ADDENDUM TO "EQUIVARIANT EMBEDDINGS OF TREES INTO HYPERBOLIC SPACES" BY BURGER–IOZZI–MONOD

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One small argument is missing in the proof of Lemma 6.5 in [1]. The statement however is correct, and we give the missing argument below.

To give some context, recall that a **Gelfand pair** (G, K) consists of a compact subgroup K of a locally compact group G such that the convolution algebra $C_c(G)^{\natural_K}$ of compactly supported bi-K-invariant continuous functions on G is commutative. For background, see e.g. [2, Sec. 24]. A sufficient – but far from necessary – condition for being a Gelfand pair is the following:

$$\forall\, g \in G : g^{-1} \in KgK. \tag{I}$$

An important property of Gelfand pairs is that for any irreducible unitary G-representation on a (complex) Hilbert space $\mathscr{H}$, the space $\mathscr{H}^K$ of K-fixed vectors has (complex) dimension at most one. This is deduced from Schur's lemma applied to the $*$-representation of $C_c(G)^{\natural_K}$ on $\mathscr{H}^K$ obtained by integration of the G-representation against a Haar measure, after checking that this $*$-representation is irreducible. Here the involution of $C_c(G)^{\natural_K}$ is given by $f^*(g) = \overline{f(g^{-1})}$ because G is unimodular, which follows from the Gelfand pair condition.

The definition of Gelfand pairs does not change if we replace $C_c(G)^{\natural_K}$ by the real $*$-subalgebra $C_c^{\mathbf{R}}(G)^{\natural_K}$ of real-valued functions. *It is not true, however,* that the $\mathbf{R}$-space of K-fixed vectors has (real) dimension at most one for all irreducible orthogonal G-representations on real Hilbert spaces. This was claimed in [1, §6.1] and constitutes a minor gap in the proof of Lemma 6.5 therein. A counterexample to the claim is provided by the irreducible representation of $G = \mathrm{SO}(2)$ on $\mathbf{R}^2$ with K trivial. The underlying cause is that the general version of a real Schur lemma leads to real division algebras, which have dimension 1 or 2 in the commutative case; this is also what comes out of an application of the complex Schur lemma to the complexification of a real representation.

The reason why this does not affect the validity of Lemma 6.5 nor of any result in [1] is that the Gelfand pairs considered there satisfy condition (I), as noted in the proof of Corollary 6.4 in [1]. It turns out that the claim does indeed hold true for this more restrictive class of Gelfand pairs:

Proposition. *If (G, K) satisfies (I), then the $\mathbf{R}$-space of K-fixed vectors has real dimension at most one for all irreducible orthogonal G-representations on real Hilbert spaces.*

Proof. We adapt the proof of the complex Schur lemma. The main observation is that under condition (I), the image of $C_c^{\mathbf{R}}(G)^{\natural_K}$ is not only closed under the involution: much more, every element individually is self-adjoint since (I) implies $f(g) = f(g^{-1})$ for all f in $C_c^{\mathbf{R}}(G)^{\natural_K}$. Therefore, given an irreducible orthogonal representation π of G on $\mathscr{H}$ and given $f \in C_c^{\mathbf{R}}(G)^{\natural_K}$, we can apply the spectral theorem to $\pi(f)$. Any spectral projector of $\pi(f)$ commutes with the representation of $C_c^{\mathbf{R}}(G)^{\natural_K}$, hence must be equal to 0 or Id. Thus $\pi(f)$ must be a multiple of the identity for any $f \in C_c^{\mathbf{R}}(G)^{\natural_K}$. By irreducibility, this implies that $\mathscr{H}^K$ has (real) dimension at most one. □

References

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