Counting hyperbolic manifolds

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Counting hyperbolic n-manifolds wrt volume bound

Let $I^n(v)$ be the number of hyperbolic n-mfds of volume $\leq v$, and $C^n(v)$ the number of these mfds up to commensurability.

Theorem (Wang's finitess, 1972)

If $n \geq 4$, then $I^n(v)$ is finite for all v.

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Given $f, g\colon \mathbb{R} \to \mathbb{R}_{\geq 0}$ non-decreasing, write $f(x) \sim g(x)$ *for* $x \gg 0$ if $\exists\, x_0 \in \mathbb{R}$ and $C > 1$ s.t.

$$g(x/C) \leq f(x) \leq g(Cx), \quad \forall x \geq x_0.$$

Theorem (Super-exponential growth)

If $n \geq 4$, then:

- $I^n(v) \sim v^v$ *for* $v \gg 0$ (Burger – Gelander – Lubotzky – Mozes 2002),
- $C^n(v) \sim v^v$ *for* $v \gg 0$ (Gelander – Levit 2014).

Geometric boundaries

A hyp mfd M^n *bounds geometrically* if $\exists$ a hyp mfd W^{n+1} with totally geodesic boundary $\partial W \cong M$. (hyp mfd = complete & finite-volume)

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Geometric boundaries are "sparse"...
E.g., for $M^3 = \partial W^4$ closed oriented, $\eta(M^3) = -\sigma(W^4) \in \mathbb{Z}$ (**Long–Reid '00**), but $\{\eta(M) \mid M^3 \text{ clsd, or, hyp}\}/\frac{1}{3}\mathbb{Z}$ is dense in $S^1 = \mathbb{R}/\frac{1}{3}\mathbb{Z}$ (**Meyerhoff '82**).

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... but still a lot:

Theorem (Kolpakov–R–Slavich)

If $n \geq 4$, the number of commensurability classes represented by a geometrically bounding hyp n-mfd of volume $\leq v$ is $\sim v^v$ for $v \gg 0$.

Main ingredients: Counting argument (**Gelander–Levit '14**), separability (**Bergeron–Haglund–Wise '11**) and embedding (**Kolpakov–Reid–Slavich '18**) results for "standard" arithmetic hyp mfds.

Signature

Let M^4 be an oriented hyperbolic 4-mfd.

If M closed then $\sigma(M) = 0$ by the Hirzebruch signature formula.
If M cusped, then "$\sigma = 0$ virtually": $\exists$ finite cover $M' \to M$ s.t. $\sigma(M'') = 0$ $\forall$ finite cover $M'' \to M'$ (**Long–Reid '00**).

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A lot of 4-mfds of any given signature:

Theorem (Kolpakov–R–Tschantz)

$\forall m \in \mathbb{Z}$, the number of commensurability classes represented by a cusped oriented hyperbolic 4-manifold of signature m and volume $\leq v$ is $\sim v^v$ for $v \gg 0$.

Main ingredients: Ideal right-angled 24-cell. Formula $\sigma(M) = -\sum_i \eta(C_i)$, where C_i is a cusp section of the i^{th} cusp (**Long–Reid '00**). Counting argument (**Gelander–Levit '14**) via some extra tricks.