Diffuse groups

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Motivation

The unit conjecture (Higman 1940, Kaplansky 1970)

Let K be a field and G a torsion-free group. Then the group ring $K[G]$ has no non-trivial units: if $\alpha, \beta \in K[G]$ such that $\alpha\beta = \beta\alpha = 1$, then $\alpha = kg$ for some $0 \neq k \in K, g \in G$.

The unit conjecture is false in general (G. 2021) but holds for *diffuse* groups, and in fact there are no known examples of non-diffuse groups for which it holds.

Delzant, 1997

If Γ acts by isometries on a δ-hyperbolic space X such that each non-trivial $\gamma \in \Gamma$ has translation length greater than 4δ, then Γ is diffuse.

For $\mathbb{H}^n$ this bound is $4 \log 2 = 2.772\ldots$

Diffuse groups

Definition (Bowditch)

Call $a \in A \subset \Gamma$ an *extremal point* of A if there is no $1 \neq \gamma \in \Gamma$ such that both $\gamma a \in A$ and $\gamma^{-1} a \in A$. The group Γ is *diffuse* if every finite subset has at least one extremal point.

Bowditch, 2000

If $\Gamma \curvearrowright \mathbb{H}^n$ with minimal translation length $> 2\log(1+\sqrt{2}) = 1.762...$ then it is diffuse.

Can we improve this bound for closed hyperbolic 3-manifolds? The Weeks manifold (systole 0.584...) is *not* diffuse (Kionke–Raimbault).

A non-diffuse hyperbolic 3-manifold with largeish systole

G., 2021

The closed hyperbolic 3-manifold `t12041(-1, 2)` has non-diffuse fundamental group. Its systole 1.265... was computed by Dunfield.