# Conformal dimension and Gromov-Hausdorff convergence

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$(X, d)$ metric space. Its conformal dimension is:

$$\mathrm{ConfDim}(X) = \inf\{\mathrm{HD}(X, d') \, : \, d' \text{ is AR and } d' \sim_{\mathrm{qs}} d\};$$

where $d' \sim_{\mathrm{qs}} d$ if $\exists$ an homeomorphism $f\colon (X, d) \to (X, d')$ s.t.

$$\frac{d'(f(x), f(y))}{d'(f(x), f(z))} \leq \eta\left(\frac{d(x, y)}{d(x, z)}\right) \qquad \forall x, y, z \in X,$$

where $\eta\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is an increasing function.

If you don't put conditions on $d'$ $\Rightarrow$ the infimum gives the *topological dimension* of $X$: a topological invariant.

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## Theorem (unpublished)

Suppose $X_n \underset{\mathrm{GH}}{\to} X_\infty$, with $X_n$ all $(L_0, \rho_0)$-quasi-selfsimilar. Then

$$\mathrm{ConfDim}(X_\infty) \geq \limsup_{n \to +\infty} \mathrm{ConfDim}(X_n).$$

$(X, d)$ is $(L_0, \rho_0)$-quasi-selfsimilar if $\forall x \in X, 0 < \rho \leq \rho_0$ $\exists \Phi \colon B(x, \rho) \to X$ s.t.

- $B(\Phi(x), \frac{\rho_0}{L_0}) \subseteq \Phi(B(x, \rho)) \subseteq B(\Phi(x), L_0 \cdot \rho_0)$;
- $\Phi \colon (B(x, \rho), \frac{\rho_0}{\rho} \cdot d) \to X$ is $L_0$-biLipschitz onto its image.

Examples of quasi-selfsimilar spaces: $\mathbb{R}^n$, manifolds, piecewise-flat simplicial complexes, self-similar fractals, boundaries of Gromov-hyperbolic groups, etc.

Adding natural geometric conditions once gets controlled quasi-selfsimilarity constants.

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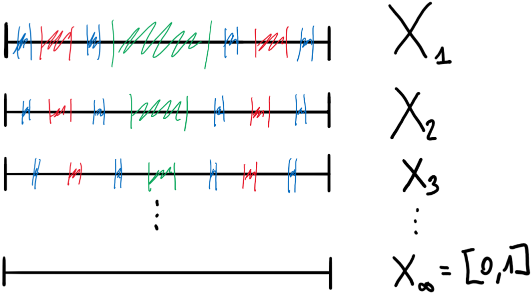
Example:

- $X_n$ is the Cantor set constructed dividing $[0,1]$ into $(2n+1)$-pieces and removing the central one;

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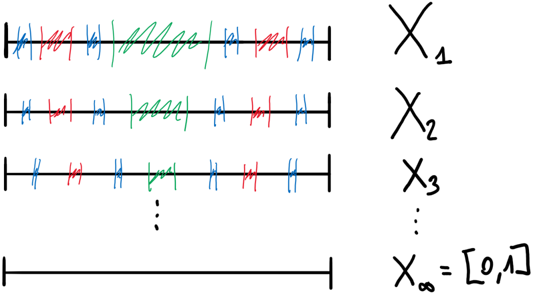


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Another example:

- $X_n$ is the Sierpinski carpet constructed dividing $[0,1] \times [0,1]$ into $(2n+1)^2$-squares and removing the central one;
- $X_n \underset{GH}{\longrightarrow} [0,1] \times [0,1]$;
- What happens to the conformal dimensions? *I don't know.*
- It is not even known the value of $\mathrm{ConfDim}(X_1)$. Recent result (Kwapisz, '19):

$$1.7652 \leq \mathrm{ConfDim}(X_1) \leq 1.8068$$

and $\mathrm{ConfDim}(X_1) \approx 1.7965$ by numerical evidence.

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