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Research Description

Quantifying residual finiteness

An element $\gamma \in \Gamma$ is *detected* by a finite quotient Q of Γ if there exists a homomorphism $\phi : \Gamma \to Q$ such that $\gamma \notin \ker \phi$. How small a finite quotient do we need to detect a given element? Viewing groups through this question unveils connections between geometry, the theory of finite groups, and the theory of infinite groups. Let us phrase this question more precisely. A group is *residually finite* if any nontrivial element in the group may be detected in some finite quotient. Let Γ be a residually finite group generated by a finite set $\mathfrak{X}$. Let $B_{\Gamma,\mathfrak{X}}(n)$ denote the metric n-ball in a group Γ with respect to the word metric $\|\cdot\|_{\Gamma,\mathfrak{X}}$. We study the following question:

Question 1. *How large of a group do we need to approximate elements in $B_{\Gamma,\mathfrak{X}}(n)$? That is, what is the smallest integer $F^{\lhd}_{\Gamma,\mathfrak{X}}(n)$ such that each nontrivial element in $B_{\Gamma,\mathfrak{X}}(n)$ is detectable by a finite group of cardinality no greater than $F^{\lhd}_{\Gamma,\mathfrak{X}}(n)$?*

There are a number of natural ways to view the function $\mathrm{F}^{\lhd}_{\Gamma,\mathfrak{X}}$. In geometry, it measures the interaction of the word metric and the profinite metric (i.e. how efficient is the embedding of a group into its profinite completion). In decision theory, this function quantifies how well the word problem may be solved through finite quotients—sometimes this is the only known way to tackle the word problem. In topology, this function measures how large a finite cover must be to lift a closed loop to a non-closed loop (this gives many pretty pictures). To get a handle on the function, we have computed the residual finiteness growth for many concrete examples, a chunk of which is evidenced by the following table.[1] Keep in mind that large values for $\mathrm{F}^{\lhd}_{\Gamma,X}$ indicate that the group is not well approximated by finite quotients.

group	upper bound	lower bound
finitely generated (fg) infinite linear groups	polynomial in n	$\log(n)$
higher rank arithmetic group G	$n^{\dim G}$	$n^{\dim \mathrm{G}}$
fg infinite groups	none	$\log\log(n)$
fg nonabelian free groups	n^3	$n^{1/3}$
fg infinite virtually nilpotent groups	polynomial in $\log(n)$	$\log(n)$
the first Grigorchuk group	2^n	2^n
lamplighter groups	polynomial in n	$n^{1/2}$

The methods used in achieving each of the estimates are very different from each other. Moreover, residual finiteness growth lives within a wide range. Relating group properties (nilpotence, solvability, linearity, etc.) to certain residual finiteness growth ranges is a major thrust of this work and much progress has been made in this direction. Finding more connections such as these and interesting combinatorial properties to quantify are things that I am very excited about. I hope that attending this conference will give me more tools and properties to aid in this endeavor.

[1] much of this work was done in collaboration with Tasho Kaletha, Ben McReynolds, and Brandon Seward.