Problems in Mean Curvature Flow

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Many of the following problems are classical, and many others have been told to me by various people over the years. Comments are welcome.

**General**

The following conjecture arises from the dimension-reducing theory of White.

1. **Partial regularity conjecture.** An embedded weak mean curvature of hypersurfaces $\mathbb{R}^{n+1}$ has a singular set of parabolic dimension at most $n - 1$.

2. **Nonsqueezing conjecture.** Let $M_0$ be a smooth, embedded, compact initial hypersurface in $\mathbb{R}^{n+1}$ with weak mean curvature flow $M_t$. Then a higher multiplicity plane cannot occur as a blowup limit of $M_t$.

3. **Uniqueness of tangent flow.** Let $N_t$ be a smooth, multiplicity-one self-similarly shrinking flow obtained as the limit of centered rescalings of a mean curvature flow. Then $N_t$ is the full limit.

4. **Positive mean curvature neighborhoods.** Let $M_t$ be a mean curvature flow and let $(x, t)$ be a singularity of positive mean curvature type (sphere or cylinder). Then there is a spacetime neighborhood of $(x, t)$ in which $M_t$ has positive mean curvature.

5. **Mean curvature flow of immersions.** Develop a theory of mean curvature flow of immersions.

We call $N_t$ a limit flow if it is the limit of a sequence of rescalings by factors $\lambda_i^{-1}$ about a sequence of points $(x_i, t_i)$. It is a blowup limit if $\lambda_i \to 0$.

6. **Self-similarity for limit flows.** Is a blowup limit always self-similarly shrinking, expanding, translating, or static?

7. **Entrance law.** (Griffeath) A random set possesses a (weak) mean curvature evolution, whose boundary has finite perimeter for $t > 0$, almost surely.

**Generic properties**

8. **Generic positive curvature singularities.** (Huisken) All singularities of a generic embedded mean curvature flow are spheres or cylinders.

Partial results by Ilmanen.

9. **Generic point singularities.** A generic weak mean curvature flow has only point singularities.

**Flow of curves with triple junctions**

10. **Networks with triple junctions.** Develop a theory of the flow by mean curvature of networks with triple junctions in the plane.
Flow of surfaces in $\mathbb{R}^3$

11. **Optimal partial regularity in dimension 3.** An embedded MCF in $\mathbb{R}^3$ satisfies $\dim_P \text{sing} \mathcal{M} \leq 1$. The singular set consists of isolated points unless $M_t$ is a tube that shrinks to a curve.

Here $\dim_P$ is the parabolic Hausdorff dimension.

12. **No cylinder conjecture.** Let $N$ be an embedded shrinking soliton in $\mathbb{R}^3$, and suppose $N$ is not the round cylinder. Can $N$ have an end asymptotic to a cylinder?

13. **Strict genus reduction conjecture.** A shrinker $N$ with mixed mean curvature has positive genus. The genus strictly decreases at any singularity modeled on $N$.

Special cases:

14. **Wiggly plane.** The only topological plane that is an embedded shrinker in $\mathbb{R}^3$ is the flat plane.

15. **Planar domains.** The only planar domain that is an embedded shrinker in $\mathbb{R}^3$ is the round cylinder.

16. **Resolution of point singularities.** Let the surface $M_0$ be possess an isolated singularity with a smooth tangent cone. Construct a a smooth evolution for a short time.

**Special solutions in $\mathbb{R}^3$**

17. **Proof of existence of shrinkers.** Prove the existence of the various shrinking solitons in $\mathbb{R}^3$ that have been found by computer:

- monkey saddle with $k$ holes,
- punctured cube,
- double cone with $k$ tubes.

18. **New shrinking solitons.** Find new families of embedded shrinking solitons in $\mathbb{R}^3$.

This can be done conceptually or by computer.

19. **Superposition problem.** When can the union of two self-shrinkers in $\mathbb{R}^3$ be desingularized along the curve of intersection by Scherk surfaces with tiny holes, to produce an infinite family of smooth, embedded shrinking flows? For example:

- sphere $\cup$ cylinder,
- plane $\cup$ cylinder?

Angenent has suggested that these constructions might produce ends asymptotic to a round cylinder.

20. **Stable shrinkers modulo symmetries.** Besides the $k$-punctured (monkey) saddle family, are there other self-shrinking surfaces that are stable modulo symmetries?

21. **Thin shrinking tubes.** Can we construct a thin shrinking tube whose final set is a given analytic curve? Can we prove the blowup curve is always, say, $C^\infty$, or analytic?

A delta-wing is a complete, convex, translating soliton of mean curvature flow in $\mathbb{R}^3$ that is not the bowl and not the Grim Reaper cross $\mathbb{R}$, if such a solution exists.

22. **Shape of a delta-wing.** Do matched asymptotics to deduce the shape of a delta-wing near infinity.

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23. **Heartbeat flow.** Prove the heartbeat mean curvature flow exists.

The heartbeat curve is an immersed curve in the plane that consists of two oppositely turning half-Yin-Yang curves in $\mathbb{R}^2$ that join in the middle. It is infinite length, periodic and passes through a singularity once per half-cycle.

*Variations on a theme*

24. **Codimension (1,1) minimal surfaces.** Develop a theory of minimal surfaces and mean curvature flow for spacelike submanifolds of codimension (1,1) surfaces in Lorentz manifolds.

25. **Acceleration of curves by mean curvature.** Develop a theory of timelike minimal surfaces in $\mathbb{R}^{2,1}$.

26. **IMCF with mixed curvature.** Make a theory of inverse mean curvature flow of curves in the plane with curvature of mixed sign.