Betti numbers of residual towers of covers of reflection groups

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Fifteen years ago Boris Okun and I were trying to prove the Singer Conjecture for right-angled Coxeter groups (RACGs). One of our ideas was to use Lück's Approximation Theorem. We had a specific tower of covers for which we could compute the Betti numbers. The only problem was we got the wrong answer: instead of proving vanishing results we were getting that all ℓ^2-Betti numbers not in the top or bottom degree were nonzero! This was the wrong answer for hyperbolic space, $\mathbf{H}^n$, $n > 2$. Eventually we realized that the reason was that the subgroups were not normal.

- Suppose W is a group generated by reflections across the facets of a right-angled fundamental polytope P on $\mathbb{H}^n$. Then W is a RACG and $\mathbb{H}^n/W = P$.
- More generally, given a RACG W, there is a Davis-Moussong complex Σ, tessellated by copies of a fundamental chamber P, so that W acts as a reflection group on Σ. Σ has the structure of a CAT(0) cube complex dual to the tessellation by chambers. (So, Σ is contractible.) If Σ is a n-manifold, then P is "polytope-like". We may as well assume P is a simple polytope.

- For $X \subset \Sigma$, let $\mathcal{H}(X)$ be the set of half-spaces of Σ containing X and put

$$\mathrm{Conv}(X) := \bigcap_{H \in \mathcal{H}(X)} H.$$

- If X is finite, then $\mathrm{Conv}(X) = P'$ is polytope-like and the subgroup $W' < W$ generated by reflections across the facets of P' is a RACG.
- It follows that W is residually finite. (If we regard W as the set of centers of chambers in Σ and X is a finite subset of W, then X embeds in $W/W' \subset \Sigma/W' = P'$.)

Residual reflection towers

So, we can find

$$P = P_0 \subset P_1 \subset \cdots P_i \subset \cdots \quad \text{with} \bigcup P_i = \Sigma.$$

where each P_i is the convex hull of a finite set. This gives a residual tower of RACGs

$$W = W_0 > W_1 > \cdots W_i > \cdots$$

N.B. The subgroup W_i is *not* normal in W.

Remark

Lück's Approximation Theorem does not work in this generality. Indeed, since P_i is contractible, $\overline{H}_*(W_i; \mathbf{Q}) = 0$. So, $b_k(W_i) = 0$ and simlarly, for the *normalized Betti numbers*, $\dfrac{b_k(W_i)}{[W : W_i]} = 0$.

(Of course, if $\Sigma = \mathbb{H}^{2k}$, then $\beta_k^{(2)}(W) \neq 0$.)

At one point we thought the problem might be caused by the fact that the W_i were not torsion-free. However, this was not the problem.

$\exists$ torsion-free subgroups $\pi_i < W_i$ of index 2^n so that $M_i = \Sigma/\pi_i$ is an n-manifold with nonzero Betti numbers. The Betti numbers depend on the combinatorics of the polytope P_i. Moreover, the normalized Betti numbers do not limit to 0.

More detail

Given a RACG W with fundamental polytope P, $\exists$ a homomorphism $W \to (\mathbf{Z}/2)^n$ with torsion-free kernel π gives a *small cover* $M = \Sigma/\pi \to \Sigma/W = P$. These can occur as the real points of a toric variety over P. Calculating the cohomology of M was the topic of my paper with Januszkiewicz twenty five years ago. It turns out that M has a perfect cell structure (over $\mathbf{Z}/2$) in the sense of Morse theory with

$$\#\{k\text{-cells}\} = h_k(P),$$

where $(h_1, \ldots, h_n)$ is the so-called *h-vector* of P

When W is a reflection group on $\mathbb{H}^n$ (and in many other cases) the k^{th} normalized Betti number satisfies:

$$\frac{b_k(M_i;\mathbb{F}_2)}{[W:\pi_i]} \geq C > 0,$$

for all $k \neq 0, n$. In favorable cases we can replace $\mathbb{F}_2$ by **Z**. So, the conclusion of Lück's Theorem again fails but for a different reason!

Theorem (Benjami-Eldan 2012)

Suppose $X \subset \mathbb{H}^n$ and $\#X = N$.

$$\mathrm{vol}(\mathrm{Conv}(X)) \leq C_n N$$

In fact,

$$C_n = \frac{2(2\sqrt{\pi})^n}{\Gamma(\frac{n}{2})}.$$

Conv(X) is a polytope: we can take X to be its vertex set.

The h vector

Let $P \subset \mathbb{R}^n$ be a simple polytope. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a generic linear form ("generic" means $f|_{\text{edge}} \neq$ constant). Then f induces an (upward-pointing) orientation on each edge of P. For each $v \in \text{Vert}(P)$, its *index*, $\iota(v)$, is defined by

$$\iota(v) = \#\{\text{inward-pointing edges at } v\}, \qquad \text{and}$$

$$h_k(P) := \#\{v \in \text{Vert}(P) \mid \iota(v) = k\}$$

- $h_0 = 1 = h_n, \quad \sum h_i = N = \#\text{Vert}(P)$
- Each $v \in \text{Vert}(P)$ of index k determines a unique k-face F_v of P s.t. $v \in F_v$ and the maximum of f on F_v is at v.

Suppose $P = P_0 < P_1 < \cdots P_i \cdots$ is the tower of reflection polytopes. Let

$m_i = \#\{\text{chambers in } P_i\}$.

If $P \subset \mathbb{H}^n$, then

$$m_i = \text{vol}(P_i)/\text{vol}(P_0).$$

Let $N(P_i)$ be the number of vertices in P_i. By Benjamini-Eldar,

$$\frac{N(P_i)}{m_i} \geq \frac{\text{vol}(P_0)}{C_n} = C > 0.$$

In fact, for each $k \neq 0, n$, $\frac{h_k(P_i)}{m_i}$ is bounded away from 0.

Proposition

For $k \neq 0, n$, $h_k \geq CN$

Proof.

$$f_i = \#\{\text{faces of codim } i+1 \text{ in } P\}$$
$$= \#\{i\text{-simplices in the dual triangulation of } S^{n-1}\}$$

This follows from two facts:

- $f_{k-1} > Cf_{n-1} = CN$, where $C = 1/\binom{n}{k}$.
- The h_i are linear combinations of the f_i, eg, $h_1 = f_0 - n$.

□

This is based on a paper of D, Januszkiewicz from twenty five years ago.

Small covers

- P a simple polytope, $\mathcal{F} = \{\text{facets of } P\}$ $(= \mathcal{F}(P))$
- Let $\lambda : \mathcal{F} \to (\mathbf{Z}/2)^n - 0$ be a function such that if $F_1, \dots F_n$ meet at a vertex, then $\lambda(F_1), \dots \lambda(F_n)$ is a basis for $(\mathbf{Z}/2)^n$. λ is called a *characteristic function*.
- The characteristic function induces a homomorphism $\overline{\lambda} : W \to (\mathbf{Z}/2)^n$ with torsion-free kernel π. Put $M = \Sigma/\pi$. The group $(\mathbf{Z}/2)^n \curvearrowright M$ with quotient P. The projection $p : M \to P$ is called a *small cover*.

- Suppose $p : M \to P$ a small cover. Then $P \subset M$ is a fundamental domain for $(\mathbf{Z}/2)^n$-action.
- Given a k-face F of P, let $M_F = p^{-1}(F)$. It is a k-manifold with $(\mathbf{Z}/2)^k$-action and a small cover of F.
- Let $\varphi = f \circ p : M \to P \to \mathbb{R}$, where f is the height function on P. Then φ is a Morse function. The critical points are at the vertices of P.
- The index of the critical point at v is $\iota(v)$.

The Morse function $\varphi : M \to \mathbb{R}$. Given $v \in \text{Vert}(P)$, let $\overset{\circ}{F}_v$ be the union of faces of F which contain v. Put

$$C_v = (\mathbf{Z}/2)^k \, \overset{\circ}{F}_v, \qquad \text{where } k = \iota(v)$$

- Then C_v is a k-cell, the *ascending submanifold at* v. Moreover,
- $\overline{C_v} = M_{F_v} := M_v$ is a (possibly non-orientable) k-manifold.

Proposition

φ is perfect in the sense of Morse theory (homology with coefficients in $\mathbb{F}_2$), i.e.,

$$b_k(M; \mathbb{F}_2) = h_k(P).$$

Proof.

Each $\overline{C_v}$ $(= M_v)$ is a manifold; hence, a mod 2 cycle. So, all incidence numbers are $0 \mod 2$. □

Remark

If all M_v are orientable manifolds, then the above proposition is true with coefficients in **Z**.

Orientability

The way to insure all the M_F are orientable is to assume that the characteristic function $\lambda : \mathcal{F} \to (\mathbf{Z}/2)^n - 0$ has image lying in $\{e_1, \dots e_n\}$ (the standard basis). In other words, the facets of P are colored by n colors. P may not always admit such a coloring; however, some simple polytopes do have such colorings. If P has such a coloring, the orientability of the M_F is assured. Also, colorability is inherited by towers $P > P_1 > \cdots$.

Review of construction

- Start with increasing sequence of convex polytopes $P < P_1 < \cdots P_i < \cdots$, which exhaust Σ and give a residual chain $W > W_1 > \cdots W_i > \cdots$, where $[W : W_i] = m_i = \#$ of copies of P in P_i. For each i, glue together 2^n copies of P_i giving a manifold M_i with fundamental group $\pi_i < W_i$ and a residual tower $M \leftarrow M_1 \leftarrow \cdots$ and a chain $W > \pi > \pi_1 > \cdots$.
- The normalized Betti numbers satisfy:

$$\frac{b_k(M_i;\mathbb{F}_2)}{2^n m_i} = \frac{h_k(P_i)}{2^n m_i} \geq \frac{C'}{2^n} \geq C.$$

- In particular,

$$\frac{1}{2^n}\sum \frac{b_k(M_i;\mathbb{F}_2)}{m_i} = \frac{1}{2^n}\frac{N(P_i)}{m_i} \geq \frac{C}{2^n}.$$