Research description for Ventotene 2013

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I'm very interested in CAT(0) cube complexes and the groups that act on them. A typical way to see that a group G acts on a CAT(0) cube complex is to find a finite collection of "immersed walls" in a presentation complex (or similar) X of G, and hope that for each such immersed wall $W \to X$, the image $\overline{W}$ of the lift $\widetilde{W} \to \widetilde{X}$ of the universal cover of W to the universal cover of X is a *wall*. Here, this means that $\widetilde{X} - \overline{W}$ has two components, each containing vertices arbitrarily far from $\overline{W} \cap \widetilde{X}^1$ (as measured by the usual graph metric on the 1-skeleton of $\widetilde{X}$). At this point, a construction of Sageev yields an action of G on a CAT(0) cube complex.

If G is word-hyperbolic, then any such G-cube complex obtained from a G-finite collection of walls in $\widetilde{X}$ will be G-cocompact, provided each wall has quasiconvex stabilizer in G. If in addition there are enough walls to "cut" every axis in $\widetilde{X}^1$, then the action on the cube complex is proper.

Recently, Dani Wise and I have put this into practice in the situation where G is a sufficiently nice ascending HNN extension of a finitely generated free group. More precisely:

Theorem 1 (H.-Wise 2013)**.** *Let $\Phi : F \to F$ be an injective endomorphism of the finite-rank free group F. Suppose that $G = F*_\Phi$ is word-hyperbolic and that Φ is irreducible. Then G acts freely and cocompactly on a CAT(0) cube complex.*

The motivating case is that in which G is (f.g. free)-by-cyclic, i.e. Φ is an automorphism. In this situation, the same techniques seem to yield a geometric G-action on a cube complex *even in the absence of the hypothesis that Φ is irreducible*, although some things remain to be sorted out in this case. It is very interesting to wonder to what extent the hypothesis of hyperbolicity can be relaxed. In such a setting, one cannot expect cocompactness, but it is plausible that the same construction of immersed walls in the mapping torus of Φ will yield enough walls to guarantee a free action on a (possibly infinite-dimensional?) CAT(0) cube complex.