L^2 BETTI NUMBERS AND THEIR APPROXIMATION BY FINITE-DIMENSIONAL ANALOGUES

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Let M be a manifold or a CW-complex whose fundamental group $\Gamma = \pi_1(M)$ is *residually finite*. That is, Γ possesses a decreasing sequence – called a *residual chain* – of normal subgroups $\Gamma_i < \Gamma$ of finite index whose intersection is trivial. By covering theory there is an associated sequence of finite regular coverings $\ldots \to M_2 \to M_1 \to M$ of M with $\pi_1(M_i) = \Gamma_i$, which we call a *residual tower of finite coverings*.

How does the size of the homology of M_i grow as $i \to \infty$?

If M happens to be aspherical, then homology of M_i is nothing else than the group homology of Γ_i.

What do we mean by size? If we measure size by Betti numbers $b_k(M_i) = \mathrm{rk}_{\mathbf{Z}} H_k(M_i; \mathbf{Z})$, there is a general answer: the limit of $b_k(M_i)/[\Gamma : \Gamma_i]$ is the k-th ℓ^2-Betti number of M by the approximation theorem of Lück. If we measure size by mod p Betti numbers or in terms of the cardinality of the torsion subgroups $\mathrm{tors}\, H_k(M_i; \mathbf{Z}) \subset H_k(M_i; \mathbf{Z})$, no general answer is available.

Surprisingly, this question turns out to be at the crossroads of a number of diverse areas of research such as topology and L^2-invariants, geometric group theory, and number theory.

The minicourse will start by a general introduction to ℓ^2-Betti numbers. We proceed by explaining in full detail an easy proof of Lück's approximation theorem, which links the homology growth in characteristic zero to L^2-Betti numbers. We also deal with approximation theorems for $\mathbf{F}_p$-Betti numbers. Finally, we will explain a version of the approximation theorem for lattices in totally disconnected groups (based on joint work with Petersen and Thom) and connect our discussion to the topic of invariant random subgroups.