Gaps between prime divisors

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Theorem <arXiv:2106.00298>

Fix $z \geq 0$ and *arbitrary* $k \geq 0$. As $x \to \infty$ we have

$$\frac{1}{x}\sum_{1\leq n\leq x}\left(\omega_z(n) - \frac{\omega(n)}{\mathrm{e}^z}\right)^k \sim \gamma\cdot(\log\log x)^{k/2},$$

where γ is an explicit constant $\gamma = \gamma(k, z)$.

Proof

- For primes p define independent Bernoulli r.v.'s

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- To justify the model I used Brun's sieve (to detect coprimality to "large" moduli.)

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The constant γ in the main theorem is

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$1 - 2z\mathrm{e}^{-z}$ a little surprising.
Independence holds only for very small or very large gaps.