Finite covers of 3-manifolds, virtual Thurston norms and simplicial volume

Manifolds and Groups

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Joint work with Stefan Friedl

Profinite completion

Let π be a finitely generated and residually finite group.

Let $\mathcal{N}(\pi)$ be the collection of all finite index subgroups Γ of π.

Equip each finite quotient $\pi/\Gamma, \Gamma \in \mathcal{N}(\pi)$ with the discrete topology.

The set $\prod_{\Gamma \in \mathcal{N}(\pi)} \{\pi/\Gamma\}$ is compact .

Let $i_\pi : \pi \to \prod_{\Gamma \in \mathcal{N}(\pi)} \{\pi/\Gamma\}$ given by $g \in \pi \to \{g\Gamma\}_{\Gamma \in \mathcal{N}(\pi)}$.

The profinite completion of π can be defined as the closure

$\widehat{\pi} = \overline{i_\pi(\pi)}$ in $\prod_{\Gamma \in \mathcal{N}(\pi)} \{\pi/\Gamma\}$.

$i_\pi : \pi \to \widehat{\pi}$ is injective since π is residually finite.

Profinite completion

$\widehat{\pi}$ is a compact topological group.

A subgroup $U < \widehat{\pi}$ is open if and only if it is closed and of finite index.

A subgroup $H < \widehat{\pi}$ is closed if and only if it is the intersection of all open subgroups of $\widehat{\pi}$ containing it.

Since π is finitely generated, a deep result of N. Nikolov and D. Segal states that *every finite index subgroup of $\widehat{\pi}$ is open*.

This means that $\widehat{\widehat{\pi}} = \widehat{\pi}$

In particular, there is a one to one correspondance between the normal subgroups with the same index in π and $\widehat{\pi}$:

$\Gamma \in \mathcal{N}(\pi) \to \overline{\Gamma} \in \mathcal{N}(\widehat{\pi})$, and $\overline{\Gamma} = \widehat{\Gamma}$.

Conversely $H \in \mathcal{N}(\widehat{\pi}) \to H \cap \pi \in \mathcal{N}(\pi)$.

Homomorphisms

An important consequence is :

Lemma

For any finite group G the map $\pi \to \widehat{\pi}$ induces a bijection $Hom(\widehat{\pi}, G) \to Hom(\pi, G)$.

A group homomorphism $\varphi : A \to B$ induces a continuous homomorphism $\widehat{\varphi} : \widehat{A} \to \widehat{B}$.

If A and B are finitely generated, any homorphism $\widehat{A} \to \widehat{B}$ is continuous.

If φ is an isomorphism, so is $\widehat{\varphi}$.

On the other hand, an isomorphism $\phi : \widehat{A} \to \widehat{B}$ is not necessarily induced by a homomorphism $\varphi : A \to B$.

There are isomorphisms $\widehat{\mathbb{Z}} \to \widehat{\mathbb{Z}}$ that are not induced by an automorphism of $\mathbb{Z}$.

Isomorphisms

Given A and B finitely generated, for any finite group G, an isomorphism $f : \widehat{A} \to \widehat{B}$ induces bijections

$$Hom(B, G) \leftarrow Hom(\widehat{B}, G) \xrightarrow{f^*} Hom(\widehat{A}, G) \to Hom(A, G).$$

Given $\beta \in Hom(B, G)$ denote by $\beta \circ f$ the resulting homomorphism in $Hom(B, G)$.

Groups A and B with isomorphic profinite completions have the same finite quotients.

The converse also holds :

Lemma

Two groups A and B have isomorphic profinite completions if and only if they have the same set of finite quotients.

3-manifold groups

In this talk M will be a compact orientable aspherical 3-manifold with empty or toroidal boundary.

By Perelman's geometrization Theorem $\pi_1(M)$ is residually finite.

The main question addressed in this talk is :

Questions

Which properties or invariants of M are detected by $\widehat{\pi_1(M)}$?

We call them profinite properties or invariants of M

There are non-isomorphic 3-manifolds groups with isomorphic profinite completions.

At the moment they are *Solv manifolds* (Stebe, L. Funar), or *Surface bundle with periodic monodromy, i.e Seifert fibered manifolds* (J. Hempel).

There are no hyperbolic examples known.

Rigidity

Questions

For M hyperbolic is $\pi_1(M)$ determined by its profinite completion, among 3-manifold groups ?

Thm (M. Bridson-A. Reid)

The figure 8 knot group is detected by its profinite completion, among 3-manifold groups.

Questions

Is a knot group determined by its profinite completion, among knot groups ?

True for torus knot groups

Goodness

Following Serre a group π is good if the following holds :

For any finite abelian group A and any representation $\alpha : \pi \to Aut_{\mathbb{Z}}(A)$

the inclusion $\iota : \pi \to \widehat{\pi}$ induces an isomorphism $\iota^* : H^j_\alpha(\widehat{\pi}; A) \to H^j_\alpha(\pi; A)$, for any j.

If π is good of finite cohomological dimension then $\widehat{\pi}$ is torsion free.

The proof of this theorem uses the Agol's virtual fibration theorem :

Thm (W. Cavendish)

The fundamental group of any compact aspherical 3-manifold is good.

Corollary

For a compact aspherical 3-manifolds the property of being closed is a profinite property.

Geometries

It is natural to ask whether the profinite completion detects geometric structures.

Results in this direction have been obtained by P. Zaleskii and H. Willton :

Thm (Zaleskii-Wilton)

Let M be a closed aspherical orientable 3-manifold, then $\widehat{\pi_1(M)}$ detects :

1. *whether M is hyperbolic.*
2. *whether M is Seifert fibered.*

Profinite completion distinguish hyperbolic geometry among Thurston's geometries because hyperbolic manifolds groups are residually non-ablian simple.

Thurston norm

We study now the relation between the profinite completion, and the Thurston norm of a 3-manifold.

M is still a compact, orientable, aspherical 3-manifold, with ∂M empty or an union of tori.

The complexity of a compact orientable surface F with connected components $F_1, \ldots, F_k$ is defined to be

$$\chi_-(F) := \sum_{i=1}^{d} \max\{-\chi(F_i), 0\}.$$

The Thurston norm of a cohomology class $\phi \in H^1(M;\mathbb{Z})$ is defined as

$$x_M(\phi) := \min\{\chi_-(F) \,|\, F \subset M \text{ properly embedded and dual to } \phi\}.$$

x_M extends to a seminorm on $H^1(M;\mathbb{R})$.

Regular isomorphism

Let M_1 and M_2 be two 3-manifolds such that there exists an isomorphism $f : \widehat{\pi_1(M_1)} \to \widehat{\pi_1(M_2)}$.

Such an isomorphism induces an isomorphism $H_1(\widehat{M_1; \mathbb{Z}}) \to H_1(\widehat{M_2; \mathbb{Z}})$.

Thus $H_1(M_1; \mathbb{Z})$ and $H_1(M_2; \mathbb{Z})$ are abstractly isomorphic.

In general the isomorphism $H_1(\widehat{M_1; \mathbb{Z}}) \to H_1(\widehat{M_2; \mathbb{Z}})$ is not induced by an isomorphism $H_1(M_1; \mathbb{Z}) \to H_1(M_2; \mathbb{Z})$.

To compare the Thurston norms of M_1 and M_2, let introduce the following definition :

An isomorphism $f : \widehat{\pi_1(M_1)} \to \widehat{\pi_1(M_2)}$ is **regular** if the induced isomorphism $H_1(\widehat{M_1; \mathbb{Z}}) \to H_1(\widehat{M_2; \mathbb{Z}})$ is induced by an isomorphism $f^* : H_1(M_1; \mathbb{Z}) \to H_1(M_2; \mathbb{Z})$.

Fibered class

A class $\phi \in H^1(N;\mathbb{R})$ is called fibered if there is a fibration $p : M \to S^1$ such that $\phi = p_* : \pi_1(M) \to \mathbb{Z}$.

Thm (B-Friedl)

Let M_1 and M_2 be two aspherical 3-manifolds with empty or toroidal boundary. If $f : \widehat{\pi_1(M_1)} \to \widehat{\pi_1(M_2)}$ is a regular isomorphism, then :

(1) for any class $\phi \in H^1(M_2;\mathbb{R})$, $x_{M_2}(\phi) = x_{M_1}(f^*\phi)$.

(2) $\phi \in H^1(M_2;\mathbb{R})$ is fibered $\iff$ $f^*\phi \in H^1(M_1;\mathbb{R})$ is fibered.

When M_1 is fibered over S^1 this result has been obtained by A. Reid and M. Bridson, by a different method.

Knots

When M_1 and M_2 have $b_1 = 1$, we do not need the regular assumption. In particular for knot spaces we obtain :

Thm (B-Friedl)

Let K_1 and K_2 be two knots in S^3. If $\pi_1(\widehat{S^3 \setminus} K_1) \cong \pi_1(\widehat{S^3 \setminus} K_2)$, then :

(1) K_1 and K_2 have the same Seifert genus.

(2) K_1 is fibered iff K_2 is fibered

(3) If Δ_{K_1} is not a product of cyclotomic polynomials, $\Delta_{K_1} = \Delta_{K_2}$

(4) If K_1 is a torus knot, $K_1 = K_2$.

(5) If K_1 and K_2 are cyclically commensurable either $K_1 = K_2$ or

Δ_{K_1} and Δ_{K_2} are product of cyclotomic polynomials.

Fiberedness and Thurston norm

For a polynomial $f(t) = \sum_{k=r}^{s} a_k t^k \in \mathbb{F}[t^{\pm 1}$ with $a_r \neq 0$ and $a_s \neq 0$ define $\deg(f(t)) = s - r$. For the zero polynomial set $\deg(0) := +\infty$.

Thm (Friedl-Vidussi, Friedl-Nagel)

Let M be a compact, aspherical, orientable 3-manifold with empty or toroidal boundary and $\phi \neq 0 \in H^1(M;\mathbb{Z})$:

(1) The class ϕ is fibered $\Leftrightarrow \Delta^{\alpha}_{M,\phi,1} \neq 0$ for all primes p and all representations $\alpha : \pi_1(M) \to GL(k, \mathbb{F}_p)$.

(2) There exists a prime p and a representation $\alpha : \pi_1(M) \to GL(k, \mathbb{F}_p)$ such that

$$x_M(\phi) = \max\left\{0, \frac{1}{k}\left(-\deg\left(\Delta^{\alpha}_{M,\phi,0}\right) + \deg\left(\Delta^{\alpha}_{M,\phi,1}\right) - \deg\left(\Delta^{\alpha}_{M,\phi,2}\right)\right)\right\}.$$

The proof is building on the work of Agol, Przytycki-Wise and Wise.

Pro-virtual abelian completion

The pro-virtually abelian completion $\widehat{\pi}_{va}$ of a group π is defined the same way as the profinite completion $\widehat{\pi}$ using virtually abelian quotients instead of finite quotients.

From the definition there exists a continuous epimorphism $\widehat{\pi}_{va} \to \widehat{\pi}$.

Co-virtually abelian normal subgroups of $\widehat{\pi}_{va}$ are open.

Any homomorphism between two finitely generated pro-virtually abelian groups is continuous.

An isomorphism between the pro-virtual abelian completions of two groups induces regular isomorphisms between the profinite completions of their corresponding finite index subroups.

Pro-virtual abelian completion and Thurston norm

A consequence is :

Proposition

The pro-virtually abelian completion $\widehat{\pi_1(M)}_{va}$ determines the Thurston norms of the finite coverings of M

For a compact orientable aspherical 3-manifold M with empty or toroidal boundary, let define :

$Vol(M) =$ sum of volumes of hyperbolic pieces in the geometric decomposition of M.

The Proposition above is the first step to show :

Thm (B-Friedl)

The vanishing of $Vol(M)$ is a pro-virtually abelian property.

Volume conjecture

The second step consists in showing that the virtual Thurston norms determines whether $Vol(M)$ vanishes or not.

Because of Perelman's geometrization theorem, this is equivalent to decide whether M is a graph manifold or not.

A strong conjecture (N. Bergeron-A. Venkatesh, W. Lück) asserts that the growth of the torsion part of the homology of the finite covers of M determines $Vol(M)$.

In this case $Vol(M)$ would be a profinite invariant.

A positive answer to this volume conjecture would answer the following question :

Question (Finiteness)

Are they only finitely many hyperbolic 3-manifolds N with $\widehat{\pi_1(N)} \cong \widehat{\pi_1(M)}$ *?*

Virtual Thurston norms

We study to which degree does the virtual Thurston norms determine the type of the JSJ-decomposition of the 3-manifold.

Given a compact orientable aspherical 3-manifold M with empty or toroidal boundary, define :

- $b_1(N) = \dim_{\mathbb{R}}(H_1(N;\mathbb{R}))$,
- $k(N) = \dim_{\mathbb{R}}(\ker(x_N))$,
- $r(N) = \frac{k(N)}{b_1(N)}$ if $b_1(N) > 0$ and 0 otherwise.

Furthermore

- $\mathcal{C}(N) = \{$the class of all finite regular covers $\tilde{N}$ of $N\}$,
- $\widehat{r}(N) = \sup_{\widetilde{N} \in \mathcal{C}(N)} r(\widetilde{N})$.

The following proposition is well-known to the experts.

Proposition

M is hyperbolic if and only if $\widehat{r}(N) = 0$.

Virtual Thurston norms

It is harder to distinguish manifolds with $Vol(M) \neq 0$ (i.e. with at least one hyperbolic JSJ-component) from graph manifolds.

We need to consider a wider class of finite coverings. A covering $f\colon \widehat{N} \to N$ is *subregular* if the covering f can be written as a composition of coverings $f_i\colon N_i \to N_{i-1}$, $i = 1, \ldots, k$ with $N_k = \widehat{N}$ and $N_0 = N$, such that each f_i is regular.

For a 3-manifold M let define :

- $\mathcal{C}^{sub}(N) =$ the class of all finite subregular covers $\widehat{M}$ of M.
- $\rho(M) = \inf\limits_{\widehat{M} \in \mathcal{C}^{sub}(M)} r(\widehat{M})$.
- $\widehat{\rho}(M) = \sup_{\widetilde{M} \in \mathcal{C}(M)} \rho(\widetilde{M})$.

The invariant $\widehat{\rho}(M)$ gives a characterization of manifolds with non vanishing volume analogous to the one for hyperbolic manifolds.

Thm

Let M be a compact, connected, orientable, aspherical 3-manifold with empty or toroidal boundary.

- *If M is a graph manifold, then $\widehat{\rho}(N) = 1$.*
- *If M admits a hyperbolic piece in its JSJ-decomposition then $\widehat{\rho}(N) = 0$.*

Corollary

A 3-manifold M is :

1. *graph manifold if and only if $\widehat{\rho}(N) = 1$,*
2. *mixed manifold if and only if $\widehat{r}(N) > \widehat{\rho}(N) = 0$.*
3. *hyperbolic manifold if and only if $\widehat{r}(N) = 0$.*

Graph manifolds

The following Proposition immediately implies the case of graph manifolds

Proposition

Let M be an aspherical graph manifold. Then $\forall\ \epsilon > 0$ there exists a finite regular cover N of M such that for any finite cover $\overline{N}$ of N we have $r(\widehat{N}) > 1 - \epsilon$.

For a compact manifold N define
$c(N) = \dim_{\mathbb{R}} coker(H_1(\partial N; \mathbb{R}) \to H_1(N; \mathbb{R}))$

Lemma

(i) Let $p\colon \widetilde{N} \to N$ be a finite covering of a manifold N. Then $c(\widetilde{N}) \geq c(N)$.

(ii) For any surface Σ we have $c(S^1 \times \Sigma) = c(\Sigma)$.

The idea is to increase by finite coverings the Euler characteristic of the bases of the Seifert pieces of the JSJ-decomposition of M much more than the numbers of JSJ-tori in order that $r(N) = \frac{k(N)}{b_1(N)} \nearrow 1$.

Proposition

Let M be a graph manifold which is not Seifert fibered or covered by a torus bundle. Then given $0 < \epsilon < 1$ M is finitely covered by a manifold N such that each JSJ-piece N_v of N is a product $S^1 \times \Sigma_v$ with $c(N_v) = c(\Sigma_v) \geq \frac{1}{\epsilon}$.

The proof uses ad hoc covering construction based on the fact that 3-manifolds groups are virtually residually p-nilpotent.

Having this finite cover N in hand, fore each product piece N_v let $f_v \in H_1(N; \mathbb{Z})$ the element determined by the S^1-factor.

Let V the set of JSJ-pieces of N. Calculation of the Thurston norm for products and the additivity along JSJ-tori show :

$\forall\ \phi \in H^1(N;\mathbb{R})\ \ x_N(\phi) = \sum_{v\in V} |\phi(f_v)| \cdot \chi_-(\Sigma_v)$.

So $\ker(x_N)) = \{\phi \mid \phi(f_v) = 0\ \forall\ v \in V\}$

$k(N) \geq b_1(N) - |V|$

Mayer-Vietoris sequence corresponding to the JSJ-splitting of N gives :

$b_1(N) \geq \sum_{v\in V} \dim_{\mathbb{R}} \left(coker\{H_1(\partial N_v;\mathbb{R}) \to H_1(N_v;\mathbb{R})\}\right) > \frac{1}{\epsilon} \cdot |V|$.

Putting the last two inequalities together gives :

$1 - r(N) \leq \frac{b_1(N)-k(N)}{b_1(N)} \leq \frac{|V|}{\frac{1}{\epsilon}|V|} = \epsilon$.

$Vol(M) \neq 0$

Since the property of being aspherical and not being a graph manifold is preserved by finite cover, it suffises to show :

Thm

If M is not a graph manifold, then given any $\epsilon > 0$, there exists a finite subregular cover N of M such that $r(N) < \epsilon$. In particular $\rho(N) = 0$.

We say that a homomorphism $\phi \colon \pi \to \mathbb{Z}$ is *large* if ϕ is non-trivial and if it factors through an epimorphism from π onto a non-cyclic free group.

Proposition

Let M be a non-graph 3-manifold. Then there exists a finite subregular cover $\overline{M}$ of M with $k \geq 1$ hyperbolic JSJ-components $H_1, \ldots, H_k$, and a class $\phi \in \hom(H_1(\overline{M}; \mathbb{Z}), \mathbb{Z}) = H^1(N; \mathbb{Z})$ such that the restriction of ϕ to each H_i is large but the restriction of ϕ to $\overline{M} \setminus (H_1 \cup \cdots \cup H_k)$ is fibered.

Given $\phi \in H^1(N;\mathbb{Z}) = \hom(\pi_1(N),\mathbb{Z})$ and $n \in \mathbb{N}$, let $\phi_n\colon \pi_1(N) \to \mathbb{Z}_n$ be the composition of ϕ with the projection $\mathbb{Z} \to \mathbb{Z}_n$.

Denote by N_{ϕ_n} the corresponding (maybe not connected) cover of N.

Lemma

Let N be a 3-manifold and let $\phi\colon \pi_1(N) \to \mathbb{Z}$ be a large homomorphism such that the restriction of ϕ to all boundary-components of N is non-trivial. Then for all but finitely many primes p we have $c(N_{\phi_p}) \geq p - 1 - 2b_0(\partial N)$

For the finite subregular cover $p : \overline{M} \to M$ given by the proposition

Mayer-Vietoris sequence gives : $k(\overline{M}) \leq b_1(\overline{M}) - \sum_{i=1}^{k} c(p^{-1}(H_i))$

One uses then the Lemma above to find subregular covers N of M with arbitrarily small $r(N)$.