Equidistribution of integer points and subconvexity of L-functions

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Sum of three squares

Let $q \geq 1$ be an integer. Then $q = x^2 + y^2 + z^2, (x, y, z) \in \mathbb{Z}^3$ (Legendre 1798; Gauss 1801) iff q is *not* of the form $4^k(8\ell - 1)$.
Denote $\mathcal{R}_q := \{(x, y, z) \in \mathbb{Z}^3 : x^2 + y^2 + z^2 = q\}$, the set of representations.
Gauss showed that

$$\#\mathcal{R}_q^* = \begin{cases} 12h(-q) & \text{if } q \equiv 1, 2 \bmod 4 \\ 8h(-q) & \text{if } q \equiv 3 \bmod 8, \end{cases}$$

where $h(-q)$ is the class number of $\mathbb{Q}(\sqrt{-q})$.
Gauss plus Dirichlet's class number formula $h(-q) = \frac{w}{2\pi} L(1, \chi_{-q}) q^{1/2}$ implies

$$\#\mathcal{R}_q^* = \frac{12}{\pi} L(1, \chi_{-q}) q^{1/2} \asymp q^{1/2+o(1)}$$

by appealing to Siegel's theorem: $L(1, \chi_{-q}) \asymp q^{o(1)}$.

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Equidistribution of representations

Deeper question: distribution of representations.
Recall $\mathcal{R}_q = \{(x,y,z) \in \mathbb{Z}^3 : x^2 + y^2 + z^2 = q\}$. Then

$$\frac{\mathcal{R}_q}{\sqrt{q}} = \{(\frac{x}{\sqrt{q}}, \frac{y}{\sqrt{q}}, \frac{z}{\sqrt{q}}) : (x,y,z) \in \mathcal{R}_q\} \subset \mathbb{S}^2 = \{x_1^2 + x_2^2 + x_3^2 = 1\}.$$

Question: How does $\vec{v} \in \frac{\mathcal{R}_q}{\sqrt{q}}$ distribute on $\mathbb{S}^2$ as $q \to \infty$?

Conjecture (Equidistribution of lattice points on the 2-sphere)

Let q be such that $q \not\equiv 0, 4, 7 \mod 8$. Let

$$\mu_q := \frac{1}{\#\mathcal{R}_q} \sum_{\vec{v} \in \frac{\mathcal{R}_q}{\sqrt{q}}} \delta_{\vec{v}}.$$

Then μ_q weak $\star$ converges to $\mu_{\mathbb{S}^2}$, as $q \to \infty$. Equivalently, for "nice" $\Omega \subset \mathbb{S}^2$, $\mu_q(\Omega) \to \mu_{\mathbb{S}^2}(\Omega)$.

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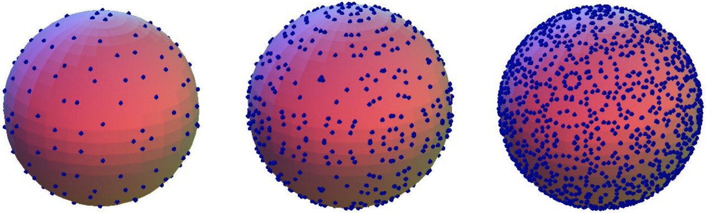
Integer points on $\mathbb{S}^2$

Examples of Equidistribution of integer points on the 2-sphere:



Integer points on spheres of radii $101, 8011, 104851$, respectively

Image credit: Ellenberg, Michel, and Venkatesh.

Equidistribution of integer points on $\mathbb{S}^2$

Linnik (1950's-60's): Equidistribution holds if q satisfies an extra congruence condition.

Theorem (Duke (1988); Golubeva–Fomenko (1990))

Let $q \to \infty$ be such that $q \not\equiv 0, 4, 7 \bmod 8$. Then

$$\frac{1}{\#\mathcal{R}_q} \sum_{\vec{v} \in \frac{\mathcal{R}_q}{\sqrt{q}}} \varphi(\vec{v}) \to \int_{\mathbb{S}^2} \varphi(y) \mathrm{d}\mu_{\mathbb{S}^2},$$

for every $\varphi \in C(\mathbb{S}^2)$.

One such proof can be reduced to a nontrivial bound of L-functions.

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From equidistribution to subconvexity

Idea of proof:

Assume $\int_{\mathbb{S}^2} \varphi(y) \mathrm{d}\mu_{\mathbb{S}^2} = 0$. Want to show

$$\frac{1}{\#\mathcal{R}_q} \sum_{\vec{v} \in \frac{\mathcal{R}_q}{\sqrt{q}}} \varphi(\vec{v}) \to \int_{\mathbb{S}^2} \varphi(y) \mathrm{d}\mu_{\mathbb{S}^2} = 0.$$

Since $\#\mathcal{R}_q \asymp q^{1/2+o(1)}$, by approximating $\varphi \in C(\mathbb{S}^2)$ by spherical harmonics, suffices to show

$$W_\varphi(q) := \sum_{\vec{v} \in \frac{\mathcal{R}_q}{\sqrt{q}}} \varphi(\vec{v}) \ll_\varphi q^{1/2-\eta},$$

for φ homogenegous polynomial of degree ν. By Waldspurger's formula,

$$|W_\varphi(q)|^2 \rightsquigarrow L(1/2, f.\chi_{-q})$$

for f a (fixed) holomorphic modular form of weight $2+2\nu$ and quadratic character χ_{-q}. The required bound follows from

$$L(1/2, f.\chi_{-q}) \ll_f q^{1/2-\eta'}.$$

A *new* proof of subconvexity

Let χ mod q be Dirichlet characters and $L(s, f.\chi) = \sum_{n\geq 1} \frac{\lambda_f(n)\chi(n)}{n^s}$ be the twisted L-functions.

Theorem (Aggarwal–Holowinsky–L.–Sun, 2020)

Let f be fixed GL_2 automorphic forms. Then

$$L(1/2, f.\chi) \ll_f (q^2)^{1/4-\delta+\varepsilon}, \text{ for } \delta = 1/16.$$

- The saving $\delta = 1/16$ represents the **Burgess**-type subconvex bounds for L-functions:

$$L(1/2, F) \ll Q(F)^{1/4-1/16+\varepsilon};$$

  such are proving ground for new methods: Bykovskiĭ, Blomer–Harcos, Munshi, etc.

- Best known exponent $\delta = 1/12$ (**Weyl**-type): Conrey–Iwaniec (2000, χ quadratic) and Petrow–Young (2020, general χ).

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References:

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