Amenable category and simplicial volume

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Simplicial volume

Let M be an oriented closed connected n-manifold.

Given $c = \sum_{j=0}^{k} a_j \cdot \sigma_j \in C_n(M; \mathbb{R})$, we define the **$\ell^1$-norm** of c as

$$\|c\|_1 := \sum_{j=0}^{k} |a_j| \ .$$

The simplicial volume of M is

$$\|M\| = \inf\{\|c\|_1 \mid [c] = [M]\} \in \mathbb{R}_{\geq 0} \ ,$$

where $[M] \in H_n(M; \mathbb{R})$ denotes the fundamental class of M.

Gromov, Thurston '78 : If M is an oriented connected closed hyperbolic n-manifold, then

$$\|M\| = \frac{\mathrm{Vol}(M)}{v_n} \ .$$

Multiplicativity w.r.t. finite coverings : If $f\colon N \to M$ is a finite covering of degree d between o.c.c. manifolds of the same dimension, then

$$\|N\| = d \cdot \|M\| \ .$$

Vanishing results :

- $\|S^1\| = 0$ and $\|T^n\| = 0$;
- **Gromov '82 :** If $\pi_1(M)$ is amenable (e.g. finite, Abelian, solvable groups, ...), then $\|M\| = 0$;
- All flat manifolds have zero simplicial volume.

Simplicial volume vs. Euler characteristic

Gromov's question $\sim$'90 : If M is an o.c.c. *aspherical* n-manifold, then do we have

$$\|M\| = 0 \quad \Longrightarrow \quad \chi(M) = 0 \text{ ?}$$

Some known examples :

- S^1 and T^n;
- **Gromov $\sim$'90 :** If $\pi_1(M)$ is amenable, then $\|M\| = \chi(M) = 0$;
- **Using multiplicativity w.r.t. finite coverings :** Every flat manifold M has both $\|M\| = \chi(M) = 0$.

Gromov's question via open covers

An open subset $U \subset M$ is called **amenable** if for every $x \in U$

$$\mathrm{im}\big(\pi_1(U, x) \hookrightarrow \pi_1(M, x)\big)$$

is amenable.

The **amenable category** of M is

$$\mathrm{cat}_{\mathrm{Am}}(M) := \min \left\{ n \in \mathbb{N} \,\middle|\, M = \bigcup_{i=1}^{n} U_i \text{ s. t. each } U_i \text{ is amenable} \right\}.$$

Vanishing theorems and counting problems

Gromov '82, Ivanov '87, Frigerio-M. '19, Sauer-Löh '20, Raptis '21 : If M is an o.c.c. manifold with $\mathrm{cat}_{\mathrm{Am}}(M) \leq \dim(M)$, then $\|M\| = 0$.

Sauer '09 : If M is an o.c.c. *aspherical* manifold with $\mathrm{cat}_{\mathrm{Am}}(M) \leq \dim(M)$, then $\chi(M) = 0$.

Question : If M is an o.c.c. *aspherical* n-manifold, then do we have

$$\|M\| = 0 \Rightarrow \mathrm{cat}_{\mathrm{Am}}(M) \leq \dim(M) \ ?$$

Gómez-González-Heil '14 : The question is affirmative for all o.c.c. 3-manifolds.

Vanishing results of fibrations

Let M be an o.c.c. fiber bundle $p\colon M \to B$ with fiber F.

Gromov '82, Lück '02 : If $\pi_1(F)$ is amenable, then $\|M\| = 0$.

Löh-M '21 : If $\mathrm{cat}_{\mathrm{Am}}(F) \leq \frac{\dim(M)}{\dim(B)+1}$, then

$$\|M\| = 0.$$

In particular, when M is aspherical, then M satisfies Gromov's question.

Example : If M is a hyperbolic n-manifold which fibers over S^1, then $2 \cdot \mathrm{cat}_{\mathrm{Am}}(F) > \dim(F) + 1$.


"That's all Folks!"