon $t = 1$ can be obtained by a full stochastic Monte Carlo simulation approach or nested simulation:



Nested simulation

- Nested simulation approach consists of two simulation sets:
  - **Outer simulation** from time 0 to time $t = 1$ represents the real-world scenarios over the risk horizon, i.e. sampling of $\mathbf{Z}(1)$ under the real-world measure $\mathbb{P}$.
  - **Inner simulation** from time $t = 1$ to time T gives the risk-neutral scenarios for the estimation of the liability value at time 1, i.e. Monte Carlo approximation of $\mathbb{E}_{\mathbb{Q}}$ by generating paths for risk factors $(\mathbf{Z}(s))_{s\geq 1}$ under $\mathbb{Q}$ and evaluating cash flows.
- Nested simulation approach is computationally inefficient (due to the scale and complexity of a life insurer's liabilities)
- Alternative methods are needed!

How to calculate a conditional expectation?

(1) Valuation Portfolio (aka Replicating Portfolio): subject of this course

(2) Explicit calculation: possible for illustrative cases, but hardly in practice

(3) Least-squares Monte Carlo simulation (LSMC): subject of this lecture. LSMC

- approximates the conditional expectation $\mathbb{E}_{\mathbb{Q}}$ at time $t+1$.
- assumes that $\mathbb{E}_{\mathbb{Q}}[\,\cdot\,|\mathcal{F}_t]$ can be represented as a linear combination of a countable set of $\mathcal{F}_t$-measurable basis functions.
- Coefficients of the linear combination are obtained via least squares.

Regulatory basis for the valuation of options

- EU: Article 79 of Directive 2009/138/EC [4]:

  When calculating technical provisions, insurance and reinsurance undertakings shall take account of the value of financial guarantees and any contractual options included in insurance and reinsurance policies.

- Switzerland: Article 9a Insurance Supervision Law [9]:

  Das risikotragende Kapital und das Zielkapital werden auf der Grundlage einer Gesamtbilanz, die sämtliche relevanten Positionen berücksichtigt, auf marktkonformer Basis ermittelt.

- Caution: Article 100 ISO (‘Deckungspflicht’):

  Versicherungsunternehmen, die Derivate einsetzen, müssen über genügend Liquidität verfügen, um die Zahlungs- und Lieferverpflichtungen, welche sich aus derivativen Finanztransaktionen ergeben können, stets erfüllen zu können.

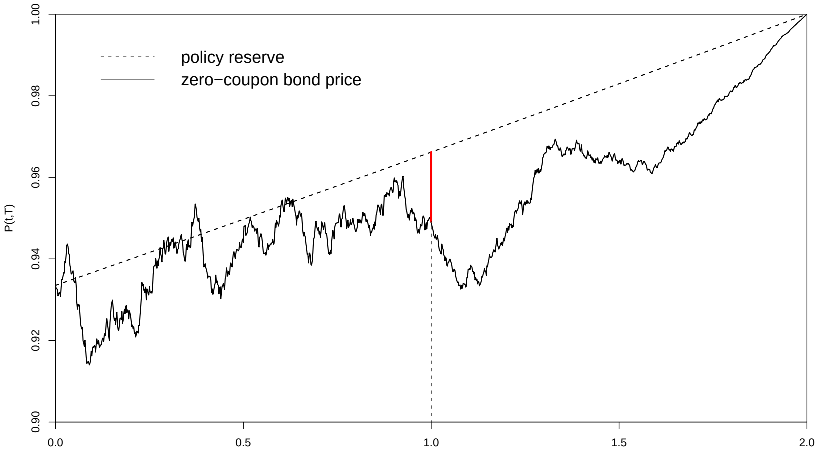
Options in life insurance contracts

- Products offered by life insurance companies such as Variable Annuities (VA) often incorporate sophisticated guarantee mechanisms and embedded options such as
  - maturity guarantees
  - rate of return guarantee (interest rate guarantee)
  - cliquet or rachet guarantees (guaranteed amounts are re-set regularly)
  - mortality aspects (guaranteed annuity options)
  - surrender possibilities
  - ...

Surrender option in a pure endowment contract

- A pure endowment contract of duration T provides for payment of the sum insured only if the policy holder survives to the end of the contract period.
- Illustrative example:
  - net single premium payment made at time $t = 0$ is invested in a zero-coupon bond with the same maturity T as the policy.
  - guaranteed interest rate r_G (technical interest rate), e.g. $r_G = 3.5\%$
  - no profit sharing
  - contract shall provide for a terminal guarantee (at $t = T$) and surrender benefit (at $t < T$), contingent on survival
  - we assume that the surrender value equals the book value of the mathematical reserves (no surrender penalty).

Visualization of the surrender option in a pure endowment contract of duration $T = 2$



Notation

- $Z(t_1), Z(t_2), \ldots, Z(t_n)$: cash flows (lump sum payments) at time t_k emanating from the life insurance contract

- $L = \{L(t) : 0 \leq t \leq T\}$ market-consistent value of the life insurance contract:

$$\boxed{L(t) = B(t)\,\mathbb{E}_{\mathbb{Q}}\left[\sum_{i}^{n} \frac{Z(t_i)\mathbf{1}_{\{t<t_i\}}}{B(t_i)} \middle| \mathcal{F}_t\right]}, \qquad (1)$$

- $V(t)$: book value of the policy reserve, $V(t) = V(0)(1 + r_G)^t$ with deterministic technical interest rate r_G (e.g. $r_G = 3.5\%$) and $V(T) = 1$.

- ${}_tp_x$: probability that an individual currently aged-x survives for t more years.

- $\tau(x)$ or τ: future lifetime of a life aged x

Closed-form expression for surrender option price

Definition of the cash flows:

- At maturity $t = T = 2$:

$$Z(2) = \mathbf{1}_{\{V(1) \leq P(1,2)\} \cap \{\tau > 2\}} \tag{2}$$

- Interpretation:
  - $Z(2) = V(2) = 1$ if the policy holder is alive at time $t = 2$ ($\tau > 2$) and has not terminated the contract at time $t = 1$. The policyholder opts for continuation at $t = 1$ if the surrender value $V(1)$ is less than the value $P(1, 2)$ of the reference portfolio.
  - $Z(2) = 0$ if the policy holder died before $t = 2$ or exercised the surrender option at time $t = 1$.

Definition of the cash flows (cont'd)

- At time $t = 1$:

$$Z(1) = V(1)\,\mathbf{1}_{\{V(1)>P(1,2)\}\cap\{\tau>1\}} \tag{3}$$

- Interpretation:
  - $Z(1) = V(1)$ in case the policyholder is alive at $t = 1$ and surrenders, thus cashing in the amount $V(1)$. Surrender occurs if the policy reserve $V(1)$ exceeds the value of the reference portfolio $P(1, 2)$.
  - $Z(1) = 0$ if the policyholder died before $t = 1$ or does not exercise the surrender option. The financial rational policy holder will not exercise the surrender option as long as the policy reserve $V(1)$ is smaller than the reference portfolio value $P(1, 2)$.

Calculation of the time-0 liability value $L(0)$ (1/4)

By means of (1) we have that

$$
\begin{aligned}
L(0) &= B(0)\,\mathbb{E}_{\mathbb{Q}}\left[\tilde{Z}(1)+\tilde{Z}(2)\middle|\mathcal{F}_0\right] \\
&= \mathbb{E}_{\mathbb{Q}}\left[\tilde{Z}(1)+\tilde{Z}(2)\right] \\
&= \mathbb{E}_{\mathbb{Q}}\left[\frac{Z(1)}{B(1)}\right]+\mathbb{E}_{\mathbb{Q}}\left[\frac{Z(2)}{B(2)}\right] \\
&= \mathbb{E}_{\mathbb{Q}}\left[\frac{V(1)}{B(1)}\,\mathbf{1}_{\{V(1)>P(1,2)\}\cap\{\tau>1\}}\right]+\mathbb{E}_{\mathbb{Q}}\left[\frac{1}{B(1)}\,\mathbf{1}_{\{V(1)\leq P(1,2)\}\cap\{\tau>2\}}\right] \\
&= {}_1p_x\,\mathbb{E}_{\mathbb{Q}}\left[\frac{V(1)}{B(1)}\,\mathbf{1}_{\{V(1)>P(1,2)\}}\right]+{}_2p_x\,\mathbb{E}_{\mathbb{Q}}\left[\frac{1}{B(2)}\,\mathbf{1}_{\{V(1)\leq P(1,2)\}}\right] \qquad (4)
\end{aligned}
$$

Calculation of the time-0 liability value $L(0)$ (2/4)

First term of (4): Set $A = \{V(1) > P(1,2)\}$ and observe that

$$
\begin{aligned}
&{}_1p_x\,\mathbb{E}_{\mathbb{Q}}\left[\frac{V(1)}{B(1)}\,\mathbf{1}_A\right] \\
&= {}_1p_x\,\mathbb{E}_{\mathbb{Q}}\left[\frac{V(1)}{B(1)}\,\mathbf{1}_A\right] - {}_1p_x\,\mathbb{E}_{\mathbb{Q}}\left[\frac{P(1,2)}{B(1)}\,\mathbf{1}_A\right] + {}_1p_x\,\mathbb{E}_{\mathbb{Q}}\left[\frac{P(1,2)}{B(1)}\,\mathbf{1}_A\right] \\
&= {}_1p_x\,\mathbb{E}_{\mathbb{Q}}\left[\left(\frac{V(1)-P(1,2)}{B(1)}\right)\mathbf{1}_A\right] + {}_1p_x\,\mathbb{E}_{\mathbb{Q}}\left[\frac{P(1,2)}{B(1)}\,\mathbf{1}_A\right] \\
&= {}_1p_x\,\mathbb{E}_{\mathbb{Q}}\left[\frac{(V(1)-P(1,2))^+}{B(1)}\right] + {}_1p_x\,\mathbb{E}_{\mathbb{Q}}\left[\frac{P(1,2)}{B(1)}\,\mathbf{1}_A\right]
\end{aligned}
$$

Calculation of the time-0 liability value $L(0)$ (3/4)

Second term of (4):

$$
\begin{aligned}
& {}_2p_x\, \mathbb{E}_{\mathbb{Q}}\left[\frac{\mathbf{1}_{A^c}}{B(2)}\right] \\
&\quad = {}_2p_x\, \mathbb{E}_{\mathbb{Q}}\left[\frac{\mathbf{1}_{A^c}}{B(2)}\right] + {}_2p_x\, \mathbb{E}_{\mathbb{Q}}\left[\frac{\mathbf{1}_{A}}{B(2)}\right] - {}_2p_x\, \mathbb{E}_{\mathbb{Q}}\left[\frac{\mathbf{1}_{A}}{B(2)}\right] \\
&\quad = {}_2p_x\, \mathbb{E}_{\mathbb{Q}}\left[\frac{1}{B(2)}\right] - {}_2p_x\, \mathbb{E}_{\mathbb{Q}}\left[\frac{\mathbf{1}_{A}}{B(2)}\right] \\
&\quad = {}_2p_x\, P(0,2) - {}_2p_x\, \mathbb{E}_{\mathbb{Q}}\left[\frac{P(2,2)}{B(2)}\,\mathbf{1}_{A}\right] \\
&\quad = {}_2p_x\, P(0,2) - {}_2p_x\, \mathbb{E}_{\mathbb{Q}}\left[\frac{P(1,2)}{B(1)}\,\mathbf{1}_{A}\right] \qquad (5)
\end{aligned}
$$

Calculation of the time-0 liability value $L(0)$ (4/4)

- Equation (5) follows because $\mathbb{Q}$ is an equivalent martingale measure for the discounted price processes $\Rightarrow$ constant mean.
- Hence:

$$L(0) = \mathbb{E}_{\mathbb{Q}}\left[\tilde{Z}(1) + \tilde{Z}(2)\right] = \text{First term} + \text{Second term}$$

$$= l_1 + l_2 + l_3,$$

where

$$l_1 = {}_2p_x P(0,2)\,, \tag{6}$$

$$l_2 = {}_1p_x\, \mathbb{E}_{\mathbb{Q}}\left[\frac{(V(1) - P(1,2))^+}{B(1)}\right]\,, \tag{7}$$

$$l_3 = ({}_1p_x - {}_2p_x)\, \mathbb{E}_{\mathbb{Q}}\left[\frac{P(1,2)}{B(1)}\,\mathbf{1}_{\{V(1)>P(1,2)\}}\right]\,. \tag{8}$$

Decomposed liability value reveals important risk management information

Interpretation of the three different components:

- First term (6): market-consistent liability value of an identical contract without surrender option.
- Second term (7): surrender option premium; equal to the price of a European put option with strike $K = V(1)$, time-to-maturity $T = 1$ written on a pure discount bond maturing at time $S = 2$ (providing protection against rising interest rates)
- Third term (8): residual term (difference of two 'neighbouring' survival probabilities and thus negligible).

Numerical example

- $x = 45$ with ${}_1p_x = 0.998971$ and ${}_2p_x = 0.997860$
- $r_G = 3.5\%$, hence $V(0) = (1 + 0.035)^{-2} = 0.9335$
- Vasicek short rate dynamics $dr(t) = (b - a\,r(t))\,dt + \sigma dW(t)$ with $a = 0.36$, $b = 0.0216$, $\sigma \in \{0.05, 0.25, 0.5\}$, yielding $r_0 = (A(0,2) - \log V(0))/B(0,2) = 0.0255$ and $P(0,2) = V(0) = 0.9335$
- For the calculation of l_2, we use the explicit formulae for European bond options in a Vasicek short rate dynamics (see Appendix)

	Standard deviation of the Vasicek dynamics					
Liability component	$\sigma = 5\%$		$\sigma = 25\%$		$\sigma = 50\%$	
l_1	0.932	97.8%	0.932	92.7%	0.932	87.1%
l_2	0.021	2.2%	0.073	7.3%	0.139	12.9%
$l_1 + l_2$	0.953	100%	1.005	100%	1.071	100%

Valuation of a pure endowment contract with $T > 2$

- The valuation of a multi-year pure endowment contract with duration $T > 2$ can no longer be carried out in closed form
- Reason is that the decision whether to surrender at time t ($t \leq T - 2$) must be made by comparing the payoff from immediate exercise with the continuation values, which in this case is non-trivial
- Idea: use LSMC
  - simulate n independent paths of the underlying asset $P(t, T)$
  - work backwards in time, starting from the contract maturity date T
  - determine the continuation value via least square regression of the option value on the current values of state variables.

American options - Recap

- **Definition**: An American option is a contract between two parties giving the buyer the right to, say, purchase one unit of a security for the amount K at any time on or before maturity T
- Recall: a European option, in contrast, can only be exercised at a fixed date
- **General facts:**
  - an American option can only be exercised once
  - the buyer of the option has the choice when to stop
  - exercise decision can only be based on price information up to the present moment ($\rightarrow$ filtration, stopping times)
  - American options are more valuable than their European counterparts
  - price of an American call option $=$ price of the European call option ($\rightarrow$ it is optimal to wait until the option expires)

Valuation framework

- Valuing an American option means
  - finding the optimal exercise rule (exercise time)
  - computing the expected discounted payoff under this rule.
- If the option seller knew in advance which stopping time τ_0 the investor will use:

$$U(0) = \mathbb{E}_{\mathbb{Q}}\left[\frac{Y(\tau_0)}{B(\tau_0)}\right], \qquad Y(t) = \big(K - S(t)\big)^+$$

- Since τ is *not* known, the option seller should prepare for the worst possible case, and charge the maximum value

$$U(0) = \sup_{\tau \in \mathcal{T}} \mathbb{E}_{\mathbb{Q}}\left[\frac{Y(\tau)}{B(\tau)}\right],$$

where $\mathcal{T}$ are the stopping times taking values in $[0, T]$

Main result

Proposition. Suppose there is $\mathbb{Q} \sim \mathbb{P}$ and define $Z = \{Z(t) : 0 \leq t \leq T\}$ by

$$Z(t) = \sup_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}_{\mathbb{Q}} \left[\frac{Y(\tau)}{B(\tau)} \,\middle|\, \mathcal{F}_t \right] B(t)\,. \tag{9}$$

Then $Z(t)/B(t)$ is the smallest $\mathbb{Q}$-supermartingale satisfying $Z(t) \geq Y(t)$. Moreover, the supremum in (9) is achieved by an optimal stopping time $\tau(t)$ that has the form

$$\tau(t) = \inf\{s \geq t : Z(s) = Y(s)\} \tag{10}$$

In other words, $\tau(t)$ maximises the right hand side of (9):

$$\mathbb{E}_{\mathbb{Q}} \left[\frac{Y(\tau(t))}{B(\tau(t))} \,\middle|\, \mathcal{F}_t \right] = \sup_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}_{\mathbb{Q}} \left[\frac{Y(\tau)}{B(\tau)} \,\middle|\, \mathcal{F}_t \right].$$

Dynamic programming formulation

- Explicit construction of $Z(t)$ by means of dynamic programming:

$$V(t) := \begin{cases} Y(t)\,, & t = T \\ \max\Big\{ Y(t),\ \underbrace{\mathbb{E}_{\mathbb{Q}}\Big[\frac{V(t+1)}{B(t+1)} \Big| \mathcal{F}_t \Big] B(t)}_{\text{expected payoff from continuation}} \Big\}\,, & t \le T-1 \end{cases} \tag{11}$$

- $V = \{V(t) : 0 \le t \le T\}$ is called snell envelope. It is the smallest supermartingale dominating Y. Thus, $Z = V$.
- Continuation value: value of holding rather than exercising the option:

$$C(t_i) = \mathbb{E}_{\mathbb{Q}}\Big[\frac{V(t_{i+1})}{B(t_{i+1})} \Big| \mathcal{F}_{t_i} \Big] B(t_i)\,. \tag{12}$$

Valuing American Options by LSMC

Longstaff and Schwarz [7] propose the following algorithm:

- **Step 1**: approximate $C(t_i)$ by a linear combination of known functions of the current state $S(t_i)$:

$$\boxed{C(t_i) = \sum_{j=0}^{\infty} \alpha_{ij} \, L_j\big(S(t_i)\big)},$$

where $\alpha_{ij} \in \mathbb{R}$ and $L_j(x)$ are basis functions (e.g. Laguerre, Legendre, Hermite polynomials, . . .)

- **Step 2**: use regression (least squares) to estimate the coefficients α_{ij} from pairs

$$\big(S(t_i, \omega), V(t_{i+1}, \omega)\big).$$

LSMC pricing algorithm (1/2)

(i) Simulate n independent paths

$$\left(S(t_1,\omega_k), S(t_2,\omega_k), \ldots, S(t_m,\omega_k)\right), \quad k = 1, 2, \ldots, n$$

under the risk neutral measure $\mathbb{Q}$

(ii) At terminal nodes, set

$$\hat{V}(t_m,\omega_k) = Y(t_m,\omega_k)$$

LSMC pricing algorithm (2/2)

(iii) Apply backward induction: for $i = m-1, \dots, 1$

- Given estimated values $\hat{V}(t_{i+1}, \omega_k)$, use regression to calculate $\hat{\alpha}_{i1}, \dots, \hat{\alpha}_{iM}$
- Set

$$\hat{V}(t_i;\omega_k) = \begin{cases} Y(t_i;\omega_k), & Y(t_i;\omega_k) \geq \hat{C}(t_i;\omega_k), \\ \hat{V}(t_{i+1};\omega_k), & Y(t_i;\omega_k) < \hat{C}(t_i;\omega_k), \end{cases}$$

with

$$\hat{C}(t_i) = \sum_{j=0}^{M} \hat{\alpha}_{ij} L_j(S(t_i)) .$$

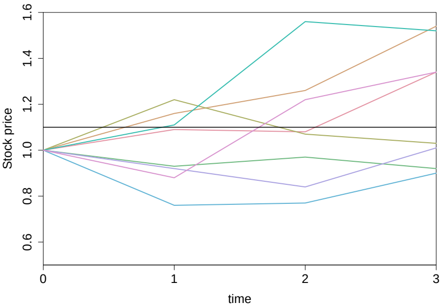
(iv) Set

$$\hat{V}(0) = \frac{1}{n} \sum_{k=1}^{n} \hat{V}(t_1, \omega_k) .$$

Illustrative example of the LSMC algorithm (1/12)

$Y(t) = (K - S(t))^+$ with $K = 1.1$ and $S(t_i, \omega_k)$, $k = 1, \ldots, 8$, $i = 0, \ldots, 3$ as follows:

	$t_0 = 0$	$t_1 = 1$	$t_2 = 2$	$t_3 = 3$
ω_1	1	1.09	1.08	1.34
ω_2	1	1.16	1.26	1.54
ω_3	1	1.22	1.07	1.03
ω_4	1	0.93	0.97	0.92
ω_5	1	1.11	1.56	1.52
ω_6	1	0.76	0.77	0.90
ω_7	1	0.92	0.84	1.01
ω_8	1	0.88	1.22	1.34



Illustrative example of the LSMC algorithm (2/12)

- At time $t = T$: $V(T) = Y(T) = \big(K - S(T)\big)^+$, where $K = 1.1$
- Cash flows occurring at time $t = T$ $(= t_3)$:

	$t_1 = 1$	$t_2 = 2$	$t_3 = 3$
ω_1			0
ω_2			0
ω_3			0.07
ω_4			0.18
ω_5			0
ω_6			0.20
ω_7			0.09
ω_8			0

- Goal: complete the above cash flow matrix!

Illustrative example of the LSMC algorithm (3/12)

- At time $t = t_2$, there are only five paths where the option is in the money, namely ω_1, ω_3, ω_4, ω_6, ω_7.
- Decide for which of these paths the option should be exercised.
- Payoff from immediate exercise: $Y(t_2) = \big(K - S(t_2)\big)^+$:

$$
\begin{aligned}
Y(t_2, \omega_1) &= 0.02 \\
Y(t_2, \omega_3) &= 0.03 \\
Y(t_2, \omega_4) &= 0.13 \\
Y(t_2, \omega_6) &= 0.33 \\
Y(t_2, \omega_7) &= 0.26.
\end{aligned}
$$

Illustrative example of the LSMC algorithm (4/12)

- We shall next determine the continuation values $\hat{C}(t_2)$
- Choose $L_0(x) = 1$, $L_1(x) = x$, $L_2(x) = x^2$ as basis functions.
- Hence: $C(t_2) = \alpha_{20} + \alpha_{21} S(t_2) + \alpha_{22} S^2(t_2)$
- Use regression to estimate the coefficients α_{20}, α_{21} and α_{22}:

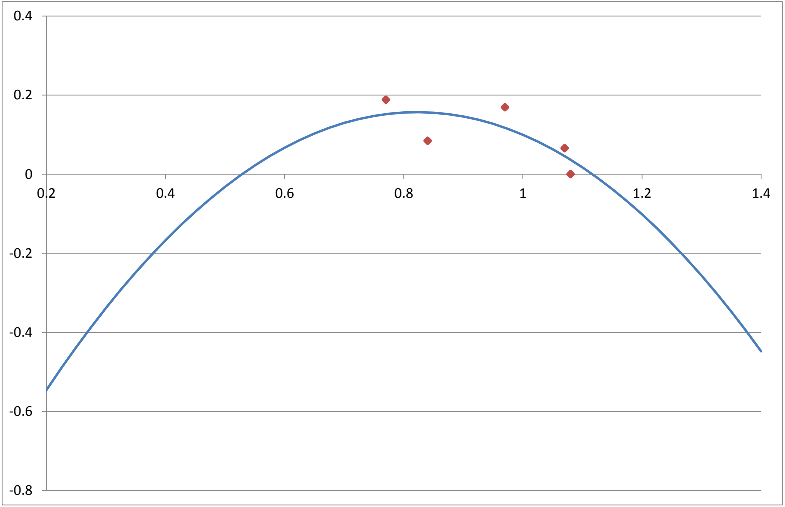
$$V(t_3, \omega_1)\, e^{-r} = \alpha_{20} + \alpha_{21} S(t_2, \omega_1) + \alpha_{22} S^2(t_2, \omega_1)$$
$$V(t_3, \omega_3)\, e^{-r} = \alpha_{20} + \alpha_{21} S(t_2, \omega_3) + \alpha_{22} S^2(t_2, \omega_3)$$
$$V(t_3, \omega_4)\, e^{-r} = \alpha_{20} + \alpha_{21} S(t_2, \omega_4) + \alpha_{22} S^2(t_2, \omega_4)$$
$$V(t_3, \omega_6)\, e^{-r} = \alpha_{20} + \alpha_{21} S(t_2, \omega_6) + \alpha_{22} S^2(t_2, \omega_6)$$
$$V(t_3, \omega_7)\, e^{-r} = \alpha_{20} + \alpha_{21} S(t_2, \omega_7) + \alpha_{22} S^2(t_2, \omega_7)$$

Illustrative example of the LSMC algorithm (5/12)

We use R to evaluate the coefficients α_{20}, α_{21} and α_{22}:

```
> S.2 = c(1.08,1.07,0.97,0.77,0.84)
> r = 0.06
> d = exp(-r)
> V = c(0,0.07,0.18,0.2,0.09)*d
> out = lm(V ~ S.2 + I(S.2^2))
> round(out$coefficients,4)
(Intercept)         S.2    I(S.2^2)
   -1.0700      2.9834    -1.8136
> # continuation values (to be compared with the payoffs from immediate exercise at t = 2):
> round(out$fitted.values,4)
     1      2      3      4      5
0.0367 0.0459 0.1175 0.1520 0.1564
>
```

Illustrative example of the LSMC algorithm (6/12)



Illustrative example of the LSMC algorithm (7/12)

- We compare the continuation values with the values from immediate exercise:

$$\hat{C}(t_2,\omega_1) = 0.0367 > 0.02 = Y(t_2,\omega_1)$$
$$\hat{C}(t_2,\omega_3) = 0.0459 > 0.03 = Y(t_2,\omega_3)$$
$$\hat{C}(t_2,\omega_4) = 0.1175 < 0.13 = Y(t_2,\omega_4)$$
$$\hat{C}(t_2,\omega_6) = 0.1520 < 0.33 = Y(t_2,\omega_6)$$
$$\hat{C}(t_2,\omega_7) = 0.1564 < 0.26 = Y(t_2,\omega_7)$$

Illustrative example of the LSMC algorithm (8/12)

- The cash flow matrix at time $t = t_2$ (and $t = t_3$) thus looks as follows:

	$t_1 = 1$	$t_2 = 2$	$t_3 = 3$
ω_1			0
ω_2			0
ω_3			0.07
ω_4		0.13	0
ω_5			0
ω_6		0.33	0
ω_7		0.26	0
ω_8			0

Illustrative example of the LSMC algorithm (9/12)

- Move one step backwards in time. The payoffs from immediate exercise $Y(t_1) = \left(K - S(t_1)\right)^+$ at time $t = t_1$ are:

$$Y(t_1,\omega_1) = 0.01 \qquad Y(t_1,\omega_6) = 0.34 \qquad Y(t_1,\omega_8) = 0.22$$
$$Y(t_1,\omega_4) = 0.17 \qquad Y(t_1,\omega_7) = 0.18$$

- Use regression to estimate the coefficients α_{10}, α_{11} and α_{12}:

$$V(t_2,\omega_1)\, e^{-r} = \alpha_{10} + \alpha_{11} S(t_1,\omega_1) + \alpha_{12} S^2(t_1,\omega_1)$$
$$V(t_2,\omega_4)\, e^{-r} = \alpha_{10} + \alpha_{11} S(t_1,\omega_4) + \alpha_{12} S^2(t_1,\omega_4)$$
$$V(t_2,\omega_6)\, e^{-r} = \alpha_{10} + \alpha_{11} S(t_1,\omega_6) + \alpha_{12} S^2(t_1,\omega_6)$$
$$V(t_2,\omega_7)\, e^{-r} = \alpha_{10} + \alpha_{11} S(t_1,\omega_7) + \alpha_{12} S^2(t_1,\omega_7)$$
$$V(t_2,\omega_8)\, e^{-r} = \alpha_{10} + \alpha_{11} S(t_1,\omega_8) + \alpha_{12} S^2(t_1,\omega_8)$$

Illustrative example of the LSMC algorithm (10/12)

Again, we use R to evaluate the coefficients α_{10}, α_{11} and α_{12}:

```
> S.1 = c(1.09,0.93,0.76,0.92,0.88)
> r = 0.06
> d = exp(-r)
> V = c(0,0.13,0.33,0.26,0)*d
> out = lm(V ~ S.1 + I(S.1^2))
> round(out$coefficients,4)
(Intercept)         S.1    I(S.1^2)
    2.0375     -3.3354      1.3565
> # continuation values (to be compared with the payoffs from immediate exercise at t = 1):
> round(out$fitted.values,4)
     1      2      3      4      5
0.0135 0.1087 0.2861 0.1170 0.1528
>
```

Illustrative example of the LSMC algorithm (11/12)

- We compare the continuation values with the values from immediate exercise

$$
\begin{aligned}
\hat{C}(t_1,\omega_1) &= 0.0135 > 0.01 = Y(t_1,\omega_1)\\
\hat{C}(t_1,\omega_4) &= 0.1087 < 0.17 = Y(t_1,\omega_4)\\
\hat{C}(t_1,\omega_6) &= 0.2861 < 0.34 = Y(t_1,\omega_6)\\
\hat{C}(t_1,\omega_7) &= 0.1170 < 0.18 = Y(t_1,\omega_7)\\
\hat{C}(t_1,\omega_8) &= 0.1528 < 0.22 = Y(t_1,\omega_8)
\end{aligned}
$$

Illustrative example of the LSMC algorithm (12/12)

- Ultimate cash flow matrix at time $t = t_1$ (and $t = t_2$ and $t = t_3$):

	$t_1 = 1$	$t_2 = 2$	$t_3 = 3$
ω_1			
ω_2			
ω_3			0.07
ω_4	0.17		
ω_5			
ω_6	0.34		
ω_7	0.18		
ω_8	0.22		

- Value of the American put option at time $t = 0$:

$$V(0) = \frac{0.07\, e^{-3 \cdot 0.06} + (0.17 + 0.34 + 0.18 + 0.22)\, e^{-0.06}}{8} = 0.1144.$$

Valuing a pure endowment contract with $T > 2$ (1/3)

(i) Simulate n independent paths

$$\left(P(t_1, T; \omega_k), P(t_2, T; \omega_k), \ldots, P(t_m, T; \omega_k)\right), \quad k = 1, 2, \ldots, n$$

under the risk neutral measure $\mathbb{Q}$ where $t_j = jT/m$ for $j = 0, 1, \ldots, m$

(ii) At terminal nodes (policy expiry date), set

$$\hat{U}(T; \omega_k) = Y(T; \omega_k) \quad (= 0)$$

with $Y(t) = D(0, t)\left(V(t) - P(t, T)\right)^+$ and $V(T) = P(T, T) = 1$. Choice of exercising or not at contract maturity T is irrelevant since – by assumption – market value of the contract equals the book value.

Valuing a pure endowment contract with $T > 2$ (2/3)

(iii) Apply backward induction: for $i = m-1, \ldots, 1$

- Given estimated values $\hat{U}(t_{i+1}; \omega_k)$, use OLS regression over all simulated sample paths to calculate the regression weights $\hat{\alpha}_{i1}, \ldots, \hat{\alpha}_{iM}$, i.e. find how the values $\hat{U}(t_{i+1}; \omega_k)$ depend on the state variables $P(t_i, T; \omega_k)$ known at time t_i

- Set

$$\hat{U}(t_i; \omega_k) = \begin{cases} Y(t_i; \omega_k), & Y(t_i; \omega_k) \geq \hat{C}(t_i; \omega_k), \\ \hat{U}(t_{i+1}; \omega_k), & Y(t_i; \omega_k) < \hat{C}(t_i; \omega_k), \end{cases}$$

with

$$\hat{C}(t_i; \omega_k) = \sum_{j=0}^{M} \hat{\alpha}_{ij} L_j\big(P(t_i, T; \omega_k)\big)$$

for some basis functions $L_j(x)$.

Valuing a pure endowment contract with $T > 2$ (3/3)

(iv) Set

$$\hat{U}(0) = \frac{1}{n} \sum_{k=1}^{n} \hat{U}(t_1; \omega_k) .$$

Numerical example: Surrender option values (absolute figures and relative to the initial mathematical reserve $V(0) = (1 + r_G)^{-T}$) with Vasicek short rate dynamics $(b - a\, r(t))\, dt + \sigma\, dW(t)$ with $a = 0.36$, $b = 0.0216$, $\sigma = 5\%$.

	Technical interest rate					
Maturity	$r_G = 1.5\%$		$r_G = 3.5\%$		$r_G = 5.5\%$	
$T = 5$	0.078	8.4%	0.059	6.9%	0.044	5.7%
$T = 10$	0.194	22.5%	0.113	16.0%	0.063	10.7%
$T = 15$	0.327	40.8%	0.151	25.3%	0.062	13.9%

Extracts from R-Codes (1/2)

```
T= 5                        # contract maturity date
t= seq(from=0,to=T,by=1)  # time instants when the contract can be surrendered
n = 100000                  # number of simulated sample paths

r= matrix(0,nrow=n,ncol=T+1)
I= matrix(0,nrow=n,ncol=T+1)       # I(t) = int_0^t r(u)du
D= matrix(0,nrow=n,ncol=T+1)       # D(t) = exp(-I(t))
r[,1] = r0
Z1     = matrix(rnorm((T-1)*n,mean=0,sd=1),nrow=n,ncol=T)
Z2     = matrix(rnorm((T-1)*n,mean=0,sd=1),nrow=n,ncol=T)

#joint simulation of (r(t),D(t)), cf. Glasserman p. 115:
for (k in 2:(T+1)){
   r[,k]= exp(-kappa*(t[k]-t[k-1]))*r[,k-1] + m*(1-exp(-kappa*(t[k]-t[k-1])))
          +sigma*sqrt(1/(2*kappa)*(1-exp(-2*kappa*(t[k]-t[k-1]))))*Z1[,k-1]
   ...
   I[,k]= I[,k-1]+mu.I[,k]+sqrt(sigma2.I[,k])*(rho.r.I[,k]*Z1[,k-1]+sqrt(1-(rho.r.I[,k])^2)*Z2[,k-1])
   D[,k]= exp(-I[,k])
   }
# corresponding bond prices:
PtT = matrix(0,nrow=n,ncol=T)
for (k in (1:T)){
   btT = (1-exp(-kappa*(T-t[k])))/kappa
   atT = (m-sigma^2/(2*kappa^2))*(btT-(T-t[k]))-sigma^2/(4*kappa)*(btT)^2
   PtT[,k] = exp(atT-btT*r[,k])
   }
PtT = cbind(PtT,1)
```

Extracts from R-Codes (2/2)

```
#surrender value price process:
U    = matrix(0,nrow=n,ncol=T)                    # surrender option value process
DU   = matrix(0,nrow=n,ncol=T)                    # one-step back discounted value process
U[,T-1] = (V[,T-1]-PtT[,T-1])*(V[,T-1]>PtT[,T-1]) # can start at T-1 since book value=market value at t=T
C        = matrix(0,nrow=n,ncol=T-1)              # continuation values
Y        = matrix(0,nrow=n,ncol=T-1)              # payoffs from immediate exercise

M        = 3                                      # number of basis functions
                                                  # [f(x) = 1, f(x) = x, f(x) = x^2]
alpha    = matrix(0,nrow=M,ncol=T-1)              # regression weights

for (i in ((T-2):1)){
   P1 = PtT[,i]
 P2 = (PtT[,i])^2
   DU[,i+1] = U[,i+1]*D[,i+1]/D[,i]
   out = lm(DU[,i+1]~ P1 + P2)
   alpha[,i]= out$coeff                           # not explicitly used
   C[,i]    = out$fitted.values
   Y[,i]    = (V[,i]-PtT[,i])*(V[,i]>PtT[,i])
   U[,i]    = Y[,i]*(Y[,i]>C[,i]) + D[,i+1]/D[,i]*U[,i+1]*(Y[,i]<C[,i])
   }

# surrender option price:
U0 = mean(U[,1]*D[,1])
round(U0,3)
```

Conclusions

- The valuation of embedded options and guarantees is a central component in the pricing and reserving of life insurance products
- Failure to take account of these options and guarantees can lead to serious financial losses
- We have evaluated the surrender option of a single premium pure endowment contract by means of (i) closed-form formulae (duration $T = 2$) and (ii) Monte Carlo simulation methods ($T > 2$)
- Surrender option becomes more valuable with e.g.
  - + increasing contract maturity date
  - + decreasing guaranteed interest rate r_G
  - + increasing volatility of the short rate dynamics
  - + lower mortality rates

Appendix: Vasicek model (1/2)

- Affine term structure: The term structure for the Vasicek model $(b - a\,r(t))\,dt + \sigma\,dW(t)$, i.e. the family of bond price processes, is given in the following result (c.f. Björk [3], Proposition 22.3, p. 334)

- **Proposition**: In the Vasicek model, bond prices are given by

$$P(t,T) = e^{A(t,T) - B(t,T)r(t)}\,, \tag{13}$$

where

$$B(t,T) = \frac{1}{a}\left(1 - e^{-a(T-t)}\right)\,,$$

$$A(t,T) = \frac{(B(t,T) - T + t)(ab - \sigma^2/2)}{a^2} - \frac{\sigma^2 B^2(t,T)}{4a}\,.$$

□

Appendix: Vasicek model (2/2)

- **Proposition**: In the Vasicek model, the price for a European call option with time to maturity T and strike price K on an S-bond is as follows:

$$\mathrm{ZBC}(t,T,K,S) = P(t,S)\Phi(d) - P(t,T)K\Phi(d-\sigma_p), \tag{14}$$

where

$$d = \frac{1}{\sigma_p}\log\left(\frac{P(t,S)}{P(t,T)K}\right) + \frac{\sigma_p}{2},$$

$$\sigma_p = \frac{1}{a}\left(1 - e^{-a(S-T)}\right)\sqrt{\frac{\sigma^2}{2a}\left(1 - e^{-2a(T-t)}\right)}.$$

□

- Reference: Björk [3], Proposition 22.9, p. 338.

References I

Bacinello, A., Biffis, E., and Millossovich, P. (2009). Pricing life insurance contracts with early exercise features. *Journal of Computational and Applied Mathematics*, 233, 27-35.

Bacinello, A., Biffis, E., and Millossovich, P. (2010). Regression-based algorithms for life insurance contracts with surrender guarantees. *Quantitative Finance*, Vol. 10, 1077-1090.

Björk, T. (2004). *Arbitrage Theory in Continuous Time*. Second edition. Oxford University Press.

European Parliament (2009). Directive 2009/138/EC of the European Parliament and of the Council of 25 November 2009 on the taking-up and pursuit of the business of Insurance and Reinsurance (Solvency II).

http://eur-lex.europa.eu/LexUriServ/LexUriServ.do?uri=OJ:L:2009:335:0001:0155:en:PDF

Furrer, H.J. (2009). Market-Consistent Valuation and Interest Rate Risk Measurement of Policies with Surrender Options. Preprint.

References II

- Glasserman, P. (2004). *Monte Carlo Methods in Financial Engineering.* Springer.
- Longstaff, F. A. and Schwartz, E. S. (2001). Valuing American Options by Simulation: A Simple Least-Squares Approach. *The Review of Financial Studies*, Vol. 14, No. 1, pp. 113-147.
- Pelsser, A. and Schweizer, J. (2016). The difference between LSMC and replicating portfolio in insurance liability modeling. *Eur. Actuar. J.*, Vol. 6, 441-494.
- Bundesversammlung der Schweizerischen Eidgenossenschaft. Bundesgesetz betreffend die Aufsicht über Versicherungsunternehmen (Versicherungsaufsichtsgesetz, VAG; 961.01). https://www.fedlex.admin.ch/eli/cc/2005/734/de

References III

Schweizerischer Bundesrat. Verordnung über die Beaufsichtigung von privaten Versicherungsunternehmen (Aufsichtsverordnung, AVO; 961.011).

https://www.fedlex.admin.ch/eli/cc/2005/735/de

FINMA (2017). Circular 2017/3 Swiss Solvency Test (SST)

https://www.finma.ch/en/~/media/finma/dokumente/dokumentencenter/myfinma/rundschreiben/finma-rs-2017-03-20201104.pdf?sc_lang=en&hash=6F57B1629F5BDB538E5568BE1E269A8C