Ergodic theory beyond amenable groups

September 13, 2013

Analytic and geometric group theory, Ventotene

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Based on joint work with Lewis Bowen

Plan

- Talk I : Amenable groups, coarse analysis and classical ergodic theory

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- Talk II : Generalizing the ergodic theory of amenable groups

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- Talk III : Ergodic theory from amenable to non-amenable groups

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- Do these averages converge ? and if so, what is their limit ?

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- For the fraction of visiting times to converge to $\mathrm{vol}(D)/\mathrm{vol}(M)$, it is therefore necessary that invariant sets be null or co-null.
- Flows satisfying this condition are called ergodic flows.

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- More generally, for an arbitrary flow, the limit is the orthogonal projection $\mathcal{E}_I f$ on the space of vectors invariant under the flow.
- For the proof, von-Neumann utilized his recently established spectral theorem.

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• and finally that the span of $\{a_s h - h\, ;\, s \in \mathbb{R}, h \in \mathcal{H}\}$ is dense in the orthogonal complement of the space of invariants.

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- We will briefly recall some of these arguments, but first let us introduce the general set-up of ergodic theorems.

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- Mean ergodic theorem for amenable groups. If $B_t \subset$ are asymptotically invariant, the limit is in fact $\int_X f d\mu$, whenever the action is ergodic, by applying Riesz's argument.

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- An extreme possibility is that for every continuous function f and for every point $x \in X$, $\beta_t f(x)$ converge uniformly to a limit $m(f)$, which is independent of x.
- This is the case if and only if Γ has a unique invariant probability measure on X.

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- Calling upon Riesz's argument again, we can at least establish

Pointwise convergence on a dense subspace

- Consider the space

$$\mathcal{K} = \text{span}\{f = \pi(g)h - h \,;\, h \in L^{\infty}(X)\,,\, g \in \Gamma\}\ ,$$

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• $\mathcal{K} + \mathcal{I}$ is dense in $L^2(X)$. Indeed, every $u \in L^2(X)$ which integrates to zero against every function in $\mathcal{K}$ is a Γ-invariant function, since L^∞ is norm-dense in every L^p.

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• But then u is measurable w.r.t. the σ-algebra of Γ-invariant functions, and if it integrates to zero against the characteristic function of every Γ-invariant set, then necessarily $u = 0$.

• Let $f = \pi(g)h - h$, and then for almost every $x \in X$

$$|\pi(\beta_t)f(x)| = \left|(\pi(\beta_t * \delta_g - \beta_t)h(x)\right| \leq \frac{2\,\|h\|_{L^\infty(X)}}{|B_t|}\,|B_t g \Delta B_t| \to 0\ .$$

Thus $\pi(\beta_t)f(x) \to \int_X f dm$ almost everywhere for every f in the dense subspace $\mathcal{K} \oplus \mathcal{I}$ of $L^2(X)$.

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- which measures the maximum deviation of the time averages from the expected value, namely the space average.

Towards the proof of the pointwise ergodic theorem

- The maximal inequality : $\mu\left\{x \in X\,;\, Mf(x) > \varepsilon\right\} \leq \frac{C}{\varepsilon}\left\|f\right\|_{L^1}$

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- Can we establish the maximal inequality by exploiting coarse analytic properties of asymptotically invariant sets ?
- Yes, but we must assume more about our Følner sequence : asymptotic invariance alone is NOT sufficient.

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- The maximal inequality : $\mu\{x \in X\,;\, Mf(x) > \varepsilon\} \leq \frac{C}{\varepsilon}\|f\|_{L^1}$
- is sufficient to complete the proof of the pointwise ergodic theorem.
- The argument uses the dense subspace on which pointwise convergence holds, and an obvious norm-approximation argument, utilizing the maximal inequality (this is an instance of the classical Banach principle).
- Can we establish the maximal inequality by exploiting coarse analytic properties of asymptotically invariant sets ?
- Yes, but we must assume more about our Følner sequence : asymptotic invariance alone is NOT sufficient.
- Before proceeding, we utilize Følner sequences to make one further important reduction.

Asymptotic invariance and the transference principle

- Transference principle. (Wiener, Calderon, Coifman-Weiss.....) Suppose B_t are balls w.r.t. a left-invariant metric. Assume β_t, $0 \leq t \leq r$ satisfy the maximal inequality for convolutions on the group, namely

$$\left| \left\{ g \in \Gamma \, ; \, \sup_{0<t\leq r} \left| F * \beta_t(g) \right| > \varepsilon \right\} \right| \leq \frac{C}{\varepsilon} \left\| F \right\|_1$$

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- for any measure preserving action of Γ on a σ-finite measure space X. Here $R > r$ is any positive number.
- In particular, if $|B_{R+r}| / |B_R| \le C$, for all $R \ge r > 1$ (say) and β_t, $0 < t < \infty$ satisfies the maximal inequality for right convolutions on the group, then β_t satisfies the maximal inequality in any measure-preserving action.

The volume doubling condition

- A sequence of balls as above has the doubling volume property if it satisfies $|B_{2t}| \leq C\,|B_t|$. This obviously holds if the volume growth is exactly polynomial : $|B_t| = Ct^d$ for a fixed exponent d.

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- Volume doubling is the typical behavior for balls on nilpotent groups, which in fact usually satisfy the stronger property that $|B_t|\,/t^d$ converges to a finite positive limit as the radius $t \to \infty$, for a fixed exponent d (Guivarc'h, Pansu, Tessera, Breuillard....)

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- Thus by our discussion so far, the proof of the pointwise ergodic theorem (including Birkhoff, Wiener, Calderon.....) for averages on the balls B_t will be complete if they satisfy the maximal inequality for convolutions on the group.

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- Wiener's covering argument. Assume d is an invariant metric on Γ, which satisfies the doubling volume condition.

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- Then every finite family of balls $\{B_i, i \in I\}$ in Γ contains a subfamily $\{B_j, j \in J \subset I\}$ of disjoint balls whose total size $|\cup_{j \in J} B_j|$, is at least $\delta \cdot |\cup_{i \in I} B_i|$, where $\delta = \delta(\Gamma) > 0$.

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- Proof. The volume doubling condition obviously implies that $|B_t| \geq \delta\, |B_{3t}|$. I being finite, choose one of the balls in the family which has maximal radius, and label it B_{i_1}. Consider now the subfamily $\{B_{i'}\, ;\, i' \in I' \subset I\}$ of all balls intersecting B_{i_1}.

- The union of all the balls $B_{i'}$, $i' \in I'$ is contained in a ball of radius at most three times that of B_{i_1}, with the same center.

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- We therefore put the index i_1 in J, and apply the same argument again to the family $\{B_i\,;\, i \in I \setminus I'\}$, which consists only of balls disjoint from B_{i_1}.
- Proceeding finitely many times, we obtain a disjoint sequence of balls whose total volume occupies at least a fraction δ of the total volume $|\cup_{i\in I} B_i|$.

Proof of the maximal inequality

- We finally show that for an invariant metric on G with balls satisfying the doubling volume condition, the ball averages β_t satisfy
$$\left|\left\{g \in \Gamma\,;\, \sup_{0<t\leq r} |F * \beta_t(g)| > \varepsilon\right\}\right| \leq \frac{C}{\varepsilon} \|F\|_1$$
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- The convolution operators are given by
$$F * \beta_t(g) = \frac{1}{|B_t|}\sum_{h\in B_t} F(gh) = \frac{1}{|B_t|}\sum_{y\in B_t(g)} F(y)$$
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- We assume $F \ge 0$, denote $F^*_\beta = \sup_{t>0} F * \beta_t(g)$, let $U_\varepsilon = \{g \in G\,;\ F^*_\beta(g) > \varepsilon\}$, and $W \subset U_\varepsilon$ finite.

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- for each $w \in W$ there is a ball $B_{t_w}(w) = wB_{t_w}(e)$ with
$$|B_{t_w}(e)| = |B_{t_w}(w)| < \frac{1}{\varepsilon}\sum_{y\in B_{t_w}(w)} F(y)$$

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- utilizing disjointness of the subfamily, we have

$$|W| \leq |\cup_{i \in I} B_i| \leq \frac{1}{\delta(\Gamma)} |\cup_{j \in J} B_j| \leq$$

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- Taking the supremum over all finite sets contained in U_ε we conclude that the same estimate holds for the size of the set U_ε and this concludes the proof of the maximal inequality.

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Every amenable group has a tempered Følner sequence !
- Regular or tempered Følner sequences satisfy a maximal inequality based on a suitable covering argument, and thus by transference satisfy also the pointwise ergodic theorem (Tempelman, Shulman, E. Linsdenstrauss, B. Weiss...).

Asymptotically invariant sequences in ergodic theory

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- Shannon-McMillan-Breiman convergence theorem (1950's) in classical entropy theory, and its further development in Ornstein-Weiss entropy theory for general amenable groups (1980's)

- The correspondence principle between sets of positive density in the integer lattice $\mathbb{Z}^d$ and ergodic systems in Multiple recurrence theory (Furstenberg, Katznelson, Weiss, Bergelson.... 1970's-80's).