Lecture 3

Ideas of proofs (and more applications)

1 - Two parallel statements

Recall the Gromov - Guth Theorem:

Th. M closed hyperbolic 3-manifold

$S^3 \xleftarrow[\deg 3]{\alpha_p} M_p \longrightarrow M$ coverings,

$K_p \subset S^3$ knot

s.t. $\mathrm{Gal}(M_p/M) = SL_2(\mathbb{F}_p)$

$S = S^{-1}$ generating $\pi_1(M)$

Because $\mathcal{G}(SL_2(\mathbb{F}_p), \text{image of } S)$ is expanding, $\text{distorsion}(K_p) \longrightarrow \infty$.

Here is another statement which has partial analogies:

Th (Ellenberg - Hall - K.)

U Riemann surface (hyperbolic, say) compact, or $U = X - \{\text{finitely many pts}\}$, X compact

$U_p \longrightarrow U$ family of Galois coverings with $\mathrm{Gal}(U_p/U) = SL_2(\mathbb{F}_p)$

$U_p \xrightarrow{\deg} \mathbb{P}^1_{\mathbb{C}} = S^2$

$S = S^{-1}$ finite generating set of $\pi_1(U)$

Because $\mathcal{G}(SL_2(\mathbb{F}_p), \text{image of } S)$ is expanding we have $\deg \beta_p \xrightarrow[p \to \infty]{} \infty$

(Cor. $\forall d$, $\exists\, p_0(d)$, $\forall p \geq p_0(d)$,
$$\Big| U_p\Big(\bigcup_{[K:\mathbb{Q}] \leq d} K \Big) \Big| < +\infty \Big)$$

It is rather interesting to see these beginning in very similar ways.

Step 1

$$\deg \beta_p \geq \frac{1}{4\pi} A_{conf}(U_p)$$

$$A_{conf}(U_p) = \inf_{\phi : U_p \to S^2 \text{ conformal}} \sup_{g} \int_{U_p} |D(g\circ\phi)|$$

(Li-Yau) see 2bis

$$A_{conf}(U_p) \geq \frac{1}{2} \mathrm{Vol}(U_p)\, \lambda_1(U_p)$$

(Li-Yau)

$$\lambda_1(U_p) \geq c\, \lambda_1(\text{—})$$

Step 1

$$\mathrm{distortion}(K_p) \geq \frac{1}{4}\, l_{conf}(K_p)$$

c (Pardon)

$$l_{conf}(K_p) = \sup_{\substack{x \in \mathbb{R}^3 \\ r > 0}} \frac{1}{r}\, \mathrm{length}[K \cap B(x,r)]$$

$$l_{conf}(K_p) \geq c\, \mathrm{Vol}(M_p)\, h(M_p)$$

(G-G)

$$h(M_p) \geq c\, h(\mathcal{G}(SL_2(\mathbb{F}_p), S_i))$$

Brooks, Burger

and we conclude

N.B. In both cases, a much weaker statement than expansion suffices, since $\mathrm{Vol}(\;)$ grows at least linearly with p

2bis

$$A_{conf}(M) = \inf_{\phi : M \to S^2} \sup_{\substack{g : S^2 \xrightarrow{\sim} S^2 \\ \text{conf. diffeo}}} \int_M |\nabla(g \circ \phi)|^2 \, d\mu$$

This is of some potential relevance because:

(1) ~~thi~~ a sufficient weaker expansion follows already from Helfgott's growth theorem or its generalizations, which are substantially easier to prove than the full Salehi-Golsefidy – Varjú Theorem

(2) For <u>other</u> graphs, we <u>might</u> have this weaker statement, but <u>not</u> expansion (eg. for S_n, $n \to \infty$, by work of Helfgott – Seress).

How to prove (say) the Burger inequality

We say some words only about $\boxed{\lambda_1^R \geq c\,\lambda_1^{comb}}$

Form the graph where:

- vertices are $\{g\mathcal{F}\}$
- edges join $g_1\mathcal{F}$ to $g_2\mathcal{F}$ $\iff \overline{g_1\mathcal{F}} \cap \overline{g_2\mathcal{F}} \neq \emptyset$

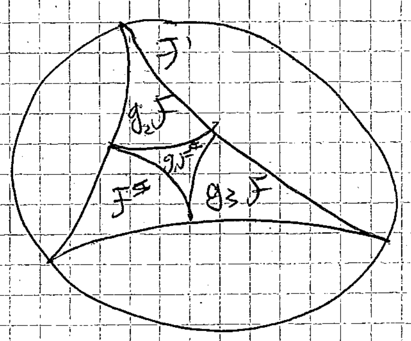
This is $\mathcal{G}(G, T)$

$U'_{\mathcal{F}'}$

G

U

for some generating set T, possibly different from S.



$\mathcal{F}'$ = fund domain for U'

$\mathcal{F}$ = ___ domain for U

But because T (as was S) is fixed, to say that $(\mathcal{C}(G_p, S))$ expands is the same as saying that $(\mathcal{C}(G_p, T))$ expands.

Recall:

$$\lambda_1(\mathcal{C}(G_p, S)) = \inf_{f \perp 1} \frac{\langle \Delta f, f \rangle}{\langle f, f \rangle}$$

$$f : G_p \longrightarrow \mathbb{C}$$

$$\lambda_1^{Rie}(U') = \inf_{\varphi \perp 1} \frac{\langle \Delta^R \varphi, \varphi \rangle}{\langle \varphi, \varphi \rangle}$$

Let ψ, $\|\psi\| = 1$, $\psi \perp 1 \equiv 1$ $\qquad \varphi : U' \longrightarrow \mathbb{C}$ smooth

$\Delta \psi = \lambda_1^R \psi$

Define the linear operator

$$\Phi : C^\infty(U') \longrightarrow \ell^2(G_p)$$

$$\varphi \longmapsto f$$

where

$$f(g) = \int_{gF} \varphi(x)\, d\mu(x)$$

Key steps: take $f = \alpha + \beta\psi$, $\alpha, \beta \in \mathbb{R}$

so $\|\nabla f\|^2 \leq \lambda_1 \|f\|^2$

(1) looking at neighbors of gF, and using Neumann eigenv., one finds $c_1 > 0$ s.t.

$$\|\nabla f\|^2 \geq c_1 \left\{ \|f\|^2 - \frac{1}{r^2 \mu(U)} \sum_{x \in U} \Big(\sum_{\substack{x' \sim x \\ x' = x}} \phi(f)(x') \Big)^2 \right\}$$

$\Rightarrow$

$$\lambda_1^R \geq c_1 \frac{\langle B\Phi(f), \Phi(f) \rangle}{\langle \Phi f, \Phi f \rangle}, \quad B = \frac{\Delta}{r^2}(2r - \Delta)$$

4bis

(2) $\exists c_2 > 0$, if $\lambda_1^R < c_2$, then

$\phi : \langle 1, \Psi \rangle \longrightarrow L^2_\theta(U)$

is surj.

$\Rightarrow$ find $\alpha + \beta\Psi \in L^2_0(U)$

$\Rightarrow \dfrac{\langle B\Phi(f), \Phi(f) \rangle}{\langle \Phi f, \Phi f \rangle} \geq \lambda_1 \left(B_{\text{comb}} \right)$

$\geq \frac{1}{n} \lambda_1$

~~then~~

2 - Something about geometry

The following result of Bazdin and Kolmogorov is the first application of expanders. It is however neither about groups nor manifolds...

Th (B-K) $\Gamma = (V, E)$ a finite graph.

Consider thick-BK-embeddings of Γ in $\mathbb{R}^3$:

- vertices are mapped to disjoint balls of radius 1
- edges are ______ smooth curves joining the centers of the balls in such a way that the 1-neighbors of ~~the~~ curves associated to disjoint edges only intersect in the balls associated to common ~~ver~~ extremities

(1) $\exists C$ s.t. $\exists$ such an embedding s.t. the union of all balls and 1-neighb of the edge-curve are contained in a ball of radius

$$\leq C\sqrt{|V|}$$

(2) $\exists c'$ s.t. for any Γ, any thick embedding has radius

$$\geq \frac{1}{2}\sqrt{h(\Gamma)}\,\sqrt{|V|}$$

Proof (1) can be done algorithmically (idea: put all vertices on $\{z=0\}$ in a grid-like configuration, join by loops...)

(2) Let $h \in \mathbb{R}$ be s.t.

$$V^{\pm} = \{\text{centers of balls ass to vertices have } x_3 \underset{\leq}{\geq} h\}$$

satisfy $|V^+| \geq \frac{|V|}{2}$, $|V^-| \geq \frac{|V|}{2}$.

Apply the definition of $h(\Gamma)$ to

$$V \supset X = \{x \text{ s.t. center of ball has } x_3 < h\} \cup (\text{enough with } x_3 = h)$$

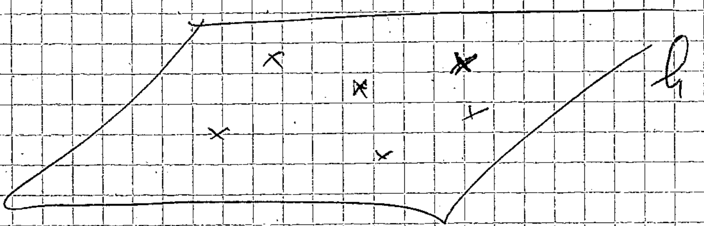
to have

$$|X| \geq \frac{|V|}{4};$$

there are ~~[illegible]~~ $|\partial X| \geq h(\Gamma)\,|X| \geq h(\Gamma)\frac{|V|}{4}$

edges joining a vertex in X to its complement.

Each curve associated to such an edge crosses $\{x_3 = h\}$. ~~Associate~~ ~~Let~~ ~~The~~ ~~The~~



The corresponding pts $\{p_e\}$ are 1-isolated and live in a disc of radius $\leq R$.

So $|\{p_e\}| \leq$ ~~[illegible]~~ R^2.

So we find

$$R \geq \sqrt{|\partial X|} \geq \sqrt{h(\Gamma)}\,\frac{\sqrt{|V|}}{2}$$

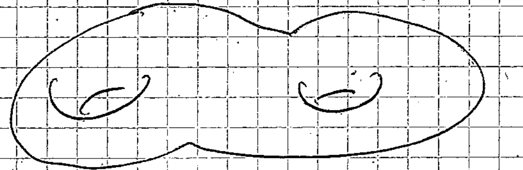
$\square$

3 – Something about torsion in H_1 of 3-manifolds

A very classical construction of (compact/closed) 3-manifolds goes as follows: [Heegaard splitting]

take $g \geq 1$

. consider a handlebody H_g



and its boundary

$$\partial H_g = \Sigma_g$$

. consider $\phi : \Sigma_g \to \Sigma_g$ diffeo.

. constructs $M_\phi = H_g \cup_\phi H_g$.

This depends only on ϕ in the mapping class group

$$\Gamma_g = \{\text{diff. } \phi, +\} / \{\text{diff. isotopic to } 1\}$$

which is a finitely generated discrete group.

There is a map

$$\Gamma_g \longrightarrow Sp_{2g}(\mathbb{Z}) \text{ "=" } Sp\left(H_1(\Sigma_g, \mathbb{Z}), \cdot\right)$$

which is a surjective homomorphism.

Dunfield-Thurston used this to define certain models of random 3-manifolds:

. Fix g, $S = S^{-1}$ s.t. $\langle S \rangle = \Gamma_g$

. Consider a random walk $\gamma_n = s_1 \cdots s_n$ (s_i in S, independent uniform)

. Form $M_n = M_{\gamma_n}$.

This is an interesting construction, although the really nice aspects are related to $g \to \infty$:

<u>Warning</u>: $\{M_n\}$ for fixed g is <u>not</u> expanding, in the sense that $\lambda_1(M_n) \longrightarrow 0$ with high prob

[<u>Lackenby</u>:

$$g \geq h(M_n)\,\mathrm{Vol}(M_n)$$

and Maher has shown that $\mathbb{E}(\mathrm{Vol}(M_n))$ grows]

Nevertheless, there are some nice things happening.

<u>Prop.</u> Consider the ~~...~~ random variables

$$X_n = \begin{cases} 0 & \text{if } H_1(M_n, \mathbb{Q}) \neq 0 \\ \dfrac{\omega(|H_1(M_n, \mathbb{Z})|) - \log n}{\sqrt{\log n}}, & \text{otherwise} \end{cases}$$

where $\omega(n)$ = nb. of primes dividing n.

Then $X_n \xRightarrow{\text{weakly}}$ (standard normal variable).

<u>Sketch</u>:

(1) It is a fact that

$$H_1(M_\phi) \simeq H_1(\Sigma_g) / \langle J, \phi J \rangle \qquad (H_1(\Sigma_g) \simeq \mathbb{Z}^{2g})$$

where $J \simeq H_1(\mathcal{H}_g)$.

and $H_1(M_\phi, \mathbb{F}_p) = \mathbb{F}_p^{2g} / \langle J, \phi J \rangle$

(2) $$p \mid |H_1(M_n, \mathbb{Z})|$$

$$\Updownarrow$$

$$\dim_{\mathbb{F}_p} H_1(M_n, \mathbb{F}_p) \geq \underline{1}$$

~~Write~~ First check that

$$\mathbb{P}(H_1(M_n, \mathbb{Q}) \neq 0) \xrightarrow[\text{exp. fast}]{} 0$$

(because generically

$$\langle J, \phi_n J \rangle = \mathbb{Z}^{2g}\text{)}$$

so we do as if this did not happen.

(3)

$$\omega(|H_1(M_n, \mathbb{Z})|)$$

$$= \sum_{p \mid |H_1(M_n)|} 1$$

$$= \sum_{p} \mathbb{1}_{p \mid |H_1(M_n)|}$$

$$= \sum_{p \leq A^n} (\quad) + \underline{(\text{bd number})}$$

(4) What is

$$\mathbb{P}(p \mid |H_1(M_n)|) \; ?$$

By <u>expansion</u> in $Sp_{2g}(\mathbb{Z})$ (i.e. Rep(T

this is <u>very</u> close to

$$\frac{1}{p} \sim \frac{|\{g \in Sp_{2g}(\mathbb{F}_p) \mid \langle J, gJ \rangle = \mathbb{F}_p^{2g}\}|}{|Sp_{2g}(\mathbb{F}_p)|}$$

Now ~~number theory~~ says that

$$\sum_{p \leq A^n} \frac{1}{p} \sim \log\log A^n$$
$$\underset{n\to\infty}{\sim} \log n$$

Group theory says that

$$\{ p \mid |H_1(M_n)| \} \text{ and}$$
$$\{ q \mid |H_1(M_n)| \}$$

are almost independent if $p \neq q$.

So $\omega(\underline{\quad})$ "looks" a lot like

$$\sum_{p \leq A^n} B_p$$

where B_p is a r.v that is either 1 or 0, and $= 1$ with proba. $\sim 1/p$, and the B_p are independent.

One knows that such a sum satisfies a CLT. From there it is not so hard to deduce it for $\omega(|H_1(M_n)|)$ ...

$\square$