Cusps of Hyperbolic 4-Manifolds and Rational Homology Spheres

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September 3, 2021

Colourings of polytopes

Definition

Let $\mathbb{X} \in \{\mathbb{S}, \mathbb{R}, \mathbb{H}\}$ and $\mathcal{P} \subset \mathbb{X}^n$ be a compact, right-angled polytope with set of facets $\mathcal{F}$ and Coxeter group $\Gamma = \Gamma(\mathcal{P})$. A $\mathbb{Z}_2^k$-*colouring* is a map $\lambda : \mathcal{F} \to \mathbb{Z}_2^k$, where $\lambda(F)$ is called the *colour* of F.

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A colouring is said to be *proper* if, at each face $f = F_{i_1} \cap ... \cap F_{i_s}$, the set $\{\lambda(F_{i_1}), ..., \lambda(F_{i_s})\}$ consists of linearly independent vectors.

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Any coloring induces a homomorphism $\lambda : \Gamma \to \mathbb{Z}_2^k$, by sending $r_F \in \Gamma$ to $\lambda(F)$. Then $\ker\lambda \triangleleft \Gamma$ determines a regular orbifold covering $M_\lambda = \mathbb{X}^n/_{\ker\lambda}$ of $\mathcal{P} = \mathbb{X}^n/_\Gamma$.

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Proposition (Davis-Januszkiewicz '91)

If λ is proper, M_λ is a closed Riemannian manifold. Moreover, if $\mathcal{P} \subset \mathbb{H}^n$ (resp. $\mathbb{R}^n$ or $\mathbb{S}^n$) then M_λ is hyperbolic (resp. flat, spherical).

Properties

Definition

Two colourings $\lambda_1, \lambda_2 : \mathcal{F} \to \mathbb{Z}_2^k$ are said to be *equivalent* if $\lambda_2 = \phi \circ \lambda_1 \circ s$, where $\phi : \mathbb{Z}_2^k \tilde{\to} \mathbb{Z}_2^k$ and $s \in \mathrm{Sym}(\mathcal{P}) \curvearrowright \mathcal{F}$.

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Orientable manifold covers of right-angled poltytopes are mirrorable.

The 3-cube

Theorem (F-Kolpakov-Slavich '20)

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The resulting manifolds are mirrorable, orientable, closed flat 3-manifolds, and thus can only be 3 diffeomorphism classes: $S^1 \times S^1 \times S^1$, $K \tilde{\times} S^1$ *and the Hantzsche-Wendt manifold.*

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Rank	3-torus	$K \tilde{\times} S^1$	HW
3	1	1	0
4	3	2	1
5	3	1	0
6	1	0	0

Ideal case and cusp sections

Proposition (Kolpakov-Slavich '16)

Let $\mathcal{P} \subset \mathbb{H}^n$ be a finite-volume polytope with some ideal vertices and λ a proper colouring of $\mathcal{P}$. Then M_λ is a finite-volume, cusped, hyperbolic manifold.

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Proposition (Kolpakov-Slavich '16)

Let λ be a rank k colouring of $\mathcal{P} \subset \mathbb{H}^n$ with ideal vertices. Then the cusp section of a cusp associated to a given ideal vertex v is diffeomorphic to the colouring induced in its vertex figure $\operatorname{lk} v$. Moreover, the number of connected components of $\pi^{-1}(\operatorname{lk} v)$ is equal to 2^{k-s}, where $\pi : M_\lambda \to \mathcal{P}$ is the covering map and s is the rank of the colouring induced on $\operatorname{lk} v$.

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Corollary

For any λ orientable colouring of $\mathcal{P} \subset \mathbb{H}^4$, M_λ has zero signature.

Proposition (F-Kolpakov-Slavich '20)

Let $\mathcal{C}$ be a 3-cube labelled with consecutive numbers on opposed faces and

$$T_1 = \{e_1 + e_2, e_3 + e_4, e_5 + e_6\} \subset \mathbb{Z}_2^6.$$

Let also λ be an orientable $\mathbb{Z}_2^k$-colouring of $\mathcal{C}$ with matrix Λ. The following cases are possible:

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Theorem (F-Kolpakov-Slavich '20)

There exists a finite-volume, cusped, orientable, hyperbolic 4-manifold such that all cusp sections are rational homology spheres.