Currents as curve functionals

Counting problems - Ventotene 2021

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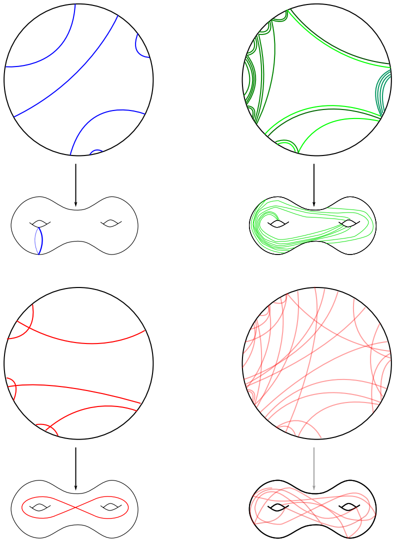
Monday September 6, 2021

joint work with Dylan Thurston

What is a geodesic current?

simple ⟶ laminations

curves ⟶ currents



1) measures on space of geodesics

2) measures on unit tangent bundle

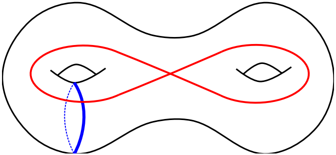
invariant under flow

Intersection number

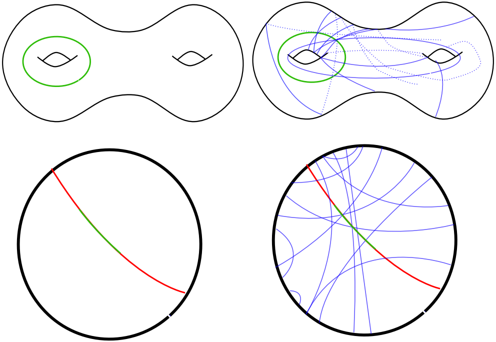
(Bonahon, 86) There exist a continuous, bilinear function

$$i\colon \text{currents} \times \text{currents} \to \mathbb{R}$$

so that for two curves C, D, with associated currents μ_C, μ_D, $i(\mu_C, \mu_D)$ is the geometric intersection number between C and D.



Curve functionals as intersection numbers

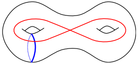


$$\ell_\Sigma(C) = i(\mathcal{L}_\Sigma, C)$$

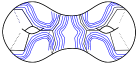
Curve functional: hyperbolic length ℓ_Σ
Geodesic current: Liouville current $\mathcal{L}_\Sigma$

Curve functionals that are intersection numbers

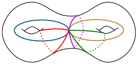
- Negatively curved lengths: (Bonahon, 86), (Otal, 90)



- Flat lengths: (Hersonsky-Paulin, 97), (Duchin-Leininger-Rafi, 10), (Constantine, 18), (Erlandsson-Leininger-Sadanand, 21)



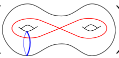
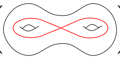
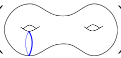
- Word length for simple generating sets: (Erlandsson, 16)



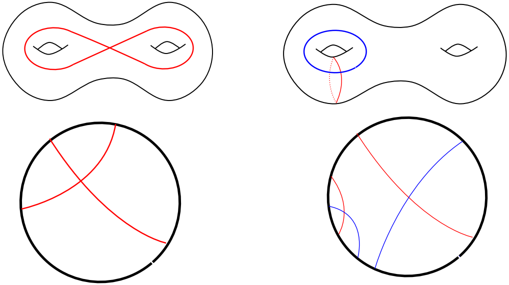
- Some Anosov “lengths”: (Martone-Zhang, 16),

Curve functionals that aren't intersection numbers

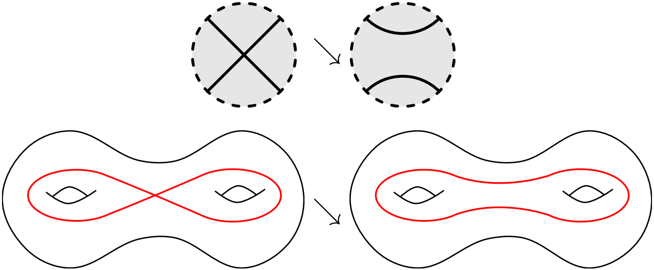
- Word length for any generating set, or any Riemannian metric (Erlandsson-Parlier-Souto, 16)
- Some Anosov "lengths": (Dreyer, 11), (Bridgeman-Canary-Labourie-Sambarino, 17)
- Extremal length: (MG-Thurston, 20)

$\sqrt{EL_\Sigma}\Bigg($  $\Bigg) < \sqrt{EL_\Sigma}\Bigg($  $\Bigg) + \sqrt{EL_\Sigma}\Bigg($  $\Bigg)$

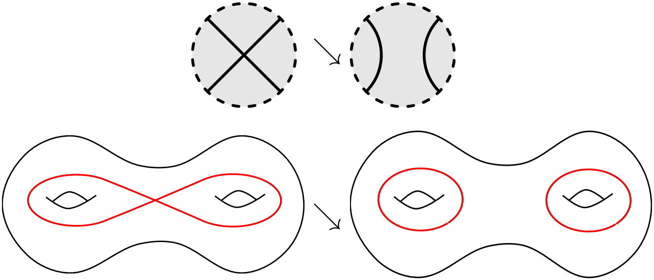
Definition of essential crossing



Definition of smoothing



Definition of smoothing



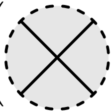
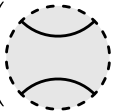
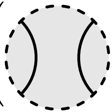
Continuous extension to geodesic currents

Theorem (MG-Thurston, 20)

Let f : curves $\to \mathbb{R}$ be a curve functional.
Suppose that

$$f(C_1 \cup C_2) \leq f(C_1) + f(C_2)$$

and, for $K > 0$,

$f\Big($  $\Big) \geq \max\Big\{ f\Big($  $\Big), f\Big($  $\Big) \Big\} - K.$

Then f extends continuously to currents.

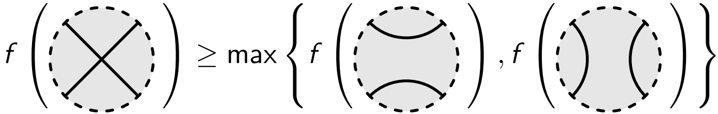
Geodesic currents as curve functionals

Theorem (MG-Thurston, in progress)

A curve functional f : curves $\to \mathbb{R}$ satisfies

$$f(C_1 \cup C_2) = f(C_1) + f(C_2)$$

and



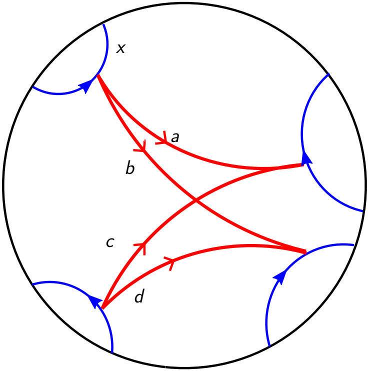
if and only if

$$f = i(\mu_f, \cdot)$$

where μ_f is a geodesic current.

Proof

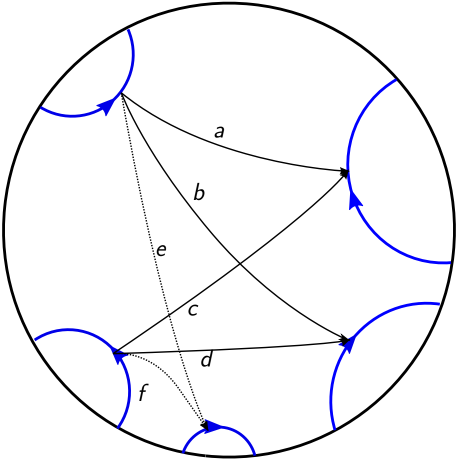
We define a pre-measure on a family of boxes of geodesics.



$$\mu_f(B_{a,b,c,d}) := \lim_{n\to\infty} f([bx^n]) + f([cx^n]) - f([ax^n]) - f([dx^n])$$

Proof

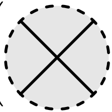
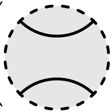
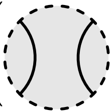
We prove that the family of boxes forms a semi-ring in the sense of measure theory, and extend to a measure by Caratheodory extension theorem.



Measured laminations as curve functionals

Theorem (MG-Thurston, in progress)

A curve functional f : curves $\to \mathbb{R}$ satisfies

$f\Big($  $\Big) = \max \Big\{ f\Big($  $\Big), f\Big($  $\Big) \Big\}$

if and only if

$$f = i(\lambda_f, \cdot)$$

where λ_f is a measured lamination.

Thanks! Questions?