Ergodic theory beyond amenable groups

September 13, 2013

Analytic and geometric group theory, Ventotene

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Based on joint work with Lewis Bowen

From amenable groups to amenable actions

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• Let us proceed to consider a natural example of a non-amenable group G where the amenable equivalence relation on $B \times X$ has a natural subrelation with an invariant measure and naturel Folner sets with the extreme Besicovich property.

The free group and its boundary

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• The boundary $\partial\mathbb{F}$ is the set of all geodesic rays emanating from the origin. Equivalently, the set of all sequences $\xi = (\xi_1, \xi_2, \ldots) \in \mathrm{S}^{\mathbb{N}}$ such that $\xi_{i+1} \neq \xi_i^{-1}$ for all $i \geq 1$. Thus $\partial\mathbb{F}$ is a subshift of finite type.

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• The probability measure ν on $\partial\mathbb{F}$ is the Markov measure satisfying for every finite sequence $t_1, \ldots, t_n$ with $t_{i+1} \neq t_i^{-1}$ for $1 \leq i < n$,

$$\nu\Big(\{(\xi_1, \xi_2, \ldots) \in \partial\mathbb{F} : \ \xi_i = t_i\,,\ 1 \leq i \leq n\}\Big) := (2r-1)^{-n+1}(2r)^{-1}.$$

Horofunctions and horospheres

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$$(t_1 \cdots t_n)\xi := (t_1, \ldots, t_{n-k}, \xi_{k+1}, \xi_{k+2}, \ldots)$$

where $t_1, \ldots, t_n \in \mathrm{S}$, $t_1 \cdots t_n$ is in reduced form and k is the largest number $\leq n$ such that $\xi_i^{-1} = t_{n+1-i}$ for all $i \leq k$.

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• A horosphere is any level set of the RN derivative, for any ξ fixed. Let H_ξ denote the horosphere (based at ξ, passing through e)

$$H_\xi = \left\{g \in \mathbb{F} :\ \frac{d\nu \circ g^{-1}}{d\nu}(\xi) = 1\right\}.$$

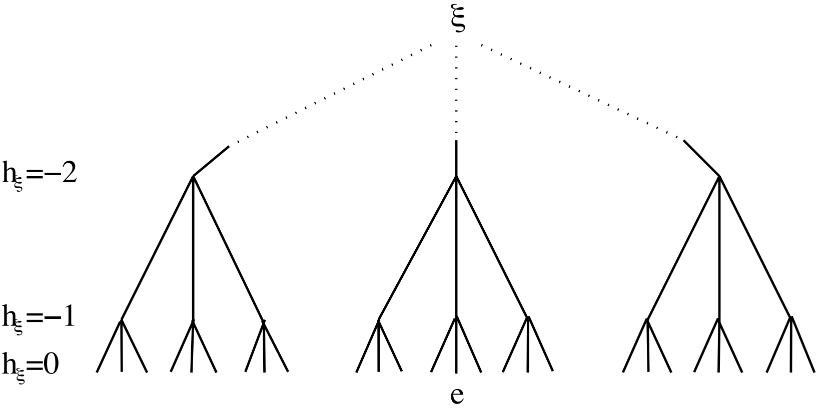

Figure : The “upper half space” model of the rank 2 free group.

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- Let $\mathcal{R}_{\partial\mathbb{F}}$ be the horospherical equivalence relation on $\partial\mathbb{F}$ where $\xi \sim \eta$ if and only if there is a $g \in \mathbb{F}$ such that $g\xi = \eta$ and $\frac{d\nu \circ g}{d\nu}(\xi) = 1$.

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• The Markov measure ν on the subshift is invariant under the synchronous tail relation.

Finite order automorphisms

• Thus we obtain the crucial fact that $\mathcal{R}_{\partial\mathbb{F}}$ is a sub-relation of the $\mathbb{F}$-orbit relation, but nevertheless, ν is an $\mathcal{R}_{\partial\mathbb{F}}$-invariant measure on the boundary !!

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• Being the synchronous tail relation, it is clear that $\mathcal{R}_{\partial\mathbb{F}}$ is an increasing union of finite equivalence relations $\mathcal{R}_{\partial\mathbb{F},n}$, namely $\mathcal{R}_{\partial\mathbb{F}}$ is hyperfinite. Just define ξ and ξ' to be $\mathcal{R}_{\partial\mathbb{F},n}$ equivalent if they coincide from the n-th place onwards.

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• Denote $\mathcal{R}_{\partial\mathbb{F},n} = \mathcal{R}_n$. The finite equivalence classes $\mathcal{R}_n(\xi)$ form a doubling Følner sequence for $\mathcal{R}$, w.r.t the countable group generated by the of inner automorphisms of the finite relations.

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• Clearly, for any $(\xi, \xi') \in \mathcal{R}_{\partial\mathbb{F}}$, there exists a map $\phi \in \mathrm{Inn}(\mathcal{R}_{\partial\mathbb{F}})$ such that $\phi(\xi) = \xi'$ and ϕ has order n for some $n < \infty$. Thus finite order inner automorphisms of $\mathcal{R}_{\partial\mathbb{F}}$ generate the relation.

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• We now define for each ξ a finite subset of the $\mathcal{R}_{\partial\mathbb{F}}$-equivalence class of ξ. We consider the set of images of ξ under the elements whose inverses lie in a horospherical ball :

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• Thus the sequence $\mathcal{B} := \{\mathcal{B}_{2n}(\xi)\}_{n=1}^{\infty}$ of horospherical balls centered at ξ in the equivalence relation is doubling, extremely Besicovich, and asymptotically invariant under the equivalence relation $\mathcal{R}_{\partial\mathbb{F}}$.

The amenable equivalence relation associated with a measure-preserving ergodic action

• Let $\mathbb{F}$ act on (X, λ) by m.p.t. , and define a hyperfinite equivalence relation $\mathcal{R}_{X\times\partial\mathbb{F}}$ on $X \times \partial\mathbb{F}$, with (x, ξ) equivalent to (x', ξ') if there exists a $g^{-1} \in H_\xi$ such that $gx = x'$ and $g\xi = \xi'$.

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• For $f \in L^1(X \times \partial\mathbb{F})$, denote by $\mathbb{E}[f|\mathcal{R}_{X\times\partial\mathbb{F}}]$ the conditional expectation of f on the σ-algebra of $\mathcal{R}_{X\times\partial\mathbb{F}}$-invariant sets.

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• Define the horospherical ball of radius n centered at (x, ξ)

$$\tilde{\mathcal{B}}_n(x, \xi) := \{(gx, g\xi) \in X \times \partial\mathbb{F} : \ g^{-1} \in H_\xi, \ |g| \leq n\}.$$

For $n \geq 0$ and $(x, \xi) \in X \times \partial\mathbb{F}$. This ball projects to $B_{2n}(\xi)$, the horospherical ball centered at ξ under the projection $X \times \partial\mathbb{F} \to \partial\mathbb{F}$.

Main convergence result for horospherical ball averages

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$$\mathbb{A}_{2n}[\tilde{\mathcal{B}}; f](x, \xi) = \frac{1}{|\tilde{\mathcal{B}}_{2n}(x, \xi)|} \sum_{(x', \xi') \in \tilde{\mathcal{B}}_{2n}(x, \xi)} f(x', \xi'). \qquad (1)$$

Then for any $f \in L^1(X \times \partial\mathbb{F})$, the sequence $\{\mathbb{A}_{2n}[\tilde{\mathcal{B}}; f]\}_{n=1}^{\infty}$ satisfies the maximal inequality, and converges pointwise a.e. and in L^1 norm to $\mathbb{E}[f|\mathcal{R}_{X \times \partial\mathbb{F}}]$.

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Proof : $\tilde{\mathcal{B}}_n(x, \xi)$ is asymptotically invariant and doubling for $\mathcal{R}_{X \times \partial\mathbb{F}}$ because they are finite subrelations in a hyperfinite relation.

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- it follows that the σ-algebra of $\mathcal{R}_{X\times\partial\mathbb{F}}$-invariant sets consists of the sets $A\times\partial\mathbb{F}$ with A in the σ-algebra of $\mathbb{F}^2$-invariant sets in X.

Ergodicity of index two subgroups

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• the limit $\mathbb{E}[f|\mathcal{R}_{X\times\partial\mathbb{F}}]$ in this case is an orthogonal projection on the two-dimensional space spanned by $1 \otimes 1$ and $h \otimes 1$, and thus different than the space average which is the orthogonal projection on the space spanned by $1 \otimes 1$.

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• The averages $\mathbb{A}_{2n}[\tilde{\mathcal{B}}; f](x, \xi)$ converge, as do the averages $\mathbb{A}'_{2n}[\tilde{\mathcal{S}}; f](x, \xi)$ supported on the difference set $\tilde{\mathcal{S}}_{2n}(x, \xi) = \tilde{\mathcal{B}}_{2n}(x, \xi) \setminus \tilde{\mathcal{B}}_{2(n-1)}(x, \xi)$, which corresponds to the intersection of a horosphere and a sphere.

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- Thus convergence holds for the following $\mathbb{F}$-supported averages :

$$A'_{2n}f(x) = \int_{\xi \in \partial\mathbb{F}} \mathbb{A}'_{2n}[\tilde{\mathcal{S}}; f](x, \xi)\eta(\xi)d\nu(\xi)$$

$$= \int_{\xi \in \partial\mathbb{F}} \frac{1}{|\tilde{\mathcal{S}}_{2n}(x, \xi)|} \sum_{(x', \xi') \in \tilde{\mathcal{S}}_{2n}(x, \xi)} f(x')\ \eta(\xi)d\nu(\xi)$$

- Taking $\eta(\xi) = 1$ we obtain pointwise ergodic theorem for the even-radius normalized spheres σ_{2n} on the free group.

Further ergodic theorems

• Take η as the normalized characteristic function of the set of boundary points starting with a fixed word, denoted w say. Then we obtain pointwise convergence for the (non-radial) averages $\sigma_{2n}^{w} f(x)$ of averaging on all words of length $2n$ starting with the word w.

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immediately implies the strong maximal inequality in $L^p(X)$, $p > 1$ and in $L\log L(X)$ for A'_{2n}, given the same maximal inequalities for $\mathbb{A}_{2n}[\tilde{S}; f](x,\xi)$ (which are true by hyperfiniteness)

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• But the additional integration interferes with the proof of the (weak-type $(1,1)$) maximal inequality in $L^1(X)$, which remains an open problem. No maximal inequality in L^1 has ever been established for any choice of ball averages on any non-amenable group!

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• Note that these average are supported on a shell (of width 2), but are NOT uniformly distribute on the shell. This is the best result that can be achieved, even) for radial averages on the free group.

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• All of this was for free in the free group case !

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4. The Γ-action on $\partial\Gamma \times \mathbb{R}$ via $g(\xi, s) = (g\xi, s - \log \frac{d\nu_c \circ g}{d\nu_c}(\xi))$ has a σ-finite Γ-invariant measure. This extension of the boundary by the RN cocycle is the space of horospheres. It is necessary when the RN derivative does not assume discrete values.

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9. The same applies to its extension by a measure preserving action of Γ on (X, μ), namely the horospherical relation on $(X \times \partial\Gamma \times [0, T])$.

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• As noted already, this result is best possible, even in the case of radial averages on the free group.