Volume of random geodesic complements

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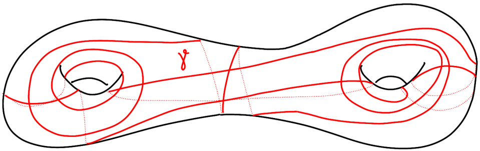
joint work with

Tommaso Cremaschi Dídac Martínez-Granado Franco Vargas Pallete

Complements of geodesics as hyperbolic three-manifolds

Let X be a closed hyperbolic surface and $\gamma \subseteq X$ a primitive filling geodesic:



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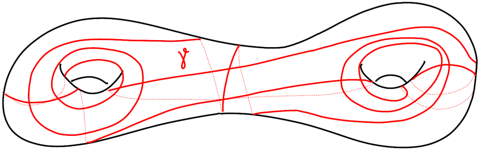

Then its complement in the projective tangent bundle

$$M_\gamma := \mathrm{PT}(X) \setminus \dot{\gamma}$$

is a finite-volume hyperbolic three-manifold (Foulon–Hasselblatt '13).

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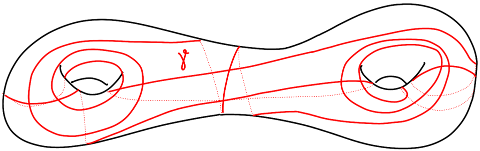

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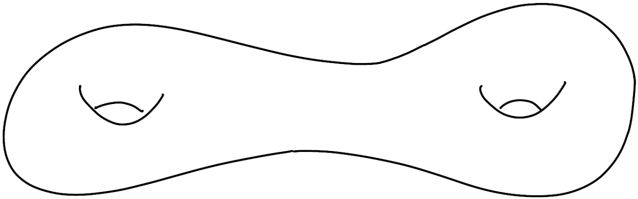
Question

What is the hyperbolic volume $\mathrm{vol}(M_\gamma)$ *for a* **generic** *geodesic* γ*?*

Random model for primitive filling geodesics



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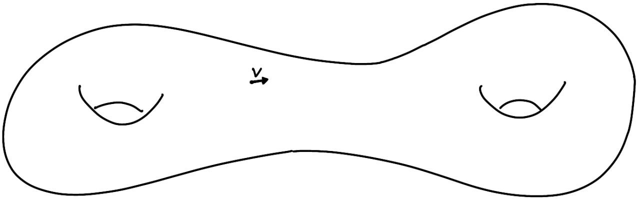

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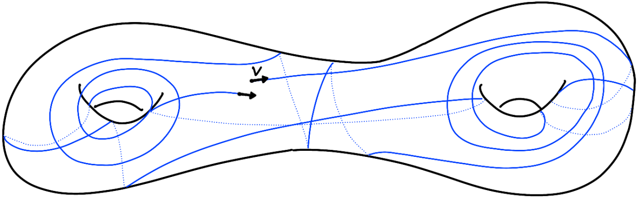

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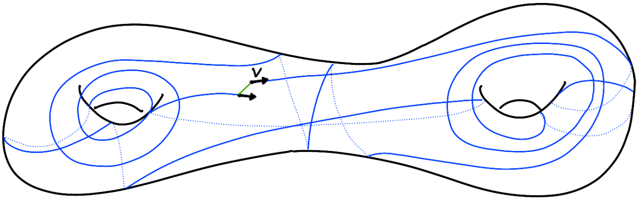

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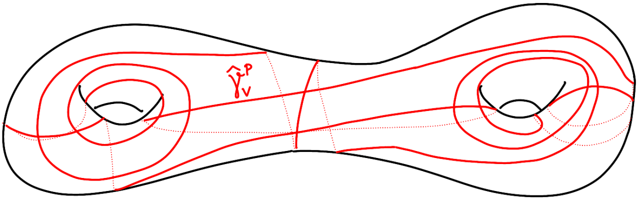

1. Sample a unit tangent vector $v \in \mathrm{UT}(X)$ uniformly at random;
2. Flow for long enough time $t = t(v)$ in the direction of v;
3. Close-up by a short arc;
4. Pull-tight to obtain a filling closed geodesic $\widehat{\gamma}_v \subseteq X$ and consider its primitive subcurve $\widehat{\gamma}_v^p$.

Result

Suppose that X is glued from isometric pairs of pants whose limit sets have Hausdorff dimension $0 < \delta < 1$.

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Theorem (Cremaschi–K–Martínez-Granado–Vargas Pallete '21)

Let $\eta > 1$. There are positive constants A, B, C, such that almost every primitive filling geodesic $\gamma \subseteq X$ satisfies

$$A \cdot \left(\frac{C \cdot \ell_X(\gamma)}{W\left(C \cdot \ell_X(\gamma)\right)} \right)^{\delta/2\eta} - B \leq \mathrm{vol}(M_\gamma),$$

where W denotes the Lambert W function.

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The **counting problem** in the proof

- There is a certain partition $\{B_i\}_{i\in\mathbb{N}}$ of $\mathrm{UT}(X)$, such that

$$C_t(v) \lesssim \mathrm{vol}(M_{\widehat{\gamma}_v^p}),$$

where

$$C_t(v) := \#\{i \in \mathbb{N} \mid \exists k \in \{0, \dots, \lfloor t \rfloor\} : \underbrace{g_k(v)}_{\text{geodesic flow}} \in B_i\},$$

that is the **number of different partition sets** that the geodesic arc starting from v visits up to time t (at integral times).

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- We estimate $C_t(v)$ by applying an **exponential multiple mixing** result for the geodesic flow.

Thank you!

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