Survey on L^2-torsion

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Outline

- Finite Hilbert chain complexes
- Classical Reidemeister and Whitehead torsion
- L^2-torsion
- Approximation
- Twisting L^2-invariants

Finite Hilbert chain complexes

- Let C_* be a finite Hilbert chain complex, i.e., a sequence of linear maps between finite-dimensional real Hilbert spaces

$$\cdots \xrightarrow{c_{n+1}} C_n \xrightarrow{c_n} C_n \xrightarrow{c_{n-1}} C_{n-2} \xrightarrow{c_{n-2}} \cdots$$

such that $c_n \circ c_{n-1} = 0$ holds for all $n \in \mathbb{Z}$ and there is a natural number N with $C_n = 0$ for $|n| \leq N$.

- Define its homology to be the Hilbert space

$$H_n(C_*) = \ker(c_n)/\operatorname{im}(c_{n+1}).$$

- Define the n-th Betti number

$$b_n(C_*) := \dim_{\mathbb{R}}(H_n(C_*)).$$

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- Define the *n*th Laplace operator

$$\Delta_n := c_{n+1} \circ c_n^* + c_n^* \circ c_{n-1} \colon C_n \to C_n.$$

- There is a baby version of the Hodge deRahm decomposition

$$\ker(\Delta_n) = \ker(c_n) \cap \operatorname{im}(c_{n+1})^{\perp}$$

which induces an isometric isomorphism

$$\ker(\Delta_n) \xrightarrow{\cong} H_n(C_*)$$

- Hence the following assertions are equivalent
  - $b_n(C_*) = 0$ for all $n \in \mathbb{Z}$;
  - C_* is acyclic, i.e., $H_n(C_*) = 0$ for all $n \in \mathbb{Z}$;
  - $\Delta_n \colon C_n \to C_n$ is bijective for all $n \in \mathbb{Z}$.

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  - $\Delta_n \colon C_n \to C_n$ is bijective for all $n \in \mathbb{Z}$.

- If C_* is acyclic, we can define its L^2-torsion

$$\rho^{(2)}(C_*) := -\frac{1}{2} \cdot \sum_{n \in \mathbb{Z}} (-1)^n \cdot n \cdot \ln(\det(\Delta_n)) \quad \in \mathbb{R}.$$

- If $f_* \colon C_* \to D_*$ is a chain homotopy equivalence of finite Hilbert chain complexes, we can define its L^2-torsion

$$\tau^{(2)}(f_*) := \rho^{(2)}(\mathrm{cone}(f_*)) \quad \in \mathbb{R}.$$

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- Additivity: If $0 \to C_* \xrightarrow{i_*} D_* \xrightarrow{p_*} E_* \to 0$ is an exact sequence of finite acyclic Hilbert chain complexes such that i_n and the induced map $p_n|\colon \ker(p_n)^{\perp} \to E_n$ are isometric for every n, then

$$\rho^{(2)}(D_*) = \rho^{(2)}(C_*) + \rho^{(2)}(E_*).$$

- Difference property: If $f_*\colon C_* \to D_*$ is a chain homotopy equivalence of acyclic finite Hilbert chain complexes, then

$$\tau^{(2)}(f_*) = \rho^{(2)}(D_*) - \rho^{(2)}(C_*).$$

- Composition formula: If $f_*\colon C_* \to D_*$ and $g_*\colon D_* \to E_*$ are chain homotopy equivalences of finite Hilbert chain complexes, then

$$\tau^{(2)}(g_* \circ f_*) = \tau^{(2)}(g_*) + \tau^{(2)}(f_*).$$

Classical Reidemeister and Whitehead torsion

- Let X be a connected finite CW-complex with fundamental group π and V be a finite-dimensional orthogonal π-representation. Then the cellular $\mathbb{Z}\pi$-basis for the cellular $\mathbb{Z}\pi$-chain complex $C_*(\widetilde{X})$ of its universal covering $\widetilde{X}$ and the Hilbert space structure on V induce the structure of a finite Hilbert chain complex on

$$C_*^{(2)}(\widetilde{X};V) := V \otimes_{\mathbb{Z}\pi} C_*(\widetilde{X}).$$

- Hence we can define the Reidemeister torsion

$$\rho^{(2)}(\widetilde{X};V) := \rho^{(2)}(C_*^{(2)}(\widetilde{X};V)) \quad \in \mathbb{R}$$

provided that $C_*^{(2)}(\widetilde{X};V)$ is acyclic.

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provided that $C_*^{(2)}(\widetilde{X};V)$ is acyclic.

- Given a homotopy equivalent $f \colon X \to Y$ of connected finite CW-complexes, one can define its Whitehead torsion

$$\tau(f) \in \mathrm{Wh}(\pi_1(Y)).$$

It vanishes if and only if f is a simple homotopy equivalence.

- One of its main applications is the

Theorem (*s*-Cobordism Theorem)

Let $(W; \partial_0 M, \partial_1 W)$ be an h-cobordism of dimension ≥ 5.

Then it is diffeomorphism relative ∂W_0 to a cylinder $\partial_o W \times [0, 1]$ if and only if the inclusion $\partial_0 W \to W$ is a simple homotopy equivalence.

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- The following conjecture is known to be true for a large class of groups, including hyperbolic groups, CAT(0)-groups, fundamental groups of 3-manifolds and lattices in almost connected Lie groups.

Conjecture

If G is torsionfree, then $\mathrm{Wh}(G)$ *is trivial.*

- Let V be unitary free $\mathbb{Z}/m$-representation. Define the associated lens space $L(V)$ to be the oriented closed Riemannian manifold of constant positive sectional curvature given by $SV/(\mathbb{Z}/m)$.

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- Using Reidemeister torsion one can show:

Theorem (Classification of lens spaces (Reidemeister, Franz))

For two lens spaces $L(V)$ and $L(W)$ the following assertions are equivalent:

- *There is an automorphism $\alpha\colon \mathbb{Z}/m \to \mathbb{Z}/m$ such that V and $\alpha^* W$ are isomorphic as orthogonal G-representations;*
- *There is an isometric diffeomorphism $L(V) \to L(W)$;*
- *There is a diffeomorphism $L(V) \to L(W)$;*
- *There is a homeomorphism $L(V) \to L(W)$;*
- *There is a simple homotopy equivalence $L(V) \to L(W)$.*

- There are lens spaces which are homotopy equivalent but not homeomorphic.

L^2-torsion

- There is a canonical choice for V which is however not finite-dimensional, namely $V = L^2(\pi)$.
- In context with Betti numbers that one can extend this notion successfully to L^2-Betti numbers using this representation $L^2(G)$, compare the talks by Sauer.
- We want to do the same for Reidemeister torsion which finally leads to L^2-torsion.

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- Recall the cellular $\mathcal{N}(\pi)$-Hilbert chain complex

$$C_*^{(2)}(\widetilde{X}) := L^2(\pi) \otimes_{\mathbb{Z}\pi} C_*(\widetilde{X})$$

associated to the universal covering $\widetilde{X}$ of a connected finite CW-complex X. It looks like

$$\ldots \xrightarrow{c_{n+1}^{(2)}} L^2(\pi)^{d_n} \xrightarrow{c_n^{(2)}} L^2(\pi)^{d_{n-1}} \xrightarrow{c_{n-1}^{(2)}} L^2(\pi)^{d_{n-2}} \xrightarrow{c_{n-2}^{(2)}} \ldots$$

where d_n is the number of n-cells in X and each $c_n^{(2)}$ is a bounded π-equivariant operator.

- There is L^2-homology

$$H_n^{(2)}(\widetilde{X}) := \ker(c_n^{(2)})/\overline{\operatorname{im}(c_{n+1}^{(2)}},$$

and the L^2-Betti number

$$b_n^{(2)}(\widetilde{X}) := \dim_{\mathcal{N}(\pi)}\big(H_n^{(2)}(\widetilde{X})\big) \quad [0, \infty).$$

- Now $\widetilde{X}$ is L^2-acyclic, i.e., $b_n^{(2)}(\widetilde{X}) = 0$ for all $n \in \mathbb{Z}$, if and only if for every $n \in \mathbb{Z}$ the nth Laplace operator

$$\Delta_n := c_{n+1}^{(2)} \circ \left(c_n^{(2)}\right)^* + \left(c_n^{(2)}\right)^* \circ c_{n-1}^{(2)} : C_n^{(2)}(\widetilde{X}) \to C_n^{(2)}(\widetilde{X})$$

is a weak isomorphism, i.e., is injective and has dense image.

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- If we want to copy the definition for Reidemeister torsion above, we have to make sense of

$$\ln(\det(f)) \in \mathbb{R}$$

for a weak isomorphism $f\colon L^2(G)^k \to L^2(G)^k$.

- Consider a positive operator $f\colon V \to V$ of finite-dimensional Hilbert spaces with trivial kernel. Let $0 < \lambda_1 < \lambda_2 < \lambda_3 < \ldots$ be its eigenvalues and μ_i be the multiplicity of λ_i.
Then

$$\ln(\det(f)) = \sum_{i \geq 1} \mu_i \cdot \ln(\lambda_i).$$

- Define the spectral density function $F\colon [0, \infty) \to [0, \infty)$ to be the right-continuous step function, which has a jump at each of the eigenvalues of height its multiplicity, and which is zero for $\lambda < 0$.

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- We can write

$$\ln(\det(f)) = \sum_{i\geq 1} \mu_i \cdot \ln(\lambda_i) = \int_{0+}^{\infty} \ln(\lambda) dF$$

  where dF is the measure on $\mathbb{R}$ associated to the monotone increasing right-continuous function F which is given by $dF\big(]a,b]\big) := F(b) - F(a)$.

- If $f\colon L^2(G)^k \to L^2(G)^k$ is a positive bounded G-equivariant operator, we define its spectral density function

$$F(f)(\lambda) := \dim_{\mathcal{N}(G)}\big(\operatorname{im}(E_\lambda^f)\big) = \operatorname{tr}_{\mathcal{N}(G)}\big(E_\lambda^f\big)$$

  where $E_\lambda^f\colon L^2(G)^k \to L^2(G)^k$ is its spectral projection for $\lambda \geq 0$.

- Now the following expression makes sense:

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- We define the logarithm of the Fuglede-Kadison determinant

$$\ln(\det(f)) := \int_{0+}^{\infty} \ln(\lambda) dF \in \mathbb{R},$$

provided that $\int_{0+}^{\infty} \ln(\lambda) dF > -\infty$ holds.

- We have $\int_{0+}^{\infty} \ln(\lambda) dF > -\infty$ if f is bijective, but there are weak isomorphisms f with $\int_{0+}^{\infty} \ln(\lambda) dF = -\infty$.

- If $G = \mathbb{Z}$ and $p(z)$ is a non-trivial complex polynomial in one variabel, then the Mahler measure as mentioned in the talk by Gelander is the square root of the Fuglede-Kadison determinant of the operator $L^2(\mathbb{Z}) \to L^2(\mathbb{Z})$ given by multiplication with $p(z) \cdot \overline{p(z)}$.

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- Now the observation comes into play that the Laplace operator coming from a cellular structure lives already over the integers and the following

Conjecture (Determinant Conjecture)

Let $A \in M_{a,b}(\mathbb{Z}G)$ be a matrix. It defines a bounded G-equivariant operator $r_A^{(2)} \colon L^2(G)^m \to L^2(G)^n$. We have

$$\ln\left(\det((r_A^{(2)})^* \circ r_A^{(2)})\right) := \int_{0+}^{\infty} \ln(\lambda)\,dF \geq 0.$$

- This conjecture is known for a very large class of groups, for instance for all sofic groups.

- Therefore we will tacitly assume that this conjecture holds and $\ln(\det(\Delta_n))$ is defined for the n-th Laplace operator Δ_n.

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Definition (L^2-torsion (L.-Rothenberg))

Let X be a connected finite CW-complex. Then we define the L^2-torsion

$$\rho^{(2)}(\widetilde{X}) := -\frac{1}{2} \cdot \sum_{n \geq 0} (-1)^n \cdot n \cdot \ln(\det(\Delta_n)) \quad \in \mathbb{R}.$$

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- The definition above extends to finite CW-complexes by taking the sum of the L^2-torsion for each path component.
- There is also an analytic definition in terms of heat kernels of the universal covering of a closed Riemannian manifold due to Matthey and Lott. Both approaches have been identified by Burghelea-Friedlander-Kappeler-Mc Donald.
- Explicit computations and the proof of some general properties are based on both approaches.
- Next we record the basic properties of L^2-torsion. It behaves similar to the Euler characteristic.

Theorem (Simple homotopy invariance)

Let $f\colon X \to Y$ be a homotopy equivalence of finite CW-complexes. Suppose that $\widetilde{X}$ and hence also $\widetilde{Y}$ are L^2-acyclic.

Then there is a homomorphism depending only on π

$$\Phi_\pi\colon \operatorname{Wh}(\pi) \to \mathbb{R}$$

sending $\tau(f)$ to $\rho^{(2)}(\widetilde{Y}) - \rho^{(2)}(\widetilde{X})$.

- If $\operatorname{Wh}(\pi)$ vanishes, the L^2-torsion is a homotopy invariant.

Theorem (Sum formula)

Let X be a finite CW-complex with subcomplexes X_0, X_1 and X_2 satisfying $X = X_1 \cup X_2$ and $X_0 = X_1 \cap X_2$. Suppose $\widetilde{X_0}$, $\widetilde{X_1}$ and $\widetilde{X_2}$ are L^2-acyclic and the inclusions $X_i \to X$ are π-injective.

Then $\widetilde{X}$ is L^2-acyclic and we get

$$\rho^{(2)}(\widetilde{X}) = \rho^{(2)}(\widetilde{X_1}) + \rho^{(2)}(\widetilde{X_2}) - \rho^{(2)}(\widetilde{X_0}).$$

Theorem (Fibration formula)

Let $F \to E \to B$ be a fibration of connected finite CW-complexes such that $\widetilde{F}$ is L^2-acyclic and the inclusion $F \to E$ is π-injective.

Then $\widetilde{E}$ is L^2-acyclic and we get

$$\rho^{(2)}(\widetilde{E}) = \chi(B) \cdot \rho^{(2)}(\widetilde{F}).$$

- By Poincaré duality we have $\rho^{(2)}(\widetilde{M}) = 0$ for every even dimensional closed manifold M, provided that $\widetilde{M}$ is L^2-acyclic.

- The L^2-torsion is multiplicative under finite coverings, i.e., if $X \to Y$ is a d-sheeted covering of connected finite CW-complexes and $\widetilde{X}$ is L^2-acyclic, then $\widetilde{Y}$ is L^2-acyclic and

$$\rho^{(2)}(\widetilde{X}) = d \cdot \rho^{(2)}(\widetilde{Y}).$$

- In particular $\widetilde{S^1}$ is L^2-acyclic and

$$\rho^{(2)}(\widetilde{S^1}) = 0.$$

Theorem (S^1-actions on aspherical manifolds (L.))

Let M be an aspherical closed manifold with non-trivial S^1-action.

Then $\widetilde{M}$ is L^2-acyclic and

$$\rho^{(2)}(\widetilde{M}) = 0.$$

Theorem (L^2-torsion and aspherical CW-complexes (L.))

Let X be an aspherical finite CW-complex. Suppose that its fundamental group $\pi_1(X)$ contains an elementary amenable infinite normal subgroup.

Then $\widetilde{X}$ is L^2-acyclic and

$$\rho^{(2)}(\widetilde{X}) = 0.$$

Theorem (S^1-actions on aspherical manifolds (L.))

Let M be an aspherical closed manifold with non-trivial S^1-action.

Then $\widetilde{M}$ is L^2-acyclic and

$$\rho^{(2)}(\widetilde{M}) = 0.$$

Theorem (L^2-torsion and aspherical CW-complexes (L.))

Let X be an aspherical finite CW-complex. Suppose that its fundamental group $\pi_1(X)$ contains an elementary amenable infinite normal subgroup.

Then $\widetilde{X}$ is L^2-acyclic and

$$\rho^{(2)}(\widetilde{X}) = 0.$$

Theorem (Hyperbolic manifolds (Hess-Schick))

There are (computable) rational numbers $r_n > 0$ such that for every hyperbolic closed manifold M of odd dimension $2n+1$ the universal covering $\widetilde{M}$ is L^2-acyclic and

$$\rho^{(2)}(\widetilde{M}) = (-1)^n \cdot \pi^{-n} \cdot r_n \cdot \mathrm{vol}(M).$$

- Since for every hyperbolic manifold M we have $\mathrm{Wh}(\pi_1(M)) = 0$, we rediscover the fact that the volume of an odd-dimensional hyperbolic closed manifold depends only on $\pi_1(M)$.
- We also rediscover the theorem that any S^1-action on a closed hyperbolic manifold is trivial.

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- We also rediscover the theorem that any S^1-action on a closed hyperbolic manifold is trivial.

- The proof is based on the fact that the analytic version of L^2-torsion is of the shape

$$\rho^{(2)}(\widetilde{M}) = \int_{\mathcal{F}} f(x)\, d\mathrm{vol}_{\mathbb{H}^{2n+1}}$$

where $\mathcal{F}$ is a fundamental domain of the π-action on the hyperbolic space $\mathbb{H}^{2n+1}$ and $f(x)$ is an expression in terms of the heat kernel $k(x,x)(t)$.

- By the symmetry of $\mathbb{H}^{2n+1}$ this function $k(x,x)(t)$ is independent of x and hence $f(x)$ is independent of x.

- If we take $r_n = (-1)^n \cdot \pi^n \cdot f(x)$ for any $x \in \mathbb{H}^{2n+1}$, we get

$$\int_{\mathcal{F}} f(x)\, d\mathrm{vol}_{\mathbb{H}^{2n+1}} = (-1)^n \cdot \pi^{-n} \cdot r_n \cdot \mathrm{vol}(\mathcal{F}) = (-1)^n \cdot \pi^{-n} \cdot r_n \cdot \mathrm{vol}(M).$$

- We have $r_1 = \frac{1}{6}$, $r_2 = \frac{31}{45}$, $r_7 = \frac{221}{70}$.

Theorem (Lott-L-,L.-Schick)

Let M be an irreducible closed 3-manifold with infinite fundamental group. Let M_1, M_2, ..., M_m be the hyperbolic pieces in its Jaco-Shalen decomposition.

Then $\widetilde{M}$ is L^2-acyclic and

$$\rho^{(2)}(\widetilde{M}) := -\frac{1}{6\pi} \cdot \sum_{i=1}^{m} \mathrm{vol}(M_i).$$

- The proof of the result above is based on the meanwhile approved Thurston Geometrization Conjecture. It reduces the claim to Seifert manifolds with incompressible torus boundary and to hyperbolic manifolds with incompressible torus boundary using the sum formula. The Seifert pieces are treated analogously to aspherical closed manifolds with S^1-action. The hyperbolic pieces require a careful analysis of the cusps.

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Approximation

- The following conjecture combines and generalizes Conjectures by Bergeron-Venkatesh, Hopf, Singer, L., and L.-Shalen.
- If G is a finitely generated group, we denote by $d(G)$ the minimal number of generators.
- We denote by RG the rank gradient introduced by Lackenby.
- A chain for a group G is a sequence of in G normal subgroups

$$G = G_0 \supseteq G_1 \supseteq G_2 \supseteq \cdots$$

such that $[G : G_i] < \infty$ and $\bigcap_{i \geq 0} G_i = \{1\}$.

Conjecture (Homological growth and L^2-invariants for aspherical closed manifolds)

Let M be an aspherical closed manifold of dimension d and fundamental group $G = \pi_1(M)$. Let $\widetilde{M}$ be its universal covering. Then

1.) *For any natural number n with $2n \neq d$ we get*

$$b_n^{(2)}(\widetilde{M}) = 0.$$

If $d = 2n$, we have

$$(-1)^n \cdot \chi(M) = b_n^{(2)}(\widetilde{M}) \geq 0.$$

If $d = 2n$ and M carries a Riemannian metric of negative sectional curvature, then

$$(-1)^n \cdot \chi(M) = b_n^{(2)}(\widetilde{M}) > 0;$$

Conjecture (Continued)

2.) *Let $(G_i)_{i\geq 0}$ be any chain. Put $M[i] = G_i\backslash\widetilde{M}$.*
Then we get for any natural number n and any field F

$$b_n^{(2)}(\widetilde{M}) = \lim_{i\to\infty} \frac{b_n(M[i];F)}{[G:G_i]} = \lim_{i\to\infty} \frac{d\big(H_n(M[i];\mathbb{Z})\big)}{[G:G_i]};$$

and for $n = 1$

$$b_1^{(2)}(\widetilde{M}) = \lim_{i\to\infty} \frac{b_1(M[i];F)}{[G:G_i]} = \lim_{i\to\infty} \frac{d\big(G_i/[G_i,G_i]\big)}{[G:G_i]}$$
$$= RG(G,(G_i)_{i\geq 0}) = \begin{cases} 0 & d \neq 2; \\ -\chi(M) & d = 2; \end{cases}$$

Conjecture (Continued)

3.) *If $d = 2n + 1$ is odd, we have*

$$(-1)^n \cdot \rho^{(2)}(\widetilde{M}) \geq 0;$$

If $d = 2n + 1$ is odd and M carries a Riemannian metric with negative sectional curvature, we have

$$(-1)^n \cdot \rho^{(2)}(\widetilde{M}) > 0;$$

Conjecture (Continued)

4.) *Let $(G_i)_{i \geq 0}$ be a chain. Put $M[i] = G_i \backslash \widetilde{M}$.*
Then we get for any natural number n with $2n+1 \neq d$

$$\lim_{i \to \infty} \frac{\ln\left(\left|\operatorname{tors}\left(H_n(M[i])\right)\right|\right)}{[G : G_i]} = 0,$$

and we get in the case $d = 2n+1$

$$\lim_{i \to \infty} \frac{\ln\left(\left|\operatorname{tors}\left(H_n(M[i])\right)\right|\right)}{[G : G_i]} = (-1)^n \cdot \rho^{(2)}\left(\widetilde{M}\right) \geq 0.$$

- The conjecture above is very optimistic, but we do not know a counterexample.
- It is related to the Approximation Conjecture for Fuglede-Kadison determinant.
- The main issue here are uniform estimates about the spectrum of the n-th Laplace operators on $M[i]$ which are independent of i.
- Abert-Nikolov have settled the rank gradient part if G contains an infinite normal amenable subgroup.
- Kar-Kropholler-Nikolov have settled the part about the growth of the torsion in the homology if G is infinite amenable.
- Abert-Gelander-Nikolov deal with the rank gradient and the growth of the torsion in the homology for right angled lattices.
- Li-Thom deal with the vanishing of L^2-torsion for amenable G.
- Bridson-Kochloukova deal with limit groups, where the limits are not necessarily zero.
- There are connections to the talks by Bader, Bridson, Boileau, Calegari, Davis, Gelander, Nikolov, and Sauer.

Theorem (L.)

Let M be an aspherical closed manifold with fundamental group $G = \pi_1(M)$. Suppose that M carries a non-trivial S^1-action or suppose that G contains a non-trivial elementary amenable normal subgroup. Then we get for all $n \geq 0$ and fields F and any chain $(G_i)_{i \geq 0}$

$$\begin{aligned} \lim_{i \to \infty} \frac{b_n(M[i]; F)}{[G : G_i]} &= 0; \\ \lim_{i \to \infty} \frac{d\big(H_n(M[i]; \mathbb{Z})\big)}{[G : G_i]} &= 0; \\ \lim_{i \to \infty} \frac{\ln\big(\big|\operatorname{tors}\big(H_n(M[i])\big)\big|\big)}{[G : G_i]} &= 0; \\ b_n^{(2)}(\widetilde{M}) &= 0; \\ \rho^{(2)}(\widetilde{M}) &= 0. \end{aligned}$$

- Let M be a closed hyperbolic 3-manifold. Then the conjecture above predicts for any chain $(G_i)_{i \geq 0}$

$$\lim_{i \to \infty} \frac{\ln\left(\left|\operatorname{tors}\left(H_1(G_i)\right|\right)}{[G : G_i]} = \frac{1}{6\pi} \cdot \operatorname{vol}(M).$$

Since the volume is always positive, the equation above implies that $|\operatorname{tors}\left(H_1(G_i)\right)|$ growth exponentially in $[G : G_i]$.

- In particular this would allow to read of the volume from the profinite completion of $\pi_1(M)$, compare with the talk by Boileau and Bridson.

Twisting L^2-invariants

- Consider a CW-complex X with $\pi = \pi_1(M)$. Suppose that $\widetilde{X}$ is L^2-acyclic.
- Fix an element $\phi \in H^1(X;\mathbb{Z}) = \hom(\pi;\mathbb{Z})$.
- For $t \in (0,\infty)$, let $\phi^*\mathbb{C}_t$ be the 1-dimensional complex π-representation given by

$$w \cdot \lambda := t^{\phi(w)} \cdot \lambda \quad \text{for } w \in \pi, \lambda \in \mathbb{C}.$$

- One can twist the L^2-chain complex of X with this representation, or, equivalently, apply the following ring homomorphism to the cellular $\mathbb{Z}G$-chain complex before passing to the Hilbert space completion

$$\mathbb{C}G \to \mathbb{C}G, \quad \sum_{g \in G} \lambda_g \cdot g \mapsto \sum_{g \in G} \lambda_g \cdot t^{\phi(g)} \cdot g.$$

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- Now one would like to know whether after twisting the Hilbert chain complex is still L^2-acyclic and whether the Fuglede-Kadison determinant is defined for the twisted Laplace operators.
- Notice that for irrational t the relevant chain complexes do not have coefficients in $\mathbb{Q}G$ anymore and the Determinant Conjecture does not apply. So the answer is only clear for rational t but not for irrational t.
- Hence the following question becomes relevant since one hopes to argue by continuity.

Question (Continuity of the regular determinant)

Let G be a group for which there exists a bound on the order of finite subgroups, e.g., G is torsionfree. Let $S \subseteq G$ be a finite subset. Put $\mathbb{C}[n,S] := \{A \in M_{n,n}(\mathbb{C}G) \mid \operatorname{supp}_G(A) \subseteq S\}$ *and equip it with the standard topology coming from the structure of a finite-dimensional complex vector space.*

Is then the function

$$\mathbb{C}[n,S] \to [0,\infty]$$

given by

$$A \mapsto \begin{cases} \det_{\mathcal{N}(\Gamma)}(r_A^{(2)})) & \text{if } r_A^{(2)} \text{ is a weak isomorphism with } \int_{0+}^{\infty} \ln(\lambda)\, dF > -\infty; \\ 0 & \text{otherwise,} \end{cases}$$

continuous?

- Suppose that G is a finitely generated abelian group. Then the answer to the question above is positive.

- If one drops in question above the condition that there is a bound on the order of finite subgroups, then there are counterexamples coming from the work of Dicks-Schick on counterexamples to the Atiyah Conjecture.

- If one discards the finite set S, one may use different topologies on the source, but for any reasonable topology one gets already counterexamples in the case $G = \mathbb{Z}$.

Theorem (L.)

Suppose that $\widetilde{X}$ is L^2-acyclic and its fundamental group is residually finite.

1. *The L^2-torsion function*

$$\rho^{(2)}(\widetilde{X};\phi)\colon (0,\infty) \to \mathbb{R}$$

is well-defined;

2. *The limits $\lim_{t\to\infty} \frac{\rho^{(2)}(\widetilde{X};\phi)(t)}{\ln(t)}$ and $\lim_{t\to 0} \frac{\rho^{(2)}(\widetilde{X};\phi)(t)}{\ln(t)}$ exist and we can define the degree of ϕ*

$$\deg(X;\phi) \in \mathbb{R}$$

to be their difference.

- Its value of $\rho^{(2)}(\widetilde{X};\phi)$ at $t = 1$ is just the L^2-torsion;
- The basic properties of the L^2-torsion such as simple homotopy invariance, sum formulas, fibration formula and so on carry over to the L^2-torsion function;
- Poincaré duality says for $d = \dim(M)$

$$\rho(\widetilde{X};\phi)(t) = (-1)^{n+1} \cdot \rho(\widetilde{X};\phi)(t^{-1}).$$

- On the analytic side this corresponds for closed Riemannian manifold M to twisting with the flat line bundle $\widetilde{M} \times_\pi \mathbb{C}_t \to M$. It is obvious that some work is necessary to show that this is a well-defined invariant since the π-action on $\mathbb{C}_t$ is not isometric.

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Definition (Thurston norm)

Let M be a 3-manifold and $\phi \in H^1(M;\mathbb{Z})$ be a class. Define its Thurston norm

$$x_M(\phi) = \min\{\chi_-(F) \mid F \text{ embedded surface in } M \text{ dual to } \phi\}$$

where

$$\chi_i(F) = \sum_{C\in\pi_0(M)} \max\{-\chi(C), 0\}.$$

- Thurston showed that this definition extends to the real vector space $H^1(M;\mathbb{R})$ and defines a seminorm on it.
- If $F \to M \xrightarrow{p} S^1$ is a fiber bundle and $\phi = \pi_1(p)$, then

$$x_M(\phi) = \chi(F).$$

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$$x_M(\phi) = \chi(F).$$

Theorem (Friedl-L.)

Let M be a 3-manifold. Then for every $\phi \in H^1(M;\mathbb{Z})$ we get the equality

$$\deg(M;\phi) = x_M(\phi).$$

- The proof of the result above is based on the meanwhile approved Thurston Geometrization Conjecture, the Virtual Fibering Theorem of Agol-Wise and approximation techniques about Fuglede-Kadison determinants.