Counting limit theorems for representations of Gromov-hyperbolic groups

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Counting problems, Ventotene 2021

joint work(s) with Stephen Cantrell (Italo Cipriano and Rhiannon Dougall)

What is in this talk?

1. Our problem(s): motivation and setting

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2. Results, objects and proof ideas

From random matrix products 1

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Furstenberg–Kesten theorem (1960) proves this generalized law of large numbers by showing that under first moment condition there exists $\lambda \in \mathbb{R}$ (called the top Lyapunov exponent) such that almost surely

$$\frac{1}{n} \log \|L_n\| \to_{n\to\infty} \lambda.$$

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More refined limit theorems stayed elusive for about 20 years, after which substantial results were obtained by Le Page, Guivarc'h–Raugi, Goldsheid–Margulis, Bougerol and several others. For example, we have

Theorem (Le Page '82: exponential decay of large deviation probabilities)

Let μ be a finitely supported probability measure on $\mathrm{SL}_d(\mathbb{R})$ such that the group generated by its support is large (e.g. Zariski dense). Then, for every $\varepsilon > 0$, we have*

$$\limsup_n \frac{1}{n} \log \mathbb{P}(|\frac{1}{n} \log \|L_n\| - \lambda| > \varepsilon) < 0.$$

Theorem (Le Page '82, Benoist–Quint '16 Central Limit Theorem)

Under the same assumptions, we have

$$\frac{\log \|L_n\| - n\lambda}{\sqrt{n}} \xrightarrow[n\to\infty]{\text{in law}} \mathcal{N}(0,\sigma^2),$$

where $\sigma^2 > 0$.

Focusing on Fustenberg–Kesten and Le Page's results: free semigroup case

To transition to counting limit theorems, notice that the support of the probability measure we saw above generates a free-semigroup $\Gamma < \mathrm{SL}_2(\mathbb{R})$ and therefore the distribution of L_n is nothing but the uniform distribution on words in $\{a, b\}$ of length n.

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In other words, looking from the perspective of the abstract group/semigroup Γ:
• we look at the sphere of radius n in the Cayley graph,
• consider the uniform measure (=normalized counting measure), and
• study its push-forward by the injection $\Gamma \hookrightarrow \mathrm{SL}_2(\mathbb{R})$ composed with operator norm.

Our problem: counting asymptotics

In view of the previous description in a very special case, **the question** arises as to whether given a general finitely generated countable subgroup $\Gamma < \mathsf{SL}_d(\mathbb{R})$ endowed with a generating set S: *do the uniform spherical averages in the Cayley graph exhibit an asymptotic behaviour similar to sums/products of random variables* (i.e. as the convolutions μ^{*n})?

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As we saw, both problems coincide when the algebraic structure of Γ is the most degenerate possible= free semigroup. However, in general, it seems non-trivial to transfer anything from one perspective to another (except in some special cases).

Main results: I-P or Zariski-dense case

Theorem (Cantrell–S '21)

Let Γ *be a non-elementary Gromov-hyperbolic group endowed with a finite symmetric generating set* S. *Let* $\rho : \Gamma \to \mathrm{SL}_d(\mathbb{R})$ *be a Zariski-dense representation. Then,*

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1. *(Convergence in expectation) There exists a constant $\lambda > 0$ such that*

$$\frac{1}{n} \sum_{g \in S_n} \frac{1}{\#S_n} \log \|\rho(g)\| \to_{n\to\infty} \lambda$$

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2. *(Exponential decay) For every* $\varepsilon > 0$*, we have*

$$\frac{1}{\# S_n} \#\{g \in S_n \mid |\frac{1}{n} \log \|\rho(g)\| - \lambda| > \varepsilon\}$$

is exponentially small.

I-P case: Theorem continued

Theorem (Cantrell-S' '21 (continued))

3. (CLT) For every $a \in \mathbb{R}$,

$$\frac{1}{\#S_n}\#\{g \in S_n \mid \frac{\log\|\rho(g)\| - n\lambda}{\sqrt{n}} \leqslant a\} \to_{n\to\infty} \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{a} e^{\frac{-t^2}{2\sigma^2}}\,dt,$$

for some $\sigma^2 > 0$.

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Remark

In joint work with Cantrell–Cipriano–Dougall, under the additional assumption of Anosov/dominated representation, we obtain more limit theorems with more quantitative versions and also for the spectral radius

Thank you

Thanks for your attention!