EQUIVARIANT MAPS AND PURELY ATOMIC SPECTRUM

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1. Introduction. Let G be a connected semisimple Lie group with no compact factors and finite center, and let Γ be an irreducible lattice in G. Under these hypotheses the Borel density theorem [B] asserts that Γ is Zariski dense in G. This purely algebraic theorem has appeared to be extremely valuable in the study of group actions on manifolds, but one of its first applications was the study of structural properties of G related to those of Γ; for instance, the following properties are straightforward from the Borel density theorem:

(1) The centralizer $C_G(\Gamma)$ of Γ in G is equal to $Z(G)$;
(2) The normalizer of Γ in G is discrete and $Z(G)\Gamma$ is discrete;
(3) The only elements with finite conjugacy classes are in $Z(G)$.

An extension of the Borel density theorem in a direction involving geometric structures has been proven in [I1]. It is hence natural to try to address similar questions relating the structural properties of G and Γ as subgroups of the diffeomorphism group of a manifold M. In particular the question which inspired this paper is the following: if M is a differentiable manifold, G and Γ are as above and $G \subset \mathrm{Diff}(M)$, what is the relation between $C_{\mathrm{Diff}(M)}(\Gamma)$ and $C_{\mathrm{Diff}(M)}(G)$? More precisely, if $\varphi\colon M \to M$ is a diffeomorphism which commutes with the action of Γ, does it commute with the action of G as well? Note that the question is answered affirmatively if M has finite invariant measure (or the G-orbits in M with compact stabilizer are not locally closed) and $\varphi_t\colon M \to M$ is a one-parameter group of diffeomorphisms. In fact, under these hypotheses, the above question is equivalent to asking if a Γ-invariant vector field (in this case the vector field $\dot{\varphi}_t$ which is the infinitesimal generator of φ_t) is G-invariant, which has been proven true (actually in greater generality for sections of suitable bundles) in [I1].

The above discussion reflects the ideas which inspired this paper; here we shall be actually working in the category of measurable maps and measurable spaces and to balance this much more general setting we are going to need some additional hypotheses, (see Definition 1.1).

Moreover we shall address a related question, as follows. Recall that two G-spaces (X_1, μ_1), (X_2, μ_2) with finite invariant measures are isomorphic (or conjugate), if

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there are conull G-invariant Borel sets $X_1' \subseteq X_1$, $X_2' \subseteq X_2$ and a measure preserving Borel isomorphism $\varphi\colon X_1' \to X_2'$ which is a G-map. The question we shall answer is when an isomorphism as Γ-spaces implies an isomorphism as G-spaces as well.

Let G, Γ be as above and let (X, μ) be a G-space with finite invariant measure.

Definition 1.1. *We say that G acts with purely atomic spectrum if the representation π induced on $L^2(X, \mu)$ by the action of G on X decomposes as a direct sum of irreducible representations.*

Then we shall prove the following:

Theorem 1.2. *Let G, Γ be as above, (X_i, μ_i), $i = 1, 2$, a G-space with finite invariant measure and let $\varphi\colon X_1 \to X_2$ be a measure preserving measurable Γ-map (i.e. $\varphi(x\gamma) = \varphi(x)\gamma$ for all $\gamma \in \Gamma$, a.e.$x \in X_1$). If G acts on the X_i's with purely atomic spectrum and either essentially freely (i.e. freely on a conull set) or essentially transitively, then φ is a G-map.*

As a direct consequence, we have:

Corollary 1.3. *Let G, Γ be as above, (X_i, μ_i), $i = 1, 2$, a G-space with finite invariant measure and assume that G acts on X_i with purely atomic spectrum and either essentially freely or essentially transitively. If the actions of Γ on the X_i's are isomorphic, the same is true for the actions of G.*

It should be noted that, although the hypothesis of purely atomic spectrum is quite restrictive, several actions with finite invariant measure do satisfy it. In particular we have the following examples:

(1) Let $\Lambda \subset G$ be a cocompact lattice; then G acts on G/Λ with purely atomic spectrum.
(2) Let H be a connected semisimple Lie group with finite center and no compact factors and let $\Lambda \subset G \times H$ be a cocompact irreducible lattice. If G acts on $G \times H$ by the first coordinate, then the G-action on $(G \times H)/\Lambda$ has purely atomic spectrum.
(3) Let $\Lambda \subset G$ be a cocompact lattice and let $\theta\colon \Lambda \to K$ be a dense homomorphism into a compact group K. If $K_0 \subset K$ is a closed subgroup, the ergodic action of Λ on $Y = K/K_0$ has purely atomic spectrum (in fact the irreducible representations in the decomposition of $L^2(Y)$ will be in this case finite dimensional) and induces an ergodic G-action on $X = (G \times Y)/\Lambda$ which has purely atomic spectrum.
(4) Let G be a semisimple algebraic group defined over $\mathbb{Q}$. Consider the group of adeles $G_{\mathbb{A}} \simeq G_{\mathbb{R}} \times H$, on which $G_{\mathbb{R}}$ acts by the first coordinate, and the discrete subgroup $G_{\mathbb{Q}}$ of the principal adeles. If $\mathbb{Q}$-rank$(G) = 0$ then $G_{\mathbb{Q}}$ is a cocompact lattice in $G_{\mathbb{A}}$ and we have an action of $G_{\mathbb{R}}$ on $G_{\mathbb{A}}/G_{\mathbb{Q}}$ with purely atomic spectrum.

Note that the hypothesis of either essential transitiveness or essential freeness is satisfied in all of these examples, as in the first case the action is essentially transitive and in the remaining ones is essentially free, [I2, Proposition 4.4]. Furthermore, this is only a particular case of a more general phenomenon in the sense that there are no known examples, so far, of actions with finite invariant measure which are neither essentially free nor essentially transitive.

A result related to this has been proven first by M. Ratner for $\mathrm{SL}(2, \mathbb{R})$ and then extended to more general Lie groups by D. Witte. We shall state here a version of their result which is most related to this paper, even though not in its greatest generality.

Theorem 1.4 ([Rt], [W]). *Let G be as above and $\Lambda \subset G$ a lattice. If $\varphi \colon G/\Lambda \to G/\Lambda$ is a measure preserving Borel map which commutes with a unipotent element in G then φ is a G-map.*

The techniques used to prove this theorem are completely different from ours. It should also be noticed that, even though Ratner-Witte's theorem is much stronger than the one we are going to prove in the cases in which both can be applied, it does not give any answer in the case of an action which is not (essentially) transitive or cannot be extended to an (essentially) transitive action of some larger Lie groups (see Examples (1) and (2) above); also, in the case of the G-action on G/Λ, if the lattice $\Gamma \subset G$ is cocompact then it is well known [Rg] that it does not have any unipotent elements, making again Ratner-Witte's theorem inapplicable. (Recall that we define an element u of a linear group to be unipotent if $u - \mathrm{Id}$ is nilpotent; an element u of a general Lie group is unipotent if $\mathrm{Ad}_G(u)$ is unipotent as a linear transformation of the Lie algebra of G.)

We shall prove also that some hypothesis on the spectrum is necessary. We shall construct, in fact, an essentially free action with finite invariant measure and no purely atomic spectrum, for which our theorem fails to be true. However such an action will not be measurably equivalent to an action on a manifold, thus leaving open the question as to whether or not such a hypothesis, or some hypothesis, is necessary in this case (see §4).

The techniques used in this paper involve mainly representation theory and ergodic theory results. In §2 we shall start the proof of Theorem 1.2, laying the foundations of it and referring to results through the paper for statements which require a proof. The crucial ingredient in the proof is a result of Tim Steger and Michael Cowling according to which an irreducible representation of G which is not in the discrete series is still irreducible when restricted to Γ, [C-S]. This, and other results in representation theory, will conclude §2 and will actually constitute a complete proof in the case in which G does not have discrete series. In §3 we study more closely the stabilizers of some actions which arise in the proof and in §4 we conclude our proof with some general results concerning the space of measurable maps with a "twisted" G-action. The following result is proven in §4 and is a direct consequence of our proof:

Corollary 4.3. *Let G, Γ as above and let H be an algebraic group. Let $P \to M$ be a principal H-bundle over a manifold M such that $G \subseteq Aut(P)$ and G preserves a probability measure on P. Let $\varphi \colon P \to P$ be a measure preserving Γ-map which preserves fibers and covers a G-map $\varphi_0 \colon M \to M$. Then φ is a G-map.*

(See §4 for a discussion about the hypotheses of this result.)

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2. Results on representation theory. Recall that an irreducible unitary representation σ of a locally compact group G is said to be in the discrete series $\mathcal{D}$ of G if its matrix coefficients are square integrable or, equivalently, if σ is contained in the regular representation. Representations in the discrete series correspond to the atomic part of the support of the Plancharel measure on the unitary dual $\hat{G}$ of G.

Given two unitary representations $\sigma_i \colon G \to \mathcal{U}(\mathcal{H}_i)$, $i = 1, 2$, we say that a unitary operator $T \colon \mathcal{H}_1 \to \mathcal{H}_2$ intertwines σ_1 and σ_2 if $\sigma_2 \cdot T = T \cdot \sigma_1$. We shall denote by (σ_1, σ_2) the vector space of unitary intertwining operators between σ_1 and σ_2.

Proof of Theorem 1.2. We shall assume, through the proof, that G acts ergodically on the X_i's, that is that the only G-invariant sets are either null or conull. There is no loss of generality in doing this, since we can always restrict our attention to a G-ergodic component in X_1; this will be also a Γ-ergodic component (see [M], [Z2]) and will be mapped, by the measure preserving Γ-map $\varphi \colon X_1 \to X_2$, to a Γ-ergodic component in X_2, on which also the action of G will be ergodic.

Since φ is measure preserving, then the operator $T_\varphi \colon L^2(X_2) \to L^2(X_1)$ defined as $T_\varphi f(x) = f(\varphi(x))$ is unitary; moreover, since φ is a Γ-map, if π^i is the representation of G on $L^2(X_i, \mu_i)$, then $T_\varphi \in (\pi^2|_\Gamma, \pi^1|_\Gamma)$. Let $(\pi^i, L^2(X_i, \mu_i)) = (\pi_0^i \oplus \pi_d^i, \mathcal{H}_0^i \oplus \mathcal{H}_d^i)$, $i = 1, 2$, where

$$\pi_d^i = \sum_{\substack{\pi_l^i \subseteq \pi^i \\ \pi_l^i \in \mathcal{D}}}^{\oplus} \pi_l^i \qquad \text{and} \qquad \pi_0^i = \sum_{\substack{\pi_j^i \subseteq \pi^i \\ \pi_j^i \notin \mathcal{D}}}^{\oplus} \pi_j^i.$$

The outline of the proof will be as follows. After observing that π^i does not consist only of discrete series representations, that is $\mathcal{H}_0^i \neq \emptyset$ (Lemma 2.2), we shall prove that $T_\varphi(\mathcal{H}_0^2) \subset \mathcal{H}_0^1$ and that the restriction of T_φ to $\mathcal{H}_0^2$ intertwines π_0^2 and π_0^1 (Corollary 2.6). The remainder of the proof will consist of showing that the fact that $T_\varphi|_{\mathcal{H}_0^2} \in (\pi_0^2, \pi_0^1)$ is "enough", in the following sense.

Let $\mathcal{A}_i$ be the measure algebra generated by elements of $\mathcal{H}_0^i$. If $\mathcal{B}(X_i)$ denotes the Boolean algebra of Borel subsets of X_i, we have that $\mathcal{A}_i \subset \mathcal{B}(X_i)$ and hence, corresponding to $\mathcal{A}_i$, there exists a factor (Y_i, ν_i) of (X_i, μ_i); in other words there exists a measure space Y_i with finite measure ν_i and a Borel map $p_i \colon (X_i, \mu_i) \to (Y_i, \nu_i)$ such that $\nu_i = p_{i*}(\mu_i)$ and $\mathcal{A}_i = \mathcal{B}(Y_i)$. Since $\mathcal{H}_0^i$ is $\pi^i(G)$-invariant then $\mathcal{A}_i$ is G-invariant and on the space (Y_i, ν_i) there is a G-action which commutes with the factor maps p_i's and with respect to which the measure ν_i is invariant. Moreover, because of Corollary 2.6.(1), we have that T_φ induces an homomorphism of σ-algebras $\mathcal{A}_2 \to \mathcal{A}_1$ and hence there exists a measure preserving Borel map $\varphi_0 \colon Y_1 \to Y_2$, such that $p_2 \circ \varphi = \varphi_0 \circ p_1$. Furthermore, φ_0 has the additional property that it commutes with the actions of G on Y_i. (In fact, by definition of Y_2, we have that $p_2(x) = p_2(\bar{x})$ if and only if $f(x) = f(\bar{x})$ for all $f \in \mathcal{H}_0^2$; since we know that $T_\varphi|_{\mathcal{H}_0^2} \in (\pi_0^2, \pi_0^1)$, then for every $f \in \mathcal{H}_0^2$, $g \in G$, we have $f(\varphi(gx)) = f(g\varphi(x))$ for a.e. $x \in X_1$, which implies that $\varphi_0(gy) = \varphi_0(gp_1(x)) = \varphi_0(p_1(gx)) = p_2(\varphi(gx)) = p_2(g\varphi(x)) = gp_2(\varphi(x)) = g\varphi_0(p_1(x)) = g\varphi_0(y)$, where $x \in X_1$ is such that $p_1(x) = y$.) For a more detailed description of such construction of factors see [Z1] or [Rm]. The next step will be the only one in which we have to distinguish between the essentially free and the essentially transitive case, the rest of the proof being completely independent of both these hypotheses. We shall prove that $Y_i \simeq X_i/A_i$, $i = 1, 2$, where either A_i is a central subgroup of G (in the essentially free case, Proposition 3.4) or A_i is the quotient of two central subgroups of G (in the essentially transitive case, Proposition 3.5). In both cases, there is an action of A_i on the fibers which commutes with the action of G (we can think of $p_i \colon X_i \to Y_i$ as a measurable principal A_i-bundle, on which G acts by principal bundle automorphisms) and the general result of Proposition 4.2 will complete the proof, showing that φ is a G-map as well. $\square$

We start with the following result on the stabilizers of actions with finite invariant measure; for actions of simple groups this can be found in [Z4] and for smooth actions of semisimple groups in [Z5].

Proposition 2.1. *Let G be a connected semisimple Lie group with finite center and no compact factors and let (X,μ) be a G-space with finite invariant measure on which G acts effectively (i.e. there are no connected normal subgroups which act trivially). Then, for almost every $x \in X$ the stabilizer G_x of the G-action is either discrete or $G_x = G$. If the action is ergodic then the G_x's are discrete for almost every $x \in X$.*

Proof. It will be enough to consider the case in which the action is ergodic. An application of the Borel density theorem as in [Z4, Theorem 5.4] shows that the Lie algebra $\mathfrak{g}_x$ of G_x is an ideal for almost every $x \in X$, hence G_x is a normal subgroup. Since G is semisimple and has finite center, then there is only a finite number of connected normal subgroups, $N_1, \dots, N_r$. For each such N_j, let $X_j \subset X$ be the set of points which are fixed under N_j; since N_j is normal, then X_j is G-invariant and hence either null or conull, by ergodicity. If $\mu(X_j) = 0$ for all j's, the proof is completed. Otherwise, let X' be the intersection of all X_j which have full measure: then the corresponding normal subgroups fix all of X' (which has full measure), contradicting the hypothesis that G acts effectively. $\square$

We shall conclude this section with the proof of the results involving representation theory. We shall assume, all through the rest of the paper, that G acts effectively, without loss of generality.

Lemma 2.2. *Let $G, (X,\mu)$ be as in Proposition 2.1 and assume that G acts on X with purely atomic spectrum. Then the representation π of G on $L^2(X,\mu)$ does not consist only of representations of the discrete series.*

Proof. Let $K \subset G$ be the maximal compact subgroup of G and suppose that $\pi = \sum^{\oplus} \pi_i$, where $\pi_i \in \mathcal{D}$. The restriction of π_i to K does not have non-trivial invariant vectors, ([S], [H-S]; see also [Kn]) hence $\pi|_K$ does not have non-trivial invariant vectors and, since X has finite invariant measure, this implies that K acts ergodically on X, which is impossible. In fact, since K is compact, K, and hence G, must act essentially transitively on X. But this is impossible since the stabilizers of the G-action are discrete (Proposition 2.1) and K has dimension lower than G. $\square$

The following result will be crucial in the proof; it was first proven by T. Steger for $\mathrm{SL}(2,\mathbb{R})$ and then extended to the general case by M. Cowling and T. Steger.

Theorem 2.3 [C-S]**.** *Suppose Γ is an irreducible lattice in a connected semisimple Lie group with finite center and suppose that σ and σ' are in $\hat{G}$. Then*

(1) *If $\sigma \notin \mathcal{D}$ then $\sigma|_\Gamma$ is irreducible*
(2) *If $\sigma|_\Gamma$ and $\sigma'|_\Gamma$ are unitarily equivalent and irreducible (e.g. $\sigma, \sigma' \notin \mathcal{D}$) then σ and σ' are unitarily equivalent.*

Corollary 2.4. *Let G, Γ be as in Theorem 2.3 and let $(\sigma, \mathcal{H}_\sigma)$, $(\sigma', \mathcal{H}_{\sigma'})$ b irreducible representations of G which are not in the discrete series. If $T \in (\sigma|_\Gamma, \sigma'|_\Gamma)$, then $T \in (\sigma, \sigma')$.*

Proof of Corollary. By Theorem 2.3.(2), since $\sigma|_\Gamma$ and $\sigma'|_\Gamma$ are irreducible and unitarily equivalent, also σ and σ' are unitarily equivalent. Let $U\colon \mathcal{H}_\sigma \to \mathcal{H}_{\sigma'}$ be the unitary operator which implements the equivalence. Then $\sigma' = U \cdot \sigma \cdot U^{-1}$ and hence $U \cdot \sigma|_\Gamma \cdot U^{-1} = T \cdot \sigma|_\Gamma \cdot T^{-1}$, that is $\sigma|_\Gamma = U^{-1}T \cdot \sigma|_\Gamma \cdot (U^{-1}T)^{-1}$. Since $\sigma|_\Gamma$ is irreducible, by Schur's Lemma, $U^{-1}T$ is a multiple of the identity operator, hence $T = cU$, where $c \in \mathbb{C}$. Since $U \in (\sigma, \sigma')$, it follows that also $T \in (\sigma, \sigma')$. $\square$

It should be noted that also the converse of Theorem 2.3.(1) holds true. In fact if $\sigma' \in \mathcal{D}$, then σ' is contained in the regular representation, as well as its restriction to Γ, and, as long as Γ is an infinite discrete subgroup, its regular representation does not contain any irreducible representations (for a proof see [C-FT]). Exactly the same argument shows the following:

Proposition 2.5. *Let G, Γ be as in Theorem 2.3 and let $(\rho, \mathcal{H}_\rho), (\rho', \mathcal{H}_{\rho'})$ be irreducible representations of G such that $\rho \in \mathcal{D}$ and $\rho' \notin \mathcal{D}$. Then $(\rho|_\Gamma, \rho'|_\Gamma) = \emptyset$, that is there is no operator $T\colon \mathcal{H}_\rho \to \mathcal{H}_{\rho'}$ which intertwines $\rho|_\Gamma$ and $\rho'|_\Gamma$.*

Proof. If such an operator T were to exist it would imply that $\rho'|_\Gamma$ is an irreducible representation contained in the left regular representation of Γ. $\square$

We are going to apply the next Corollary to the operator T_φ defined in the proof of Theorem 1.2.

Corollary 2.6. *Let G, Γ be as in Theorem 2.3, $\pi^i = \pi^i_0 \oplus \pi^i_d$, $i = 1, 2$, be unitary representations of G on $\mathcal{H}^i_0 \oplus \mathcal{H}^i_d$, where*

$$\pi^i_0 = \sum_{\pi^i_j \notin \mathcal{D}}^{\oplus} \pi^i_j \qquad \textit{and} \qquad \pi^i_d = \sum_{\pi^i_l \in \mathcal{D}}^{\oplus} \pi^i_l,$$

and let $T\colon \mathcal{H}^2_0 \oplus \mathcal{H}^2_d \to \mathcal{H}^1_0 \oplus \mathcal{H}^1_d$ be a unitary operator such that $T \in (\pi^2|_\Gamma, \pi^1|_\Gamma)$. Then

(1) *$T(\mathcal{H}^2_0) \subset \mathcal{H}^1_0$, $T(\mathcal{H}^2_d) \subset \mathcal{H}^1_d$ and*
(2) *$T|_{\mathcal{H}^2_0} \in (\pi^2_0, \pi^1_0)$.*

Proof. (1) This is just Proposition 2.5.
(2) For $1 = 1, 2$, let V^i_j be the Hilbert space of the representation $\pi^i_j \subset \pi^i_0$ and let $P^i_j\colon \mathcal{H}^i_0 \to V^i_j$ be the projection operator. Since $T(\mathcal{H}^2_0) \subset \mathcal{H}^1_0$ and $T|_{\mathcal{H}^2_0} \in (\pi^2_0|_\Gamma, \pi^1_0|_\Gamma)$ this, together with the equalities $\pi^i_0 P^i_j = P^i_j \pi^i_0 = \pi^i_j$, shows that $P^1_j T P^2_i$ intertwines the representations $\pi^2_l|_\Gamma$ and $\pi^1_j|_\Gamma$. If $\pi^2_l|_\Gamma \sim \pi^1_j|_\Gamma$ then apply Corollary 2.3; if $\pi^2_l|_\Gamma \not\sim \pi^1_j|_\Gamma$ then $P^1_j T P^2_i$ has to be the zero operator, hence concluding the proof. $\square$

3. Ergodic theory results. Recall that we define a Borel space to be *tame* if there exists a countable family of Borel subsets which separates points. For a topological space tameness is equivalent to being T_0 together with having a countable base for the topology. If G is a locally compact group and S is an ergodic G-space, it can be proven that every G-invariant Borel function $f\colon S \to T$, where T is a tame space,

T is tame if both T and the space of orbits T/G are tame. For continuous actions on a separable metric space tameness is equivalent to the the conditions that all orbits be locally closed or that every ergodic measure is supported on an orbit. We should point out that tameness is a very natural condition; for instance, compact groups always act tamely, [Z3].

Let G be as in Proposition 2.1, (X,μ) an ergodic G-space with finite invariant measure and (Y,ν) a G-factor of X. If K is any subgroup of G and $\mathcal{O}\subset Y$ is a K-orbit, then obviously $p^{-1}(\mathcal{O})$ is K-invariant (since the map $p\colon X\to Y$ commutes with the G-action), but in general will not be a K-orbit, not even if $K=G$. However the next result will show that this is the case if G acts with purely atomic spectrum on X, (Y,ν) is a factor of X defined as the (Y_i,ν_i)'s in proof of Theorem 1.2 and $K\subset G$ is the maximal compact subgroup.

Denote by K_y be the stabilizer in K of $y\in Y$, i.e. $K_y=K\cap G_y$, where G_y is the stabilizer in G of $y\in Y$.

Lemma 3.1. *Let $G,K,(X,\mu),(Y,\nu)$ be as above. For almost every $y\in Y$, K_y is transitive on $p^{-1}(y)$.*

Corollary 3.2. *For almost every $y\in Y$ the fiber $p^{-1}(y)$ can be identified with K_y/K_x, where $p(x)=y$; in particular its cardinality $|p^{-1}(y)|$ is finite and constant almost everywhere.*

Proof of Corollary. If $x\in X$ is such that $p(x)=y$, then, because of Lemma 3.1, the fiber $p^{-1}(y)$ over y can be identified with K_y/K_x (and hence with G_y/G_x). This implies, of course, that for almost every $y\in Y$, $|p^{-1}(y)|=|K_y/K_x|<\infty$. Moreover the measurable map $f\colon Y\to\mathbb{R}$ defined as $f(y)=|p^{-1}(y)|$ is G-invariant; in fact, since $p(gy)=gp(y)=gx$, $f(gy)=|p^{-1}(gy)|=|G_{gy}/G_{gx}|=|gG_yg^{-1}/gG_xg^{-1}|=|G_y/G_x|=f(y)$. By ergodicity of G on X and hence on Y, we have that f is essentially constant. $\square$

Proof of Lemma 3.1. Since Y is defined as the measure space whose σ-algebra of Borel sets is the measure algebra generated by the non-discrete series part $\mathcal{H}_0$ of the representation π of G on $L^2(X)$ (see the proof of Theorem 1.2 at the beginning of §2) and $p_*(\mu)=\nu$, there is an isometric embedding $L^2(Y)\hookrightarrow L^2(X)$ given by $f\mapsto f\circ p$ and $L^2(Y)$ can be identified as the subspace of $L^2(X)$ which consists of all the functions which are measurable with respect to $\mathcal{A}$. Let $\mathcal{H}_1=L^2(Y)^\perp=L^2(X)\ominus L^2(Y)=\{f\in L^2(X)\colon \int_X f\tilde{h}=0 \text{ for every } h\in L^2(Y), \tilde{h}=h\circ p\}$ and let π_1 be the subrepresentation of π which acts on $\mathcal{H}_1$. Since $\mathcal{H}_0\hookrightarrow L^2(Y)$, we have that $\mathcal{H}_1\subset\mathcal{H}_d$, so that π_1 is a direct sum of representations in the discrete series of G. As in the proof of Lemma 2.2, we have that $\pi_1|_K$ does not have non-trivial invariant vectors. We shall prove that if there were a set of positive measure $B\subset Y$ such that, for every $y\in B$, K_y is not transitive on $p^{-1}(y)$, then we could construct a non-constant function in $\mathcal{H}_1$ which would be invariant under the action of $\pi_1(K)$. Clearly if K_y is not transitive on $p^{-1}(y)$, then, for each $\tilde{y}$ in the K-orbit through y, we have that $K_{\tilde{y}}$ is not transitive on $p^{-1}(\tilde{y})$; this implies that if $\mathcal{O}\subset Y$ is the K-orbit of y, then K is not transitive on $p^{-1}(\mathcal{O})$. If $\mathcal{O}$ had positive measure, then we could construct the function f as follows. Let $S\subset p^{-1}(\mathcal{O})$ be a K-invariant set which is neither null nor conull. Let $\mu=\int_Y^\oplus m_y d\nu(y)$ be the essentially unique decomposition of the measure μ with respect to ν, where m_y is a measure on X supported on $p^{-1}(y)$, [Z1]; since μ is G-invariant, the m_y's have the property that, of S, implies that the function $M\colon \mathcal{O} \to \mathbb{R}$, defined as $M(y) = m_y(S \cap p^{-1}(y))$ is K-invariant on $\mathcal{O}$ and hence constant, say $M(y) = M_{\mathcal{O}}$. Then the function $f_{\mathcal{O}}(x) = \dfrac{M_{\mathcal{O}}}{M_x} - \chi_S(x)$, where M_x is the $m_{p(x)}$-measure of the fiber $p^{-1}(p(x))$ through x and $\chi_S(x)$ is the characteristic function of the set S, is clearly K-invariant and is in $\mathcal{H}_1$, since, for every $h \in L^2(Y)$,

$$\int_X f_{\mathcal{O}}(x)\tilde{h}(x)d\mu(x) = \int_Y \left(\int_{p^{-1}(y)} f_{\mathcal{O}}(x)\tilde{h}(x)dm_y(x)\right) d\nu(y) =$$
$$= \int_Y \tilde{h}(x) \left(\int_{p^{-1}(y)} f_{\mathcal{O}}(x)dm_y(x)\right) d\nu(y) = 0.$$

The problem is that this function $f_{\mathcal{O}}$ is trivial, being supported on a K-orbit, which is a null set. However, since K is compact, it acts tamely on Y and hence the remainder of the proof will consist just of "glueing" together the functions $f_{\mathcal{O}}$ defined on single K-orbits.

If $\eta\colon Y \to Y'$ is the natural projection onto the space $Y' = Y/K$ of K-orbits in Y, and $q = \eta \circ p\colon X \to Y'$, let $B' = q(B)$, where B is the set defined above. Consider the Borel space $\mathcal{F} = \{S \subset X\colon\ S$ is measurable and K-invariant, $S \subset q^{-1}(y')$ for some $y' \in B'$ and if $\eta(y) \in y'$, $S \cap p^{-1}(y)$ is neither null nor conull in $p^{-1}(y)\}$. Because of the definition of B, there is an obvious surjective map $\mathcal{F} \to B'$ and hence a Borel section $\theta\colon B' \to \mathcal{F}$. Define a measurable function f on X as $f(x) = \dfrac{m_{p(x)}(\theta(q(x)))}{M_x} - \chi_{\theta(q(x))}(x)$, where M_x is the $m_{p(x)}$-measure of the fiber through x; the same argument as in the case of a single K-orbit shows that f is K-invariant (in fact $m_{p(x)}(\theta(q(x))) = M_{\mathcal{O}}$, where $p(x) \in \mathcal{O}$) and, since B has positive measure, is non-trivial. Moreover as above, since $\int_{p^{-1}(p(x))} f(x)dm_{p(x)}(x) = 0$, it follows that $f \in \mathcal{H}_1$, and this is the contradiction which completes the proof. $\square$

At this point in the proof we have to distinguish between the essentially free and the essentially transitive case; we shall deal with the former first.

Recall that, as for compact groups, an algebraic action of the real points of a real algebraic group on the real point of a real algebraic variety is always tame, [B-S]. The first step to be able to use this nice regularity property of algebraic groups will be to observe that given any connected semisimple Lie group G with finite center, $\mathrm{Ad}_G(G)$ (which we shall denote by G^*) is of finite index in a connected (semisimple) real algebraic group defined over $\mathbb{Q}$, which we shall denote by $\tilde{G}$, and $G/Z(G)$ is isomorphic (as a Lie group) to G^*.

Recall that S is an extension of T if T is a factor of S; a countable (respectively finite) extension is such that the fiber over almost every point is countable (respectively finite). For such extensions we have the following Lemma, which will be used in the proof of Proposition 3.4 to deal with some of the complications due to the presence of a center in G.

Lemma 3.3. *If S is a countable extension of a tame G-space T, where G is a locally compact group, S is tame.*

Proof. Let $\phi\colon S \to T$ be the factor map and let μ be an ergodic measure on S; then $\phi_*(\mu)$ is an ergodic measure on T and, since T is tame, $\phi_*(\mu)$ is supported on an orbit. But then μ is supported on a countable union of orbits and, since it is ergodic, it must be supported on a single orbit, hence proving tameness. $\square$

Proposition 3.4. *Let $G, (X, \mu)$ be as in Proposition 2.1 and (Y, ν) be a factor of (X, μ). If the fiber over almost all points in Y has finite constant cardinality and G acts essentially freely on X, then there exists a central subgroup $A \subset Z(G)$ which is the stabilizer of almost every $y \in Y$ and, in fact, $Y \simeq X/A$.*

Proof. Because of Corollary 3.2, we have that for almost every $y \in Y$, $|G_y| = |G_y/G_x| = |p^{-1}(y)| = n$. Let $\mathcal{S}_G^n$ be the space of subsets of G of cardinality n, on which G acts by conjugation. If G algebraic we could deduce that the G-action on $\mathcal{S}_G^n$ is tame and that the stabilizers are isomorphic to algebraic groups. We need instead a little more work.

Note that, since $Z(G)$ acts trivially on $\mathcal{S}_G^n$, the action of G on $\mathcal{S}_G^n$ by conjugation factors through an action of G^*. Let $\mathcal{S}_{G^*}^{\leq n}$ denote the space of subsets of G^* of cardinality less then or equal to n, and likewise for $\mathcal{S}_{\tilde{G}}^{\leq n}$ from $\tilde{G}$; since $\mathcal{S}_{\tilde{G}}^{\leq n} = \bigcup_{k=1}^n \mathcal{S}_{\tilde{G}}^k$, then $\mathcal{S}_{\tilde{G}}^{\leq n}$ consists of the real points of a variety defined over $\mathbb{Q}$, on which $\tilde{G}$ acts algebraically and hence tamely. Since the $\tilde{G}$-orbits are finite union of homeomorphic copies of G^*-orbits, then G^* acts tamely on $\mathcal{S}_{\tilde{G}}^{\leq n}$ and thus on $\mathcal{S}_{G^*}^{\leq n}$. The map $\mathrm{Ad}_G\colon G \to G^*$ induces a map $\alpha\colon \mathcal{S}_G^n \to \mathcal{S}_{G^*}^{\leq n}$ which has finite fiber and hence, by Lemma 3.3, the G^*-action (and hence the G-action) on $\mathcal{S}_G^n$ is tame. Define a map $\Phi\colon Y \to \mathcal{S}_G^n$ as $\Phi(y) = G_y$; it is straightforward to check that Φ is a G-map. If $q\colon \mathcal{S}_G^n \to \mathcal{S}_G^n/G$ is the natural projection and we define $\tilde{\Phi} = q \circ \Phi\colon Y \to \mathcal{S}_G^n/G$, we have that $\tilde{\Phi}$ is a G-invariant map and hence, by ergodicity of G on Y and tameness of $\mathcal{S}_G^n/G$, $\tilde{\Phi}$ is essentially constant. This is equivalent to saying that $\Phi\colon Y \to \mathcal{S}_G^n$ actually takes values in a single G-orbit, that is we can write $\Phi\colon Y \to G/G_A$, where $A = G_y \in \mathcal{S}_G^n$ is the stabilizer of some $y \in Y$ and G_A is the stabilizer in G of A, that is the normalizer in G of G_y. It is clear that G_A^*, the stabilizer in G^* of A, is isomorphic to $G_A/Z(G)$, so that $G/G_A \simeq G^*/G_A^*$. Since Φ is a G-map, the finite invariant measure ν on Y pushes forward to a finite G-invariant measure $\Phi_*(\nu)$ on G/G_A, which we can think of as a finite G^*-invariant measure on G^*/G_A^*. If we knew that G_A^* were algebraic, the Borel density theorem would imply that $G_A^* = G^*$, so that $G_A = G$ and hence $G_y = A$ for almost every $y \in Y$ and A, being normal and discrete, would be central, hence completing the proof. But $\mathcal{S}_G^n$ is not a variety, hence we do not know whether or not G_A^* is algebraic and we can only conclude that G_A^* is Zariski dense in G^*.

To proceed with the proof, observe that $G_{\alpha(A)}^*$ is of finite index in the algebraic group $\tilde{G}_{\alpha(A)}$ (since the action of $\tilde{G}$ on $\mathcal{S}_{\tilde{G}}^{\leq n}$ is algebraic) and that $G_{\alpha(A)}^* = G_{AZ(G)}^*$. But since $G_A^* \subseteq G_{\alpha(A)}^* = G_{AZ(G)}^*$, by passing to the Zariski closure, we have that $G_{AZ(G)}^* = G^*$. This means that G^* acts (continuously) on $AZ(G)$, which is discrete, and hence trivially. In particular it fixes $A \subseteq Z(G)$, hence showing that $G_A^* = G^*$. Reasoning as above we can now conclude the proof. $\square$

For essentially transitive actions we have the following:

Proposition 3.5. *Let $G, (X, \mu)$ be as in Proposition 2.1, $K \subset G$ be the maximal compact subgroup and let (Y, ν) be a G-factor of (X, μ). Assume that G acts essentially transitively on (X, μ) and that, for almost every $y \in Y$, K_y acts essentially transitively on $p^{-1}(y)$, where $p\colon X \to Y$ is the factor map and K_y is the stabilizer* *of y in K. Then there are two fixed central subgroups A^X, A^Y such that for almost every $y \in Y$, $p^{-1}(y) \simeq A^Y/A^X$, and hence $Y \simeq X/A$, where $A = A^Y/A^X$.*

Proof. It will be enough to prove the following:

Lemma 3.6. *Let G be a connected semisimple Lie group with no compact factors and finite center, $K \subset G$ its maximal compact subgroup and let S be an essentially transitive G-space with finite invariant measure. Then there is a subgroup $A \subset K$ which is central in G, such that K/A acts essentially freely on S. In other words the stabilizer in K of almost every $s \in S$ is a central subgroup $A \subset K$ (which does not depend on $s \in S$).*

To deduce Proposition 3.5 from this Lemma it suffices to observe that since G is essentially transitive on X, it must be essentially transitive on Y too. Then applying Lemma 3.6 twice with $S = X$ and $S = Y$ and using Corollary 3.2, we get that, for almost every $y \in Y$, $p^{-1}(y) \simeq A^Y/A^X$. where $A^Y, A^X \subset Z(G)$.

Proof of Lemma 3.6. Let $F = K_s = K \cap G_s \neq \{e\}$, where K_s and G_s are the stabilizers in K and G, respectively, of a fixed $s \in S$; since G_s is discrete (Proposition 2.1), F is a finite group and we can also find a neighborhood U of K such that $U \cap G_s = F$. Let V be a symmetric neighborhood of $e \in G$ with the following properties:

i) $gKg^{-1} \subset U$ for all $g \in V$

ii) If $F = \{e, f_1, \dots, f_n\}$ then $gf_ig^{-1} \neq f_j$ for all $i \neq j, g \in V$.

Let $\mathcal{F}$ be the collection of all subsets of F. If $E \in \mathcal{F}$ define $V_E = \{g \in V \colon gKg^{-1} \cap G_s \supset E\} = \{g \in V \colon gKg^{-1} \supset E\}$. Because of the choice of V and because of the property that if $E_1 \subseteq E_2$ then $V_{E_1} \subseteq V_{E_2}$, V_E is exactly the subset of points in V which fix E pointwise, that is $V_E = \{g \in V \colon gfg^{-1} = f$ for all $f \in E\}$. If we knew that for $f \in E$, $gfg^{-1} = f$ for all $g \in G$, then we would have that $E \subset Z(G)$; then our next goal would be to show that on a neighborhood of s, the stabilizer of almost every point is a central subgroup which does not depend on the point.

It is clear that all of the V_E's are symmetric (since V is symmetric) and closed in V: in fact if $\phi_f \colon V \to G$ is the continuous map defined as $\phi_f(g) = g^{-1}fg$ for each $f \in F$, then $\phi_f^{-1}(K) = \{g \in V \colon g^{-1}fg \in K\} = \{g \in V \colon f \in gKg^{-1}\}$ is closed in V and $V_E = \{g \in V \colon gKg^{-1} \supset E\} = \bigcap_{f \in E} \phi_f^{-1}(K)$ is closed. Moreover, we have that $V = \bigcup_{E \in \mathcal{F}} V_E$. Since $m(V) > 0$ (where m is the Haar measure on G) and the union is finite, then at least for one $E \in \mathcal{F}$ we must have $m(V_E) > 0$; choose the largest subset E of F such that $m(V_E) > 0$. If $<E>$ denotes the subgroup generated by E, we have that $V_E = V_{<E>}$ and hence, by maximality of E, E must be a group. Moreover, since V_E is closed, symmetric and has positive measure, its powers generate G; it follows that if $f \in E$ then $gfg^{-1} = f$ for every $g \in G$, hence E is a central subgroup.

But $E \subset Z(G) \cap F \subset Z(G) \cap G_s = \bigcap_{g \in G}(gKg^{-1} \cap G_s)$ and hence $E = g^{-1}Eg \subset g^{-1}(gKg^{-1} \cap G_s)g = K \cap G_{g^{-1}s} = K_{g^{-1}s}$ for all $g \in G$, that is E is a central subgroup which is contained in the stabilizer in K of almost all $s \in S$ (since G is essentially transitive on S). Furthermore, because of the choice of E, we have that if $E \subsetneq E'$ then $m(V_{E'}) = 0$, hence $V'_E = V_E \setminus \bigcup_{E \subsetneq E'} V_{E'}$ is a symmetric set of positive measure with non-empty interior, such that for $g \in V'_E$ we have that $E = gKg^{-1} \cap G_s$ and hence $E = K_s$.

Repeating the above argument, we have hence shown that, given any $s \in S$ such that $K_s \neq \{e\}$, there exist an open neighborhood W_s and a subgroup $E_s \subset Z(G)$ such that $E_s \subset K_t$ for almost all $t \in S$ and that for almost all $g \in W_s$, $K_{gs} = E_s$. From this we see that there is a fixed $E \subset Z(G)$ such that $\{t \in S\colon K_t \neq E\} \cap \{W_s s\}$ has measure zero for almost every $s \in S$. Because G is second countable and acts essentially transitively on S we have that for almost every $s \in S$, $K_s = E$. $\square$

4. Measurable cocycles. Let G be a locally compact group and S be a Borel G-space. Recall that a Borel map $\alpha\colon S \times G \to H$ into a locally compact group is a measurable cocycle if, for $g, h \in G$, $\alpha(s, gh) = \alpha(s, g)\alpha(gs, h)$ for almost every $s \in S$. Two cocycles α, β are said to be equivalent, $(\alpha \sim \beta)$, if there is a measurable map $\theta\colon S \to H$ such that, if $g \in G$, $\theta(s)^{-1}\alpha(s, g)\theta(gs) = \beta(s, g)$, for almost every $s \in S$. Let $F(S, H)$ denote the space of measurable functions from S into H. Then, given the cocycle $\alpha\colon S \times G \to H$, we can define a "twisted" G-action on $F(S, H)$ as $(g \cdot f)(s) = f(gs)\alpha(s, g)^{-1}$, for almost every $s \in S$. A function $f \in F(S, H)$ is said to be α-invariant if for every $g \in G$, $f(gs) = f(s)\alpha(s, g)$ for almost every $s \in S$. It is easy to prove that there exists an α-invariant function in $F(S, H)$ if and only if α is equivalent to the trivial cocycle, [Z3, 4.2.18]. Also the following much stronger result holds true; we denote here by α_Γ the restriction of the measurable cocycle α to the action of Γ.

Theorem 4.1 [I1, Theorem 3.3]**.** *Let G be a connected semisimple Lie group with finite center and no compact factors, H an algebraic group, (S, μ) a G-space with finite invariant measure and $\alpha\colon S \times G \to H$ a measurable cocycle such that α_Γ is equivalent to the trivial cocycle. Then every α_Γ-invariant function $f \in F(S, H)$ is α-invariant.*

The following Proposition will conclude the proof of our main result.

Proposition 4.2. *Let G, Γ be as in Proposition 2.1, (X_i, μ_i), $i = 1, 2$, be a G-space with a finite invariant measure and let (Y_i, ν_i) be a factor of X_i such that $Y_i \simeq X_i/A_i$, where A_i is an algebraic group whose action on X_i commutes with the G_i-action. Assume that $\varphi\colon X_1 \to X_2$ is a measure preserving Γ-map which commutes with the factor maps $p_i\colon X_i \to Y_i$ and covers a G-map $\varphi_0\colon Y_1 \to Y_2$. Then φ is a G-map.*

The following result is just a reformulation of the last Proposition in terms of principal bundles.

Corollary 4.3. *Let G, Γ be as in Proposition 2.1 and let H be an algebraic group. Let $P \to M$ be a principal H-bundle over a manifold M such that $G \subseteq Aut(P)$ and G preserves a probability measure on P. Let $\varphi\colon P \to P$ be a measure preserving Γ-map which preserves fibers and covers a G-map $\varphi_0\colon M \to M$. Then φ is a G-map.*

Remark. It should be noted that if there is a G-invariant probability measure on M and H is compact, then there is a G-invariant probability measure on P. Thus, for instance, if $P \to M$ is the frame bundle, the hypotheses of Corollary 4.3 are verified if G preserves a probability measure on M and a measurable Riemannian metric on M. On the other hand, if we exclude the case in which there are locally closed G-orbits in P with compact stabilizer, we need only the existence of a quasi-invariant measure on P, not necessarily finite. In fact Theorem 4.1, and hence Proposition 4.2, hold true under these less restrictive hypotheses, (see [I1] for details).

Proof of Corollary 4.3. We can choose a measurable section $s\colon M \to P$ and measurably trivialize the bundle, $P \simeq M \times H$, (note that different sections give equivalent trivializations). We can then apply Proposition 4.2 to $M \times H \to M = Y_1 = Y_2$. $\square$

Proof of Proposition 4.2. Identify X_i with $Y_i \times A_i$. We have an action of A_i on $Y_i \times A_i$ on the right, $(y_i, a_i)a_i' = (y_i, a_i a_i')$ and a G-action on $Y_i \times A_i$ which commutes with the A_i-action and will be given by a measurable cocycle; in other words there exists a measurable cocycle $\alpha_i\colon Y_i \times G \to A_i$ such that for every $g \in G$, $a_i \in A_i$, $g(y_i, a_i) = (gy_i, a_i\alpha_i(y_i, g))$ for almost every $y_i \in Y_i$. Since $\varphi\colon X_1 \to X_2$ has the property that $p_2 \circ \varphi = \varphi_0 \circ p_1$, its expression, after the identification $X_i \simeq Y_i \times A_i$, becomes $\varphi(y_1, a_1) = (\varphi_0(y_1), \varphi_1(y_1, a_1))$, where $\varphi_1 \in F(Y_1 \times A_1, A_2)$. Since φ is a Γ-map, we have that for every $\gamma \in \Gamma$, a.e. $y_1 \in Y_1$

$$\begin{aligned}(\varphi_0(\gamma y_1), \varphi_1(\gamma(y_1, a_1))) = \varphi(\gamma y_1, a_1\alpha_1(y_1, \gamma)) = \varphi(\gamma(y_1, a_1)) = \gamma\varphi(y_1, a_1) = \\ = \gamma(\varphi_0(y_1), \varphi_1(y_1, a_1)) = (\gamma\varphi_0(y_1), \varphi_1(y_1, a)\alpha_2(\varphi_0(y_1), \gamma))\end{aligned}$$

which shows that the map $\varphi_1 \in F(Y_1 \times A_1, A_2)$ is β_Γ-invariant, where $\beta\colon (Y_1 \times A_1) \times G \to A_2$ is the measurable cocycle defined as $\beta((y_1, a_1), g) = \alpha_2(\varphi_0(y_1), g)$ and $\beta_\Gamma = \beta|_{(Y_1 \times A_1) \times \Gamma}$. We know already that φ_0 commutes with the G-action and to complete the proof it will be enough to prove that φ_1 is β-invariant. But this follows from Theorem 4.1, after observing that, since φ_1 is β_Γ-invariant, β_Γ is equivalent to the trivial cocycle. $\square$

Example. We want to show now with an example how some hypotheses of "discreteness" on the spectrum is necessary.

Let G, Γ be as above and let $\sigma\colon G \to \mathcal{U}(\mathcal{H})$ be a unitary representation of G which does not have purely atomic spectrum, that is a unitary representation that cannot be decomposed as direct sum of unitary irreducible representations. Since G is semisimple, hence of type I, there is a unique decomposition of σ as direct integral of irreducible representations; however, if Γ is a lattice in G, Γ is not of type I or equivalently the decomposition of $\sigma|_\Gamma$ as direct integral of irreducible representations need not be unique any more, thus implying that $(\sigma_\Gamma, \sigma_\Gamma) \supseteq (\sigma, \sigma)$.

We want to describe now the construction of an essentially free ergodic G-space associated to $\sigma\colon G \to \mathcal{U}(\mathcal{H})$ (known in the literature as Gaussian space; for a more detailed description see [Ku] or [Z3, 5.2.13]) in such a way that every $B \in (\sigma_\Gamma, \sigma_\Gamma)$ which is not in (σ, σ) will correspond to a measure preserving Γ-map $\varphi_B\colon X \to X$ which is not a G-map.

Let (X, μ) be the (countable) infinite product of copies of $(\mathbb{R}, \nu)$, where $d\nu = \dfrac{1}{2\pi}e^{-x^2/2}dx$ is the Gaussian measure. If $\{e_i\}$ is an orthonormal basis of $\mathcal{H}$, let us consider the map $T\colon \mathcal{H} \to L^2(X)$ defined as $T(e_i) = p_i$, where $p_i\colon X \to \mathbb{R}$ is the projection onto the i-th component. T is an isometry onto $T(\mathcal{H})$ and, for every unitary operator $U\colon \mathcal{H} \to \mathcal{H}$, we can define a measure preserving map $\varphi_U\colon X \to X$ such that the diagram

$$\begin{tikzcd} \mathcal{H} \arrow[r, "T"] \arrow[d, "U"'] & L^2(X) \arrow[d, "\varphi_U^*"] \\ \mathcal{H} \arrow[r, "T"] & L^2(X) \end{tikzcd}$$

commutes. Since $\sigma\colon G \to \mathcal{U}(\mathcal{H})$ is unitary we can hence define a G-action on X which will be measure preserving, essentially free and ergodic and such that, if we denote by π the representation of G on $L^2(X)$, $T \in (\sigma, \pi)$. To conclude this brief description we need only to observe that two unitary operators $U, V \colon \mathcal{H} \to \mathcal{H}$ commute if and only if the corresponding maps $\varphi_U, \varphi_V \colon X \to X$ commute and that, since σ does not have purely atomic spectrum, then the representation π on $L^2(X)$ does not have purely atomic spectrum.

We want to observe, however, that such Gaussian action will not be measurably equivalent to an action on a compact manifold, as (by construction) the former will have elements which act with infinite entropy and the latter cannot have this property.

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