Lattices and Invariant Random Subgroups

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The Chabauty space of closed subgroups

Let G be a locally compact second countable group.

$\mathrm{Sub}_G = \{\text{closed subgroups of } G\} +$ the Chabauty topology,

$$d(H_1, H_2) = \int_0^\infty d_{\mathrm{H}}\big(H_1 \cap B_r(id_G), H_2 \cap B_r(id_G)\big) e^{-r} dr.$$

Definition (The Chabauty topology)

The topology is generated be sets of the following types:

- $O_1(K) = \{H : H \cap K = \emptyset\}$ for $K \subset G$ compact.
- $O_2(U) = \{H : H \cap U \neq \emptyset\}$ for $U \subset G$ open.

The Chabauty space of closed subgroups

Exercise

Show that a sequence $H_n \in Sub_G$ converges to a limit H iff

- *for any $h \in H$ there is a sequence $h_n \in H_n$ such that $h = \lim h_n$, and*
- *for any sequence $h_{n_k} \in H_{n_k}$, with $n_{k+1} > n_k$, which converges to a limit, we have $\lim h_{n_k} \in H$.*

Examples:

- $\mathrm{Sub}_{\mathbb{R}} \sim [0, \infty]$.
  Every proper non-trivial closed subgroup of $\mathbb{R}$ is of the form $\alpha\mathbb{Z}$ for some $\alpha > 0$. $\alpha = 0$ corresponds to $\mathbb{R}$, and $\alpha = \infty$ to $\{0\}$.
- $\mathrm{Sub}_{\mathbb{R}^2}$ is homeomorphic to the sphere S^4 (Hubbard, Pourezza).

Problem

Describe Sub_G for $G = SL_2(\mathbb{R})$.

Compactness

Proposition (Exercise)

Sub_G is compact.

One can use Sub_G in order to compactify certain sets of closed subgroups.

For instance one can study the Chabauty compactification of the space of lattices in G. In particular, it is interesting to determine the points of that compactification:

Problem

Determine which subgroups of $SL_3(\mathbb{R})$ are limits of lattices.

This problem might be more accessible if we replace $SL_3(\mathbb{R})$ with a group for which the congruence subgroup property is known for all lattices.

When is G isolated?

Exercise

A discrete group Γ is isolated in Sub_Γ iff it is finitely generated.

What about non-discrete groups?

Exercise

Show that if G surjects on S^1, then G is not isolated in Sub_G.

Claim

For connected Lie groups the converse is true as well.

Theorem (Zassenhaus)

A Lie group G admits an identity neighborhood U such that for every discrete group $\Gamma \leq G$, $\langle \log(\Gamma \cap U) \rangle$ is a nilpotent Lie algebra.

G is isolated

Exercise

Consider $G = SL_n(\mathbb{Q}_p)$ and show that G is an isolated point in Sub_G.

Hint: Use the following facts:

- $SL_n(\mathbb{Z}_p)$ is a maximal subgroup of $SL_n(\mathbb{Q}_p)$.
- The Frattini subgroup of $SL_n(\mathbb{Z}_p)$ is open, i.e. of finite index.

Proposition

Let $G = \mathbb{G}(k)$ be the group of k points of a simple algebraic group, where k is a local field. Then G is an isolated point in Sub_G.

In the non-archimedean case the proof relies on:

1. A maximal compact subgroup is maximal (Tits) and open.
2. Pink's criterion: A closed subgroup is open iff it is
   - Zariski dense
   - non-discrete, and
   - not contained in the rational points of a proper subfield.

Invariant measures on Sub_G

The group G acts on Sub_G by conjugation and it is natural to consider the invariant measures on this compact G-space.

Definition

An Invariant Random Subgroup (hereafter IRS) is a Borel probability measure on Sub_G which is invariant under conjugations.

First examples and remarks

Definition

An IRS is a conjugacy invariant probability measure on Sub_G.

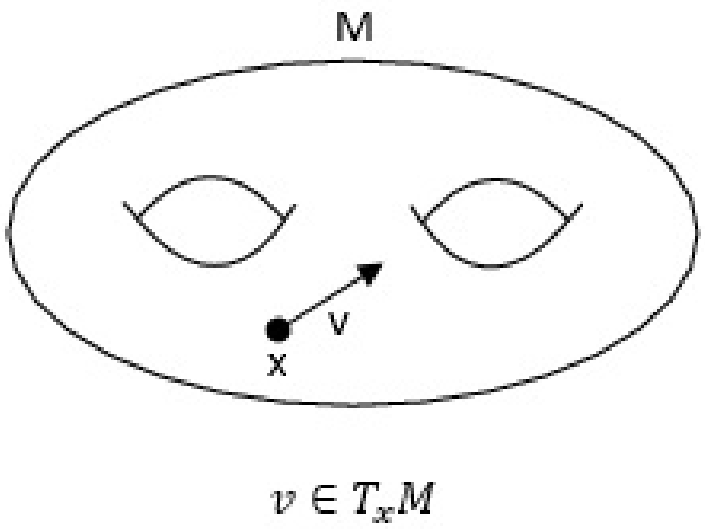
First examples and remarks:

1. The Dirac measures correspond to normal subgroups.
2. Let $\Gamma \leq G$ be a lattice (or more generally a closed subgroup of finite co-volume).
   Let
   $$\psi : G/\Gamma \to \mathrm{Sub}_G,\ g \mapsto g\Gamma g^{-1},$$
   Let m be the normalized measure on G/Γ, and set $\mu_\Gamma := \psi_*(m)$.

   Note that μ_Γ is supported on (the closure of) the conjugacy class of Γ.

An hyperbolic surface is an IRS

For instance let Σ be a closed hyperbolic surface and normalize its Riemannian measure. Every unit tangent vector yields an embedding of $\pi_1(\Sigma)$ in $PSL_2(\mathbb{R})$. Thus the probability measure on the unit tangent bundle corresponds to an IRS of type (2) above.



Induction

Let again $\Gamma \leq_L G$ and let $N \lhd \Gamma$ be a normal subgroup of Γ. As in (2) the G-invariant probability measure on G/Γ can be used to choose a random conjugate of N in G via the map

$$\psi : G/\Gamma \to \mathrm{Sub}_G,\ g \mapsto gNg^{-1}.$$

This is an IRS supported on the (closure of the) conjugacy class of N.

More generally, every IRS on Γ can be induced to an IRS on G. Intuitively, the random subgroup is obtained by conjugating Γ by a random element from G/Γ and then picking a random subgroup in the corresponding conjugate of Γ.

Connection with p.m.p. actions

Let $G \curvearrowright (X, m)$ be a probability measure preserving action.

The stabilizer of almost every point in X is closed in G (Varadarajan) and the stabilizer map

$$X \to \mathrm{Sub}_G, \ x \mapsto G_x$$

is measurable.
Hence m defines an IRS on G. In other words the random subgroup is the stabilizer of a random point in X.

The study of p.m.p. G-spaces can be divided to

- the study of stabilizers (i.e. IRS),
- the study of orbit spaces

and the interplay between the two.

Connection with p.m.p. actions

The connection between IRS and p.m.p. actions goes also in the other direction:

Theorem

Let G be a locally compact group and μ an IRS in G. Then there is a probability space (X, m) and a measure preserving action $G \curvearrowright X$ such that μ is the push-forward of the stabilizer map $X \to Sub_G$.

The first thing that comes to mind is to take the given G action on (Sub_G, μ), but then the stabilizer of a point $H \in \mathrm{Sub}_G$ is $N_G(H)$ rather than H.

To correct this consider the larger space Cos_G of all cosets of all closed subgroups, as a measurable G-bundle over Sub_G. Define an appropriate invariant measure on $\mathrm{Cos}_G \times \mathbb{R}$ and replace each fiber by a Poisson process on it.

The space of IRS's

Definition

We shall denote by $\mathrm{IRS}(G)$ the space of IRS on G equipped with the w^*-topology.

$$\mathrm{IRS}(G) := \mathrm{Prob}(\mathrm{Sub}_G)^G$$

By Alaoglu's theorem $\mathrm{IRS}(G)$ is compact.

Existence

An interesting yet open question is whether this space is always non-trivial.

Question

Does every non-discrete locally compact group admit a non-trivial IRS?

A counterexample, if exists, should be a simple group without lattices. The candidate is the Neretin group.

Question

1. *Does the Neretin group admit a (non-discrete) closed subgroup of finite co-volume?*
2. *Does the Neretin group admit a non-trivial discrete IRS?*

Remark

There are many discrete groups without nontrivial IRS, for instance $PSL_n(\mathbb{Q})$.

IRS and Lattices

Viewing IRS as a generalization of lattices there are two directions toward which one is tempted to go:

1. Extend classical theorems about lattices to general IRS.
2. Use the compact space IRS(G) in order to study its special 'lattice' points.

Remarkably, the approach (2) turns out to be quite fruitful in the theory of asymptotic properties of lattices.
We shall see later on an example of how rigidity properties of G-actions yield interesting data of the geometric structure of locally symmetric spaces $\Gamma\backslash G/K$ when the volume tends to infinity.

Here is another direction in the spirit of (2), this time with a fixed volume:

The IRS compactification of moduli spaces

Let Σ be a closed surface of genus ≥ 2. Every hyperbolic structure on Σ corresponds to an IRS in $PSL_2(\mathbb{R})$. Taking the closure in IRS(G) of the set of hyperbolic structures on Σ, one obtains an interesting compactification of the moduli space of Σ.

Problem

Analyse the IRS compactification of $Mod(\Sigma)$.

Stuck–Zimmer theorem

Perhaps the first result about IRS and certainly one of the most remarkable, is the Stuck–Zimmer rigidity theorem, which is a (far reaching) generalisation of Margulis normal subgroup theorem.

Theorem (SZ, 1994)

Every ergodic p.m.p. action of $SL(3, \mathbb{R})$ is either free or transitive.

Corollary

The non-trivial ergodic IRS in $SL(3, \mathbb{R})$ correspond to lattices.

Corollary

Every IRS of $SL(3, \mathbb{Z})$ is supported on finite index subgroups.

Soficity of IRS

Definition

Let us say that an IRS μ is *co-sofic* if it is a weak-$*$ limit in IRS(G) of ones supported on lattices.

The following question can be asked for any locally compact group G, however I find the 3 special cases of $G = SL_2(\mathbb{R}), SL_2(\mathbb{Q}_p)$ and $\mathrm{Aut}(T)$ particularly intriguing:

Question

Is every IRS in G co-sofic?

Exercise

1. Show that the case $G = F_n$, the discrete rank n free group, is equivalent to the Aldous–Lyons conjecture that every unimodular network is a limit of ones corresponding to finite graphs.
2. A Dirac mass δ_N, $N \lhd F_n$ is co-sofic iff the corresponding group $G = F_n/N$ is sofic.

Lattices

Let G be a locally compact group.

Definition

A closed subgroup $H \leq G$ is *co-finite* if the homogeneous space G/H admits a finite G-invariant measure. A *lattice* is a discrete co-finite subgroup.

Simple analytic groups over local fields

Let $G = \mathbb{G}(k)$ be a simple algebraic group over a local field k.
i.e.

- k is $\mathbb{R}, \mathbb{C}$, a finite extension of $\mathbb{Q}_p$ or the field $\mathbb{F}_q((t))$ of formal Laurent series over a finite field
- $\mathbb{G}$ is a simple k-algebraic group, and G is the group of k rational points.

You may think of the example $G = SL(n, \mathbb{R})$. Or $G = SL(n, \mathbb{R}_p)$.

Arithmetic Groups

Examples:

- $SL(n, \mathbb{Z})$ is a non-uniform lattice in $SL(n, \mathbb{R})$.
- Let $Q(x, y, z, w) = x^2 + y^2 + z^2 - \sqrt{2}w^2$, let $\mathbb{G} = \mathbb{SO}(Q)$ and let $\Gamma = \mathbb{G}(\mathbb{Z}[\sqrt{2}])$. Then $\Gamma = \mathbb{H}(\mathbb{Z})$ for some algebraic group $\mathbb{H}$, and

$$\mathbb{H}(\mathbb{R}) \cong SO(3, 1) \times SO(4).$$

  Γ is a lattice in $\mathbb{H}(\mathbb{R})$. It projects to a lattice in $SO(3, 1)$.

Definition

A subgroup $\Gamma \leq G$ is called *arithmetic* if there is a $\mathbb{Q}$-algebraic group $\mathbb{H}$ and a surjective map $f : \mathbb{H}(\mathbb{R}) \to G$ with compact kernel, such that $f(\mathbb{H}(\mathbb{Z}))$ is commensurable with Γ.

Theorem (Borel–Harish-Chandra)

Suppose that G is simple. Then every arithmetic group is a lattice.

Some classical results

Suppose that G is a simple non-compact group over a local field.

Theorem (Borel Density)

A proper co-finite subgroup of G is discrete and Zariski dense.

Theorem (Kazhdan–Margulis)

There is an identity neighbourhood $U \subset G$ such that every lattice $\Gamma \leq G$ admits a conjugate which intersects U trivially.

Theorem (Margulis' arithmeticity)

If the k-rank of $\mathbb{G}$ is ≥ 2 then every lattice is arithmetic.

Borel density theorem for IRS

Theorem

Let G be a non-compact simple algebraic group over a local field. Let μ be a non-atomic IRS in G. Then a μ random subgroup is discrete and Zariski dense.

The idea (in the Archimedean case) is to consider the maps $\mathrm{Sub}_G \to Gr(\mathrm{Lie}(G))$

$$H \mapsto \mathrm{Lie}(H), \text{ and } H \mapsto \mathrm{Lie}(\overline{H}^Z),$$

and push the invariant measure to one on $Gr(\mathrm{Lie}(G))$. By Furstenberg's lemma every such measure is trivial.

Borel density for IRS

Theorem (Borel density for IRS)

Let G be a simple algebraic group over a local field. Let μ be a non-atomic IRS in G. Then a μ random subgroup is discrete and Zariski dense.

Note that if G is simple, the only possible atoms are at the trivial group $\{1\}$ and at G. Since G is an isolated point in Sub_G, it follows that

$$\mathrm{IRS}_d(G) := \{\mu \in \mathrm{IRS}(G) : \text{a } \mu\text{-random subgroup is a.s. discrete}\}$$

is a compact space. We shall refer to the points of $\mathrm{IRS}_d(G)$ as *discrete* IRS.

Uniform Discreteness

Definition

A family $\mathcal{F}$ of lattices (or discrete subgroups) of G is said to be *uniformly discrete* (UD) if there is an identity neighbourhood $\Omega \subset G$ which intersects trivially every conjugate of a member of $\mathcal{F}$.

Conjecture (Margulis)

The family of all co-compact arithmetic lattices in G is UD.

Or equivalently

Conjecture

There is an identity neighbourhood $\Omega \subset G$ whose intersection with every arithmetic lattice in G consists of unipotent elements only.

Weak Uniform Discreteness

Definition

A family of IRS, $\mathcal{F} \subset IRS(G)$ is said to be weakly uniformly discrete if for every $\epsilon > 0$ there is an identity neighbourhood $\Omega \subset G$ such that for every $\mu \in \mathcal{F}$,

$$\mu(\Gamma \in \mathrm{Sub}(G) : \Gamma \cap \Omega \text{ is non-trivial}) < \epsilon.$$

A variant of the Margulis' uniform-discreteness conjecture:

Conjecture

For G as above (simple over local field) the full space $IRS_d(G)$ is weakly uniformly discrete.

Some evidence:

- True for p-adic groups.
- True for real Lie groups of rank≥ 2.
- Seems to hold also rank one Lie groups (at least for t.f. IRS).

Stuck–Zimmer rigidity theorem

Theorem (SZ94)

Let G be a connected simple Lie group of real rank ≥ 2. Then every ergodic p.m.p. action of G is either (essentially) free or transitive.

Thus: *every non-atomic ergodic IRS in G is of the form μ_Γ for some lattice $\Gamma \leq G$.*

Remark

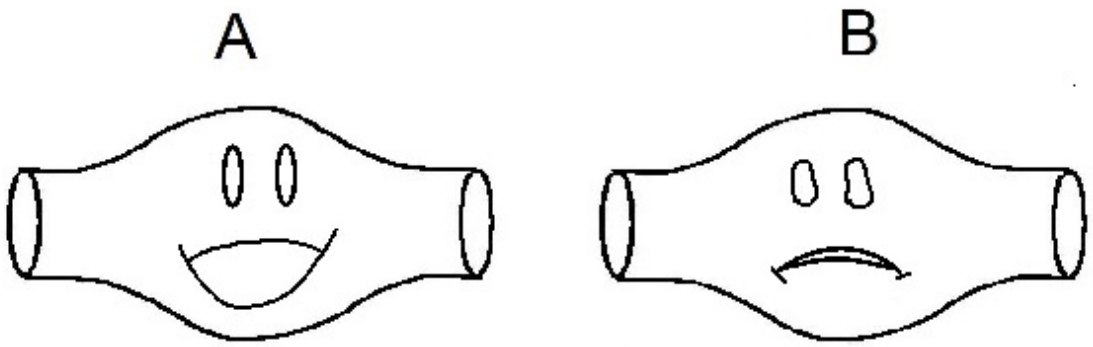
1. *The theorem holds for the wider class of higher rank semisimple groups with property (T). The situation for certain groups, such as $SL_2(\mathbb{R}) \times SL_2(\mathbb{R})$ is still unknown.*
2. *Recently A. Levit proved the analog result for groups over non-archmedean local fields.*

Exotic IRS in rank one

In the lack of Margulis' normal subgroup theorem one can construct IRS supported on subgroups of infinite co-volume.

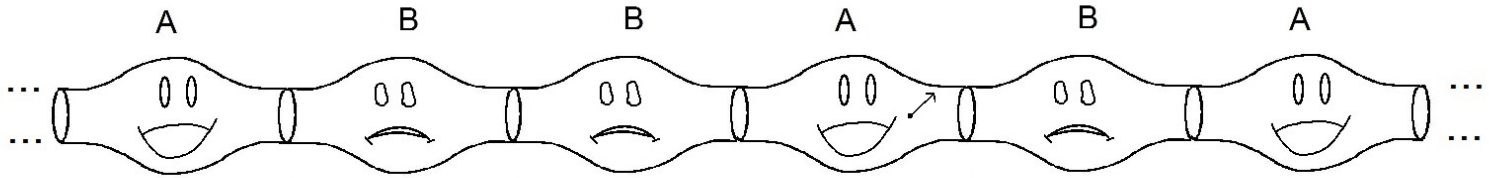
Here is a more exotic example (taken from the [7s] paper)
The 7 samurai: Abert, Bergeron, Biringer, G, Nikolov, Raimbault, Sammet

Let A, B be two copies of a surface with 2 open discs removed equipped with distinguishable hyperbolic metrics.



Exotic IRS in $SO(n,1)$

Consider the space $\{A,B\}^{\mathbb{Z}}$ with the Bernoulli measure $(\frac{1}{2},\frac{1}{2})^{\mathbb{Z}}$. Any element $\alpha \in \{A,B\}^{\mathbb{Z}}$ is a two sided infinite sequence of A's and B's and we can glue copies of A, B 'along a bi-infinite line' following this sequence. This produces a random surface M^{α}.



Existence of non-arithmetic manifolds

Theorem (Gromov and Piatetski-Shapiro, 1987)

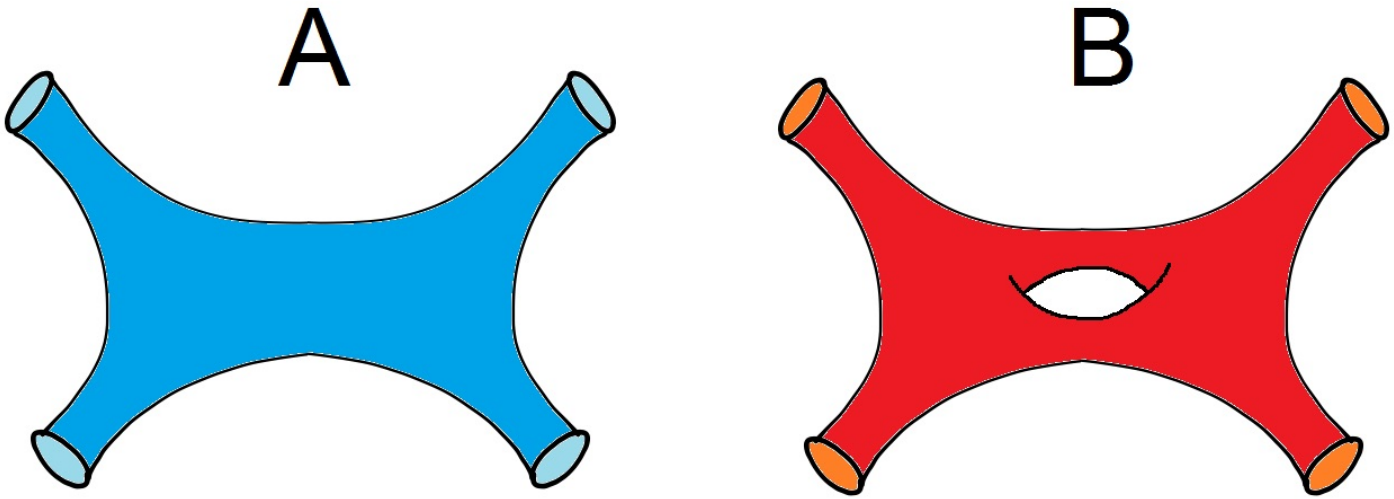
There exists a non-arithmetic finite volume complete hyperbolic manifold of any dimension $d \geq 2$.

The idea of the proof (in odd dimension) is to cut and glue together two non-commensurable arithmetic manifolds.
The proof relies on two lemmas:

- Let Γ_1, Γ_2 be two arithmetic subgroups of G such that $\Gamma_1 \cap \Gamma_2$ is Zariski dense. Then they are commensurable.
  (Follows from Borel–Harish-Chandra theorem.)
- Let M be a finite volume hyperbolic manifold and $N \subset M$ a submanifold of the same dimension with totally geodesic boundary. Then $\pi_1(N)$ is Zariski dense.

Most hyperbolic manifolds are non-arithmetic

Using pieces of non-commensurable arithmetic manifolds with 4 (isometric) boundary components, one can obtain plenty of hyperbolic manifolds modeled over 4-regular finite graph.



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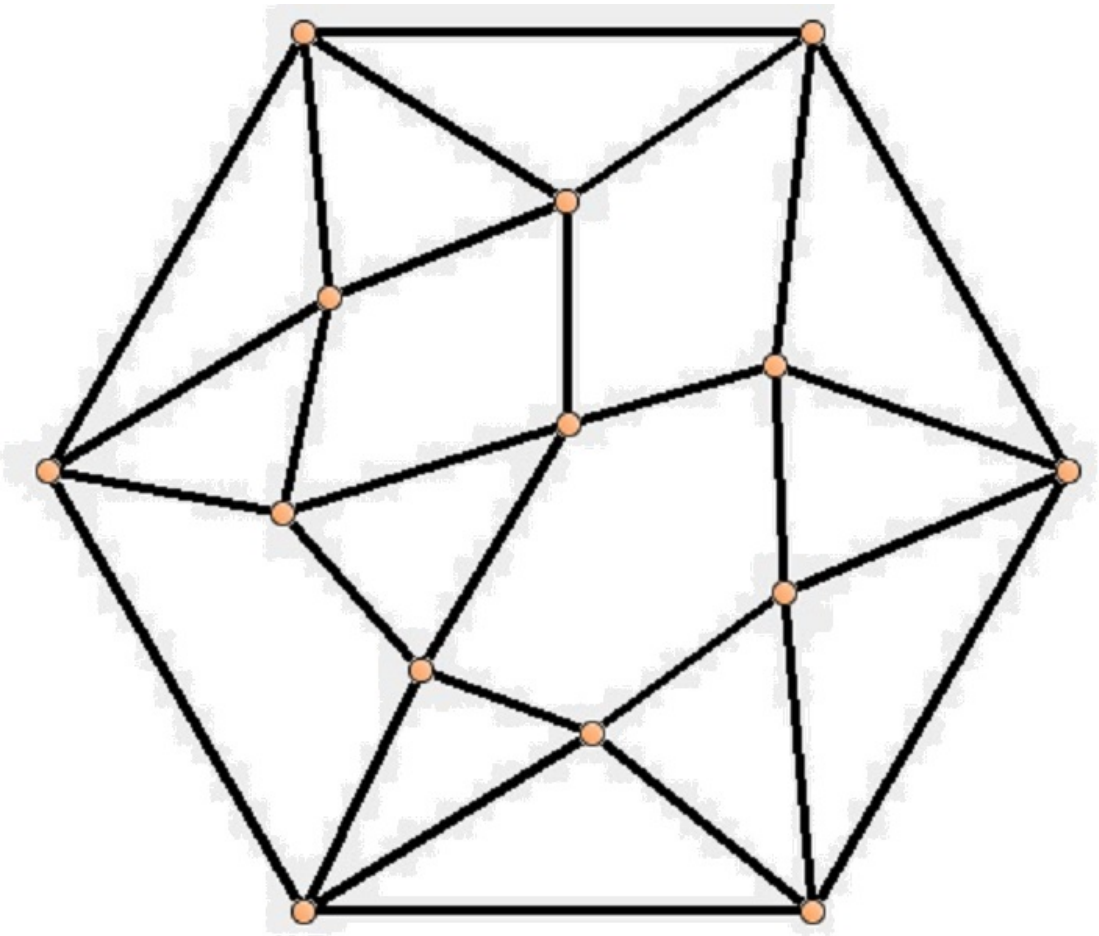
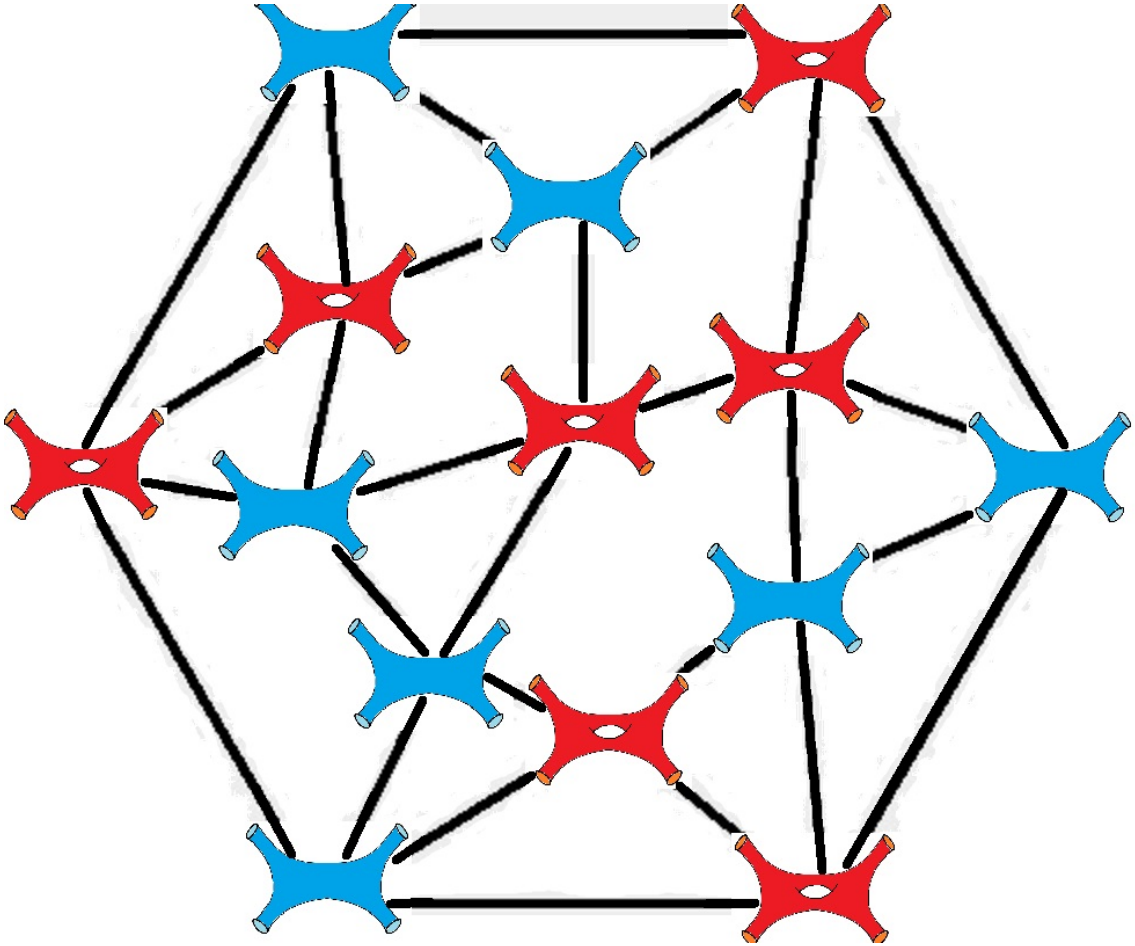


Most hyperbolic manifolds are non-arithmetic



Most hyperbolic manifolds are non-arithmetic

Recall that two manifolds are said to be commensurable if they admit a common finite cover.

Theorem (G, Levit 2014)

For $d \geq 4$ and any V sufficiently large, there are about V^V pairwise non-commensurable hyperbolic n-manifolds of volume $\leq V$.

Remark

- *The upper bound was proved in [Burger,G,Lubotzky,Mozes,2002].*
- *The same estimate holds when counting up to QI of π_1.*
- *Among those only polynomially many are arithmetic.*

Most hyperbolic manifolds are non-arithmetic

Some issues that come up in the proof:

- There are enough 4-regular graphs but they are all commensurable.
- How to produce the 6 building blocks.

The hardest case is for compact in even dimension, where we use the forms

$$Q_p = px_1^2 + x_2^2 + \ldots + x_n^2 - \sqrt{2}x_{n+1}^2, \text{ with } p = 17, 41, 97, 137, 193, 241.$$

The proof that the associated manifolds are non-commensurable relies of the following:

Theorem (Gauss)

For prime $p \equiv 1(mod 4)$ the equation $x^4 = 2$ has an integer solution modulo p iff $p = x^2 + 64y^2$ for some $x, y \in \mathbb{N}$.

The Benjamini–Schramm space

Recall the *Hausdorff distance* $d_{\mathrm{H}}(A, B)$ between two closed subsets of a compact metric space Z

$$d_{\mathrm{H}}(A, B) := \inf\{\epsilon : N_\epsilon(A) \supset B \text{ and } N_\epsilon(B) \supset A\},$$

and the *Gromov distance* $d_{\mathrm{G}}(X, Y)$ between two compact metric spaces X, Y

$$d_{\mathrm{G}}(X, Y) := \inf_{X, Y \hookrightarrow Z} d_{\mathrm{H}}(X, Y).$$

The Benjamini–Schramm topology

If $(X,p),(Y,q)$ are pointed compact metric spaces, we define the Gromov distance

$$d_{\mathrm{G}}((X,p),(Y,q)) := \inf_{X,Y\hookrightarrow Z}\{d_{\mathrm{H}}(X,Y)+d(p,q)\}.$$

The *Gromov–Hausdorff distance* between two pointed proper spaces $(X,p),(Y,q)$ can be defined as

$$d_{\mathrm{GH}}((X,p),(Y,q)) := \int_{r>0} d_{\mathrm{G}}(B_r(p),B_r(q))e^{-r}dr,$$

where $B_r(p)$ is the ball of radius r in around p in X.

The Benjamini–Schramm topology

Let $\mathcal{M}$ be the be the space of all (isometry classes of) pointed proper metric spaces equipped with the Gromov–Hausdorff topology.

Given an integer valued function $f(\epsilon, r)$, let $\mathcal{M}_f$ consist of those spaces for which the ϵ entropy of the r ball is bounded by $f(r, \epsilon)$. This is a compact space.

We define the Benjamini–Schramm space $\mathcal{BS} = \mathrm{Prob}(\mathcal{M})$ to be the space of all Borel probability measures on $\mathcal{M}$ equipped with the weak-$*$ topology. Given f as above, we set $\mathcal{BS}_f := \mathrm{Prob}(\mathcal{M}_f)$. Note that $\mathcal{BS}_f$ is compact.

Examples:

An example of a point in $\mathcal{BS}$ is a measured metric space. A particular case is a finite volume Riemannian manifold — in which case we normalize the Riemannian measure to be one, and then randomly choose a point and a frame.

The interplay between $\mathcal{BS}$ and IRS

Thus a finite volume locally symmetric space $M = \Gamma\backslash G/K$ produces both a point in the Benjamini–Schramm space and an IRS in G. This is a special case of a more general analogy.

Let $G = \mathbb{G}(k)$ be a non-compact simple analytic group over a local field k. Let X be the associated Riemannian symmetric space or Bruhat–Tits building.

$\mathcal{M}(X) =$ the space of all pointed (or framed) complete metric spaces of the form $\Gamma\backslash X$.

$\mathcal{BS}(X) = \mathrm{Prob}(\mathcal{M}(X))$ the corresponding subspace of the Benjamini–Schramm space.

The interplay between $\mathcal{BS}$ and IRS

There is a natural map

$$\{\text{discrete subgroups of } \ G\} \to \mathcal{M}(X), \ \Gamma \mapsto \Gamma \backslash X.$$

this map is continuous, hence inducing a continuous map

$$\psi : \mathrm{IRS}_d(G) \to \mathcal{BS}(X).$$

The latter map is one to one, and since $\mathrm{IRS}_d(G)$ is compact, it is an homeomorphism to its image.

Exercise (Invariance under the geodesic flow)

Given a tangent vector $\overline{v}$ at the origin (the point corresponding to K) of $X = G/K$, define a map $\mathcal{F}_{\overline{v}}$ from $\mathcal{M}(X)$ to itself by moving the special point using the exponent of $\overline{v}$ and applying parallel transport to the frame. This induces a homeomorphism of $\mathcal{BS}(X)$. Show that the image of $IRS_d(G)$ under the map above is exactly the set of $\mu \in \mathcal{BS}(X)$ which are invariant under $\mathcal{F}_{\overline{v}}$ for all $\overline{v} \in T_K(G/K)$.

The interplay between $\mathcal{BS}$ and IRS

Thus we can view geodesic-flow invariant probability measures on framed locally X-manifolds as IRS on G and vice versa, and the Benjamini--Schramm topology on the first coincides with the IRS-topology on the second.

Exercise

Show that the analogy above can be generalised, to some extent, to the context of general locally compact groups: Given a locally compact group G, fixing a right invariant metric on G, we obtain a map $Sub_G \to \mathcal{M},\ H \mapsto G/H$, where the metric on G/H is the induced one. Show that this map is continuous and deduce that it defines a continuous map $IRS(G) \to \mathcal{BS}$.

Farber condition

Let $\mu_n \in \mathrm{IRS}(G)$ be a sequence of IRS and let $\nu_n = \psi(\mu_n) \in \mathcal{BS}(X)$ be the corresponding sequence in the Benjamini–Schramm space.

Definition

μ_n is a Farber sequence if $\mu_n \overset{w^*}{\to} \delta_{\langle 1 \rangle}$.
Equivalently, ν_n is Farber if $\nu_n \overset{\mathcal{BS}}{\to} X$.

For an X-manifold M (or simplicial complex, in the non-archimedean case) and $r > 0$, we denote by $M_{\geq r}$ the r-thick part in M:

$$M_{\geq r} := \{x \in M : \mathrm{InjRad}_M(x) \geq r\}.$$

Lemma

A sequence M_n of finite volume X-manifolds BS-converges to X iff

$$\frac{vol((M_n)_{\geq r})}{vol(M_n)} \to 1, \ \forall r > 0.$$

Asymptotic cohomology

Theorem (7s)

In the Riemannian case. If M_n is a uniformly discrete Farber sequence of locally X manifolds than for all $k \leq \dim(X)$

$$\frac{b_k(M_n)}{vol(M_n)} \to \beta_k^{(2)}(X).$$

Here

$$\beta_k^{(2)}(X) = \begin{cases} \frac{\chi(X^d)}{\mathrm{vol}(X^d)} & k = \frac{1}{2}\dim X, \\ 0 & \text{otherwise,} \end{cases}$$

where X^d is the compact dual of X.

Remark

For sequences of congruence covers we prove effective estimates.

Betti numbers

Let M be a Riemannian manifold. Let d, d^* be the boundary and coboundary operators on differential forms on M. Then

$$\Delta := dd^* + d^*d$$

is the Laplace operator on differential forms. Let

$$\mathcal{H}^k(M) = \{w \in \Omega_k(M) : \Delta w = 0\}$$

denote the space of harmonic k-forms on M, then the k'th betti number is

$$b_k(M) = \dim \mathcal{H}^k(M).$$

Equivalently,

$$b_k(M) = \dim Ker(d_k)/Im(d_{k-1}).$$

The heat kernel on M

Let $e^{-t\Delta_k}$ denote the heat kernel on k-forms on M, i.e. the fundamental solution to the heat equation on M. The corresponding bounded integral operator in $\mathrm{End}(\Omega_k(M))$ is given by

$$e^{-t\Delta_k}(f)(x) = \int_M e^{-t\Delta_k}(x,y)f(y)dy, \ \forall f \in \Omega_k(M),$$

Then

$$b_k(M) = \lim_{t\to\infty} \int_M \mathrm{tr}\ e^{-t\Delta_k}(x,x)dx.$$

The heat kernel on X

Similarly, let $e^{-t\Delta_k^{(2)}}$ denote the heat kernel on L_2-differential forms on X. The relation between the kernels is given by:

$$e^{-t\Delta_k}(x,y) = \sum_{\gamma\in\Gamma}(\gamma_{\tilde{y}})^* e^{-t\Delta_k^{(2)}}(\widetilde{x},\gamma\widetilde{y}), \tag{1}$$

where $\widetilde{x},\widetilde{y}$ are lifts of x,y to X and by $(\gamma_y)^*$, we mean pullback by the map $(x,y)\mapsto(\tilde{x},\gamma\tilde{y})$.

Lemma

Let $m>0$. There exists a positive constant $c=c(G,m)$ such that

$$\|e^{-t\Delta_k^{(2)}}(x,y)\| \leq ct^{-d/2}e^{-d(x,y)^2/5t},\ 0<t\leq m.$$

The difference is controlled by the infectivity radius

Lemma

Let $m > 0$. There exists a positive constant $c = c(G, m)$ such that

$$\|e^{-t\Delta_k^{(2)}}(x,y)\| \leq ct^{-d/2}e^{-d(x,y)^2/5t}, \text{ for } t \in [0,m].$$

Corollary

$$\left|\mathrm{tr}\ e^{-t\Delta_k}(x,x) - \mathrm{Tr}e^{-t\Delta_k^{(2)}}\right| \leq c \cdot \mathrm{InjRad}_{M_n}(x)^{-d}.$$

The L_2 betti numbers

The L_2 betti numbers of X are given by

$$\beta_k(X) = \lim_{t \to \infty} \text{Trace } e^{-t\Delta_k^{(2)}}.$$

They all vanish except in the middle dimension.

Proving that the normelized betti numbers converges

Proof.

Let $k \neq \frac{d}{2}$.

- Pick t sufficiently large so that Trace $e^{-t\Delta_k^{(2)}} \leq \epsilon$.
- Pick r so that $c \cdot r^{-d} \leq \epsilon$.
- Pick n sufficiently large so $M = M_n$ satisfies $\frac{\mathrm{vol}(M_{\leq r})}{\mathrm{vol}(M)} \leq \epsilon\delta^d/c$.

Then

$$\frac{1}{\mathrm{vol}(M)} b_k(M) \leq \frac{1}{\mathrm{vol}(M)} \int_M \mathrm{tr}\ e^{-t\Delta_k}(x,x)dx \leq$$

$$\mathrm{Tr}\ e^{-t\Delta_k^{(2)}}(x,y) + \frac{1}{\mathrm{vol}(M)} \int_{M_{\leq r} \cup M_{\geq r}} \|\mathrm{tr}\ e^{-t\Delta_k}(x,x) - \mathrm{Tr}\ e^{-t\Delta_k^{(2)}}\| \leq 3\epsilon$$

Now for $k = d/2$ we can use the Euler characteristic trick. □

Asymptotic cohomology

Theorem (Petersen, Thom, Sauer)

Let G be a totally disconnected locally compact group and $\Gamma_n \leq G$ a Farber sequence of lattices. Then:

- $\liminf \frac{b_i(\Gamma_n)}{vol(G/\Gamma_n)} \geq b_i^{(2)}(G, \mu)$.
- *If the sequence is UD then* $\lim \frac{b_i(\Gamma_n)}{vol(G/\Gamma_n)} = b_i^{(2)}(G, \mu)$.

Higher rank and Rigidity

Let X be a higher rank irreducible symmetric space.

Theorem (7s)

Let $M_n = \Gamma_n \backslash X$ be any sequence of distinct finite volume X-manifolds with $vol(M_n) \to \infty$. Then M_n is Farber (i.e. $M_n \overset{\mathcal{BS}}{\to} X$).

Equivalently:

Theorem (7s)

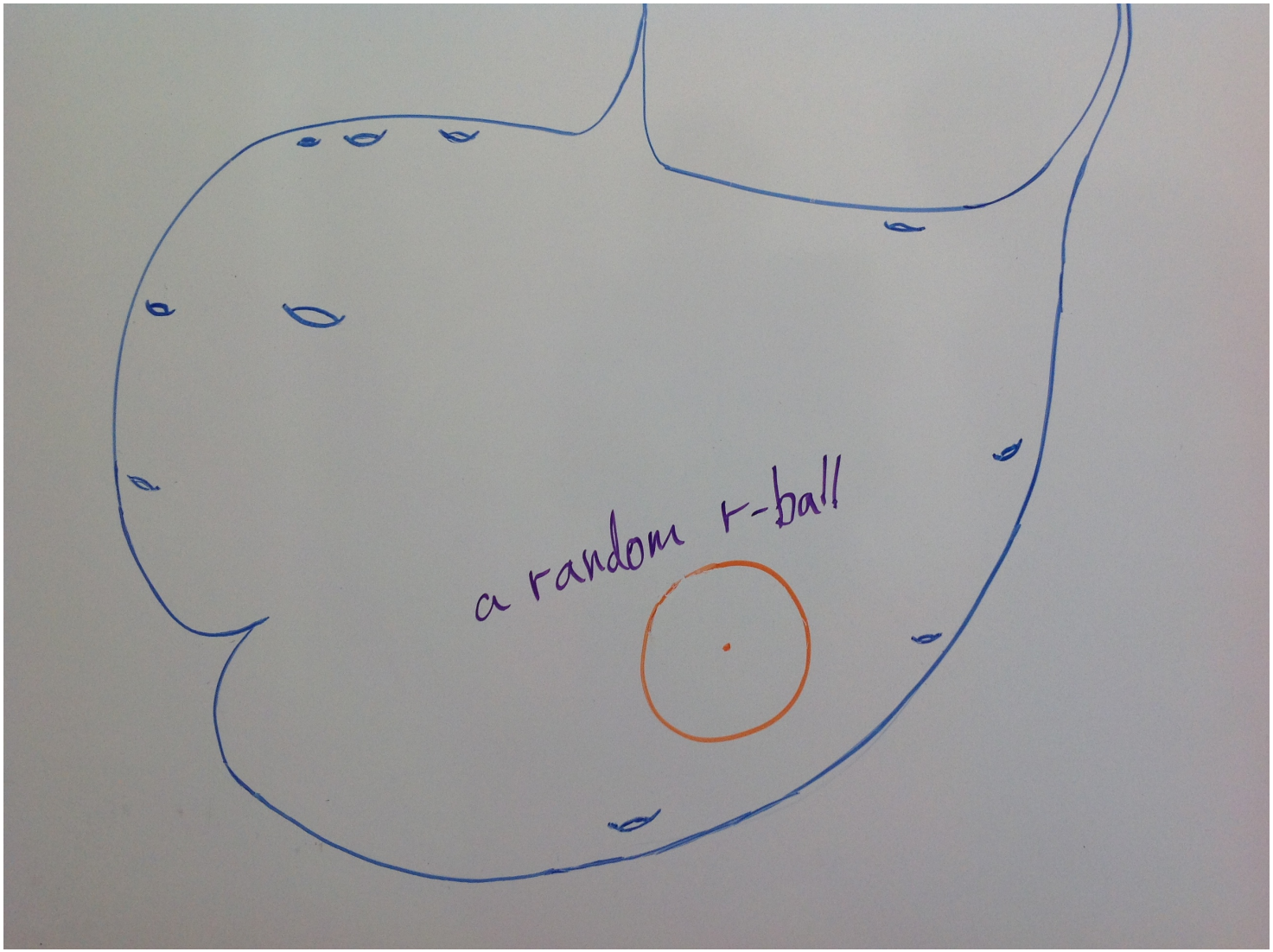
For every r and ϵ there is V such that for any X-manifold M of volume $> V$ we have

$$vol(M_{\geq r}) > (1-\epsilon) vol(M).$$

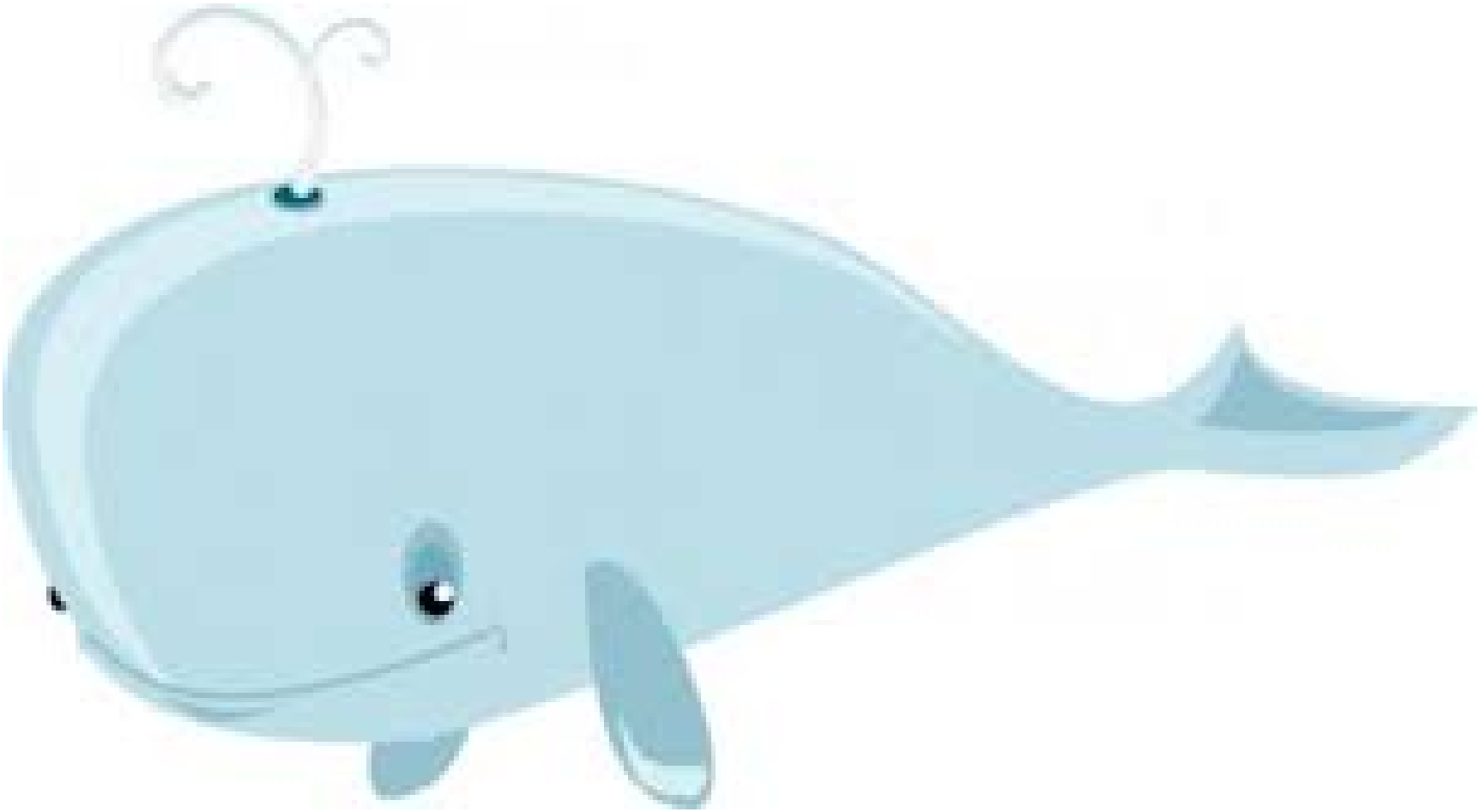
Jointly with A. Levit we extended this to Bruhat–Tits buildings.

1. The p-adic case is simpler than the real case (one can avoid Property (T)).
2. The positive characteristic case is more involved. We assumed WUD.

Manifolds of large volume are fat



Manifolds of large volume are fat



An application of Stuck–Zimmer theorem

Proposition

The only ergodic IRS on G are δ_G, δ_1 and μ_Γ for $\Gamma \leq G$ a lattice.

Proof.

Let μ be an ergodic IRS on G. We have seen that μ is the stabilizer of some p.m.p. action $G \curvearrowright (X, m)$. By [SZ] the latter action is either essentially free, in which case $\mu = \delta_1$, or transitive, in which case the (random) stabilizer is a subgroup of co-finite volume. The Borel density theorem implies that in the latter case, the stabilizer is either G or a lattice $\Gamma \leq G$. $\square$

The role of Property (T)

Theorem (Glasner–Weiss)

Let G be a group with property (T) acting by homeomorphisms on a compact Hausdorff space Ω. Then the set of ergodic G-invariant probability Borel measures on Ω is w^*-closed.

Thus, the main theorem is equivalent to

Theorem

The only accumulation point of $\{\delta_1, \delta_G, \mu_\Gamma, \Gamma \leq_L G\}$ is δ_1.

The role of Property (T)

Since G is isolated in Sub_G, δ_G is isolated in $\mathrm{IRS}(G)$.

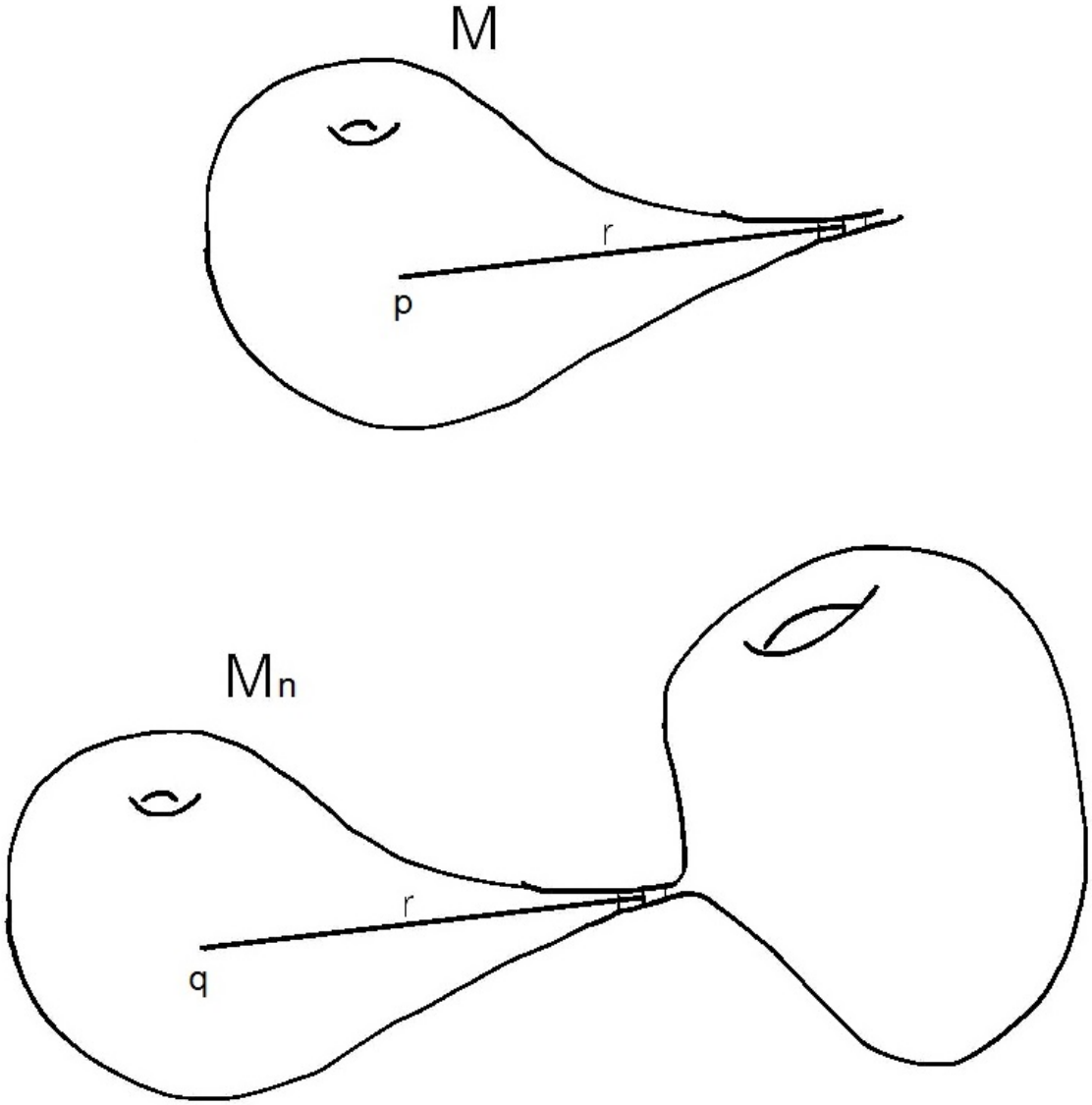
Hence we need only to exclude the case that μ_{Γ_n} converges to μ_Γ for $\Gamma \leq_L G$ a lattice.

Let

$$M = \Gamma \backslash X, \ M_n = \Gamma_n \backslash X.$$

By property (T), the Cheeger constant of X-manifolds is uniformly bounded below.

A picture of the proof



Interesting cases that are still open

Open cases:

1. Rank one groups with property (T), such as $Sp(n,1)$.
2. Higher rank groups without property (T), such as $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$.

   Property (T) is used in: 1. Stuck–Zimmer theorem, 2. Glasner–Weiss theorem and 3. The Cheeger constant argument (avoidable).
3. In any G (even $SL(2,\mathbb{R})$), suppose that Γ_n is a sequence of non-commensurable arithmetic groups, does it necessarily converge to μ_1?

Convergence of Plancherel measures

Suppose now that $k = \mathbb{R}$. For a uniform lattice $\Gamma \leq G$ define the *relative Plancherel measure* associated with $L_2(\Gamma \backslash G)$

$$\nu_\Gamma = \frac{1}{\mathrm{Vol}(G/\Gamma)} \sum_{\pi \in \hat{G}} m(\pi, \Gamma) \delta_\pi$$

where $m(\pi, \Gamma)$ is the multiplicity of π in $L_2(\Gamma \backslash G)$.
Let ν^G denote the Plancherel measure of the right regular representation $L^2(G)$.

Theorem (7s)

Let (Γ_n) be a uniformly discrete Farber sequence of lattices in G. Then for any relatively compact ν_G-regular open subset $S \subset \hat{G}$ or $S \subset \hat{G}_{temp}$ we have

$$\nu_{\Gamma_n}(S) \to \nu_G(S).$$

Convergence of Plancherel measures

For $\pi \in \hat{G}$ let $d(\pi)$ denote the *formal degree* of π in the regular representation. Thus $d(\pi) = 0$ unless π is a discrete series representation.

Corollary (7s)

Let (Γ_n) be a uniformly discrete Farber sequence of lattices in G. Then for all $\pi \in \hat{G}$, we have

$$\frac{m(\pi, \Gamma)}{vol(\Gamma \backslash G)} \to d(\pi).$$

Remark

The result concerning normalized Betti numbers could be deduced from the theorem above, but there is also a cheaper trick to prove it.

Asymptotic of some non-analytic invariants

For a f.g. group Γ let $d(\Gamma)$ denote its 'algebraic rank', i.e. the minimal size of a generating set.

Theorem

Let G be a connected non-compact simple Lie group. There is a constant $C = C(G, \mu)$ such that

$$d(\Gamma) \leq C \cdot vol(G/\Gamma)$$

for every lattice $\Gamma \leq G$.

Conjecture

If $rank_{\mathbb{R}}(G) \geq 2$ the algebraic rank $d(\Gamma)$ is sub-linear w.r.t $vol(G/\Gamma)$.

Asymptotic of some non-analytic invariants

The following results were recently obtained by [Abert,G,Nikolov]:

Theorem

Let G a simple Lie group with $\mathrm{rank}_{\mathbb{R}}(G) \geq 2$ and $\Gamma \leq G$ a 'right angled' lattice. Then for every sequence of finite index subgroups $\Gamma_n \leq \Gamma$ with $|\Gamma : \Gamma_n| \to \infty$, we have

$$\frac{d(\Gamma_n)}{|\Gamma : \Gamma_n|} \to 0.$$

Definition

A group Γ is said to be *right angled if it admits a finite generating set $\{\gamma_1, \ldots, \gamma_n\}$ consisting of non-torsion consecutively commuting elements. I.e. $[\gamma_i, \gamma_{i+1}] = 1,\ i = 1, \ldots, n-1$.*

Theorem

$G = SL(n, \mathbb{R}),\ n \geq 3$ admit right angled co-compact lattices.

Thank you for listening!

Questions?

Another example of a manifold of large volume.