New examples in the bounded cohomology of finitely generated groups

Ventotene V – Lightning talk

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Definition

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- The *bounded simplicial cochain complex* is:

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Bounded cohomology of groups

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- The *bounded cohomology* $\mathrm{H}_b^\bullet(\Gamma)$ is the cohomology of the subcomplex of Γ-invariants.

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Bad news: Very hard to compute.

Computing bounded cohomology

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Acylindrically hyperbolic groups have *large bounded cohomology* in degrees 2 and 3: that is $\dim_{\mathbb{R}} \mathrm{H}_b^n(\Gamma) = |\mathbb{R}|$ for $n = 2, 3$ (Brooks, Soma, Hull–Osin, Frigerio–Pozzetti–Sisto).

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It is unknown whether $\mathrm{H}_b^n(F) = 0$ for $n \geq 4$, where F is a non-abelian free group...

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Are there non-boundedly acyclic groups for which it is possible to compute $\mathrm{H}_b^n(\Gamma)$ for every $n \geq 1$?

YES: First (countable) examples: infinite direct sums of free groups and 3-manifold groups (Löh).

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YES: First (countable) examples: infinite direct sums of free groups and 3-manifold groups (Löh). These examples have *large bounded cohomology* that is $\dim_{\mathbb{R}} \mathrm{H}_b^n(\Gamma) = |\mathbb{R}|$ for all $n \geq 2$.

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Are there *finitely generated* non-boundedly acyclic groups for which it is possible to compute $\mathrm{H}^n_b(\Gamma)$ for every $n \geq 1$?

The answer two both questions is **YES**:

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- *There exists a finitely presented non-amenable boundedly acyclic group.*

Theorem (FF–Löh–Moraschini)

There exist continuum many 10*-generated groups with large bounded cohomology.*

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Find an example of a finitely presented group with large bounded cohomology.

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Question 2

Find an example of a finitely presented group with large bounded cohomology.

Our example relies on the existence of a group isomorphic to its own direct square (Meier). It is a long-standing open problem whether finitely presented groups with this property exist.

Thank you for the attention!