# Ergodic theory beyond amenable groups

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Analytic and geometric group theory, Ventotene

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Based on joint work with Lewis Bowen

# Plan

- Talk I : Amenable groups, coarse analysis and classical ergodic theory

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- Talk II : Generalizing the ergodic theory of amenable groups

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- Talk III : From amenable to non-amenable groups

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- Using these finite sets we defined a sequence of averaging operators $\beta_t f(x) = \frac{1}{|F_t \cdot x|} \sum_{y \in F_t \cdot x} f(y)$ (assuming now that the action is essentially free)
- using asymptotic invariance we showed that the averages converge in the mean, and using doubling : $\left|F_t^{-1} F_t\right| \le C\,|F_t|$, or regularity : $\left|\cup_{t \le r} F_t^{-1} F_r\right| \le C\,|F_t|$ or temperedness : $\left|\cup_{m \le n} F_m^{-1} F_{n+1}\right| \le C\,|F_{n+1}|$, they satisfy a covering argument, a maximal inequality and a pointwise ergodic theorem.

# From measure-preserving group actions to measured equivalence relations

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• A Borel map $\phi : X \to X$ is called an inner automorphism of $\mathcal{R}$ if it is invertible with Borel inverse and its graph is contained in $\mathcal{R}$.

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$$(\mu \times c)(W) = \int_{x \in X} |\{y \in X\,;\,(x,y) \in W\}|\, d\mu(x)$$

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- Equivalently if $K : \mathcal{R} \to \mathbb{R}$ is a function then

$$\int_{y \in X} \sum_{z \in X} K(y,z)\, d\mu(y) = \int_{z \in X} \sum_{y \in X} K(y,z)\, d\mu(z)$$

# Relation-invariant sets and ergodicity

• A Borel set $E \subset X$ is $\mathcal{R}$-invariant if $(E \times X) \cap \mathcal{R} = E \times E$. This means that $E$ consists of entire equivalence classes : if $x \in E$ and $(x, y) \in \mathcal{R}$ then $y \in E$.

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• A set $\Phi \subset \mathrm{Inn}(\mathcal{R})$ generates $\mathcal{R}$ if for almost every pair $(x, y) \in \mathcal{R}$ (w.r.t. $\mu \times c$), there exists $\phi$ in the group generated by $\Phi$ such that $\phi(x) = y$.

# Subset functions

- A subset function for $\mathcal{R}$ is a map $\mathcal{U}$ on $X$ such that $\mathcal{U}(x) \subset \mathcal{R}(x)$ for all $x \in X$ (where $\mathcal{R}(x)$ denotes the equivalence class of $x$).

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• The union of subset functions $\{\mathcal{U}_i\}_{i \in I}$ is the subset function $\cup_{i \in I}\mathcal{U}_i$ defined by

$$(\cup_{i \in I}\mathcal{U}_i)(x) := \bigcup_{i \in I}\mathcal{U}_i(x).$$

# Asymptotic invariance

• Given a measured equivalence relation $\mathcal{R}$, a Borel family of subset functions $\mathcal{F} = \{\mathcal{F}_r\}_{r \in \mathbb{I}}$ is a family of subset functions $\mathcal{F}_r$ indexed by a set $\mathbb{I} \in \{\mathbb{N}, \mathbb{R}_+\}$ such that $\{(x, y, r) \in X \times X \times \mathbb{I} : \ y \in \mathcal{F}_r(x)\}$ is a Borel subset of $\mathcal{R} \times \mathbb{I}$.

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• We will always assume that $\cup_{s \leq r} \mathcal{F}_s(x) \subset \mathcal{R}(x)$ is finite for almost every $x \in X$ and $r \in \mathbb{I}$. As a result we also have that $\bigcup_{t \leq r} \mathcal{F}_t^{-1} \mathcal{F}_r(x)$ is finite for every $r \in \mathbb{I}$ and almost every $x \in X$.

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• Let $\mathcal{F}$ be a Borel family of subset functions, and let us use them to generalize some of the concepts of ergodic group theory.

• $\mathcal{F}$ is called asymptotically invariant (or *Følner*) if there exists a countable set $\Phi \subset \mathrm{Inn}(\mathcal{R})$ which generates $\mathcal{R}$ and

$$\lim_{n\to\infty} \frac{|\mathcal{F}_n(x)\Delta\phi(\mathcal{F}_n(x))|}{|\mathcal{F}_n(x)|} = 0$$

for all $\phi \in \Phi$, $\mu$-a.e. $x \in X$.

# Doubling, regularity, temperedness

• $\mathcal{F}$ is doubling if $\mathcal{F}$ is a monotone family (namely for $s < r$ we have $\mathcal{F}_s(x) \subset \mathcal{F}_r(x)$ a.e.) and $\mathcal{F}$ satisfies for almost every $x \in X$, every $r > 0$ and a fixed constant $C_d$

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• $\mathcal{F}$ is tempered if the index set $\mathbb{I} = \mathbb{N}$ and there is a constant $C_t$ such that for $\mu$-a.e. $x \in X$ and every $n > 0$

$$\left|\bigcup_{m \leq n} \mathcal{F}_m^{-1}\mathcal{F}_{n+1}(x)\right| \leq C_t|\mathcal{F}_{n+1}(x)|.$$

# Ergodic averages

• Averaging in measured equivalence relations. Given a family of subset functions $\mathcal{F} = \{\mathcal{F}_t\}_{n\in\mathbb{I}}$, for a function $f$ on $X$, consider the averaging operators $\mathbb{A}_t[f|\mathcal{F}]$ given by :

$$\mathbb{A}_t[f|\mathcal{F}](x) := \frac{1}{|\mathcal{F}_t(x)|} \sum_{y\in\mathcal{F}_t(x)} f(y).$$

# Statement of ergodic theorems for equivalence relations

- Theorem 1. If $\mathcal{F}$ is asymptotically invariant, then $\mathcal{F}$ is a mean ergodic family in $L^p$, $1 \leq p < \infty$. Namely, for every $f \in L^p(X, \mu)$, $\mathbb{A}[f|\mathcal{F}_r]$ converges in $L^p$-norm to $\mathbb{E}[f|\mathcal{I}(\mathcal{R})]$ as $r \to \infty$.

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- Theorem 2. If $\mathcal{F}$ is asymptotically invariant and regular then $\mathcal{F}$ is a pointwise ergodic family in $L^1$. Namely, for every $f \in L^1(X, \mu)$, $\mathbb{A}[f|\mathcal{F}_r]$ converges pointwise almost everywhere to $\mathbb{E}[f|\mathcal{I}(\mathcal{R})]$ as $r \to \infty$.

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- Theorem 2 generalizes the ergodic theorem for a regular Følner sequence on an amenable group (Birkhoff, Wiener, Calderon, Tempelman...) to general ergodic equivalence relation with an invariant probability measure.

- We simply let $\mathcal{R} = \mathcal{O}_G$ be the orbit equivalence relation of a m.p. free action of an amenable group $G$ on $(X, \mu)$, and $\{F_t\}_{t \in \mathbb{I}}$ regular Folner subsets in $G$. Then $\mathcal{F}_t(b) = \{gb\,;\, g \in F_t\}$ is asymptotically invariant and regular for the equivalence relation.

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• A subset function $\mathcal{F}$ is uniform if there are constants $C_u, a_r, b_r > 0$ (for $r \in \mathbb{I}$) such that

- $b_r \leq C_u a_r$ for every $r \in \mathbb{I}$,
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- Theorem 3. If $\mathcal{F}$ is asymptotically invariant, uniform and tempered then $\mathcal{F}$ is a pointwise ergodic family in $L^1$. Namely, for every $f \in L^1(X, \mu)$, $\mathbb{A}[f|\mathcal{F}_r]$ converges pointwise almost everywhere to $\mathbb{E}[f|\mathcal{I}(\mathcal{R})]$ as $r \to \infty$.

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- Similarly, Theorem 3 generalizes the ergodic theorem for tempered Følner sequences (Shulman, E. Lindenstrauss, B. Weiss) to general ergodic equivalence relation with an invariant probability measure.

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• Mean convergence and a dense subspace of good functions. We use a variant of Riesz's argument to show that if $\mathcal{F}$ is asymptotically invariant then it is mean ergodic, and there exists a dense subset $\mathcal{G} \subset L^1(X)$ such that for all $f \in \mathcal{G}$, $\mathbb{A}[f|\mathcal{F}_r]$ converges pointwise a.e. to $\mathbb{E}[f|\mathcal{I}(\mathcal{R})]$ as $r \to \infty$. As in the classical case, this is a consequence of asymptotic invariance only.

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• Mean convergence and a dense subspace of good functions. We use a variant of Riesz's argument to show that if $\mathcal{F}$ is asymptotically invariant then it is mean ergodic, and there exists a dense subset $\mathcal{G} \subset L^1(X)$ such that for all $f \in \mathcal{G}$, $\mathbb{A}[f|\mathcal{F}_r]$ converges pointwise a.e. to $\mathbb{E}[f|\mathcal{I}(\mathcal{R})]$ as $r \to \infty$. As in the classical case, this is a consequence of asymptotic invariance only.

• For $f \in L^1(X)$, let $\mathbb{M}[f|\mathcal{F}] = \sup_{r \in \mathbb{I}} \mathbb{A}[|f||\mathcal{F}_r]$. $\mathbb{M}[\cdot|\mathcal{F}]$ is the maximal operator associated to the family of averages $\{\mathbb{A}[\cdot|\mathcal{F}_r]\}_{r \in \mathbb{I}}$.

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• The maximal function satisfies the maximal inequality. Suppose that $\mathcal{F}$ is either regular or (asymptotically invariant, uniform and tempered). Then there exists a constant $C > 0$ such that for any $f \in L^1(X)$ and any $\varepsilon > 0$,

$$\mu\left(\{x \in X :\ \mathbb{M}[f|\mathcal{F}] > \varepsilon\}\right) \leq \frac{C||f||_1}{\varepsilon}.$$

# The simple, but important, hyperfinite case

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• This is an extreme form of the Besicovich covering property, which states that from every finite covering family of a set, it is possible to extract a subfamily covering the same set with multiplicity bounded by a fixed constant (independent of the set and the covering family !)

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• Moreover, denoting $\mathcal{R}'_n(x) = R_n(x) \setminus \mathcal{R}_{n-1}(x)$, and $\mathbb{A}[f|\mathcal{R}'_n]$ the average on $\mathcal{R}'_n(x)$, if there is a constant $C > 1$ such that

$$|\mathcal{R}_n(x) \setminus \mathcal{R}_{n-1}(x)| \geq \frac{1}{C}|\mathcal{R}_n(x)|$$

for almost every $x$ and every $n$, then $\mathbb{A}[f|\mathcal{R}'_n]$ also converges pointwise a.e. to $\mathbb{E}[f|\mathcal{R}]$ as $n \to \infty$. .

# Proof of the maximal inequality in the hyperfinite case

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- Let $\rho' : D_{n,\varepsilon} \to \mathbb{N}$ be the function $\rho'(x) = m$ if $m \le n$ is the smallest integer such that $\mathbb{A}[|f||\mathcal{R}_m](x) > \varepsilon$. (Possibly, $\mathcal{R}_{\rho'(x)}(x) \subsetneq \mathcal{R}_{\rho'(y)}(y)$).

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• Now let $\rho : D_{n,\varepsilon} \to \mathbb{N}$ be the function $\rho(x) = m$ where $m$ is the largest number $\le n$ so that there exists $x' \in D_{n,\varepsilon}$ with $m = \rho'(x')$ and $x \in \mathcal{R}_{\rho'(x')}(x')$. (Now, if $\mathcal{R}_{\rho(x)}(x) \cap \mathcal{R}_{\rho(y)}(y)$ intersect, they coincide !!)

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• Therefore each $x \in D_{n,\varepsilon}$ belongs to a unique set of the form $\mathcal{R}_{\rho(z)}(z)$, with $z \in D_{n,\varepsilon}$. Thus the cover of $D_{n,\varepsilon}$ by the sets $\mathcal{R}_{\rho(z)}(z)$ is a disjoint cover !

- Note also that by definition $\mathbb{A}[|f||\mathcal{R}_{\rho(x)}](x) > \varepsilon$ for every $x \in D_{n,\varepsilon}$. So

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- Let $K : X \times X \to \mathbb{R}$ be the function

$$K(y,z) = \frac{|f(y)|}{|\mathcal{R}_{\rho(z)}(z)|}$$

if $z \in D_{n,\varepsilon}$ and $y \in \mathcal{R}_{\rho(z)}(z)$. Let $K(y,z) = 0$ otherwise.

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- Since $\mu \times c|_{\mathcal{R}} = c \times \mu|_{\mathcal{R}}$,

$$\int_{D_{n,\varepsilon}} |f(y)|\, d\mu(y) = \int \sum_{z \in D_{n,\varepsilon}} K(y,z)\, d\mu(y)$$

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• Now suppose there is a constant $C > 1$ such that

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• Note that if $\mathbb{A}[|f||\mathcal{R}'_n](x) > \varepsilon$ then $\mathbb{A}[|f||\mathcal{R}_n](x) > \varepsilon/C$. Therefore,

$$\mu(\{x \in X :\ \mathbb{M}'[f](x) > \varepsilon\}) \leq \mu(\{x \in X :\ \mathbb{M}[f](x) > \varepsilon/C\}) \leq \frac{C\|f\|_1}{\varepsilon}.$$

This completes the proof of the maximal inequality.

# Some comments on the hyperfinite case

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• The proof in the regular and tempered cases is considerably more involved technically, and the argument above demonstrates the method in the simple special case of hyperfinite actions.

# The classical ratio ergodic theorem

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• Assume that $T : (X, \eta) \to (X, \eta)$ preserves the $\sigma$-finite measure $\eta$ and is conservative and ergodic. Then the ratio theorem states that for $f, g \in L^1(X, \eta)$ with $g > 0$, the ratios

$$\mathcal{Q}_n[f, g] := \frac{\sum_{k=0}^{n} T^k f}{\sum_{k=0}^{n} T^k g}$$

converge almost everywhere as $n \to \infty$ to the constant $\frac{\int f \, d\eta}{\int g \, d\eta}$

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• In the case of $\mathbb{Z}^d$-actions for $d > 1$ the ratio ergodic theorem is of recent vintage (Feldman 2007, Hochman 2009). It was shown by Hochman that the Besicovich property is necessary and sufficient for the validity of the ratio ergodic theorem in this case.

# Ratio ergodic theorem for non-singular equivalence relations

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• Let $(X, \eta)$ be any Borel space which is non-singular under the equivalence relation $\mathcal{R}$, with Radon-Nikodym derivative $\Delta(b', b)$.

• The ratio ergodic operators $\mathcal{Q}_n[\cdot, \cdot]$ are defined, given $f, g \in L^1(X)$ with $g > 0$, by

$$\mathcal{Q}_n[f, g](x) = \frac{\sum_{x' \in \mathcal{F}_n(x)} f(x')\Delta(x', x)}{\sum_{x' \in \mathcal{F}_n(x)} g(x')\Delta(x', x)}$$

Theorem 4 : Ratio ergodic theorem for equivalence relations. If $\mathcal{R}$ is hyperfinite, for any $f, g \in L^1(X)$ with $g > 0$, the sequence $\{\mathcal{Q}_n[f,g]\}_{n=1}^{\infty}$ converges pointwise almost everywhere.

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• It is possible to identify the limit also in the general case.

# Extensions of a measured equivalence relation

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3. for almost every $\tilde{\mathcal{R}}$-equivalence class $\tilde{\mathcal{R}}(\tilde{x}) \subset \tilde{\mathcal{R}}$, $\pi$ restricted to $\tilde{\mathcal{R}}(\tilde{x})$ is a bijection onto the $\mathcal{R}$-equivalence class $\mathcal{R}(\pi(\tilde{x}))$.

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• Suppose $\mathcal{F} = \{\mathcal{F}_r\}_{r \in \mathbb{I}}$ is a family of subset functions for $(X, \mu, \mathcal{R})$. Then we may lift this family to a family of subset function $\tilde{\mathcal{F}}$ by

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Proposition. Let $P$ be one of the properties : asymptotically invariant, uniform, doubling, regular, or tempered. If $\mathcal{F}$ has property $P$ then $\tilde{\mathcal{F}}$ also has property $P$.