Ventotene

6 – 13 . 9 2015

Ref "Thin groups and superstrong approximation", MSRI Publ 61 2014

Coverings and expanders

Outline :

1) Introduction → { def of expander, ~~equivalent definitions~~

Expanders and coverings : where do the links come from ?

Examples of results :

- $\left\{ \begin{array}{l} \lambda_1^R \longleftrightarrow \lambda_1^{comb} \\ h^R \longleftrightarrow h^{comb} \end{array} \right.$

- gonality ⟷ expansion

- Heegaard genus ⟷ "

- knot distorsion ⟷ "

- Sieve (?)

2) What graphs are expanders ?

~~[illegible]~~

- Property (T)

- ——— (τ)

↳ { Selberg, etc, Clozel, $\boxed{\text{Thin groups}}$ Bourgain – Varjú – Salehi - Golsefidy

Key ingredients :
- Growth
- Nb of short loops
- Quasi-randomness

some ideas used in proving expansion

3) Indications on some applications

Burger?
Gromov–Guth?
Gonalicty?

Lecture 1

Introduction

1 – Expanders

We will discuss in this lecture the definition of expanders, and survey the remarkable geometric results arithmetic that this <u>combinatorial</u> definition implies in a variety of settings through links with coverings.

We start with the base definition: intuitively, a <u>family</u> of graphs is <u>an expander</u> if it is:

(1) growing in size (nb. of vertices)

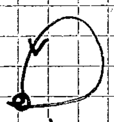
(2) "sparse" (valency bounded)

(3) "highly and robustly" connected: it is <u>not</u> possible to disconnect the graphs by cutting only "a few" edges.

Before formalizing this, a few words on what we mean by a graph:

(1) our graphs are <u>not</u> oriented

(2) we allow multiple edges and loops



[(3) our graphs are <u>purely</u> combinatorial]

We will say that a graph is <u>finite</u> if both vertex and edge sets are finite. The degree/valency of a vertex x is the number of edges from x. A graph ~~will be~~ is <u>regular</u> if the degree is the same at every x.

(but we write $|\Gamma| = |\{\text{vertices}\}|$, $x \in \Gamma$ for x vertex)

<u>Def.</u> (Expander family)

A family $(\Gamma_i)_{i \in I}$ of finite graphs is called an <u>expander</u> family $\iff$

(1) $|\Gamma_i| \to \infty$

(2) $\exists\, C \geqslant 1,\ \forall i\ \forall x \in \Gamma_i,\ \deg(x) \leq C$

(3) $\exists\, \delta > 0,\ \forall i,\ h(\Gamma_i) \geqslant \delta > 0$

where

$$h(\Gamma) = \inf_{\substack{S \subset \Gamma \\ 1 \leq |S| \leq \frac{|\Gamma|}{2}}} \frac{|\partial S|}{|S|}$$

$|\{$edges with one extremity in S, one outside $S\}|$

3 bis

Exemple :

$$G = \mathfrak{S}_n, \quad n \geq 2$$

$$S_n = \{(1,2),\ (1,2,\dots,n),\ (1,\dots,n)^{-1}\}$$

$$\dim \mathcal{G}(\mathfrak{S}_n, S_n) \simeq \frac{n^2}{(\log n)^2}$$

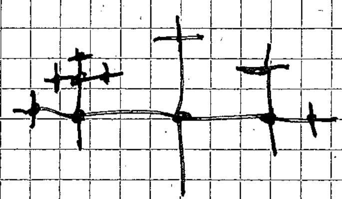
$$h(\quad) \ll$$

Ex.

$$\mathcal{G}(\mathbb{Z}^2, \{\pm e_i\})$$

$$\mathcal{G}(\mathbb{Z}/n\mathbb{Z}, \{\pm 1\})$$

$$\mathcal{G}(F_2, \{a^{\pm 1}, b^{\pm 1}\})$$



Ex.

$$\Gamma = SL_m(\mathbb{Z})$$

$$\Gamma_q = \ker(SL_m(\mathbb{Z}) \to SL_m(\mathbb{Z}/q\mathbb{Z}))$$

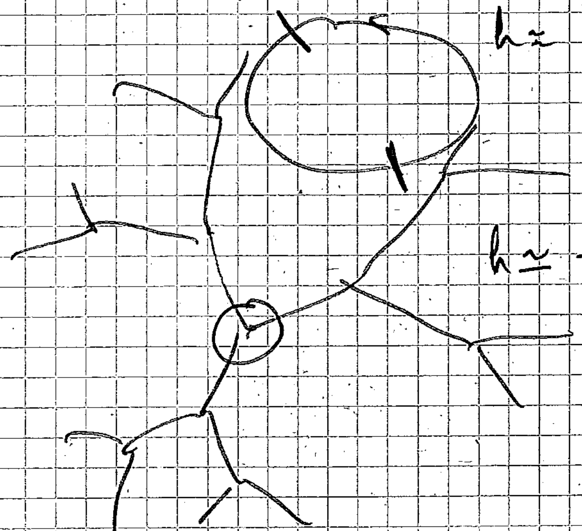
Examples. At first the existence of expanders is by no means clear because $h(\Gamma)$ is a rather complicated invariant to compute and simple examples of sequences of graphs have $h(\Gamma) \to 0$ as $|\Gamma| \to \infty$.

$h \simeq \frac{2}{n/2} \simeq \frac{4}{n}$

Ex. – cycles

– trees

(despite having small diameter!)



$h \simeq \frac{3}{n/3} \simeq \frac{9}{n}$

– [illegible]

Remark. If (Γ_i) is a ~~seque~~ family of expanders, it is an elementary exercise that

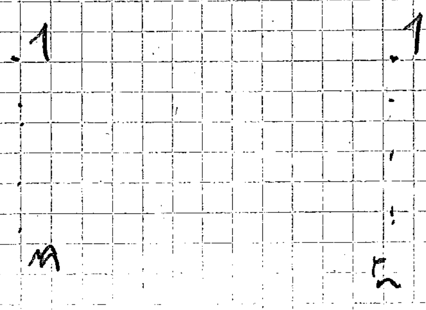
$\exists C \geq 1, \quad \mathrm{diam}(\Gamma_i) \leq C \log |\Gamma_i|$.

This is optimal order of magnitude for sparse graphs.

However, there are lots of expanders.

Th. (Barzdin–Kolmogorov, Pinsker)

Consider $n \geq 1$, form a graph $\Gamma_{\sigma_1, \sigma_2 \sigma_3}$ with vertex set



and edges ~~from~~ joining

$$i \longleftrightarrow \sigma_1(i)$$

$$i \longleftrightarrow \sigma_2(i)$$

$$i \longleftrightarrow \sigma_3(i)$$

where $(\sigma_1, \sigma_2, \sigma_3) \in \mathfrak{S}_n$ are permutations chosen independently at random. Then

$$\exists \delta > 0, \quad \liminf_{n \to \infty} \quad \mathbb{P}\left(h\left(\Gamma_{\sigma_1, \sigma_2, \sigma_3}\right) \geq \delta \right) > 0.$$

(In fact, for δ small enough this lim $\to 1$).

Our interest will however be in cases where the graphs we want to be expanders are not to be chosen as we wish: they are given to us by applications of various kinds. Either these are, or are not, expanders...

Before ~~we~~ we go in this direction, which has ~~seen~~ seen extraordinary progress during the last 10 years, we need to see why we should ~~care~~ care.

2. Coverings and expanders

The link between graphs and coverings and then expanders has to do with Cayley and Schreier graphs.

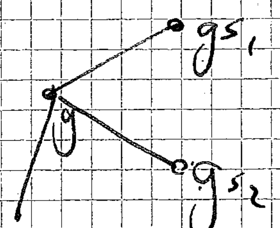
We recall again the definition:

G is a group

$S \subset G$ is a subset, which for simplicity we <u>always</u> assume to be symmetric ($S = S^{-1}$)

Then $\mathcal{G}(G, S)$ is the graph with

- vertices G
- edges



for $s_i \in S$

So it is a $|S|$-regular graph.

We will be particularly interested in families of Cayley graphs constructed as follows:

G is an infinite discrete group

$S \subset G$ is a symmetric, <u>finite</u>, generating set

$(H_i)_{i \in I}$ is a family of <u>finite-index</u> subgroups $H_i \lhd G$

$\pi_i : G \longrightarrow H_i$ is the projection

<u>and</u> we look at

$$\left(\mathcal{G}\left(G/H_i, \pi_i(S) \right) \right)_{i \in I}$$

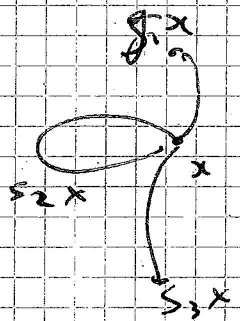
These are <u>connected</u> graphs because $\pi_i(S)$ generates G/H_i

The following variants are equally useful:

G group acting on X

$S \subset G$ symmetric

$\mathcal{G}(X, S)$ = graph with vertices X and edges



and then:

G discrete, $S = S^{-1}$ generating

$H_i < G$ subgroups of finite index

$$\left(\mathcal{G}\left(G/H_i, S\right)\right)_{i \in I}$$

is a family of $\underbrace{\text{connected}}_{\text{finite}}$ graphs

Now apply this to a (connected) manifold M with fundamental group G of finite type, generated by $S \subset G$ symmetric

(case $H_i \lhd G$)

$$\begin{tikzcd}
\hat{M} \arrow[d, "H_i"] \\
M_i \arrow[d, "G/H_i"] \\
M
\end{tikzcd}$$

G

Galois covering

or

$$\begin{tikzcd}
\hat{M} \arrow[d, "H"] \\
M_i \arrow[d, "\text{“}G/H_i\text{”}"] \\
M
\end{tikzcd}$$

(or more generally, do this for any category of objects in which it makes sense to speak of (Galois) coverings:

- field extensions
- schemes
- ...
)

The basic principle is:

Principle: if $(M_i \longrightarrow M)$ is such a family and if $\left(\mathcal{G}\left(G/H_i, \pi_i(S)\right)\right)_{i \in I}$ is expanding then ... something will happen. ~~Precisely~~ Very slightly more precisely, M_i will be very complicated for i large, in its / varied quantitative senses.

3 _ Examples

1) We can certainly begin with graphs.

Of course being expander is already rather special. But we can expect other characterizations. This is indeed true.

Th. (Dodziuk, Alon ①, Tanner, Alon-Milman ②)

For a finite k-regular graph Γ, let $\ell^2(\Gamma)$ denote the space of functions $f : V_\Gamma \to \mathbb{C}$ with inner product

$$\langle f, g \rangle = \sum f(x)\overline{g(x)}$$

and let

$$\Delta : \ell^2(V) \to \ell^2(V)$$
$$f \longmapsto \Delta f$$

where

$$\Delta f(x) = f(x) - \underbrace{\frac{1}{k}\sum_{y \sim x} f(y)}_{(Mf)(x)}$$

Then Δ is a self-adjoint operator on $L^2(\Gamma)$ with (real) spectrum $\subset [0,2]$

Let

$$0 = \lambda_0 \leq \lambda_1 \leq \dots$$

be the spectrum. Then we have

$$\frac{h^2(\Gamma)}{2k^2} \overset{①}{\leq} \lambda_1 \overset{②}{\leq} \frac{2}{k} h(\Gamma)$$

In particular, a family (Γ_i) with $|\Gamma_i| \to \infty$ is an expander iff

$$\exists\, \varepsilon > 0, \forall i, \lambda_1(\Gamma_i) \geq \varepsilon > 0$$

Sketch of proof(s)

(1) for $\leq$ [which is easier and "proves" existence of expanders]

Recall that from elementary spectral theory we have

$$\lambda_1 = \underset{\underbrace{0 \neq f \perp (\mathrm{cte})}_{L^2_0}}{\mathrm{Inf}} \frac{\langle \Delta f, f \rangle}{\langle f, f \rangle}$$

Take $X \subset V$ and

$$f = 1_X - \frac{|X|}{|V|} \in L^2_0$$

Then a simple computation gives

$$\langle \Delta f, f \rangle = \frac{1}{2k|V|} \sum_{x \sim y} (f(x) - f(y))^2$$

$$= \frac{1}{k|V|} \sum_{\substack{x \in X \\ y \notin X \\ x \sim y}} 1 = \frac{1}{k|V|} |\partial X|$$

whereas

$$\|f\|^2 = \|1_X\|^2 - \left(\frac{|X|}{|V|}\right)^2 = \frac{|X| \, (|V - X|)}{|V|^2}$$

so ~~for all~~ X

$$\lambda_1 \leq \frac{|\partial X|}{k|V|} \times \frac{|V|^2}{|X| \, |V - X|}$$

$$= \frac{|\partial X|}{k} \frac{|V|}{|X| \, |V - X|} \leq \frac{2}{k} \frac{|\partial X|}{|X|}$$

(but $|V - X| \geq \frac{|V|}{2}$)

so $\lambda_1 \leq \frac{2}{k} h(\Gamma)$

Sketch of proof (ii)

(ii) for $\geq$: this is trickier; we just give a hint of a well-motivated proof by L. Trevisan:

given $f : V \longrightarrow \mathbb{R}$

we "test" expansion using

$$X = X_{f,t} = \{ x \mid f(x) \leq t \}$$

and we choose t at random according to some prob. distribution.

Precisely working with t selected on

$$[\min f, \max f]$$

according to

$$d\nu = \frac{1}{S} | t - t_0 | \, dt$$

normalizing

median of $\{ f(x) \}$

turns out to lead to the desired bound

$\square$

(Other choices may be more efficient in some cases)

(2) Now let's go to <u>geometry</u>.

Historically this may be the first connection between expanders and coverings.

Recall that if M is a Riemannian manifold (wt boundary), it comes with

- a Laplace operator Δ
- a natural measure $d\mu$, from which "area" measures on submanifolds may be deduced

Δ is ≥ 0 and formally self-adjoint on $C_0^\infty(M)$ at least.

If $\mu(M) < +\infty$ we can define

$$L_0^2(M) = \{ f \in L^2(M) \mid \int f = 0 \}$$

and a "first eigenvalue"

$$\lambda_1 = \inf_{0 \neq f \in L_0^2(M)} \frac{\langle \Delta f, f \rangle}{\langle f, f \rangle}$$

<u>Ex</u> $\dim M = 2$, curvature $= -1$, compact / finite covol

so $M = \mathbb{H}/\Gamma$ — torsion-free, cocompact

$$\Rightarrow \left.\begin{aligned} \Delta &= -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \\ \mu &= \frac{dx\,dy}{y^2} \end{aligned}\right\} \text{ down to } M$$

Then λ_1 <u>is</u> indeed the first non-zero eigenvalue of Δ on M.

Th (Brooks–Burger)

M compact, S generating $\pi_1(M)$

$(M_i \xrightarrow{G_i} M)_i$ family of coverings as before, $|G_i| \to \infty$.

Then

$\liminf \lambda_1(M_i) > 0$

$\Updownarrow$

$h(M_i) > 0$

$\Updownarrow$

$\lambda_1(\mathcal{G}(G_i, S)) > 0 \Big\} \iff (\mathcal{G}(G_i, S))_i$ expand

$\Updownarrow$

$h(\;\_\_\_\_\;) \longrightarrow 0$

$$\| \qquad \inf_{\substack{S \subset M \\ \text{hypersurface} \\ \text{s.t. } M \setminus S = A \cup B, \\ \text{connected}}} \frac{\operatorname{area}(S)}{\min(\operatorname{vol}(A), \operatorname{vol}(B))}$$

None of these facts are easy. For us we view (i) $\iff$ (iii) or (iv) or (ii) $\iff$ (iii) or (iv) as instances of the "expander–covering" principle.

We (may) sketch the proof of (iii) $\Rightarrow$ (i) following Burger later.

(3) Gromov - Guth

(Knot distortion)

Consider some subset $n \geq 2$

$$K \subset \mathbb{R}^n$$

[Interest will be K = knot in $\mathbb{R}^3$]

and d_K the "intrinsic" distance on K:

$$d_K(x,y) = \inf_{\substack{\gamma \text{ path} \\ \text{on } K \\ \text{from } x \text{ to } y}} \ell(\gamma)$$

Now for a knot $K \subset \mathbb{R}^3$ let

$$\text{distorsion}(K) = \inf_{\substack{\text{isotopies} \\ K \sim K'}} \sup_{\substack{x \neq y \\ \text{in} \\ K'}} \frac{d_{K'}(x,y)}{\|x-y\|_{\text{eucl}}}$$

Q. (Gromov, ~ 1980)

Is distorsion unbounded over all knots?

Answer 1: (Pardon, 2011) Yes: torus knots $T_{p,q}$, distorsion $\geq$ (cte) $\min(p,q)$ $p,q \geq 2$

Answer 2: (Gromov - Guth) Yes, under some expanding assumptions

The argument goes as follows:

(1) According to Hilden and Montesinos (indep.) if M is a compact oriented connected 3-manifold ~~with~~ without boundary

there exists a map

$$M \xrightarrow{\alpha} S^3$$

s.t. (1) $\deg \alpha = 3$

(2) the "ramification locus" is a knot $K \subset S^3$

(3) $M - \alpha^{-1}(K) \longrightarrow S^3 - K$ is a covering of degree 3

Th. (G - G) M, K as above

$$\text{distorsion}(K) \geq c\, h(M)\, \mathrm{vol}(M)$$

This applies to K' isotopic to K:

$$\text{distorsion}(K') \geq c\, h(M)\, \mathrm{vol}(M)$$

(by considering $\varphi : S^3 \longrightarrow S^3$ diffeo s.t. $\varphi(K) = K'$ and $\varphi \circ \alpha : M \longrightarrow S^3$ ramified over K')

Cor. If $(M_i \xrightarrow{G_i} M)$ is a family of closed oriented 3-manif. coverings with $|G_i| \to \infty$, $\lambda_1(\mathcal{G}(G_i, S)) \geq \varepsilon > 0$, $\forall i$, then $\text{distorsion}(K_i) \longrightarrow \infty$.

(4) Heegaard ~~g~~ genus (Lackenby)

Let M be a compact oriented 3-mfld without boundary.

A basic fact (going back to the "prehistory of 3-manifold theory) is that M admits a Heegaard splitting:

$\exists\ g \geq 2$,

$\exists\ \phi : H_g \xrightarrow{\sim} H_g$ diffeo

H_g handlebody of genus g

s.t. $M \simeq H_g \cup_\phi H_g$

(this is "easy" if M is triangulated, which is however a big deal...)

However: the smallest $g = g(M)$, the Heegaard genus of M, is not easy to find (because the decomposition is not unique).

Nevertheless, this can be very useful and can allow you to prove "thms" about 3-mflds without knowing much about them or their fundamental grps

Th (Lackenby)

M 3-mfld compact [or complete hyperbolic with finite covolume]

$$h(M) \leq \frac{4\pi\,(2g(M)-2)}{\mathrm{Vol}(M)}$$

In particular: if $(M_i \to M)$ is a tower with $\underline{\lim}\, h(M_i) > 0$, then

$$g(M_i) \longrightarrow \infty$$

(in fact linearly wrt volume !)